Essential Topic: Continuous cash flows
Chapters 2 and 3

The Mathematics of Finance: A Deterministic Approach
by S. J. Garrett
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SUMMARY
CONTINUOUS PAYMENT STREAMS

- Continuously paid cash flows are an important theoretical construction. Although they do not appear in practice, a frequently paid discrete cash flow can be approximated by a continuous payment stream over the long term.

- For example, consider a pension paid weekly for an extended time period, say, 20 years. The natural time unit here is the week and we should use nominal rates $i^{(52)}(t)$ per annum in the present-value calculation involving $20 \times 52 = 1040$ payments.

- Recalling that

$$\delta(t) = \lim_{\delta \to \infty} i^{(p)}(t)$$

we see that $\delta \approx i^{(52)}$ and the importance of the force of interest in the approximation of very frequently paid cash flows is clear.
CONTINUOUS PAYMENTSTREAMS

- We define the *rate of payment* of a continuously paid cash flow stream, \( \rho(t) \), such that

\[
\rho(t) = \frac{dM(t)}{dt} \quad \text{for all } t
\]

where \( M(t) \) is the total payment between time 0 and \( t \).

- Between times \( t \) and \( t + dt \) with \( dt \to 0 \), the total payment is therefore such that

\[
\lim_{dt \to 0} \{M(t + dt) - M(t)\} = \rho(t)dt.
\]
PRESENT VALUE AND ACCUMULATION

- We consider the payment stream as a collection of payment elements, \( \rho(t)dt \), at each time \( t \) within the cash flow.
- In its general form, each element has present value

\[
\nu^t \rho(t)dt = \rho(t) \exp \left[ - \int_0^t \delta(s)ds \right] dt
\]

- We then integrate (i.e. sum) these to give the total present value of the stream

\[
\int_0^\infty \nu^t \rho(t)dt = \int_0^\infty \rho(t) \exp \left[ - \int_0^t \delta(s)ds \right] dt
\]

- The equivalent expression for the accumulation of the stream to time \( t = n \) is

\[
\int_0^n \rho(t) \exp \left[ \int_t^n \delta(s)ds \right] dt
\]
EXAMPLE

If the force of interest at time $t$ is given by

$$\delta(t) = \begin{cases} 
0.02 & \text{for } 0 \leq t < 5 \\
0.02 \times (t - 5) & \text{for } t \geq 5
\end{cases}$$

calculate

a.) the value at time 0 of £1000 due at time $t = 10$,
b.) the accumulated value at time $t = 20$ of a payment stream of rate $\rho(t) = £1.5t$ paid continuously between $t = 8$ and $t = 10$. 
**EXAMPLE**

a.) In this case we have a single payment and should evaluate the present value by breaking the period into two subintervals

\[
\frac{1000}{A(0, 10)} = \frac{1000}{A(0, 5) \times A(5, 10)}
\]

We have

\[
A(0, 5) = \exp(0.02 \times 5) = e^{0.10}
\]

\[
A(5, 10) = \exp \left( \int_{5}^{10} 0.02 \times (t - 5) \, dt \right) = e^{0.25}
\]

Therefore

\[
\frac{1000}{A(0, 10)} = \frac{1000}{e^{0.10+0.25}} = £704.69
\]
**Example**

b.) Here we need to consider the payment stream, \( \rho(t) = 1.5t \). Similar arguments to those above can be used to construct the accumulation as

\[
\int_8^{10} \rho(t) \exp \left( \int_t^{10} \delta(s) \, ds \right) \, dt
\]

\[
= \int_8^{10} 1.5t \exp \left( \int_t^{10} 0.02(s - 5) \, ds \right) \, dt = £29.39
\]

Note that here the piecewise nature of \( \delta(t) \) does not matter as the integrals are constrained within \( 8 \leq t \leq 10 \) where \( \delta(t) = 0.02 \times (t - 5) \).
CONTINUOUSLY PAID ANNUITIES

- Consider the particular case that $\delta(t) = \delta$ and the payment stream is $\rho(t) = 1$ for $0 \leq t \leq n$.
- This payment stream represents a continuously paid, unit $n$-year annuity.
- In order to construct an expression for the present value of the annuity we follow the procedure above

$$\int_0^n 1 \times \exp \left( - \int_0^t \delta ds \right) dt = \frac{1 - e^{-\delta n}}{\delta}$$

- It is usual to denote the present value of this annuity as

$$\bar{a}_n = \frac{1 - \nu^n}{\delta}$$
ACCUMULATED VALUE

- The accumulated value at time $n$ of the continuously paid annuity, $\bar{s}_{\overline{n}}$, can be calculated in a number of ways.
- One approach is to construct the accumulation of the payment stream as above

$$\bar{s}_{\overline{n}} = \int_0^n 1 \times \exp \left( \int_t^n \delta ds \right) dt = \frac{(1 + i)^n - 1}{\delta}$$

- Alternatively, one can simply accumulate the present value at time $t = 0$ to time $t = n$

$$\bar{s}_{\overline{n}} = (1 + i)^n \times \bar{a}_{\overline{n}} = \frac{(1 + i)^n - 1}{\delta}$$
**Example**

If the effective rate of interest is \( i = 4\% \) per annum, calculate the following quantities.

a.) \( 3\bar{a}_{10} \)

b.) \( 10\bar{s}_5 \)

**Answers**

a.) This is the present value of a continuous annuity paying 3 per annum for 10 years,

\[
3\bar{a}_{10} = 3 \times \frac{1 - \nu^{10}}{\delta} = 3 \times \frac{1 - (1.04)^{-10}}{\ln(1.04)} = 24.816
\]

b.) This is the accumulated value at \( t = 5 \) of a continuous annuity paying 10 per annum for 5 years,

\[
10\bar{s}_5 = 10 \times \frac{(1 + i)^5 - 1}{\delta} = 10 \times \frac{(1.04)^5 - 1}{\ln(1.04)} = 55.239
\]
THE PRESENT VALUE AND ACCUMULATION OF A CONTINUOUSLY PAID CASH FLOW CAN BE CALCULATED BY CONSTRUCTING INTEGRAL EXPRESSIONS IN TERMS OF $\rho(t)$ AND $\delta(t)$.

A CONTINUOUSLY PAID UNIT $n$-YEAR ANNUITY IS THE PARTICULAR CASE THAT $\rho(t) = 1$ FOR $0 \leq t \leq n$ YEARS. IF $\delta(t) = \delta$, WE DENOTE THE PRESENT VALUE OF THIS STREAM AS

$$\bar{a}_n = \frac{1 - \nu^n}{\delta}$$

AND THE ACCUMULATED VALUE AT TIME $t = n$ AS

$$\bar{s}_n = \frac{(1 + i)^n - 1}{\delta}$$