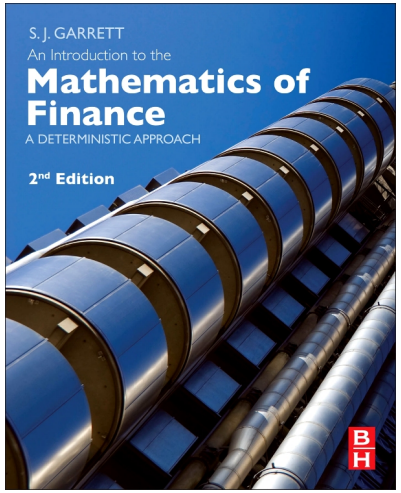
Essential Topic: Continuous cash flows

Chapters 2 and 3



Continuous Payment streams

- Continuously paid cash flows are an important theoretical construction. Although they do not appear in practice, a frequently paid discrete cash flow can be approximated by a continuous payment stream over the long term.
- For example, consider a pension paid weekly for an extended time period, say, 20 years. The natural time unit here is the week and we should use nominal rates $i^{(52)}(t)$ per annum in the present-value calculation involving $20 \times 52 = 1040$ payments.
- Recalling that
$$\delta(t) = \lim_{\delta \to \infty} i^{(p)}(t)$$
we see that $\delta \approx i^{(52)}$ and the importance of the force of interest in the approximation of very frequently paid cash flows is clear.

Continuous payment streams

- We define the *rate of payment* of a continuously paid cash flow stream, $\rho(t)$, such that

$$\rho(t) = \frac{\mathrm{d}M(t)}{\mathrm{d}t} \text{ for all } t$$

  where $M(t)$ is the total payment between time 0 and t.
- Between times t and $t + \mathrm{d}t$ with $\mathrm{d}t \to 0$, the total payment is therefore such that

$$\lim_{\mathrm{d}t \to 0} \{M(t + \mathrm{d}t) - M(t)\} = \rho(t)\mathrm{d}t.$$

Present value and accumulation

- We consider the payment stream as a collection of payment elements, $\rho(t)\mathrm{d}t$, at each time t within the cash flow.
- In its general form, each element has present value

$$\nu^t \rho(t)\mathrm{d}t = \rho(t)\exp\left[-\int_0^t \delta(s)\mathrm{d}s\right]\mathrm{d}t$$

- We then integrate (i.e. sum) these to give the total present value of the stream

$$\int_0^\infty \nu^t \rho(t)\mathrm{d}t = \int_0^\infty \rho(t)\exp\left[-\int_0^t \delta(s)\mathrm{d}s\right]\mathrm{d}t$$

- The equivalent expression for the accumulation of the stream to time $t = n$ is

$$\int_0^n \rho(t)\exp\left[\int_t^n \delta(s)\mathrm{d}s\right]\mathrm{d}t$$

Example

If the force of interest at time t is given by

$$\delta(t) = \begin{cases} 0.02 & \text{for } 0 \leq t < 5 \\ 0.02 \times (t-5) & \text{for } t \geq 5 \end{cases}$$

calculate

a.) the value at time 0 of £1000 due at time $t = 10$,

b.) the accumulated value at time $t = 20$ of a payment stream of rate $\rho(t) = £1.5t$ paid continuously between $t = 8$ and $t = 10$.

Example

a.) In this case we have a single payment and should evaluate the present value by breaking the period into two subintervals

$$\frac{1000}{A(0,10)} = \frac{1000}{A(0,5) \times A(5,10)}$$

We have

$$A(0,5) = \exp\left(0.02 \times 5\right) = e^{0.10}$$
$$A(5,10) = \exp\left(\int_5^{10} 0.02 \times (t-5)\mathrm{d}t\right) = e^{0.25}$$

Therefore

$$\frac{1000}{A(0,10)} = \frac{1000}{e^{0.10+0.25}} = £704.69$$

Example

b.) Here we need to consider the payment stream, $\rho(t) = 1.5t$. Similar arguments to those above can be used to construct the accumulation as

$$\int_8^{10} \rho(t) \exp\left(\int_t^{10} \delta(s)\mathrm{d}s\right) \mathrm{d}t$$
$$= \int_8^{10} 1.5t \exp\left(\int_t^{10} 0.02(s-5)\mathrm{d}s\right) \mathrm{d}t = £29.39$$

Note that here the piecewise nature of $\delta(t)$ does not matter as the integrals are constrained within $8 \leq t \leq 10$ where $\delta(t) = 0.02 \times (t-5)$.

Continuously paid annuities

- Consider the particular case that $\delta(t) = \delta$ and the payment stream is $\rho(t) = 1$ for $0 \leq t \leq n$.
- This payment stream represents a *continuously paid, unit n-year annuity*.
- In order to construct an expression for the present value of the annuity we follow the procedure above

$$\int_0^n 1 \times \exp\left(-\int_0^t \delta \mathrm{d}s\right) \mathrm{d}t = \frac{1 - e^{-\delta n}}{\delta}$$

- It is usual to denote the present value of this annuity as

$$\bar{a}_{\overline{n|}} = \frac{1 - \nu^n}{\delta}$$

Accumulated value

- The accumulated value at time n of the continuously paid annuity, $\bar{s}_{\overline{n|}}$, can be calculated in a number of ways.
- One approach is to construct the accumulation of the payment stream as above

$$\bar{s}_{\overline{n|}} = \int_0^n 1 \times \exp\left(\int_t^n \delta \mathrm{d}s\right) \mathrm{d}t = \frac{(1+i)^n - 1}{\delta}$$

- Alternatively, one can simply accumulate the present value at time $t = 0$ to time $t = n$

$$\bar{s}_{\overline{n|}} = (1+i)^n \times \bar{a}_{\overline{n|}} = \frac{(1+i)^n - 1}{\delta}$$

EXAMPLE

If the effective rate of interest is $i = 4\%$ per annum, calculate the following quantities.

a.) $3\bar{a}_{\overline{10|}}$

b.) $10\bar{s}_{\overline{5|}}$

Answers

a.) This is the present value of a continuous annuity paying 3 per annum for 10 years,

$$3\bar{a}_{\overline{10|}} = 3 \times \frac{1-\nu^{10}}{\delta} = 3 \times \frac{1-(1.04)^{-10}}{\ln(1.04)} = 24.816$$

b.) This is the accumulated value at $t = 5$ of a continuous annuity paying 10 per annum for 5 years,

$$10\bar{s}_{\overline{5|}} = 10 \times \frac{(1+i)^5 - 1}{\delta} = 10 \times \frac{(1.04)^5 - 1}{\ln(1.04)} = 55.239$$

Summary

- The present value and accumulation of a continuously paid cash flow can be calculated by constructing integral expressions in terms of $\rho(t)$ and $\delta(t)$.
- A continuously paid unit n-year annuity is the particular case that $\rho(t) = 1$ for $0 \leq t \leq n$ years. If $\delta(t) = \delta$, we denote the present value of this stream as

$$\bar{a}_{\overline{n|}} = \frac{1 - \nu^n}{\delta}$$

and the accumulated value at time $t = n$ as

$$\bar{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{\delta}$$