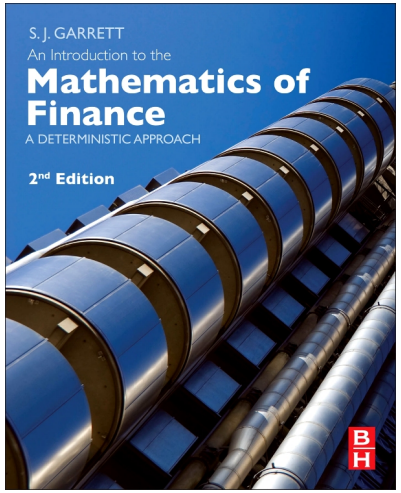
Essential Topic: Redington Immunization

Chapter 9



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Summary

Motivation

- Actuaries are typically concerned with assets held to cover future liabilities.
- For example, a pension scheme holds assets to cover the future pension payments when they become due.
- It is therefore useful to consider the *surplus* which is simply defined as

$$\text{value of assets} - \text{value of liabilities}$$

Surplus

- The surplus at a particular time can be defined in terms of the *net present values* of future cash flows arising from the assets and liabilities, NPV_A and NPV_L.
- In the simple case that the NPVs depend on the constituent cash flows discounted at a <u>constant</u> force of interest δ, we can define the surplus to be

$$S(\delta) = NPV_A(\delta) - NPV_L(\delta)$$

- We could equally have defined this in terms of a constant interest rate, i, but using δ will make the mathematics more straightforward to illustrate.
- Recall that for fixed interest rate quantities $i = e^{\delta} - 1$.
- Of course the surplus is a function of more than just δ, but here we focus on changes in the prevailing interest rate only.

Immunization

- We begin by assuming that the surplus is zero at the prevailing δ_0 and are concerned with the behaviour of the surplus for *small* changes in δ, i.e. $\delta_0 \to \delta_0 + \epsilon$.
- In particular, we require that the surplus remains non-negative to all small changes in δ_0, positive or negative.
- Mathematically, we require

$$\begin{aligned} S(\delta_0) =& 0 \\ S(\delta_0 + \epsilon) \geq& 0 \end{aligned}$$

- When this is true, the surplus is said to be *immunized* against changes in δ.

Redington immunization

- Redington derived three conditions for a surplus to be immunized against *small* changes in δ, as follows.
- We consider the Taylor expansion about δ_0 and consider each term

$$S(\delta_0+\epsilon) = \underbrace{S(\delta_0)}_{\text{defined}=0} + \underbrace{\epsilon}_{\pm} \times \underbrace{\left.\frac{\mathrm{d}S}{\mathrm{d}\delta}\right|_{\delta_0}}_{\text{fix }=0} + \underbrace{\frac{\epsilon^2}{2!}}_{>0} \times \underbrace{\left.\frac{\mathrm{d}^2S}{\mathrm{d}\delta^2}\right|_{\delta_0}}_{\text{fix}>0} + \underbrace{O\left(\delta^3\right)}_{\approx 0}$$

- As ϵ could be positive or negative, we arrive at the above conditions for the derivatives of $S(\delta)$.
- These are Redington's conditions for immunization of a surplus against small changes in δ:

$$S(\delta_0) = 0 \qquad S'(\delta_0) = 0 \qquad S''(\delta_0) > 0$$

Redington immunization

- Redington's three conditions can be stated in terms of the net present values of the assets and liabilities:

$$NPV_A(\delta_0) = NPV_L(\delta_0)$$
$$NPV'_A(\delta_0) = NPV'_L(\delta_0)$$
$$NPV''_A(\delta_0) \geq NPV''_L(\delta_0)$$

- In words:
  1. the NPV of the assets equals the NPV of the liabilities
  2. the *volatility* of the assets equals the volatility of the liabilities
  3. the *convexity* of the assets exceeds the convexity of the liabilities

  at δ_0.
- Although the conditions are stated in terms of derivatives with respect to δ, they can be recast in terms of derivatives with respect to i and ν, for example.

Volatility

- The *volatility* of a cash flow gives the proportionate change in the net present value to changes in δ.
- It is defined mathematically as

$$T(\delta_0) = -\frac{NPV'(\delta_0)}{NPV(\delta_0)} = \left.\frac{\sum tc_t e^{-\delta t} + \int_0^\infty t\rho(t)e^{-\delta t}\mathrm{d}t}{\sum c_t e^{-\delta t} + \int_0^\infty \rho(t)e^{-\delta t}\mathrm{d}t}\right|_{\delta_0}$$

- The right-hand expression shows that the volatility is also the *weighted average term* of the cash flow.
- The volatility is therefore also known as the *discounted mean term*, *Macauley duration* or simply *duration*.
- T is a measure of the sensitivity of the NPV to changes in δ. The sensitivity to changes in i can be derived as

$$-\frac{1}{NPV}\frac{\mathrm{d}NPV}{\mathrm{d}i}\bigg|_{i_0} = \left(-\frac{1}{NPV}\frac{\mathrm{d}NPV}{\mathrm{d}\delta} \times \frac{\mathrm{d}\delta}{\mathrm{d}i}\right)\bigg|_{i_0} = \frac{T(i_0)}{1+i_0}$$

Convexity

- The *convexity* of a cash flow is given by

$$\frac{NPV''(\delta_0)}{NPV(\delta_0)}$$

- The convexity gives the sensitivity of the volatility (or duration) of the cash flow to changes in δ.
- It can be recast in terms of the sensitivity to changes in i.
- Redington's conditions can be thought of in terms of sensitivities
  1. the NPVs of the assets and liabilities are balanced
  2. the two NPVs have equal sensitivity to changes in δ
  3. the volatility of the assets is more sensitive to changes in δ than that of the liabilities.

  at δ_0.

Example

An investor has liabilities of £1m and £2m due at times $t = 4$ and 5, respectively. He currently holds assets X and Y. If each unit of X produces an income of £1m at $t = 4$ and each unit of Y produces £1m at $t = 8$, construct a portfolio consisting of the liabilities and amounts of X and Y that is immune to small changes in interest rate at $i = 5\%$ per annum.

Answer

We require holdings x and y such that Redington's conditions hold at $i = 5\%$. We choose to work in terms of δ and the conditions are stated as

1. $1e^{-4\delta} + 2e^{-5\delta} = xe^{-4\delta} + ye^{-8\delta}$
2. $-4e^{-4\delta} - 10e^{-5\delta} = -4xe^{-4\delta} - 8ye^{-8\delta}$
3. $16e^{-4\delta} + 50e^{-5\delta} < 16xe^{-4\delta} + 64ye^{-8\delta}$

Example

Conditions 1. and 2. form simultaneous equations that determine x and y. This are stated as

1. $2.3898 = 0.8227x + 0.6768y$
2. $11.1261 = 3.2908x + 5.4147y$

and are solved with $x = 2.4287$ and $y = 0.5787$. We can easily check that these values satisfy condition 3.

3. $52.3395 < 58.0388$

A portfolio consisting of the liabilities and £2.4287m of asset X and £0.5787m of asset Y has zero NPV at $i = 5\%$ and is immunized against small changes in i and δ.

Example

Calculate the value of the portfolio formed above at

a.) $i = 5\%$ per annum

b.) $i = 6\%$ per annum

c.) $i = 4\%$ per annum

Answer

a.) At $i = 5\%$, the value is
$0.8227 \times 2.4287 + 0.6768 \times 0.5787 - 2.3898 = £0m$

b.) $i = 6\%$, the value is
$0.7921 \times 2.4287 + 0.6274 \times 0.5787 - 2.2866 = £0.0003m$

c.) $i = 4\%$, the value is
$0.8548 \times 2.4287 + 0.7307 \times 0.5787 - 2.4987 = £0.0003m$

We see that the value is zero at $i = 5\%$ (as intended) and has a positive value following changes in i in either direction.

Summary

- Assets are typically held to back liabilities. We define the surplus at a particular i to be

  NPV of the future asset cash flows − NPV of future liabilities

- A surplus is at risk to changes in the market conditions that affect the assets and liabilities in a different way.
- A zero surplus at i_0 is said to be immunized to changes in i if the surplus remains non-negative following a change in i.
- Redington's conditions for a surplus to the immunized to small changes in i are
  1. the NPV of future cash flows are balanced
  2. the volatility (duration) of the asset and liability cash flows are balanced
  3. the asset cash flows have great convexity than the liability cash flows

  at i_0.