Web Notes, Chapter 4: Reference to Footnote 1

Consider the simple consumption allocation problem:

$$\max - c_1^{-\gamma_1} c_2^{-\gamma_1}, \quad \gamma_1 > 0 \;\; \gamma_2 > 0$$

s.t. $p_1 c_1 + p_2 c_2 \le Y, \quad p_1,\ p_2,\ Y > 0$, given

$$= \max - c_1^{-\gamma_1} \left(\frac{Y - p_1 c_1}{p_2} \right)^{-\gamma_2}$$

N/S first order condition:

$$\frac{\partial u(\,)}{\partial c_1} = 0 = \gamma_1 c_1^{-\gamma_1 - 1} \left(\frac{Y - p_1 c_1}{p_2} \right)^{-\gamma_2} + \gamma_2 c_1^{-\gamma_1} \left(\frac{Y - p_1 c_1}{p_2} \right)^{-\gamma_2 - 1} \left(\frac{-p_1}{p_2} \right)$$

$$\gamma_1 c_1^{-\gamma_1 - 1} \left(\frac{Y - p_1 c_1}{p_2} \right)^{-\gamma_2} = \gamma_2 c_1^{-\gamma_1} \left(\frac{Y - p_1 c_1}{p_2} \right)^{-\gamma_2 - 1} \frac{p_1}{p_2}$$

$$\gamma_1 c_1^{-1} = \gamma_2 \left(\frac{Y - p_1 c_1}{p_2} \right)^{-1} \frac{p_1}{p_2}$$

$$\frac{\gamma_1}{c_1} = \gamma_2 \left(\frac{\cancel{p_2}}{Y - p_1 c_1} \right)^{-1} \frac{p_1}{\cancel{p_2}}$$

$$\frac{\gamma_1}{c_1} = \frac{\gamma_2 p_1}{Y - p_1 c_1}$$

$$\gamma_1 Y - \gamma_1 p_1 c_1 = \gamma_2 p_1 c_1$$

$$\gamma_1 Y = (\gamma_1 p_1 + \gamma_2 p_1) c_1$$

$$p_1 c_1 = \left(\frac{\gamma_1}{\gamma_1 + \gamma_2} \right) Y$$

$$p_2 c_2 = \left(\frac{\gamma_2}{\gamma_1 + \gamma_2} \right) Y$$

Therefore, at the optimum, utility is given by:

$$-c_1^{-\gamma_1} c_2^{-\gamma_2} = -\left[\left(\frac{\gamma_1}{\gamma_1+\gamma_2}\right)\frac{Y}{p_1}\right]^{-\gamma_1}\left[\left(\frac{\gamma_2}{\gamma_1+\gamma_2}\right)\frac{Y}{p_2}\right]^{-\gamma_2}$$

$$= -\left[\left(\frac{\gamma_1}{\gamma_1+\gamma_2}\right)\frac{1}{p_1}\right]^{-\gamma_1}\left[\left(\frac{\gamma_2}{\gamma_1+\gamma_2}\right)\frac{1}{p_2}\right]^{-\gamma_2} \quad Y_1^{-(\gamma_1+\gamma_2)}$$

$$\underset{\text{(same ordering)}}{\equiv} \quad -\frac{1}{\gamma_1+\gamma_2}\underbrace{\kappa(\gamma_1, \gamma_2, P_1, P_2)}_{\text{positive}}Y^{-(\gamma_1+\gamma_2)}$$

Let γ be defined by

$$1-\gamma = -(\gamma_1+\gamma_2) \Rightarrow \gamma = 1+\gamma_1+\gamma_2 \geq 1$$

$$\equiv \quad \frac{1}{1-\gamma}\kappa(\gamma_1, \gamma_2, p_1, p_2)Y^{1-\gamma}$$

$$\kappa(\gamma_1+\gamma_2, p_1, p_2) = \left[\left(\frac{\gamma_1}{\gamma_1+\gamma_2}\right)\frac{1}{p_1}\right]^{-\gamma_1}\left[\left(\frac{\gamma_2}{\gamma_1+\gamma_2}\right)\frac{Y}{p_2}\right]^{-\gamma_2} > 0$$

From the consumer's perspective, $\kappa(\gamma_1, \gamma_2, p_1, p_2)$ is a positive constant.