CODES FOR CHAPTER 16

`P1sor.f90`, `P1sor.cpp`

Subroutine `P1sor` provides the solution to equation (16.38) with its boundary condition (16.49) for a two-dimensional (rectangular or axisymmetric cylinder) enclosure with reflecting walls and an absorbing, emitting, linear-anisotropically scattering medium.

Input:

`II`	=	Number of nodes in x-direction
`JJ`	=	Number of nodes in y- or r-direction
`KK`	=	`0` for rectangular, `KK=1` for cylindrical enclosure
`IRE`	=	Radiative equilibrium identifier; `IRE=0`: no equilibrium; `IRE=1`: radiative equilibrium
`L`	=	Length of enclosure (in cm)
`R`	=	Height (rectangle) or radius (cylinder) of enclosure (in cm)
`EPSX`	=	Wall emittances, `EPSX(1)` at `X=0, EPSX(2)` at `X=L`
`EPSR`	=	Wall emittances, `EPSR(1)` at `Y=0` (for rectangle only), `EPSY(2)` at `Y,r=R`
`SX`	=	Sources at x-direction walls: `SX(1,j=1,2,...JJ)` source at $x = 0$ for varying y/r-nodes `SX(2,j=1,2,...JJ)` source at $x = L$ for varying y/r-nodes (for a standard, gray application `SX` = $4\sigma T^4$, in W/cm^2)
`SR`	=	Sources at y, r-direction walls: `SR(1,i=1,2,...II)` source at $y = 0$ for varying x-nodes (for rectangle only) `SR(2,i=1,2,...II)` source at $y, r = R$ for varying x-nodes (for a standard, gray application `SR` = $4\sigma T^4$, in W/cm^2)
`KT`	=	Absorption coefficient for all internal nodes (in cm^{-1})
`ST`	=	Scattering coefficient for all internal nodes (in cm^{-1})
`A1`	=	Linear anisotropy factor for all internal nodes
`SS`	=	Sources for all internal nodes (in cm^{-1}) (for a standard, gray application `SS` = $4\sigma T^4$, in W/cm^2)

Output:

`G`	=	Incident radiation for all internal nodes, (in W/cm^2)
`QX`	=	Fluxes at x-direction walls: `QX(1,j=1,2,...JJ)` flux at $x = 0$ for varying y/r-nodes `QX(2,j=1,2,...JJ)` flux at $x = L$ for varying y/R-nodes (positive into positive x-direction, in W/cm^2)
`QR`	=	Fluxes at x-direction walls: `QR(1,i=1,2,...II)` flux at $y = 0$ for varying x-nodes (for rectangle only) `QR(2,i=1,2,...II)` flux at $y, r = R$ for varying x-nodes (positive into positive r, y-direction, in W/cm^2)

Calculations can be done for a gray medium or, on a spectral basis, for a nongray medium. For a gray medium the user may either specify a temperature field (`IRE=0`) by supplying `SS=` $4n^2\sigma T^4$, or radiative equilibrium may be invoked (`IRE=1`), in which case the heat generation term `SS=` $\dot{Q}'''$ must be input. Note that radiative equilibrium is not possible on a spectral level.

Width `L` is broken up into `II` equally spaced nodes with spacing `Δx = L/(II − 1)`; similarly height/radius `R` is broken up into `JJ` equally spaced nodes with spacing `Δr = R/(JJ − 1)`.

For each of the `II`×`JJ` nodes each of the radiative properties (κ = `KT`, σ_s = `ST`, A_1 = `A1`) must be input, as well as the local radiative source `SS` (= $4\pi I_b$ if `IRE=0`, or = $\dot{Q}'''$ if `IRE=1`). In addition, for each surface an emittance must be specified [$\epsilon(x=0)$ = `EPSX(1)`, $\epsilon(x=L)$ = `EPSX(2)`; $\epsilon(y=0)$= `EPSR(1)` for rectangular enclosures only, and $\epsilon(r_{\rm or} y=R)$ = `EPSR(2)`], as well as radiation sources [$4\pi I_{bw}(x=0)$ = `SX(1)`, $4\pi I_{bw}(x=L)$ = `SX(2)`; $4\pi I_{bw}(y=0)$ = `SR(1)` for rectangular enclosures only, and $4\pi I_{bw}(r_{\rm or} y=R)$ = `SR(2)`]. Insulated boundaries can be treated by setting the emittance of that surface to zero. One-dimensional problems can be treated by setting two opposing emittances to zero; for better efficiency the number of nodes in the cross-direction should be set to one. Thus, `EPSR(1) = EPSR(2) = 0` and `JJ = 1` makes the problem a one-dimensional slab, while `EPSX(1) = EPSX(2) = 0` and `II = 1` makes a one-dimensional cylinder.

Upon return `P1sor` provides the solution array `G` (incident radiation G for all `II`×`JJ` nodes), as well as flux vectors `QX` (for radiative fluxes at the two surfaces $x=0$ and $x=L$) and `QY` (radiative fluxes at $y=0$ for a rectangle, and $r_{\rm or} y=R$). The solution is found by *successive over-relaxation*, with over-relaxation parameter `OM`, which is optimized by an implementation of algorithm 9-6.1 given in [1].

Code Details

For a two-dimensional problem equation (16.38) may be rewritten as

$$\begin{aligned} -\frac{1}{3}\frac{1}{r^k}\frac{\partial}{\partial r}\left(\frac{r^k}{\beta^*}\frac{\partial G}{\partial r}\right)+\frac{\partial}{\partial x}\left(\frac{1}{\beta^*}\frac{\partial G}{\partial x}\right) &= \kappa(4\pi I_b - G) \text{ temperature specified,}\\ &= \dot{Q}''' \text{ radiative equilibrium,} \end{aligned} \tag{CC-16-1}$$

where $\beta^* = \beta - A_1\sigma_s/3$; `KK = 0` makes it a rectangular enclosure, and `KK = 1` makes it an axisymmetric cylinder. Standard central finite differencing with equal spacing `Δr = R/(JJ − 1)` and `Δx = L/(II − 1)` and $\lambda = \Delta x/\Delta r$ produces an equation for each (internal and boundary) node:

$$A_{ij}G_{i-1,j} + B_{ij}G_{i+1,j} + C_{ij}G_{i,j-1} + D_{ij}G_{i,j+1} - E_{ij}G_{ij} = -F_{ij}, \tag{CC-16-2}$$

where

$$\begin{aligned}
A_{ij} &= \frac{\beta^*_{ij}}{\beta^*_{i-1/2,j}} \simeq \frac{2\beta^*_{ij}}{\beta^*_{i-1,j}+\beta^*_{ij}}\\
B_{ij} &= \frac{\beta^*_{ij}}{\beta^*_{i+1/2,j}} \simeq \frac{2\beta^*_{ij}}{\beta^*_{ij}+\beta^*_{i+1,j}}\\
C_{ij} &= \lambda^2\frac{\beta^*_{ij}}{\beta^*_{i,j-1/2}}\left(\frac{r_{j-1/2}}{r_j}\right)^k \simeq \lambda^2\frac{2\beta^*_{ij}}{\beta^*_{i,j-1}+\beta^*_{ij}}\left(1-\frac{1}{2(j-1)}\right) \text{ since } r_j=(j-1)\Delta r\\
D_{ij} &= \lambda^2\frac{\beta^*_{ij}}{\beta^*_{i,j+1/2}}\left(\frac{r_{j+1/2}}{r_j}\right)^k \simeq \lambda^2\frac{2\beta^*_{ij}}{\beta^*_{ij}+\beta^*_{i,j+1}}\left(1+\frac{1}{2(j-1)}\right)\\
E_{ij} &= \begin{cases} 3\kappa_{ij}\beta^*_{ij}\Delta x^2 + A_{ij}+B_{ij}+C_{ij}+D_{ij} & \text{temperature specified,}\\ A_{ij}+B_{ij}+C_{ij}+D_{ij} & \text{radiative equilibrium,}\end{cases}\\
F_{ij} &= \begin{cases} 3\kappa_{ij}\beta^*_{ij}\Delta x^2 \mathtt{SS}_{ij} & \text{temperature specified } (\mathtt{SS}_{ij}=4\pi I_{bij}),\\ 3\beta^*_{ij}\Delta x^2 \mathtt{SS}_{ij} & \text{radiative equilibrium } (\mathtt{SS}_{ij}=\dot{Q}'''_{ij}).\end{cases}
\end{aligned}$$

Boundary conditions equation (16.49) are written as, and finite-differenced using artificial nodes ($i=0$ at

$x = 0$, $i = \texttt{II}$ at $x = L$, $j = 0$ at $r = 0$ and $j = \texttt{JJ}$ at $r = R$)

$$\begin{aligned}
x = 0: &\quad \frac{\partial G}{\partial x} - \texttt{bx}(1)\beta^* \left[G - \texttt{SX}(1)\right] = 0 \quad \text{where} \quad \texttt{bx}() = \frac{3}{2}\frac{\epsilon}{2-\epsilon}, \quad \texttt{SX}() = 4\pi I_{bw} \\
x = L: &\quad \frac{\partial G}{\partial x} + \texttt{bx}(2)\beta^* \left[G - \texttt{SX}(2)\right] = 0 \\
r = 0: &\quad \frac{\partial G}{\partial r} - \texttt{br}(1)\beta^* \left[G - \texttt{SR}(1)\right] = 0 \quad \text{(rectangular enclosure, } \texttt{KK} = \texttt{0}\text{, only)} \\
r = R: &\quad \frac{\partial G}{\partial r} - \texttt{br}(2)\beta^* \left[G - \texttt{SR}(2)\right] = 0
\end{aligned}$$

or, with $\beta^* = \texttt{BT}$

$$\begin{aligned}
x = 0\ (i = 1): &\qquad G_{i-1,j} - G_{i+1,j} + 2\texttt{bx}(1)\,\Delta\texttt{x}\,\texttt{BT}_{ij}\left(G_{ij} - \texttt{SX}_j(1)\right) = 0 \\
x = L\ (i = \texttt{II}): &\qquad G_{i+1,j} - G_{i-1,j} + 2\texttt{bx}(2)\,\Delta\texttt{x}\,\texttt{BT}_{ij}\left(G_{ij} - \texttt{SX}_j(2)\right) = 0 \\
r = 0\ (j = 1): &\qquad G_{i,j-1} - G_{i,j+1} + 2\texttt{br}(1)\,\Delta\texttt{r}\,\texttt{BT}_{ij}\left(G_{ij} - \texttt{SR}_i(1)\right) = 0 \quad (\texttt{KK} = \texttt{0}\text{ only}) \\
r = R\ (j = \texttt{JJ}): &\qquad G_{i,j+1} - G_{i,j-1} + 2\texttt{br}(2)\,\Delta\texttt{r}\,\texttt{BT}_{ij}\left(G_{ij} - \texttt{SR}_i(2)\right) = 0
\end{aligned}$$

Eliminating the artificial nodes between internal node and boundary node equations yields the updated values

$$\begin{aligned}
i = 1: &\quad A'_{ij} = 0, B'_{ij} = A_{ij} + B_{ij}, E'_{ij} = E_{ij} + 2\texttt{bx}(1)\,\Delta\texttt{x}\,\texttt{BT}_{ij}A_{ij} \\
&\quad F'_{ij} = F_{ij} + 2\texttt{bx}(1)\,\Delta\texttt{x}\,\texttt{BT}_{ij}A_{ij}\texttt{SX}_j(1) \\
i = \texttt{II}: &\quad B'_{ij} = 0, A'_{ij} = A_{ij} + B_{ij};\ E'_{ij} = E_{ij} + 2\texttt{bx}(2)\,\Delta\texttt{x}\,\texttt{BT}_{ij}B_{ij} \\
&\quad F'_{ij} = F_{ij} + 2\texttt{bx}(2)\,\Delta\texttt{x}\,\texttt{BT}_{ij}B_{ij}\texttt{SX}_j(2) \\
j = 1: &\quad C'_{ij} = 0, D'_{ij} = C_{ij} + D_{ij}, E'_{ij} = E_{ij} + 2\texttt{br}(1)\,\Delta\texttt{r}\,\texttt{BT}_{ij}C_{ij} \\
&\quad F'_{ij} = F_{ij} + 2\texttt{br}(1)\,\Delta\texttt{r}\,\texttt{BT}_{ij}C_{ij}\texttt{SR}_j(1) \\
j = \texttt{JJ}: &\quad D'_{ij} = 0, C'_{ij} = C_{ij} + D_{ij}, E'_{ij} = E_{ij} + 2\texttt{br}(2)\,\Delta\texttt{r}\,\texttt{BT}_{ij}D_{ij} \\
&\quad F'_{ij} = F_{ij} + 2\texttt{br}(2)\,\Delta\texttt{r}\,\texttt{BT}_{ij}D_{ij}\texttt{SR}_j(2)
\end{aligned}$$

For a cylindrical enclosure ($\texttt{KK} = 1$) the boundary condition at $r = 0$ ($\texttt{J} = 1$) becomes

$$r = 0, (j = 1): \frac{\partial G}{\partial r} = 0 \quad \text{or} \quad G_{i,j-1} = G_{i,j+1}.$$

Also, the governing equation (CC-16-1) becomes indeterminate. Expanding the radial derivative and using De l'Hopital's rule, we obtain

$$\begin{aligned}
\lim_{r\to 0}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\beta^*}\frac{\partial G}{\partial r}\right) &= \frac{1}{\beta^*}\frac{\partial^2 G}{\partial r^2} - \frac{1}{\beta^{*2}}\frac{\partial G}{\partial r}\frac{\partial \beta^*}{\partial r} + \lim_{r\to 0}\frac{1}{r\beta^*}\frac{\partial G}{\partial r} = \frac{2}{\beta^*}\frac{\partial^2 G}{\partial r^2} \\
&= \frac{4}{\beta_{i1}\Delta r^2}(G_{i2} - G_{i1})
\end{aligned}$$

Thus, for $\texttt{KK} = 1$ and $\texttt{J} = 1$

$$C_{ij} = 0, \quad D_{ij} = 4\lambda^2$$

`P1-2D.f90`, `P1-2D.cpp`

Program `P1-2D` is a front end for subroutine `P1sor`, setting up the problem for a gray medium with spatially constant radiative properties (dimensions, radiative properties, and sources from known temperatures); may be used as a starting point for more involved applications. After calling `P1sor` the program also generates appropriate output. As given, `P1-2D` simulates the case of a two-dimensional axisymmetric cylinder (`KK=1`) of $R = 10\,\mathrm{cm}$ radius and $L = 20\,\mathrm{cm}$ length, using `JJ=21` nodes in the radial direction and `II=41` nodes in the axial direction (i.e., $\Delta x = \Delta r = 0.5\,\mathrm{cm}$), with a cold ($T_{ij}$ = `TM` = 0) gray medium, with constant absorption and scattering coefficients ($\kappa = \sigma_s = 0.1\,\mathrm{cm}^{-1}, A_1 = 0$); bounding walls are black and cold except for the face at $x = 0$, which is gray (`EPSX(1)=0.5`) and hot (`TX(1)=2000` K). Since the temperature field is specified, we have `IRE=0`. Running `P1-2D` we find from screen output that the calculation requires 97 iterations with a residual 2-norm error of 0.1354×10^{-4}.

The output is in file `P1-2Dsor.dat`, giving:

```
        GENERAL DATA
        ************

        CYLINDER RADIUS (R-DIR):     10.00
        CYLINDER LENGTH (X-DIR):     20.00
        TEMPERATURE AT r=R(j=J):      0.00K,  EMITTANCE  1.00
        TEMPERATURE AT x=0(i=1):   2000.00K,  EMITTANCE  0.50
        TEMPERATURE AT x=L(i=I):      0.00K,  EMITTANCE  1.00

   MEDIUM TEMPERATURE TM (K)

  \J   1     3     5     7     9    11    13    15    17    19    21
 I
 1    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
 3    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
 5    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
 7    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
 9    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
11    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
13    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
15    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
17    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
19    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
21    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
23    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
25    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
27    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
29    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
31    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
33    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
35    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
37    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
39    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
41    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.

   INCIDENT RADIATION G (W/SQCM)

  \J   1     3     5     7     9    11    13    15    17    19    21
 I
 1  99.6  99.4  99.0  98.3  97.1  95.5  93.1  89.7  84.7  77.0  64.0
 3  76.3  76.2  75.7  75.0  73.7  72.0  69.5  65.9  60.9  53.5  42.7
 5  58.3  58.1  57.7  56.9  55.7  54.0  51.6  48.3  43.7  37.6  29.6
 7  44.3  44.1  43.7  43.0  41.9  40.3  38.2  35.3  31.5  26.7  20.9
 9  33.5  33.4  33.0  32.4  31.4  30.0  28.2  25.8  22.8  19.2  14.9
```

11	25.3	25.2	24.8	24.3	23.4	22.3	20.8	18.9	16.6	13.8	10.8
13	19.0	18.9	18.6	18.2	17.5	16.5	15.3	13.8	12.1	10.0	7.8
15	14.2	14.2	13.9	13.5	13.0	12.2	11.3	10.2	8.8	7.3	5.7
17	10.6	10.6	10.4	10.1	9.6	9.1	8.3	7.5	6.5	5.3	4.1
19	7.9	7.9	7.7	7.5	7.2	6.7	6.1	5.5	4.7	3.9	3.0
21	5.9	5.9	5.8	5.6	5.3	5.0	4.5	4.0	3.5	2.9	2.2
23	4.4	4.4	4.3	4.1	3.9	3.7	3.3	3.0	2.6	2.1	1.6
25	3.3	3.2	3.2	3.1	2.9	2.7	2.5	2.2	1.9	1.5	1.2
27	2.4	2.4	2.4	2.3	2.1	2.0	1.8	1.6	1.4	1.1	0.9
29	1.8	1.8	1.7	1.7	1.6	1.5	1.3	1.2	1.0	0.8	0.6
31	1.3	1.3	1.3	1.2	1.2	1.1	1.0	0.9	0.7	0.6	0.5
33	1.0	1.0	1.0	0.9	0.9	0.8	0.7	0.6	0.6	0.5	0.3
35	0.7	0.7	0.7	0.7	0.6	0.6	0.5	0.5	0.4	0.3	0.3
37	0.5	0.5	0.5	0.5	0.5	0.4	0.4	0.4	0.3	0.2	0.2
39	0.4	0.4	0.4	0.4	0.4	0.3	0.3	0.3	0.2	0.2	0.1
41	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2	0.2	0.1	0.1

WALL FLUXES AT X=O AND X=L (W/SQCM)

J	1	3	5	7	9	11	13	15	17	19	21
Q	43.9	43.9	44.0	44.1	44.3	44.6	45.0	45.5	46.4	47.6	49.8
Q	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

RADIAL FLUXES TO CYLINDER WALL (W/SQCM)

I	QR
1	32.0
2	25.9
3	21.3
.	
.	
.	

Had we defined IRE=1 the same case would be calculated, but for radiative equilibrium with $\dot{Q}''' = 0$ (since TM was set to zero). This results in (now requiring 137 iterations):

```
GENERAL DATA
************

CYLINDER RADIUS (R-DIR):    10.00
CYLINDER LENGTH (X-DIR):    20.00
TEMPERATURE AT r=R(j=J):     0.00K,  EMITTANCE  1.00
TEMPERATURE AT x=0(i=1):  2000.00K,  EMITTANCE  0.50
TEMPERATURE AT x=L(i=I):     0.00K,  EMITTANCE  1.00
```

MEDIUM TEMPERATURE TM (K)

\J I	1	3	5	7	9	11	13	15	17	19	21
1	1611.	1610.	1606.	1600.	1592.	1579.	1563.	1540.	1510.	1466.	1393.
3	1555.	1554.	1550.	1542.	1532.	1517.	1497.	1470.	1433.	1381.	1302.
5	1499.	1497.	1493.	1484.	1472.	1455.	1432.	1402.	1361.	1306.	1228.
7	1442.	1441.	1435.	1426.	1413.	1394.	1370.	1337.	1295.	1238.	1163.
9	1386.	1384.	1379.	1369.	1355.	1335.	1309.	1276.	1233.	1177.	1105.
11	1331.	1329.	1323.	1313.	1298.	1278.	1251.	1218.	1175.	1121.	1051.
13	1277.	1275.	1268.	1258.	1243.	1223.	1196.	1163.	1121.	1068.	1002.

15	1224.	1222.	1215.	1205.	1190.	1169.	1143.	1110.	1069.	1019.	955.
17	1172.	1170.	1164.	1153.	1138.	1118.	1093.	1061.	1021.	972.	911.
19	1122.	1120.	1114.	1104.	1089.	1069.	1044.	1013.	975.	928.	870.
21	1074.	1072.	1066.	1056.	1041.	1022.	998.	968.	931.	886.	830.
23	1027.	1025.	1019.	1009.	995.	977.	953.	924.	889.	846.	793.
25	982.	980.	974.	965.	951.	933.	911.	883.	849.	807.	757.
27	938.	936.	930.	921.	908.	891.	869.	843.	810.	771.	722.
29	895.	893.	888.	879.	867.	850.	829.	804.	773.	735.	689.
31	853.	852.	847.	838.	826.	810.	790.	766.	736.	700.	656.
33	812.	811.	806.	798.	786.	771.	752.	729.	701.	666.	624.
35	772.	770.	765.	758.	747.	732.	714.	692.	665.	633.	593.
37	730.	729.	725.	717.	707.	693.	676.	655.	630.	599.	561.
39	688.	686.	682.	675.	665.	653.	636.	617.	593.	564.	528.
41	642.	641.	637.	630.	621.	609.	594.	576.	553.	526.	493.

INCIDENT RADIATION G (W/SQCM)

\J I	1	3	5	7	9	11	13	15	17	19	21
1	152.8	152.4	151.1	148.8	145.5	141.1	135.2	127.6	117.8	104.7	85.5
3	132.7	132.2	130.8	128.4	124.9	120.1	113.9	105.9	95.6	82.4	65.1
5	114.5	114.0	112.5	110.1	106.5	101.7	95.5	87.6	77.9	65.9	51.5
7	98.2	97.7	96.3	93.8	90.3	85.7	79.8	72.5	63.7	53.3	41.5
9	83.8	83.3	81.9	79.6	76.4	72.1	66.7	60.1	52.4	43.6	33.8
11	71.2	70.7	69.5	67.4	64.4	60.5	55.6	49.9	43.2	35.8	27.7
13	60.2	59.9	58.7	56.8	54.1	50.7	46.4	41.5	35.8	29.5	22.8
15	50.8	50.5	49.5	47.8	45.4	42.4	38.7	34.5	29.7	24.4	18.9
17	42.8	42.5	41.6	40.1	38.1	35.5	32.3	28.7	24.6	20.3	15.6
19	36.0	35.7	34.9	33.7	31.9	29.6	27.0	23.9	20.5	16.8	13.0
21	30.1	29.9	29.3	28.2	26.7	24.8	22.5	19.9	17.0	14.0	10.8
23	25.2	25.0	24.5	23.5	22.3	20.6	18.7	16.6	14.2	11.6	9.0
25	21.1	20.9	20.4	19.6	18.6	17.2	15.6	13.8	11.8	9.6	7.4
27	17.5	17.4	17.0	16.3	15.4	14.3	13.0	11.4	9.8	8.0	6.2
29	14.6	14.4	14.1	13.6	12.8	11.8	10.7	9.5	8.1	6.6	5.1
31	12.0	11.9	11.7	11.2	10.6	9.8	8.9	7.8	6.7	5.5	4.2
33	9.9	9.8	9.6	9.2	8.7	8.0	7.3	6.4	5.5	4.5	3.4
35	8.0	8.0	7.8	7.5	7.1	6.5	5.9	5.2	4.4	3.6	2.8
37	6.5	6.4	6.2	6.0	5.7	5.2	4.7	4.2	3.6	2.9	2.2
39	5.1	5.0	4.9	4.7	4.4	4.1	3.7	3.3	2.8	2.3	1.8
41	3.8	3.8	3.7	3.6	3.4	3.1	2.8	2.5	2.1	1.7	1.3

WALL FLUXES AT X=O AND X=L (W/SQCM)

J	1	3	5	7	9	11	13	15	17	19	21
Q	35.0	35.1	35.3	35.7	36.2	37.0	37.9	39.2	40.8	43.0	46.2
Q	1.9	1.9	1.9	1.8	1.7	1.6	1.4	1.2	1.1	0.9	0.7

RADIAL FLUXES TO CYLINDER WALL (W/SQCM)

I	QR
1	42.7
2	37.0
.	
.	
.	

Finally, if we set IRE=1, EPSR=0 and JJ=1, we obtain the results for a one-dimensional slab at radiative equilibrium:

```
        GENERAL DATA
        ************

        CYLINDER RADIUS (R-DIR):    10.00
        CYLINDER LENGTH (X-DIR):    20.00
        TEMPERATURE AT r=R(j=J):     0.00K,  EMITTANCE  0.00
        TEMPERATURE AT x=0(i=1):  2000.00K,  EMITTANCE  0.50
        TEMPERATURE AT x=L(i=I):     0.00K,  EMITTANCE  1.00

   MEDIUM TEMPERATURE TM (K)

  \J  1
 1 1829.
 3 1809.
 5 1788.
 7 1767.
 9 1745.
11 1722.
13 1698.
15 1673.
17 1646.
19 1619.
21 1590.
23 1559.
25 1527.
27 1492.
29 1454.
31 1414.
33 1369.
35 1320.
37 1264.
39 1201.
41 1124.

   INCIDENT RADIATION G (W/SQCM)

  \J  1
 1 253.7
 3 242.8
 5 232.0
 7 221.1
 9 210.2
11 199.3
13 188.4
15 177.5
17 166.7
19 155.8
21 144.9
23 134.1
25 123.2
27 112.3
29 101.5
31  90.6
33  79.7
35  68.9
37  58.0
39  47.1
```

```
 41  36.2


     WALL FLUXES AT X=O AND X=L (W/SQCM)

     J    1
     Q  18.2
     Q  18.1


     RADIAL FLUXES TO CYLINDER WALL (W/SQCM)

          I     QR

          1     0.0
          2     0.0
          .
          .
          .
```

Of course, the matrix for this case could have easily been inverted by a tridiagonal matrix solver (instead of using 181 iterations as done here), or could have been found analytically using Example 15.5 (but for a gray wall).

`Delta.f90`:
Program `Delta` is a stand-alone program to calculate the rotation matrix $\Delta^n_{mm'}(\alpha,\beta,\gamma)$ required for the boundary conditions of higher-order P_N-approximations, as given by equations (16.64) through (16.67); here set for $2l = N-1 = 4$ (P_5). Results for the case of a backward rotation with $-\gamma(=$ `alpha`$) = -\pi/2$, $-\beta(=$ `beta`$) = \pi/2$, $-\alpha(=$ `gamma`$) = \pi/2$ (a surface at $y =$ const facing toward larger y, with $\bar{x} = x$) are calculated and stored in `delta.dat`. For incorporation into a general P_N-code the stand-alone program can easily be converted into a subroutine calculating a single or all rotation Δ-values for a given set of angles α, β, γ.

`pnbcs.f90`:
Program `pnbcs` is a stand-alone program to calculate the Legendre half-moments $p^m_{n,j}$ and coefficients $u^m_{li}, v^m_{li}, w^m_{li}$, which are required for the boundary conditions of higher-order P_N-approximations, as given by equations (16.71) through (16.72). Calculations use the recursion relationships described in [2], Eqs. (27) through (32). As provided, $N =$ `NN` $= 5$, i.e., the $p^m_{n,j}, u^m_{li}, v^m_{li}$ and w^m_{li} are calculated up to $n = 5$ (P_5-approximation). Output is directed to `PNbc.dat`, containing all the $p^m_{n,j}$ data for Table 16.2 (i.e., normalized by 10^{-m}), and the corresponding u, v, w. Higher orders may be implemented by changing `NN` (however, output format would need adjustment beyond P_{19}).

References

1. Hageman, L. A., and D. Young: *Applied Iterative Methods*, Academic Press, 1981.
2. Modest, M. F.: "Further developments of the elliptic P_N-approximation formulation and its boundary conditions," *Numerical Heat Transfer – Part B: Fundamentals*, vol. 62, no. 2–3, pp. 181–202, 2012.