Problem Sets

Chapter 1

Problem 1.1

In this chapter we discussed the dominant two-step processes through which O^+ recombines. These reaction sets, along with their corresponding reaction rates, are:

$$O^+ + O_2 \rightarrow O_2^+ + O, \quad k_1 = 2.82 \times 10^{-11} - 7.74 \times 10^{-12}(T_{eff}/300) + 1.073 \times 10^{-12}(T_{eff}/300)^2 - 5.17 \times 10^{-14}(T_{eff}/300)^3 + 9.65 \times 10^{-16}(T_{eff}/300)^4 \text{ cm}^3 \text{ s}^{-1}$$

$$O_2^+ + e \rightarrow O + O, \quad k_2 = 1.8 \times 10^{-7}(T_e/300)^{-0.55} \text{ cm}^3 \text{ s}^{-1}$$

and

$$O^+ + N_2 \rightarrow NO^+ + N, \quad k_3 = 1.53 \times 10^{-12} - 5.92 \times 10^{-13}(T_{eff}/300) + 8.6 \times 10^{-14}(T_{eff}/300)^2 \text{ cm}^3 \text{ s}^{-1}$$

$$NO^+ + e \rightarrow N + O, \quad k_4 = 4.3 \times 10^{-7}(300/T_e)^{0.85} \text{ cm}^3 \text{ s}^{-1}$$

where $T_{eff} = 0.667T_i + 0.333T_n$.

Determine how to use the online MSIS-E-90 neutral atmosphere model (http://modelweb.gsfc.nasa.gov/atmos/msise.html) to obtain the neutral densities and temperature as a function of altitude for typical equinoctial, midlatitude, nighttime, and solar minimum conditions.

Ignoring any production,

$$\frac{d[O^+]}{dt} = -\{k_1[O_2] + k_3[N_2]\}[O^+] = -\beta[O^+].$$

Plot the time constant for recombination, β^{-1}, for 100–500 km using $T_i = T_n$. A semilog axis may be appropriate.

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Problem 1.2

Using the formulas in problem 1.1, determine the loss time constants for NO^+ and O_2^+ at the E region heights of 100 and 120 km in solar maximum using $[O_2^+] = [NO^+] = 0.5[e] = 5 \times 10^4\ \mathrm{cm}^{-3}$. Use the table in Appendix B to determine the electron temperature, taking $T_e = T_n$.

Problem 1.3

This problem demonstrates behavior that one might expect in a sporadic E layer. Suppose that the nighttime E region after sunset is composed of 50% NO^+ and 50% Mg^+ with an electron density of $2 \times 10^5\ \mathrm{cm}^{-3}$. Let the recombination coefficients be $\alpha_{NO^+} = 5 \times 10^{-7}\ \mathrm{cm}^3\ \mathrm{s}^{-1}$ and $\alpha_{Mg^+} = 8 \times 10^{-12}\ \mathrm{cm}^3\ \mathrm{s}^{-1}$. What is the initial recombination rate of each ion species? What is the ion composition after one hour? How long does it take for 99% of the NO^+ to disappear? (You may assume that one species goes away quickly and solve for it, assuming that the other density is constant. Use a numerical integration scheme or graphical approach. Then, solve for the slow species decay. Or you can try to solve the coupled equations numerically.) If the initial ion composition is all NO^+, what would be the initial recombination rate? In this case, what would be the density after one hour? Comment on the two cases of different composition.

Problem 1.4

Height-integrated quantities are of some importance to a number of ionospheric calculations, as we will study in future chapters. Consider the so-called α-Chapman layer for the nighttime F region:

$$n(z') = n_0 \exp\left[\frac{1}{2}\left(1 - z' - e^{-z'}\right)\right].$$

Assume a constant neutral scale height H and use the fact that $z' = (z - z_0)/H$ where z_0 is the height of the peak. Use a numerical integration scheme for the following calculations:

1. Show that the height-integrated plasma content, N, also called the total electron content (TEC), can be written as

$$N = \int n(z')dz \approx 4.1 n_0 H.$$

2. The density-weighted collision frequency is defined as

$$\langle \nu_{in} \rangle = \frac{1}{N} \int n(z')\nu_{in}(z')dz.$$

Assuming $\nu_{in}(z') = \nu_0 e^{-z'}$, show that $\langle \nu_{in} \rangle \approx \nu_0 = \nu_{in}(z_0)$.

3. Show that the height-integrated conductivity, $\sum_P = \int \sigma_P dz$, can be expressed as

$$\sum\nolimits_P \approx 4.1 H \sigma_P(z_0).$$

We introduce the Pedersen conductivity, σ_P, in the next chapter. Use $\sigma_P = (nM\nu_{in})/B^2$ and the form of the collision frequency in part 2.

Problem 1.5

Find the current density implied by the tail's magnetic field configuration if it is modeled in the form

$$B_x = -B_0 \tanh(z/h)$$

using $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. For $B_0 = 10\,\mathrm{nT}$ and $h = 1\mathrm{R_e}$, find the maximum value of J in $\mathrm{A/m^2}$.

Problem 1.6

If reconnection in the nightside takes place after plasma in the ionosphere convects across the polar cap ($\approx 30°$ of latitude) under the influence of a 50 mV/m electric field at ionospheric heights, how long is the magnetic tail? Assume that the shocked solar wind accelerates instantly to a 500 km/s solar wind speed due to the connection process, and take the ionospheric magnetic field to be 0.5×10^{-4} T.

Chapter 2

Problem 2.1

Derive (2.37b) from (2.34).

Problem 2.2

Show that $\mathbf{J} \times \mathbf{B} = -\rho \nu_{in} \mathbf{U}$ in the F region using $\mathbf{E} = 0$. Use the approximation for σ_p in (2.40b) and $\mathbf{J}_\perp = \sigma_p \mathbf{E}'_\perp$. Here $\mathbf{J} \times \mathbf{B}$ is referred to as ion drag.

Problem 2.3

Compare the collisionless $\mathbf{E} \times \mathbf{B}$ drift in a 0.5×10^{-4} T magnetic field due to a 10 mV/m electric field to the ion diamagnetic drift caused by a 200 K temperature change in a plasma with a constant density of $10^5\,\mathrm{cm^{-3}}$ if the temperature change takes place over 10 km. If the 10 mV/m field were due to a thermoelectric effect in which the temperature change was measured in millielectron volts, what would the temperature change be in Kelvin over a 10 km distance? This is a very unrealistic temperature change and shows how large ionospheric electric fields actually are compared to temperature changes.

Problem 2.4

Suppose that the high-latitude, Northern Hemisphere electric field is southward with a value of 50 mV/m and that the magnetic field is vertical with a value of 0.5×10^{-4} T. If the E region density is 10^5 cm^{-3}, find the current density vector using (2.38) and the fact that $\sigma_H >> \sigma_P$. What particles carry the current? Now, transform to a frame moving with the $\mathbf{E} \times \mathbf{B}$ drift. Show that if you use the electric field and neutral wind in the new frame, (2.38) gives the same value for the current vector. What particles carry the current in the new reference frame?

Problem 2.5

In this problem we model the ionosphere as a superposition of a Chapman layer in the F region and a narrow Gaussian layer in the E region, which might be typical of nighttime, midlatitude conditions. A Chapman layer is of the form

$$n_F(z') = n_{F0} \exp\left[\frac{1}{2}\left(1 - z' - e^{-z'}\right)\right]$$

where $z' = (z - z_{F0})/H, H$ is the neutral scale height, and z_{F0} is the height of the F peak. Use a peak height of 300 km, a peak density of 4×10^5 cm^{-3}, and a scale height of 50 km. In the E region, model the layer as a Gaussian with a characteristic scale of $\sigma = 10$ km, a peak height of 10^5 km, and a peak density of 10^5 cm^{-3}. So, the E layer will be of the form

$$n_E(z) = n_{E0} \exp\left[-(z - z_{E0})^2/2\sigma^2\right].$$

In the F region, assume there is only atomic oxygen with a neutral density given by

$$n_O(z) = n_{O0} \exp[-(z - 100)/H]$$

with a density of 5×10^{11} cm^{-3} at 100 km. In the E region, take the neutral density to be constant with a value of 6×10^{12} cm^{-3}. Use $B = 0.5 \times 10^{-4}$ T. Find the height-integrated conductivity tensor contributions from both layers, ignoring the Hall term in the F region (explain why this is justified). Be sure to make use of the simplifications discussed in the chapter.

Problem 2.6

Solve the single particle equations of motion for an ion in crossed electric and magnetic fields (assume vacuum conditions).

Problem 2.7

Derive (2.38) and (2.39a–c).

Problem 2.8

A simple pendulum (point mass attached to a massless, stiff string of length l with no friction) is very slowly shortened. Find the associated adiabatic invariant, assuming that the angular displacement is small. (**Hint**: For a photon the energy equals $\hbar\omega$ and $\hbar$ is the invariant quantity. Find the pendulum energy in terms of θ_{max} and ω for a given length of a pendulum and divide them.) If a 1 m length pendulum of mass 10 g has a maximum angular displacement of 10°, find its maximum angular displacement after an adiabatic shortening to 0.5 m.

Problem 2.9

The action is defined as $J = \int pdq$ over a full cycle of a periodic motion, where p denotes momentum. For a pendulum, $p = ml(d\theta/dt)$ with $\theta = \theta_0 \cos\omega t$ and $\omega = \sqrt{g/l}$. Find J and show that it agrees with E/ω in the previous problem to within a numerical constant.

Problem 2.10

Radial diffusion brings a 50 keV proton from $8\,R_e$ to $3\,R_e$ in a dipole field. Find the final energy if initially the 50 keV proton was moving entirely in the perpendicular direction.

Problem 2.11

The magnetopause current effectively terminates the earth's magnetic field at the subsolar point. To effect this, the field (B_{MB}, where MB stands for "magnetospheric boundary") is double that of the dipole field at the same location without a solar wind. Set the associated magnetic field pressure ($B_{MB}^2/2\mu_0$) equal to the solar wind kinetic pressure, $2\rho v_{sw}^2$, where ρ is the hydrogen plasma mass density and v_{sw} is the solar wind speed. Note that we have assumed perfect reflection to obtain the factor of 2. Find an expression for the magnetopause stand-off distance and evaluate it for $v_{sw} = 400\,\text{km/s}$ and $n = 2\,\text{cm}^{-3}$. Take the magnetic field at the surface to be 0.25×10^{-4} T.

Problem 2.12

A derivative of Poynting's theorem has been given in this chapter. Show that the integrated Poynting flux ($\mathbf{E} \times \mathbf{H}$) into a resistor is equal to I^2R. Ignore fringing effects on the magnetic field and take $\mathbf{E}$ to be uniform throughout the resistive material. A useful formula is $R = L/\sigma A$ where σ is conductivity, L is length, and A is the cross-sectional area of the resistor.

Problem 2.13

Show that in the magnetic tail the direction of the earth's magnetic field and of the interplanetary electric field is such that $\mathbf{E} \times \mathbf{H}$ is inward for a southward interplanetary magnetic field. Model the tail as a rectangular solid $20R_e \times 20R_e \times 200R_e$ in size with an average magnetic field strength at the boundary of 15 nT. (Of course, a cylinder would make a better model, but this will do for an estimate.) Let the magnetic field at the surface of the solid be toward or away from the sun, depending on the hemisphere, and let the electric field be in the dawn-to-dusk direction. Let the magnetospheric configuration be static so that the time rate of change of the magnetospheric magnetic field vanishes. Using the energy dissipation value at the magnetic storm peak from Fig. B.8 in Appendix B ($\approx 3 \times 10^{19}$ ergs/s) and Poynting's theorem, find the average electric field at the surface and the total potential across the system. If half of the energy is dissipated in the current systems driven by that potential, find the associated current.

Problem 2.14

Use dimensional arguments to show that $\nu = (\text{thermal speed})^2/(\text{collision frequency})$.

Chapter 3

Problem 3.1

Show that dropping the advective derivative on the left-hand side of Eq. (3.5) is justified compared to terms involving $\partial/\partial t$, F, and $\eta\tilde{u}/\rho H^2$ in Fig. 3.7 for a diurnal tide with maximum speed of 100 m/s at 300 km. Make sure that your units are the same when comparing the terms.

Problem 3.2

Using Fig. 3.7, find the steady-state wind at 300 km altitude on the dayside of the terminator if F is balanced by ion drag via ν_{ni}. Now suppose that an eastward electric field due to the prereversal enhancement lifts the ionosphere by one neutral scale height such that the plasma density is reduced by a factor of 3 at 300 km. Find the new wind speed. Estimate the viscous force for such a wind difference across one scale height and compare it to F. This shows that viscosity would limit the acceleration of the neutral wind "under" the lifted ionosphere.

Problem 3.3

What plasma density corresponds to $\nu_{ni} = 4 \times 10^{-4}\,\text{s}^{-1}$ at 300 km for solar max and solar min conditions? Show that ν_{ni} is independent of neutral density and derive an expression for it, assuming the ions and neutrals are O^+ and O, respectively.

Problem 3.4

In a simple F-layer equatorial dynamo with an eastward neutral wind of magnitude $u = 150$ m/s, which is independent of altitude, find the vertical electric field if $\sum_P^F = 5$ mho and $\sum_P^E = 1$ mho in each hemisphere, using Eq. (3.11) and $B = 0.25 \times 10^{-4}$ T. Estimate the gradient of $\sum_P^F$ at 450 km using Fig. 3.11, and find the magnitude and direction of the field-aligned current at the top of the Northern Hemispheric E region if the electric field is constant with height at the value found above. Use $B = 0.25 \times 10^{-4}$ T.

Problem 3.5

Make a model for the measured currents in Fig. 3.17a, using a Gaussian form with z having a characteristic scale of 5 km (i.e., proportional to $\exp[-(z - z_0)^2/2\sigma^2]$ with $\sigma = 5$ km) and a peak value of 9 μA/m^2 at 108 km. Let $E_{east} = 1.0$ mV/m. Give an expression for the Cowling conductivity and compare it to Fig. 3.15. Find the height-integrated current (A/m).

Problem 3.6

Let the current density in Fig. 3.17a also have a Gaussian dependence with latitude with a characteristic scale of $\sigma = 100$ km. Model the current *locally* at any given latitude as being due to two uniform current sheets, one eastward and equal to the overhead value and the other a mirror current in the opposite direction beneath the earth at a depth of 108 km. Find the magnitude and direction of the magnetic field perturbation at the surface of the earth as a function of distance from the magnetic equator. (**Hint**: Use the integral form of Amperes Law per meter and the height-integrated current found in Problem 3.5.)

Problem 3.7

Compare the various ion *horizontal* drift velocities 300 km above the magnetic equator where $B_0 = 0.25 \times 10^{-4}$ T. Let $E_z = -2$ mV/m, $E_{east} = 0.5$ mV/m, $T_i = T_e = 1000$ K, $g = 9\,\text{m/s}^2$, and let the density have a characteristic vertically upward gradient scale length of 20 km. (Don't forget the Pedersen drift due to E_{east} and use the sunspot maximum values from Appendix B or MSIS for equinox, sunspot maximum.)

Problem 3.8

Verify that the rotation matrix R is indeed a rotation of the coordinate system by I, where I is a rotation angle counterclockwise, looking down the x-axis.

Problem 3.9

Derive the equation set (3.16).

Problem 3.10

Explain why there are two lunar gravitational high tides and two lunar low tides in the ocean each day (i.e., a semi-diurnal tide). What dominant tidal mode do you expect from solar heating? Use an argument based on the Fourier series for a half-wave rectifier to explain why higher-order tides are expected for solar heating.

Problem 3.11

Compare the magnitude of the lunar and solar gravitational tidal forces. Show that, for the earth, their ratio is equal to the relative average density of the moon and the sun. (**Hint**: Use the fact that the angular size of the sun and moon are equal as viewed from the earth.) Use this result to estimate the relative tidal influence of the sun and the moon. (**Hint**: The mean density of the sun and moon are different.) How large is Mars's tidal influence compared to that of the moon at its closest approach to the earth? What about Jupiter's?

Problem 3.12

In Fig. 3.17b, panel d, resolve the discrepancy between J/ne and $\mathbf{E}/\mathbf{B}$ by hypothesizing a zonal neutral wind-driven ion current. Describe the zonal wind versus altitude. What is the vertical electric field at 98 km in the neutral frame of reference?

Problem 3.13

What is the charge density at 1800 and 2000 LT, implied by the dashed curve in Fig. 3.18, if $B_0 = 0.25 \times 10^{-4}$ T?

Problem 3.14

Use the slab model in Fig. 3.16 and apply a uniform neutral wind in the eastward direction without any initial eastward electric field. Set $J_z = 0$ and find the total eastward current, including any polarization electric field. Go into the frame moving with the wind and show that you get the same value for the current.

Chapter 4

Problem 4.1

A rough model for the center of the quasi-sinusoidal scattering layer in Fig. 4.1 is of the form:

$$h(t) = \{500 - 100(t - 19.33) + 100 \sin[2\pi(t - 19.33)]\}\,\text{km}$$

where t is measured in decimal hours and h in kilometers. Assuming that $h(t)$ represents the effect of a zonal electric field, take its time derivative and evaluate the associated eastward electric field for $B = 0.25 \times 10^{-4}\,\text{T}$. Using $L = 20\,\text{km}$, $g = 9\,\text{m/s}^2$, and $\nu_{in}(h) = 1\text{e}^{[(300-z)/50]}\,\text{s}^{-1}$, calculate and plot the two contributions to the growth rate and the total growth rate γ where

$$\gamma = \frac{E}{BL} + \frac{g}{\nu_{in}(h)L}$$

How many e-folds occur between 1900 and 1945 when the first "apogee" plume occurs?

Problem 4.2

In Fig. 4.14, estimate the separation distance for the side-plumes on the western wall of the plasma structure. Show that this side of the structure is unstable to an eastward neutral wind. Estimate the wind contribution to the linear growth rate if $u_{east} = 150\,\text{m/s}$, $B = 0.25 \times 10^{-4}\,\text{T}$, $E_z = 3\,\text{mV/m}$, and the bottomside plasma density scale height is 15 km (use the figure to evaluate the tilt angle). Compare this contribution to the values found in Fig. 4.11.

Problem 4.3

Use Fig. 3.12b (measurement at 20:25) to determine an average dV/dz. Estimate the plasma Richardson number in the postsunset bottomside if $L_N = 15\,\text{km}$. Using Fig. 3.33, estimate the neutral atmospheric Richardson number at 95 km if the Brunt-Väisälä period is 3 minutes (use the total horizontal velocity change over the 90–100 km height range in this figure). Which condition is closer to instability?

Problem 4.4

Derive (4.14).

Problem 4.5

Model a plasma plume as two vertical parallel planes with internal density n_i and external density n_0. Using current continuity, show that the internal eastward electric field is

$$\delta E_i = \left(\frac{n_0 - n_i}{n_i}\right)\frac{gB}{\nu_{in}}$$

Estimate $\delta E_i/B$ for the height of 322 km in Fig. 4.16b at 1000 s, taking $\nu_{in} = 2\,\mathrm{s}^{-1}$ and using this formula. Compare this velocity to the height change of the 1.0 E_i contour in the next 200 seconds.

Problem 4.6

In the text, we discuss two possible explanations for the tilted plumes. One involves shears in the background plasma flow due to an altitude variation in the F region dynamo. Let the zonal wind be constant in height, the E region conductivity be uniform in latitude, and allow for a possible E region dynamo to exist. Explain how (3.11) could be used to explain a C-shaped pattern with westward drifts at low altitudes, a peak in the flow in the F region, and a decreasing electric field/plasma flow above the peak. Remember in the slab model that the z direction in the E region represents latitude as well.

Problem 4.7

A possibility for the tilt of plumes toward the west involves defining an effective gravity that is not vertical. We know that there is a $\mathbf{J} \times \mathbf{B}$ force on the neutral gas caused by the currents that flow. By Newton's Third Law, this force must act back on the plasma. Show that, for a zonal wind greater than the eastward $\mathbf{E} \times \mathbf{B}$ drift, the sum of this force and the gravitational force creates an effective gravity, $\mathbf{g}'$, that is consistent with the tilt of upward bubbles moving antiparallel to $\mathbf{g}'$.

Problem 4.8

Derive (4.39b) by evaluating the determinant in (4.39a).

Problem 4.9

Derive (4.40a) and (4.40b) using (4.39a) and the assumption $\gamma << \omega_r$ and $\gamma << \nu_{in}$.

Problem 4.10

Ignoring any neutral wind for the moment, Fig. 3.17b indicates that $E/B = 400$ m/s. For $C_s = 360$ m/s, $\nu_i/\Omega_i = 36$, $\nu_e/\Omega_e = 0.01$, $B = 0.25 \times 10^{-4}$ T and an ion mass of 32 Amu, find γ using (4.40b). Are the waves growing or are they damped? What is the minimum electric field required for wave growth?

Problem 4.11

Estimate the neutral wind (u_i) at 105 km from Fig. 3.17b. Using $\mathbf{V}_{\mathrm{Di}} = \mathrm{u_i}\hat{a}_{\mathrm{x}}$, find the new growth rate using the two-stream term in (4.48b). Let $\mathbf{k} = k\hat{a}_x$. Are the waves growing or damped? Repeat the calculation for a westward neutral wind of 100 m/s. Are the waves growing or damped? What is the necessary magnitude and direction of the neutral wind for instability?

Problem 4.12

Show that (4.59b) satisfies the differential equation (4.59a).

Chapter 5

Problem 5.1

Consider a hydrogen test ion subject to the electric field in (5.4c) and gravity, using the oxygen ion mass. Ignoring collisions, find the magnitude and direction of the acceleration felt by the particle parallel to the field line if $I = 45°$.

Problem 5.2

Show that the solution given in (5.17c) satisfies (5.17b), using the definition of λ given.

Problem 5.3

Show for the solution given in (5.17c) that $W_D = g/\nu_{in}$ everywhere, where W_D is given in (5.13) and ν_{in} is evaluated at the F peak.

Problem 5.4

Show that (5.16) is a solution of (5.14) if the loss term vanishes.

Problem 5.5

Show that $\langle \nu_{in} \rangle = \nu_{in}(h_{\max})$ for a Chapman alpha-layer.

Problem 5.6

The spaceship *Enterprise* is inside the earth's nightside plasmasphere with an orbit that would revolve around the sun in 365¼ days. A cloud of low temperature xenon plasma is injected by a thruster that is less dense than the plasmaspheric density. Using a xenon ion as a tracer, describe its subsequent motion qualitatively.

Problem 5.7

Using a numerical evaluation of (5.24a,b), calculate and plot the altitude of a Chapman alpha-layer as a function of time if its velocity is given by $W_D = (g/ <\nu_{in}>) \sin^2 I$. Let the layer start at 400 km at midnight and simply fall down the field lines. Let $g = 9\,\mathrm{m/s^2}, I = 45°$, and $<\nu_{in}> = 0.1e^{(400-z)/50}\,\mathrm{s^{-1}}$.

Problem 5.8

Repeat Problem 5.6 including a constant southward wind of 30 m/s. Verify that the layer reaches the expected equilibrium height.

Problem 5.9

Repeat 5.6 using a meridional wind of the form

$$u_s = (30 + 20\cos\omega t)\mathrm{m/s}$$

with $\omega = 2\pi/3$ rad/hr.

Problem 5.10

For the event in Fig. 5.17, let the eastward electric field between 04 LT and 05 LT be given by $5\sin\pi(t4)$ mV/m and zero thereafter. At $t = 04, h_{\max} = 450$ km. Find $h_{\max}$ for $04 \le t \le 08$.

Problem 5.11

The ion layer trajectories that reach 90 km around 00:00 LT in Fig. 5.23 repeat every 24 hours. If this is indeed a diurnal tide, find the implied vertical wavelength from its phase velocity (ω/k_z) by estimating the velocity in the height range 110–140 km, where the descent seems to be at a constant velocity. Does the corresponding wavelength agree with the spacing between the corresponding layers in the figure?

Problem 5.12

The four layers on January 5/6 in Fig. 5.23 seem to be a 6-hour tide. Repeat Problem 5.12 for this case.

Problem 5.13

The classic diffusion equation for a cylindrical minor constituent of radius r is $r^2(t) = 4Dt + r_0^2$ where D is the diffusion coefficient. The thin traces in Fig. 5.25b, after 82 seconds, have $r = 50$ m, whereas the thick region has $r = 1000$ m. Find D in both cases and compare them to molecular and eddy diffusion rates at 95 km. Let $r_0 = 1$ m.

Chapter 6

Problem 6.1

Derive the linearized equations (6.2a–d).

Problem 6.2

Derive Eq. (6.3).

Problem 6.3

Find the polarization relationships for $\delta p/p_0, v$, and w in terms of $\delta\rho/\rho_0$.

Problem 6.4

Show that $\omega_b^2 = 0$ for an adiabatic temperature lapse rate.

Problem 6.5

Show that the right-hand side of (6.4) is pure real if and only if $k_z'' = 1/2H$.

Problem 6.6

Estimate the viscous damping time constant for the two waves in Fig. 6.3 (the set at 120 km and the higher altitude waves). Assume $\delta\rho_N/\rho_N = 5\delta n_e/n_e$ and use the results of 6.3 above to find v and w. Compare the damping rate to the wave period in the earth frame for the low altitude wave using reasonable atmospheric parameters.

Problem 6.7

As an approximation to Eq. (6.8), we often drop the $1/4H^2$ term. Show that the resulting equation is the oscillation frequency of a parcel of air moving adiabatically on a frictionless inclined plane perpendicular to **k**.

Problem 6.8

In this problem we will investigate the differential equation of layer formation for molecular ions in the upper E region.

(a) Write out the equation for this process including transport effects due to ambipolar diffusion (neglecting gravity and any thermal gradients) and an altitude-dependent meridional wind as well as an appropriate recombination term.

(b) Assume a wind profile of the form

$$u(z) = u_0 \frac{(z - z_0)}{H} e^{-(z-z_0)/2H^2}$$

Plot $u(z)$ for $u_0 = 100$ m/s and $H = 5$ km and describe qualitatively what will happen for $u_0 < 0$ and $u_0 > 0$, and also the expected effects of recombination and diffusion.

(c) Solve the equation derived in (a) numerically using the neutral wind profile from (b) with an appropriate sign for gathering ions, using $I = 45°, u_0 = 100$ m/s, $H =$ 5 km, and appropriate ambipolar diffusion and recombination coefficients. Solve the equation for four cases: neglecting recombination and diffusion, including diffusion only, including recombination only, and including both recombination and diffusion. Discuss/compare these results.

Problem 6.9

Ignoring recombination, use the same wind profile as in the previous problem to find the layer gradient $(1/n(dn/dz)^{-1})$ in equilibrium for metallic ions. What value of D_A^* is needed to yield the observed gradients on the order of 1 km?

Problem 6.10

The main source of metastable oxygen atoms excited in the ^1D state is the dissociative recombination of O_2^+, which has a rate coefficient of $\sim 2 \times 10^{-7}$ cm^3 s^{-1} and a quantum yield close to 1. Assuming that O_2^+ is only produced by the charge exchange reaction of O_2 with O^+, which has a rate coefficient of $\sim 10^{-11}$ cm^3 s^{-1}, relate the O_2^+ density to the O^+ density, and show that $O^+ \approx$ [e]. Using this approximation, write an expression for the volume emission rate of $O(^1D)$ atoms (the 630.0 nm red-line emission) including quenching by O (rate of $\sim 8 \times 10^{12}$ cm^3 s^{-1}), O_2 (rate of $\sim 3 \times 10^{11}$ cm^3 s^{-1}), and N_2 (rate of $\sim 2 \times 10^{11}$ cm^3 s^{-1}), and assuming a transition coefficient from the ^1D to the ^3P state of 10^{-2} s^{-1}. Assuming a Chapman profile for the electrons, and barometric profiles with appropriate scale heights ($T = 1000$ K) for the neutral species, plot and discuss the integrated intensities in Rayleighs as a function of the Chapman scale and h_mF2 for a peak electron density of 5×10^5 cm^{-3} and neutral densities of $[O_2] = 3 \times 10^{10}$ cm^{-3}, $[O] = 8 \times 10^{10}$ cm^{-3}, and $[N_2] = 3 \times 10^{11}$ cm^{-3}at 120 km.

Problem 6.11

Show that the group and phase velocities of internal gravity waves have the opposite vertical signs. Use (6.8) and ignore the $(1/2H)^2$ term.

Problem 6.12

Solve the differential equation (6.20b) for $\sum(t)$ if $E_x(t) = E_0 \cos \omega t$, given $E_0 = 3\,\text{mV/m}, B = 0.4\,\text{G}$, $H = 50\,\text{km}$, and a Chapman profile for the electrons with peak density of $10^5\ \text{cm}^{-3}$ (and a Chapman scale height equal to the neutral scale height). Show solutions for $\tau = 2\pi/\omega = 30, 60$, and 120 minutes.

Problem 6.13

Suppose a constant 100 m/s westward wind blows on a Gaussian-shaped sporadic E layer with a characteristic scale of 2 km and a peak density of $10^5\ \text{cm}^{-3}$, which is infinitely elongated in the east-west direction. The layer is at 105 km and the dip angle is 45°. Find the magnitude and direction of the polarization field and the total current, ignoring F region shorting. Repeat the calculation for a southward-directed wind.

Problem 6.14

Derive Eqs. (6.30a–d).

Chapter 7

Problem 7.1

Use the equations of Chapter 6 (also see Problem 6.2) to relate δu in a gravity wave to δT. Discuss/evaluate the expression for winter and summer mesospheric conditions.

Problem 7.2

Derive Eq. (7.2) from the result in Problem 7.1 using the definition of the potential temperature.

Problem 7.3

Derive Eqs. (7.3) and (7.5).

Problem 7.4

Show that the wave-breaking conditions $w > \omega/m$ and $u > \omega/k$ are identical. You may use $\nabla \cdot \mathbf{u} = 0$ for the wave perturbation.

Problem 7.5

A deep water wave has a 10 m wavelength. At what amplitude (in meters) will white caps form (deep water wave breaking)? (**Hint**: The phase velocity for a deep water wave is given by $c = \sqrt{g/k}$.)

Problem 7.6

The deep water wave in the previous problem approaches a shore with $\delta u = 1$ m/s. Find the depth at which the wave will break. (**Hint**: The phase velocity of a shallow water wave is $\omega/k = \sqrt{gh}$ where h is the depth.)

Problem 7.7

A typical jet stream wind speed is eastward at 100 km/hour. If $m = 2\pi/1000\,\text{km}^{-1}$, find and plot the wave period as a function of k for all eastward propagating waves that will be absorbed. Use the approximation $\omega^2 = [N^2k^2/(k^2 + m^2)]$ and a Brunt-Väisälä period of 5 minutes.

Problem 7.8

Use the dashed curve in Fig. 7.7b as a representative background electron density at 85 km and the measured electron density in the bite-out region. Referring to Reid's calculations for a production rate of $10\,\text{cm}^{-3}\,\text{s}^{-1}$ (7.15a,b), which size particle seems more appropriate? Estimate the number density of such particles present.

Problem 7.9

A small ($b >> a$ in what follows) isolated 90 amu dust grain is immersed in a plasma with $T_e = T_i = T$. In the following, neglect any secondary charging mechanisms.

(a) Write down expressions for the ion and electron currents to the dust grains. For zero surface potential, would the grain become positively or negatively charged?
(b) Derive expressions for the negative and positive surface potentials.
(c) The formula for the capacity of a vacuum spherical capacitor is given by $4\pi\varepsilon_0$ $(ab/b - a)$ where $a(b)$ is the inner (outer) radius. For $T_e = 300$ K, find the radius at which $Q = e$. How does this result depend on the charge number of the grain?

Problem 7.10

Assuming spherical particles with a mass density of 1 gram/cm^{-3}, calculate the fall speed versus altitude for 10 and 50 nm radius particles in the 80–90 km height range. The equation of motion for a particle of mass m and diameter d, neglecting winds and assuming non-Brownian particles, can be written as

$$m\frac{dv}{dt} = -3\pi\mu dv + mg$$

where μ is the viscosity coefficient. Assume that the particle has no initial velocity.

Problem 7.11

We wish to model the fair weather electric field as a leaky spherical capacitor with an exponentially increasing conductivity, $\sigma = \sigma_0 \exp(z/H)$, with a constant current density as a function of z, and with a surface electric field of $-E_0$ V/m. For this system, find $E(z)$ and the total voltage V between the ground and infinity. Find the total current, I, to the surface of the earth and the effective resistance, R_{eff}. Compare your expression to the classical formula for a resistor of length l, area A, and conductivity σ.

Problem 7.12

The time constant for discharge of charge in a conducting fluid is $\tau = \varepsilon_0/\sigma$. Use this result coupled with the effective resistance found in the previous problem to determine an expression for the effective capacity of the earth, C_{eff}. Model the earth as a spherical capacitor and use the fact that $R_e >> H$. What values for the inner and outer radius yield the same C_{eff} just derived?

Problem 7.13

Using $H = 6$ km, find V, I, R_{eff}, C_{eff}, and the discharge time constant for the earth if $E_0 = 100$ V/m and $\sigma_0 = 4 \times 10^{-12}$ mho/m.

Problem 7.14

A lightning strike brings -10 C to earth from a height of 2 km. Using the concept of image charges, how large is the electrostatic field change at 80 km due to this charge?

Chapter 8

Problem 8.1

Estimate the length of the x-line (at which dayside connection to the interplanetary magnetic field takes place) by comparing the typical cross-polar cap potential to the dawn-dusk potential across the magnetosphere. Compare the amount of power from the solar wind getting into the atmosphere to the Poynting flux times the area subtended by the earth (without a magnetosphere).

Problem 8.2

Treat the solar wind MHD generator as a voltage source and the polar cap as a resistive medium with value Σ_P^{-1}. Compare the power dissipated in the winter and summer polar caps for which Σ_P is equal to 1 and 8 mhos, respectively. Which season should have larger region 1 currents in this model? Does this agree with the measurements?

Problem 8.3

Use the height-integrated conductivities from Problem 2.5 (E region only) and a Gaussian auroral arc elongated in the zonal direction with a characteristic Gaussian shape and a scale of 5 km in the meridional direction. Plot the field-aligned current density and integrate the upward and downward current sheets if the electric field is constant at 50 mV/m.

Problem 8.4

What is the magnetic field disturbance (magnitude and sign) due to the Hall current modeled in Problem 8.3 directly below the current and ±50 km north and south? Assume that the current is infinitely long and can be modeled as a thin wire. Ignore earth currents and the Pedersen current. The latter is shielded from detection at the ground.

Problem 8.5

Suppose 1000A flows across the earth's northern polar cap during a connection to the solar wind and that this current is closed in a region of the solar wind $10 \times 50\ R_e$ in size. Let $B_z = 5\,\mathrm{nT}$ in the solar wind, $n = 10\,\mathrm{cm}^{-3}$, and $V_{sw} = 500\,\mathrm{km/s}$. How much does the solar wind slow down in this interaction? (**Hint**: The time for connection is $50\,R_e/500\,\mathrm{km/s}$.)

Problem 8.6

The auroral zone ($L = 7$) is observed to have a 50 mV/m zonally westward electric field with $B = 0.5$G. Find the magnitude and direction of the plasma velocity in the magnetospheric equatorial plane using (2.48b). Use the dipole approximation with an equatorial field at the surface of 0.25G. A 1 MeV electron gradient-drifts from 10 pm LT to 02 am LT. Compare the $\nabla \mathbf{B}$ drift speed to the $\mathbf{E} \times \mathbf{B}$ drift. How much energy does the electron gain due to the electric field? (You may assume the electric field is uniform and dawn to dusk.)

Problem 8.7

Find the Poynting flux across the front of the magnetosphere using a $20R_e \times 20R_e$ area. Compare this to the ionospheric dissipation in the two polar caps if $E = 50\,\text{mV/m}$ and $\Sigma_P = 7\,\text{mho}$. Use a "square" polar cap with 2000 km on a side. What is the efficiency of the MHD generator?

Problem 8.8

A satellite magnetometer measures a 100 nT (δB_y) step function when crossing two current sheets in the auroral oval. If the field line-integrated conductivity is 10 mho, find the electric field (δE_x) in the E region below. Let $\delta E_x = 0$ outside the perturbed zone. Show that the Poynting flux toward the earth is equal to the Joule heat.

Problem 8.9

From Fig. 8.3b estimate the Poynting flux into the auroral oval and polar cap. Estimate the average height-integrated Pedersen conductivity from these data.

Problem 8.10

What region 1 current sheets would be needed to support the average polar cap δB in Fig. 8.3b? Assume an infinite current sheet model at the region 1/region 2 boundary. Using the first term in (8.15), estimate Σ_P for the two reversals of the electric fields.

Problem 8.11

As plasma flows back toward the sun on closed field lines in Fig. 8.2(b), let E be constant with a value of 1 mV/m, B be given by 100 nT, and $\nabla B = B/2R_e$. The magnetic field increase causes a perpendicular current that may close the region 1 and region 2 current systems across B in the magnetosphere. Take Σ_P in the ionosphere to be 5 mho and the density in the equatorial plane to be $1\,\text{cm}^{-3}$ with a Gaussian thickness along B of $2R_e$. How large a $J_{||}$ can be supported at ionospheric levels for $L = 6$ (region 2)? Take the latitude range for $J_{||}$ to be 1° at ionospheric heights and the equatorial current to be constant over 1 R_e length perpendicular to the equatorial plane. The other length dimension scales as $L^{3/2}$ between the ionosphere and the equatorial plane.

Problem 8.12

Compare analytically $\delta E/\delta H$ in an Alfvén wave and a field-aligned current sheet system. Estimate these ratios for an Alfvén wave at 500 km in the high-latitude F region with $n = 10^5\,\text{cm}^{-3}$ and compare this with typical E region height-integrated conductivity (Σ_p) in the E region.

Problem 8.13

In the lower panel of Fig. 8.8b, model the electric field in the polar cap using one-half of a sine wave between ±70° latitude with $B = 0.5$ G, and determine the cross polar cap potential. Using (8.10b) with $n = 3\,\mathrm{cm}^{-3}$, $\xi = 4$, $B_{sw} = 10\,\mathrm{nT}$, $B_z = -5\,\mathrm{nT}$, and $V_{sw} = 400\,\mathrm{km/s}$, find the polar cap conductivity.

Chapter 9

Problem 9.1

Find the escape velocity from a height of 300 km. How long will it take for H^+ to be accelerated from 0 m/s to escape velocity in the presence of the ambipolar electric field, $M\mathbf{g}/2q_e$, where M is the O^+ mass? Assume **B** is parallel to **g**. For a temperature of 1500 K, what percentage of O^+, He^+, and H^+ atoms are in the loss cone (i.e., can escape)?

Problem 9.2

In hydrostatic equilibrium the velocity terms in Eq. (9.3) vanish. Show in this case that

$$E = \frac{(n_i M_i + n_j M_j)g}{2ne}$$

where n is the plasma density.

Problem 9.3

A shaped charge Ba^+ beam is observed to accelerate from 4 km/s to 70 km/s between 3000 and 5000 km altitude in the auroral zone. How much of this can be explained by conservation of the adiabatic invariant if the average initial launch angle is 10° from the magnetic field direction? Let $|\mathbf{B}|$ decrease by a factor of two. What is the total potential drop? What must the electric potential be to explain the difference?

Problem 9.4

The trough is collocated with a very large poleward electric field during substorms. If $E = 300$ mV/m and $B = 0.5$G, calculate the plasma density decrease as the plasma flows from midnight to the dusk terminator. Start with an initial longitudinal profile calculated assuming the F peak is at 300 km and decays as $e^{-\beta(300)t}$. Take $n(N_2) = 5 \times 10^8\,\mathrm{cm}^{-3}$ and use k_1 only.

Problem 9.5

Model the trough electric field as a Gaussian with characteristic scale of 50 km and a maximum value of 300 mV/m and assume an initial uniform plasma density with $\sum_p^F$ of 2 mho. Find the field-aligned current pairs. What are the peak values of $J_{||}$?

Problem 9.6

Show that the ion Pedersen drift peaks for $\nu_i = \Omega_i$ and find its magnitude compared to E/B for this condition. Find the neutral wind acceleration time constant under these conditions as a function of electron density.

Problem 9.7

In 9.5 find the divergence of the ion Pedersen flow velocity at the height where that velocity peaks ($\nu_i = \Omega_i$). From the continuity equation, find and plot the time constant versus distance for plasma density changes at this altitude due to the ion pump effect.

Problem 9.8

Ignoring the magnetospheric electric field, at what altitude in earth radii would corotation exceed the Alfvén speed in the equatorial plane? Take $n = 100\,\text{cm}^{-3}$, hydrogen ions to be dominant, and the magnetic field of the form $0.25\ R_E^3/(R_E + z)^3$ Gauss. Do the same calculations for Jupiter using its 10 hour rotation period, the magnetic field at the equator of $4R_J^3/(R_J + z)^3$ Gauss, a density of 10^4cm^{-3}, and $R_J = 71{,}800\,\text{km}$.

Problem 9.9

Let the topside F region density be of the form $n_0 e^{-(z-200)/H}$, where $n_0 = 10^5\text{cm}^{-3}$ and $H = 100\,\text{km}$ and consider an upward field-aligned current of $10\,\mu A/\text{cm}^2$ at 200 km. The magnetic field is of the form $B(z) = (0.5)[R_E/(z + R_E)]^3$ Gauss. Above 200 km there is no cross-field current, so the current density decreases in direct proportion to the magnetic field strength squared. For $T_e = 10T_i = 5000\,\text{K}$, find the altitudes above which (9.22) and (9.23) are satisfied if the ions are stationary and only the thermal electrons carry the current.

Chapter 10

Problem 10.1

Calculate the lifetime for plasma clouds of 1 and 10 km wavelength structures due to diffusion across the magnetic field. Take $D_\perp = 20\,\text{m}^2/\text{s}$, which corresponds to the ion diffusion coefficient. Would these structures survive transport across the polar cap if $E = 50$ mV/m and $B = 0.5$G? Assume the polar cap to be 2000 km across.

Problem 10.2

In Figure 10.3a, estimate the field-aligned current on each edge of the electric field structure if $\sum_P^E$ is uniform and equal to 1 mho. Take the satellite velocity to be 8000 m/s. Does this calculation agree with the measurements of $J_{||}$ plotted? Estimate the magnitude and sign of the charge density on each edge of the structure.

Problem 10.3

Find the shear frequency in Fig. 10.3 at ionospheric and equatorial altitudes. Compare the latter to the "pendulum" frequency in the equatorial plane using an effective gravity for a 50 keV proton at 4 Re($\sqrt{\text{“}g\text{”}/L}$). Does the condition for the Kelvin-Helmholtz instability hold?

Problem 10.4

Estimate the $\mathbf{E} \times \mathbf{B}$ instability growth rate in Figure 10.10a if the electric field 150 km north of Chatanika is eastward and 20 mV/m ($B = 0.5$ G).

Problem 10.5

Suppose the F peak density at 200 km is 10^{11} m^{-3} and falls off linearly until it reaches 10^7 m^{-3} at 1 R_e with height and then stays constant. Let the field-aligned current be carried by the electrons with velocity V_d and be equal to 20 $\mu A/m^2$ at 200 km. Let $T_e = 2000$ K be constant. Assume $R_e = 6360$ km. Find the height at which $V_d = V_e^{th}$.

Problem 10.6

If $J_{||} = 10\mu A/m^2$ and $\Sigma_p = 10$ mho, what is the plasma vorticity?

Problem 10.7

Estimate the plasma Richardson number corresponding to Fig. 10.3b where $R_{ip} = (g/L)/(V/L)^2$.

Problem 10.8

Estimate the field line-integrated conductivity in Fig. 10.18 for the E and F regions. Find γ for the E′ × B instability if $E_0' = 20$ mV/m, $L = 50$ km, and $B = 0.5 \times 10^{-4}$ T. Ignore diffusion.