Using a Spreadsheet for Numerical Methods

Several of the numerical methods discussed Chapter 7 can be implemented easily using an electronic spreadsheet program. Only a few of the spreadsheet program's facilities are needed. We refer to them in the text with a capital letter. If you are not familiar with these facilities, use the Help screens to find them and learn about their properties. Some information about Excel ® is included. Modifications for different spreadsheets, if necessary at all, will be straightforward.

Recalculate. Calculates all the formulas in the spreadsheet. Options may include Mode (Automatic/Manual), Number of iterations, Order (Natural/ Column-wise/Row-wise) and others. Not all spreadsheet programs have all options. In Manual mode, recalculation takes place when you press the Recalculation key (typically F9). Excel: Starting from the standard toolbar, select Tools – Options and the Calculation tab. Select Calculation-Manual. Check Iteration, and fill in Maximum Iterations 25, Maximum Change 0.001. (You can change these at any time.)

Copy. Copies a formula (or value) from a specified cell to other cells. Excel: Enter the desired formula or value in a cell. Select that cell. Place the cursor in the solid square in the lower right; it becomes a +. Drag to adjacent cells to fill them.

To copy to a whole block: copy first to a column (or row). Drag the + to adjacent rows (or columns).

Fill. Fills a specified block of cells with a sequence of numbers. The user specifies the starting number and the step. Excel: Enter the desired values in two adjacent cells. Select the two cells. Place the cursor in the solid square in the lower right; it becomes a +. Drag to adjacent cells. You can also use CTRL + C and CTRL + V.

1. Boundary value problems.

Suggested Layout

Use row 1 to register values of the independent variable x.
Use row 2 for values of and formulas for the approximate solution.
Use row 3 to store any constants necessary, such as Δx or n.
Use column A for labels.

Preparation

Set the Recalculation Mode to Manual, and Iterations to 25.

Setting up the Equations

- Fill a string of cells in row 1 with values of x_i for $i = 0, \cdots, n$.
- Write out the generic replacement equation at x_i, and solve it (on paper) algebraically for u_i.
- Insert the equivalent formula, using cell addresses, into a cell of row 2.

- Copy that formula into the cells of row 2 that correspond to the unknowns $u_2, \cdots, u_{n-1}$.
- Insert appropriate boundary conditions:

  For conditions specifying values, such as $u(0) = T$ or $u(1) = U$, where T and U are numbers, insert the numbers T or U into the appropriate cells.
  For conditions involving the derivative at 0 or 1, the replacement of the boundary condition becomes the equation for a cell outside the x-range.

Example 1.1. Find an approximate solution, with $n = 5$, of the problem (see Eqs. (1) to (7) in Section 7.1)

$$u'' - 12xu = -1, \qquad 0 < x < 1,$$
$$u(0) = 1, \qquad u(1) = -1.$$

First, follow the instructions for Preparation, setting the Recalculation Mode to Manual and Iterations to 25.

We are solving with $n = 5$, $\Delta x = 0.2$. Therefore, in row 1, fill cells *B1* to *G1* with the values 0, 0.2, 0.4, 0.6, 0,8, 1.0. The easiest way to do this is to use the Fill command with initial value 0 and step 0.2 in cells *B1..G1*. The spreadsheet looks like this:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1		0	0.2	0.4	0.6	0.8	1.0		
2									
3									

The generic replacement equation is given in Eq. (3) of Section 7.1. With $\Delta x = 0.2$, it becomes

$$25\,(u_{i+1} - 2u_i + u_{i-1}) - 12x_i u_i = -1 \tag{1}$$

for $i = 1, \cdots, n-1$. When this equation is solved algebraically for u_i we find

$$u_i = (1 + 25(u_{i+1} + u_{i-1}))/(50 + 12x_i). \tag{2}$$

Now, we record appropriate translations of this equation in the spreadsheet. Cells $B2..G2$ are going to carry values $u_0, \cdots, u_5$, so cell $C2$ carries the value u_1 (just below cell $C1$, which carries x_1). Equation (2), with $i = 1$, is $u_1 = (1 + 25(u_2 + u_0))/(50 + 12x_1)$. Cells $D2$ and $B2$ carry $u_{1+1} = u_2$ and $u_{1-1} = u_0$, respectively. Thus, we record this formula in cell $C2$:

$$C2 := (1 + 25 * (D2 + B2))/(50 + 12 * C1).$$

Now Copy the formula from $C2$ to the block $C2..F2$. Because the spreadsheet uses relative addresses, it interprets $B2$ as "the contents of the cell to the left" and $C1$ as "the contents of the cell above," so it puts the correct formula into each cell. For instance, cell $D2$ acquires the formula

$$(1 + 25 * (E2 + C2))/(50 + 12 * D1).$$

Next, insert the boundary conditions: set cell $B2$ to value 1 and cell $G2$ to value -1.

The last step is to perform the calculations. Press the Recalculation key. The numbers in the cells can be seen changing while the 25 iterations are being calculated. Press the Recalculation key again. If you see any changes, press it again. When no more changes occur, the numbers in row 2 satisfy the replacement equations. The spreadsheet looks like this:

	A	B	C	D	E	F	G	H	I
1		0	0.2	0.4	0.6	0.8	1.0		
2		1	0.634	0.290	-0.039	-0.419	-1		
3									

The numbers in row 2 can be used with the graphing facility to produce a graph of the approximate solution. □

Example 1.2. Solve the problem consisting of the same differential equation and the same boundary condition at $x = 0$, but with the boundary condition $u'(1) = -1$.

The replacement equation for the differential equation remains the same as the foregoing, but now it applies for $i = n$ as well. The second boundary condition is replaced by

$$(u_{n+1} - u_{n-1})/2\Delta x = -1. \tag{3}$$

Simplifying and using $n = 5$ and $\Delta x = 0.2$, we obtain the condition $u_6 = u_4 - 0.4$.

Now, in the spreadsheet, the replacement equation that applies to cells $C2..F2$ must be copied into cell $G2$ as well. That equation will contain a reference to the contents of cell $H2$, which corresponds to u_{n+1} (that is, u_6, since $n = 5$). The boundary condition requires that $u_6 = u_4 - 0.4$. Thus the formula for cell $H2$ is

$$H2 := F2 - 0.4.$$

(Remember that the spreadsheet program interprets an entry beginning with a letter as being a label. The leading equal sign tells the program that $F2$ means a cell address.)

We are ready to calculate the solution of the replacement equations. Simply press the Recalculation key several times, until no changes are seen in the numbers in the cells. The solution is shown in the following. Of course, we are interested only in the values in cells $B2..G2$.

	A	B	C	D	E	F	G	H	I
1		0	0.2	0.4	0.6	0.8	1.0		
2		1	0.720	0.468	-0.267	0.103	-0.062	-0.297	
3									

2. One-dimensional heat problems.

Suggested Layout

Use row 1 to register values of the independent variable x.
Use row 2 for the initial values, $u(x_i, 0)$.
Use column A for values of the independent variable t.

Preparation

Set the Recalculation Mode to Automatic.

Setting up the Equations

- Fill a string of cells in row 1 with values of x_i.
- Fill a string of cells in column A with the values of t_m.
- Apply appropriate boundary conditions.

  For conditions specifying values, such as $u(0,t) = T(t)$ or $u(1,t) = U(t)$, where U, T are given functions of t, fill the column under $x = 0$ or $x = 1$ with the appropriate formula. Because values of t appear in column A, the formulas will be easy to write.
  For conditions involving the derivative at 0 or 1, find the replacement for the boundary condition. Solve algebraically for $u_{-1}(m)$ or $u_{n+1}(m)$ and use as the replacement equation for the corresponding cell.

- Apply the initial condition $u(x,0) = f(x)$, by filling the first row of u's with the formula for $f(x)$, appropriately translated to spreadsheet formulas.
- Write out the generic replacement equation and solve it (on paper) for $u_i(m+1)$.
- Insert the equivalent formula, using cell addresses, into a cell of row 3.
- Copy the formula to the cells corresponding to unknown values of $u_1(m)$.

Example 2.1. Find a numerical solution of the heat problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \qquad 0 < t \tag{1}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad 0 < t \tag{2}$$

$$u(x,0) = x, \quad 0 < x < 1 \tag{3}$$

(Eqs. (4) to (7), Section 7.2). Use $\Delta x = 1/4$ and $r = 1/2$.

First, follow the instructions for Preparation. Next, Fill cells $C1..G1$ with the x-values 0, 0.25, 0.5, 0.75, 1, and Fill some cells in column A (say, $A2..A30$) with t-values. Because $\Delta t = r(\Delta x)^2$, we must have $\Delta t = 1/32 = 0.03125$. When these steps are completed, the top of the spreadsheet looks like this:

	A	B	C	D	E	F	G	H	
1			0	0.25	0.5	0.75	1		
2	0								
3	0.03125								
4	0.0625								
5	0.09375								
6	0.125								

Next, fill in the initial condition, $u(x, 0) = x$, in cells $C2..G2$, and the boundary conditions, $u(0, t) = 0$, $u(1, t) = 0$ in cells $C3..C30$ and $G3..G30$. Note that we have made the initial condition override the boundary condition. Now the spreadsheet looks like this:

	A	B	C	D	E	F	G	H	I
1			0	0.25	0.5	0.75	1		
2	0		0	0.25	0.5	0.75	1		
3	0.03125		0				0		
4	0.0625		0				0		
5	0.09375		0				0		
6	0.125		0				0		

The generic replacement equation for the simple heat equation is Eq. (8), Section 7.2. With $r = 0.5$, it becomes

$$u_i(m + 1) = 0.5(u_{i-1}(m) + u_{i+1}(m)).$$

In words, the value of u in any cell is the average of the values of u to the left and right in the row above it. Thus, we put in cell $D3$ the formula

$$D3 := 0.5 * (C2 + E2) \tag{5}$$

and copy this into cells $D3..F30$. The calculations are carried out automatically. The spreadsheet is filled with approximations to the solution of the original problem. The first few lines are the same as Table 4 of Section 7.2.

	A	B	C	D	E	F	G	H	I
1			0	0.25	0.5	0.75	1		
2	0		0	0.25	0.5	0.75	1		
3	0.03125		0	0.25	0.5	0.25	0		
4	0.0625		0	0.25	0.25	0.25	0		
5	0.09375		0	0.125	0.25	0.125	0		
6	0.125		0	0.125	0.125	0.125	0		

Example 2.2. Find a numerical solution of the heat problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \qquad 0 < t \tag{6}$$

$$u(0,t) = 100t, \quad \frac{\partial u}{\partial x}(1,t) = 0, \qquad 0 < t \tag{7}$$

$$u(x,0) = 0, \qquad 0 < x < 1. \tag{8}$$

Let us choose $\Delta x = 1/4$ and $r = 1/2$. We follow the same procedure as in the first example for filling in x's in the first row and t's in the first column. In row 2, we record the initial values of 0. In column C, record the left boundary condition $u(0, t_m) = 100t_m$. This can be done in several ways, but the one closest to the spirit of the problem is to treat t as a function of t (which it certainly is) by inserting in cell $C2$ the formula $C2 := 100 * A2$ and Copying to the block $C2..C30$. The top of the spreadsheet now looks like this:

	A	B	C	D	E	F	G	H	I
1			0	0.25	0.5	0.75	1		
2	0		0	0	0	0	0		
3	0.03125		3.125						
4	0.0625		6.25						
5	0.09375		9.375						
6	0.125		12.5						
7	0.15625		15.625						

The right-hand boundary condition requires zero gradient at $x = 1$. The replacement equation for this condition is

$$(u_{n+1}(m) - u_{n-1}(m))/2\Delta x = 0, \tag{9}$$

or $u_5(m) = u_3(m)$, because $n = 4$ here. Therefore, in cell $H2$ we place the formula $H2 := F2$, and Copy it down column H.

Finally, the replacement equation is the same as in previous cases,

$$u_i(m+1) = 0.5(u_{i-1}(m) + u_{i+1}(m)). \tag{10}$$

The equivalent of this equation is inserted into, say, cell $D3$ (see Eq. (5)) and Copied to all relevant cells in columns D through G. Numbers appear immediately throughout the block containing the formulas. The resulting spreadsheet looks like this (only two decimals are shown).

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1			0	0.25	0.5	0.75	1		
2	0		0	0	0	0	0	0	
3	0.03125		3.125	0	0	0	0	0	
4	0.0625		6.25	1.56	0	0	0	0	
5	0.09375		9.375	3.13	0.78	0	0	0	
6	0.125		12.5	5.08	1.56	0.39	0	0.39	
7	0.15625		15.625	7.03	2.73	0.78	0.39	0.78	

3. One-dimensional wave problems.

Suggested Layout

Use row 1 to register values of the independent variable x.
Use rows 2 and 3 for the initial values, $u(x_i, 0)$, $u(x_i, t_1)$.
Use column A for values of the independent variable t.

Preparation

Set the Recalculation Mode to Automatic.

Setting up the Equations

- Fill a string of cells in row 1 with values of x_i.
- Fill a string of cells in column A with the values of t_m.
- Fill row 2 with the initial condition, $u_i(0) = f(x_i)$.
- Fill row 3 using the starting equation (see Eq. (7), Section 7.3).
- Insert appropriate boundary conditions.

For conditions specifying boundary values, such as $u(0,t) = T(t)$ or $u(1,t) = U(t)$, fill the column under $x = 0$ or $x = 1$ with the appropriate formula. Because values of t appear in column A, the formulas will be easy to write.
For conditions involving the derivative at 0 or 1, find the replacement for the boundary condition. Solve algebraically for $u_{-1}(m)$ or $u_{n+1}(m)$ and use as the replacement equation for the corresponding cell.

- Write out the generic running equation. Insert the equivalent formula, using cell addresses, into a cell of row 4. Copy the formula to the cells corresponding to unknown values of $u_i(m)$.

Example 3.1. Find a numerical solution of this wave problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \qquad 0 < t \tag{1}$$

$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad 0 < t, \tag{2}$$

$$u(x,0) = 1 - |2x - 1|, \quad \frac{\partial u(x,0)}{\partial x} = 0, \qquad 0 < x < 1. \tag{3}$$

(See Eqs. (1) to (3), Section 7.3.) Use $\Delta x = \Delta t = 1/4$.

First, follow the instructions for Preparation. Next, Fill cells $C1..G1$ with the x-values 0, 0.25, 0.5, 0.75, 1, and Fill cells $A2..A30$ with t-values. When these steps are completed, the top of the spreadsheet looks like this:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1			0	0.25	0.5	0.75	1		
2	0								
3	0.25								
4	0.5								
5	0.75								

Next, fill in the initial condition in cells $C2..G2$, and the boundary conditions, $u(0,t) = 0$, $u(1,t) = 0$ in cells $C3..C30$ and $G3..G30$. Note that there is no conflict between initial and boundary conditions. Now the spreadsheet looks like this:

	A	B	C	D	E	F	G	H	I
1			0	0.25	0.5	0.75	1		
2	0		0	0.5	1	0.5	0		
3	0.25		0				0		
4	0.5		0				0		
5	0.75		0				0		

The starting equation, Eq. (7) of Section 7.3, with $g(x) = 0$, and $\rho = 1$ becomes

$$u_i(1) = 0.5(f(x_{i-1}) + f(x_{i+1})). \tag{4}$$

We can put the equivalent equation into cell $D2$,

$$D2 := 0.5 * (C1 + E1) \tag{5}$$

and Copy it to cells $D2..F2$. Calculation takes place immediately, and the top of the spreadsheet is

	A	B	C	D	E	F	G	H	I
1			0	0.25	0.5	0.75	1		
2	0		0	0.5	1	0.5	0		
3	0.25		0	0.5	0.5	0.5	0		
4	0.5		0				0		
5	0.75		0				0		

The general running equation for $\rho = 1$ is given by Eq. (9), Section 7.3,

$$u_i(m+1) = u_{i-1}(m) + u_{i+1}(m) - u_i(m-1). \tag{6}$$

Thus, we put in cell $D4$ the formula

$$D4 := C3 + E3 - D2 \tag{7}$$

and Copy this into cells $D4..F30$. The calculations are carried out automatically, and the spreadsheet is filled with approximations to the solution of the original problem. The first few lines are the same as Table 6.

	A	B	C	D	E	F	G	H	I
1			0	0.25	0.5	0.75	1		
2	0		0	0.5	1	0.5	0		
3	0.25		0	0.5	0.5	0.5	0		
4	0.5		0	0	0	0	0		
5	0.75		0	-0.5	-0.5	-0.5	0		
6	1.0		0	-0.5	-1	-0.5	0		
7	1.25		0	-0.5	-0.5	-0.5	0		

Example 3.2. Obtain a numerical solution of the following problem for t up to 4 using $\Delta x = \Delta t = 1/5$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \qquad 0 < t \tag{8}$$

$$u(0,t) = \sin(\pi t), \quad u(1,t) = 0, \qquad 0 < t \tag{9}$$

$$u(x,0) = 0, \qquad \frac{\partial u(x,0)}{\partial t} = 0, \qquad 0 < x < 1. \tag{10}$$

As Δx and Δt are both 0.2, we must construct anew the top row and left column of the spreadsheet. The boundary condition on the right, $u(1,t) = 0$, is satisfied by putting 0's into column H. To satisfy the boundary condition on the left, we insert this formula into cell $C2$,

$$C2 := \sin(PI() * A2), \tag{11}$$

and Copy it into cells $C2..C22$. The constant π is generated as a function with no argument.

Because both initial conditions are 0, the starting equation provides zero values for time t_1. Thus, cells $D2..G2$ and $D3..G3$ are filled with 0's. The top of the spreadsheet is

	A	B	C	D	E	F	G	H	I
1			0	0.2	0.4	0.6	0.8	1	
2	0		0	0	0	0	0	0	
3	0.2		0.588	0	0	0	0	0	
4	0.4		0.951					0	
5	0.6		0.951					0	
6	0.8		0.588					0	
7	1		0					0	
8	1.2		-0.588					0	

Finally, the block $D3..G22$ must be filled with the running equation. Because $\Delta x = \Delta t$, the running equation is the same as in Example 1. (See Eq. (7).) When the block receives the correct formulas, calculation takes place immediately, and the first few rows of the spreadsheet are:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1			0	0.2	0.4	0.6	0.8	1	
2	0		0	0	0	0	0	0	
3	0.2		0.588	0	0	0	0	0	
4	0.4		0.951	0.588	0	0	0	0	
5	0.6		0.951	0.951	0.588	0	0	0	
6	0.8		0.588	0.951	0.951	0.588	0	0	
7	1		0	0.588	0.951	0.951	0.588	0	
8	1.2		-0.588	0	0.588	0.951	0.951	0	
9	1.4		-951	-0.588	0	0.588	0.363	0	
10	1.6		-.951	-.951	-0.588	0	-0.363	0	

Inspection of a longer piece of the spreadsheet clearly shows that the function u grows steadily in amplitude. Figure 1 shows calculated values of $u(0.4, t)$ for $0 \leq t \leq 6$.

4. Potential problems.

Solving the potential equation and other related equations in two-dimensional regions is one of the most satisfying applications of spreadsheets to numerical methods. The problem can be set up in a simple natural way, and the numerical solution is displayed within the region where the problem is to be solved.

For simplicity, we start with problems on a rectangle. Then it will be easy to modify our procedures for nonrectangular regions.

Suggested layout

Use row 1 and column 1 to register values of the coordinates x and y.

Use a rectangular region for values of and formulas for the approximate solution.

Preparation

Set the Default Recalculation Mode to Manual and Iterations to 25.

Setting up

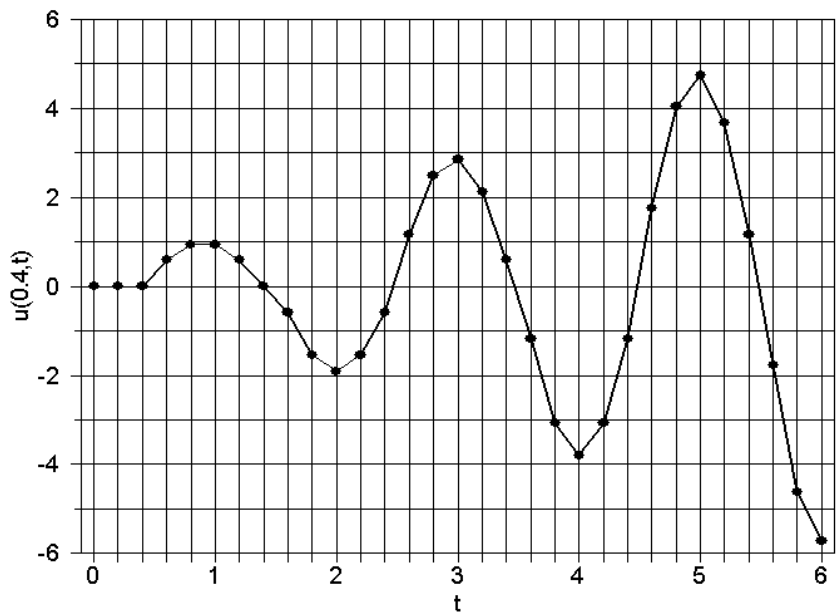

Figure 1: Approximate values of $u(0.4, t)$, solution of Eqs. (8) to (10).

- Fill strings of cells in row 1 and column 1 with values of the independent variables x_i and y_j. These define a rectangular block that will contain the approximate solution.
- Write out the generic replacement equation and solve it algebraically (on paper) for the unknown $u_{i,j}$.
- Insert the equivalent formula, using cell addresses, into a cell of the region and Copy it into the rest of the cells of the region.
- Fill in boundary conditions:

  For Dirichlet conditions (values of u given at boundary points) simply fill the appropriate cells with the given values.
  For conditions involving the normal derivative at boundary points, the replacement of the boundary condition becomes the equation for a cell outside the region.

Example 4.1. Solve this potential equation in the square numerically, using $\Delta x = \Delta y = 1/4$. (See Eqs. (3) to (6), Section 7.4.)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1, \qquad 0 < y < 1 \tag{1}$$

$$u(0, y) = 0, \qquad u(1, y) = 0, \qquad 0 < y < 1 \tag{2}$$

$$u(x, 0) = f(x), \quad u(x, 1) = f(x), \qquad 0 < x < 1 \tag{3}$$

where $f(x) = 1 - |2x - 1|$. (This function's graph is an isosceles triangle.)

First, follow the instructions for Preparation. Next, place values of the independent variables into row 1 and column 1. It is convenient to leave some empty space around the block that will contain the solution. Thus let us specify cell $C3$ (you might prefer $D4$) as the upper left corner of the region. Thus, we put the x-values into row 1 starting with cell $C1$ and y-values into column 1, starting with cell $A3$. Because $\Delta x = \Delta y = 1/4$, the spreadsheet now looks like this:

	A	B	C	D	E	F	G	H	I
1			0	0.25	0.5	0.75	1		
2									
3	1								
4	0.75								
5	0.5								
6	0.25								
7	0								

Now we fill in columns C and G with the zero boundary conditions. In rows 3 and 7, we fill in the nonzero boundary conditions. In cell $C3$ place the formula

$$C3 := 1 - ABS(2 * C\$1 - 1)$$

and copy it to cells $C3..G3$, then to cells $C7..G7$. The form of address $C\$1$ means that the *column* is a relative address, but the *row* is an absolute address. Thus, when this formula is copied, cell $D3$ contains the formula

$$D3 := 1 - ABS(2 * D\$1 - 1)$$

and cell $D7$ contains exactly the same formula. Computation of values takes place immediately and the spreadsheet looks like this.

	A	B	C	D	E	F	G	H	I
1			0	0.25	0.5	0.75	1		
2									
3	1		0	0.5	1	0.5	0		
4	0.75		0				0		
5	0.5		0				0		
6	0.25		0				0		
7	0		0	0.5	1	0.5	0		

Finally, the generic replacement for the potential equation is

$$u_N + u_S + u_E + u_W - 4u_i = 0 \tag{4}$$

(See Eq. (7), Section 7.4.) Solved algebraically for u_i it becomes

$$u_i = (u_N + u_S + u_E + u_W)/4. \tag{5}$$

The equation must be translated into an equivalent, using cell addresses. We insert into cell $D4$ the formula

$$D4 := 0.25 * (D3 + D5 + C4 + E4)$$

and copy this into the entire block $D4..F6$. Numbers appear immediately. Press the Recalculation key. In this case, the correct approximate solution appears after one cycle of recalculations. The spreadsheet looks like this (compare Fig. 5):

	A	B	C	D	E	F	G
1			0	0.25	0.5	0.75	1
2							
3	1		0	0.5	1	0.5	0
4	0.75		0	0.3125	0.5	0.3125	0
5	0.5		0	0.25	0.375	0.25	0
6	0.25		0	0.3125	0.5	0.3125	0
7	0		0	0.5	1	0.5	0
8							

Example 4.2. Solve the problem made from the one in Example 4.1 by changing the boundary condition at $y = 0$ to

$$\frac{\partial u(x,0)}{\partial y} = 0, \qquad 0 < x < 1.$$

The same setup can be reused with two changes. First, the replacement for the potential equation must be applied to cells $D7..F7$, which comprise the bottom boundary of the region. Second, we must find and apply the replacement of the boundary condition. If point number i is a point on the bottom boundary, then the boundary condition there is

$$(u_N - u_S)/2\Delta x = 0, \quad or \quad u_N = u_S \tag{6}$$

Thus, we put into cell $D8$ the formula $D8 := D6$ and Copy it into cells $D8..F8$. (Recall that a formula must not begin with a letter, so we use the leading plus sign to indicate that the letter D is a cell address.)

Numbers appear (or change) immediately in the rectangle of unknowns. After a few Recalculation cycles, these numbers stabilize to the correct approximate solution, and the spreadsheet looks like this (only three decimals are shown here):

	A	B	C	D	E	F	G	H	I
1			0	0.25	0.5	0.75	1		
2									
3	1		0	0.5	1	0.5	0		
4	0.75		0	0.268	0.436	0.268	0		
5	0.5		0	0.138	0.206	0.138	0		
6	0.25		0	0.077	0.112	0.077	0		
7	0		0	0.060	0.086	0.060	0		
8				0.077	0.112	0.077			

The numbers in cells $D8..F8$ have no interpretation in terms of the original problem.

Example 4.3. Torsion stresses in a prismatic beam can be found from the solution of Poisson's equation, $\nabla^2 u = -H$, on a region that is the cross section of the beam. The boundary condition is $u = 0$ on the boundary. (The region must be simply connected. Stresses are partial derivatives of u.) Appropriate scaling allows us to replace the constant H by 1 or by any other convenient constant. To find torsion stresses in a beam whose cross section is a particular L-shaped region $\mathcal{R}$, we would solve this problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = -H \quad \text{in} \quad \mathcal{R} \tag{7}$$

$$u(x, y) = 0 \quad \text{for} \quad \text{points} \quad (x, y) \quad \text{on} \quad \text{the} \quad \text{boundary}. \tag{8}$$

Assume that the L-shaped region is a 1×1 square from which a $1/2 \times 1/2$ corner has been removed. Solve numerically with $\Delta x = \Delta y = 1/10$.

It will be convenient for us to assume that the constant H is 100. It turns out that the values of u will lie between 0 and 10. Thus the display will not contain many leading 0's. The replacement for the partial differential equation is

$$(u_N + u_S + u_E + u_W - 4u_i)/(\Delta x)^2 = -100. \tag{9}$$

Solved for u_i, this equation becomes

$$\begin{aligned} u_i &= (100(\Delta x)^2 + u_N + u_S + u_E + u_W)/4, \quad \text{or} \\ u_i &= (1 + u_N + u_S + u_E + u_W)/4. \end{aligned} \tag{10}$$

Here, we have used the fact that $\Delta x = 1/10$ and $100(\Delta x)^2 = 1$.

In the spreadsheet, set up values of x and y in row 1 and column 1. We will assume that the removed quarter of the region is in the lower right corner.

The jobs to be done are: set the boundary conditions and fill internal cells with the equivalent of Eq. (10). Of course there are many ways to accomplish these, but the shape of the region suggests these steps.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1			0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
2													
3	0		0	0	0	0	0	0	0	0	0	0	0
4	0.1		0	1.08	1.66	1.94	2.03	2.00	1.90	1.74	1.46	0.96	0
5	0.2		0	1.66	2.62	3.07	3.18	3.07	2.87	2.59	2.14	1.37	0
6	0.3		0	1.94	3.07	3.56	3.54	3.23	2.93	2.61	2.15	1.37	0
7	0.4		0	2.03	3.18	3.54	3.20	2.36	2.02	1.77	1.47	0.96	0
8	0.5		0	2.00	3.07	3.23	2.36	0	0	0	0	0	0
9	0.6		0	1.90	2.87	2.93	2.02	0	0	0	0	0	0
10	0.7		0	1.74	2.59	2.61	1.77	0	0	0	0	0	0
11	0.8		0	1.46	2.14	2.15	1.47	0	0	0	0	0	0
12	0.9		0	0.96	1.37	1.37	0.96	0	0	0	0	0	0
13	1		0	0	0	0	0	0	0	0	0	0	0
14													
15													

(1) Fill the block $C3..M13$ with 0's.

(2) Put the equivalent of Eq. (10) into cell $D4$, the upper left cell of the interior of the L-shaped region. The formula is

$$D4 := 0.25 * (1 + D3 + D5 + C4 + E4).$$

(3) Copy this formula into the block $D4..L12$. At this point, we are ready to find the numerical solution of the Poisson equation in a square.

(4) Fill the "removed corner," $H8..M13$, with 0's.

Now the spreadsheet is ready for the computation. After a few Recalculation cycles, the spreadsheet contains the values of the approximate solution, as shown below. The graph in Fig. 2 was made from this solution.

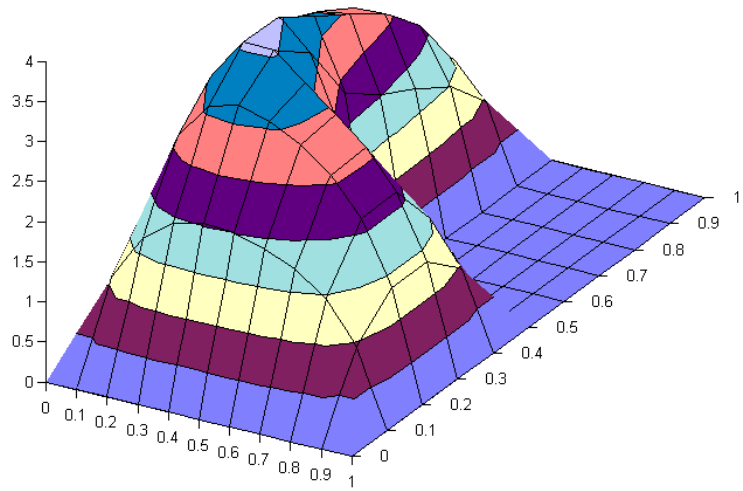

Figure 2: Numerical solution of Poisson equation on an L-shaped region.

Exercises

1. A current-carrying wire is exposed in its middle section to convection but is encased by ceramic insulation near its ends. (See Fig. 3.) The temperature $u(x)$ in the wire is governed by different differential equations in different parts of the wire:

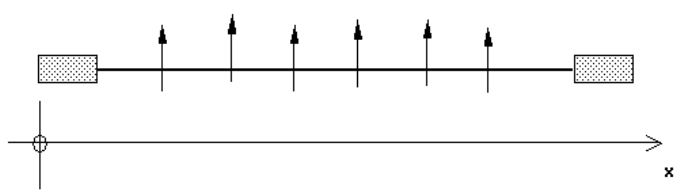

Figure 3: Current-carrying wire exposed to convection in the middle, with ceramic insulators at the ends.

$$
\begin{aligned}
\frac{d^2u}{dx^2} &= -H, && 0 < x < aL, \\
\frac{d^2u}{dx^2} &= -H - b^2(T-u), && aL < x < (1-a)L \\
\frac{d^2u}{dx^2} &= -H, && (1-a)L < x < L.
\end{aligned}
$$

Note that u'' jumps in value at aL and $(1-a)L$, but u and u' are continuous there.

In these equations, H is a constant that represents resistance heating, b^2 contains a heat transfer coefficient and geometric parameters, and a is the fraction of the wire that is covered at each end by the ceramic insulation. For boundary conditions, we will assume that the temperature at the endpoints of the wire is the same as that of the surrounding air,

$$u(0) = T, \qquad u(L) = T.$$

Because of the obvious symmetry with respect to the midpoint $x = 1/2L$, we can simplify the problem to

$$\begin{aligned}
&\frac{d^2u}{dx^2} = -H, && 0 < x < aL,\\
&\frac{d^2u}{dx^2} = -H - b^2(T-u), && aL < x < \tfrac{1}{2}L\\
&u(0) = T, && \frac{du(\frac{1}{2}L)}{dx} = 0.
\end{aligned}$$

Now, remove the dimensions by defining $X = x/L$, $K = b^2L^2$, $U = (u-T)(b^2/H)$, and the problem becomes

$$\begin{aligned}
&\frac{d^2U}{dX^2} = -K, \qquad 0 < X \le a,\\
&\frac{d^2U}{dX^2} = -K(1-U), \qquad a < X < 0.5\\
&U(0) = 0, \qquad \frac{dU(0.5)}{dX} = 0.
\end{aligned}$$

Solve this problem numerically, taking $\Delta X = 1/20$, $a = 0.125$, and $K = 0.25/\Delta X^2$. Graph the solution. Other interesting values for K are $0.5\Delta X^2$ and $0.1\Delta X^2$.

2. In Chapter 0, we derived the following nonlinear problem as a model of the displacement of a hanging cable loaded by its own weight.

$$\begin{aligned}
\frac{d^2u}{dx^2} &= (w/T)\sqrt{1+\left(\frac{du}{dx}\right)^2}, \qquad 0 < x < a\\
u(0) &= h_0, \qquad u(a) = h_1.
\end{aligned}$$

Remove the dimensions by changing variables to $X = x/a$, $U = (u-h_0)/a$. Then the problem becomes

$$\begin{aligned}
\frac{d^2u}{dx^2} &= K\sqrt{1+\left(\frac{du}{dx}\right)^2}, \qquad 0 < x < 1,\\
U(0) &= 0, \qquad U(1) = H,
\end{aligned}$$

where $K = wa/T$ and $H = (h_1 - h_0)/a$. Set up the replacement equations for the numerical solution of this problem.

3. Use a spreadsheet program to solve numerically the problem in Exercise 2. Take $n = 10$, $K = 1$ and $H = 0$. Compare the numerical solution to the exact solution found in Exercises 4 and 5 of Chapter 0, Section 3.

4. Problems in cylindrical coordinates often present difficulties even when it is possible to write out the solution in terms of Bessel functions. One such problem describes the steady-state temperature $U(r)$ in an annulus:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) - k^2(U - T) = 0, \qquad a < r < b$$

$$U(a) = T_0, \qquad U(b) = T_1.$$

(Compare Chapter 5, Section 5, Exercise 9.)

Introduce dimensionless variables $x = (r - a)/(b - a)$ and $u = (U - T)/M$, where M is the largest of the three quantities $T_0 - T$, $T_1 - T$, $T_1 - T_0$. Then the problem becomes

$$\frac{1}{s + x}\frac{d}{dx}\left((s + x)\frac{du}{dx}\right) - AU = 0, \qquad 0 < x < 1$$

$$u(0) = h_0, \qquad u(1) = h_1$$

where $A = k^2(b - a)^2$, and $s = a/(b - a)$ is a shape parameter.

a. If $M = 0$, what is the solution of the dimensional problem?

b. If $M \neq 0$, find h_0 and h_1 in terms of other parameters, and show that they are both between -1 and 1.

5. Obtain a numerical solution of the problem in Exercise 4 using $\Delta x = 1/10$, $A = 20$, $h_0 = 1$, $h_1 = 0$. Try the two values $s = 0.1$ and $s = 1.0$; sketch annuli for these shape parameters.

6. A problem of diffusion in a material with variable properties may be expressed as

$$D(x)\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad 0 < t$$

$$u(0, t) = 0, \qquad u(1, t) = 0, \quad 0 < t$$

$$u(x, 0) = 100, \quad 0 < x < 1$$

with $D(x) = 1 + \alpha x$. If $\alpha = 0$, this problem can be solved analytically as in Chapter 2, Section 3. An important question is this: if $\alpha \neq 0$, how different in the solution? In particular, how noticeable is the induced asymmetry?

Study these questions by solving numerically with $\alpha = 0$, 0.5, 1. Use $\Delta x = 1/10$, $\Delta t = 1/400$ ($r = 1/4$).

7. Miscellaneous Exercises 18 to 20 of Chapter 3 give equations that model the phenomenon of water hammer. Initially, water is flowing steadily through a pipe from a large reservoir to open air. At time 0, a valve is closed suddenly at the end of the

pipe, causing violent changes in pressure and velocity in the water. If the equations are combined, we can obtain this problem for dimensionless pressure:

$$\begin{aligned}
&\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2}, && 0 < x < 1, \quad 0 < t \\
&p(0,t) = 0, \quad \frac{\partial p}{\partial x}(1,t) = 0, && 0 < t, \\
&p(x,0) = -x, \quad \frac{\partial p}{\partial t}(x,0) = 0, && 0 < x < 1.
\end{aligned}$$

Solve this problem numerically using $\Delta x = \Delta t = 1/10$ for time 0 to 2.

8. Find the steady-state temperature in a square plate that is insulated along the lower half of each vertical side and has controlled temperature on the remaining parts of its sides, as shown in Fig. 4. Because of symmetry, we can solve the problem in just one-half of the plate and use the symmetry condition

$$\frac{\partial u}{\partial x}(1,y) = 0$$

along the center line. The full problem is

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, && 0 < x < 2, \quad 0 < y < 2, \\
&u(x,2) = 100, \quad u(x,0) = 0, && 0 < x < 2, \\
&u(0,y) = 100, \quad u(2,y) = 100, && 1 \le y < 2, \\
&\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(2,y) = 0, && 0 < y < 1.
\end{aligned}$$

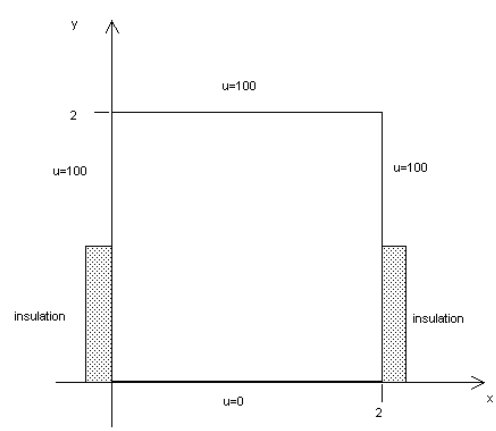

Figure 4: Partially insulated plate.

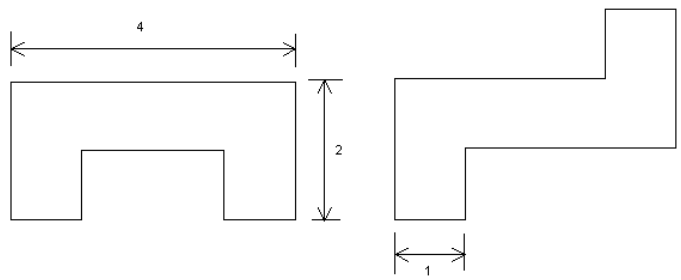

Figure 5: Beam sections.

9. Part of a certain structure is a beam subjected to torsion. The engineer designing the structure must choose one of the two cross sections shown in Fig. 5. The criterion is to be torsional rigidity, which is proportional to the integral over the cross section of the function u that satisfies Poisson's equation $\nabla^2 = -1$ in the cross section and $u = 0$ on the boundary.

For each cross section: (1) solve the Poisson equation numerically using $\Delta x = \Delta y = 1/4$; (2) approximate the required integral as $\Delta x \Delta y$ times the sum of the values of u.

Which section gives greater rigidity?