CODES FOR CHAPTER 19

`transPN.f90`

Program `transPN` calculates energy from a pulsed collimated laser source transmitted through an absorbing, isotropically scattering slab as a function of time, using the P_1 and $P_{1/3}$ methods. Following Example 19.3 the equations for the P_1- and $P_{1/3}$-approximations for a nonemitting and isotropically scattering, one-dimensional medium, reduce to

$$\frac{\partial G}{\partial t^*} + \frac{\partial q}{\partial \tau} = -(1-\omega)G + \omega G_c, \qquad \text{(CC-19-1)}$$

$$3a\frac{\partial q}{\partial t^*} + \frac{\partial G}{\partial \tau} = -3q, \qquad \text{(CC-19-2)}$$

where $a = 1$ for P_1 and $a = 1/3$ for $P_{1/3}$, and G and q have been normalized as $G = G_d/q_o$ and $q = q_d/q_o$. These two equations are subject to the initial and boundary conditions

$$t^* = 0: \qquad G(0,\tau) = q(0,\tau) = 0, \qquad \text{(CC-19-3)}$$

$$\tau = 0: \quad -2q(t^*,0) = G(t^*,0), \qquad \text{(CC-19-4)}$$

$$\tau = \tau_L: \quad +2q(t^*,\tau_L) = G(t^*,\tau_L). \qquad \text{(CC-19-5)}$$

The normalized isotropic scattering source is immediately found from equations (19.25) and (19.18) for a nonreflecting boundary. For the top-hat profile of Example 19.3 this results in a total nondimensional pulse energy of t_p^* and

$$G_c(t^*,\tau) = \left[H(t^*-\tau) - H^*(t^*-\tau-t_p^*)\right]e^{-\tau}. \qquad \text{(CC-19-6)}$$

If a clipped Gaussian source is used [1], then

$$q_0(0,t) = q_0\left[H(t) - H(t-2t_c)\right]\exp\left[-\left(\frac{t-t_c}{t_p}\right)^2\right], \qquad \text{(CC-19-7)}$$

and the total nondimensional pulse energy is

$$\begin{aligned}\int_0^\infty \frac{q_0(0,t)}{q_0}\beta c\,dt &= \int_0^\infty \left[H(t^*) - H(t^*-2t_c^*)\right]\exp\left[-\left(\frac{t^*-t_c^*}{t_p^*}\right)^2\right]dt^* \\ &= \int_0^{2t_c^*}\exp\left[-\left(\frac{t^*-t_c^*}{t_p^*}\right)^2\right]dt^* = \sqrt{\pi}t_p^*\,\mathrm{erf}\left(\frac{t_c^*}{t_p^*}\right). \qquad \text{(CC-19-8)}\end{aligned}$$

Thus, to run `transPN` with equal pulse strengths, one must use

$$t^*_{p,\mathrm{TH}} = \sqrt{\pi}\,\mathrm{erf}\left(\frac{t_c^*}{t_{p\mathrm{G}}^*}\right)t_{p\mathrm{G}}^* \simeq \sqrt{\pi}t_{p\mathrm{G}}^*, \qquad \text{(CC-19-9)}$$

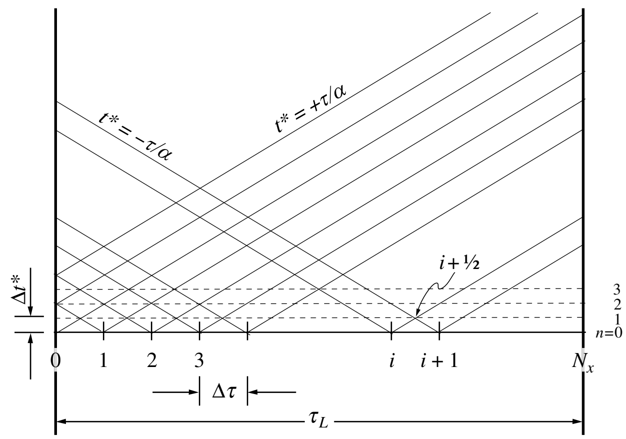

FIGURE 1
Time-space nodal system for `transPN.f90`.

the latter assuming $t_c \gtrsim 2t_{p\mathrm{G}}$. For the clipped Gaussian pulse the source term then becomes

$$G_c(t^*, \tau) = [H(t^* - \tau) - H(t^* - 2t_c^* - \tau)] \exp\left[-\tau - \left(\frac{t^* - t_c^* - \tau}{t_p^*}\right)^2\right]. \tag{CC-19-10}$$

The hyperbolic nature of this set of equations becomes obvious, if q is eliminated from them (by differentiating the first with respect to t^* and the second with respect to τ), leading to

$$\frac{\partial^2 G}{\partial t^{*2}} - \frac{1}{3a}\frac{\partial^2 G}{\partial \tau^2} + \left(1 - \omega + \frac{1}{a}\right)\frac{\partial G}{\partial t^*} + \frac{1-\omega}{a}G - \frac{\omega}{a}G_c - \omega\frac{\partial G_c}{\partial t^*} = 0, \tag{CC-19-11}$$

which has a signal velocity of $\alpha = 1/\sqrt{3a}$ (nondimensional in terms of speed of light, c), as already indicated in the formulation for the P_a methods. Eliminating q also from initial and boundary conditions gives

$$t^* = 0: \quad G(0, \tau) = \frac{\partial G}{\partial t^*}(0, \tau) = 0, \tag{CC-19-12}$$

$$\tau = 0: \quad 3\left(G(t^*, 0) + a\frac{\partial G}{\partial t^*}(t^*, 0)\right) - 2\frac{\partial G}{\partial \tau}(t^*, 0) = 0, \tag{CC-19-13}$$

$$\tau = \tau_L: \quad 3\left(G(t^*, 0) + a\frac{\partial G}{\partial t^*}(t^*, 0)\right) + 2\frac{\partial G}{\partial \tau}(t^*, 0) = 0. \tag{CC-19-14}$$

This second-order hyperbolic equation is readily solved by the method of characteristics [2] along the characteristic lines $\tau = \pm\alpha t^*$. Using subscript notation, i.e., $G_x = \partial G/\partial\tau$, etc., equation (CC-19-11) may be rewritten as

$$G_{tt} - \alpha^2 G_{xx} + (1-\omega)G_t + 3\alpha^2\left[G_t + (1-\omega)G - \omega G_c'\right] = 0, \tag{CC-19-15}$$

where $G_c' = G_c + \partial G_c/\partial t^*$. Along the two characteristic lines $\tau = \pm\alpha t^*$ we have [2]

$$\pm\alpha dG_t - \alpha^2 dG_x \pm \left\{(1-\omega)G_t + 3\alpha^2\left[G_t + (1-\omega)G - \omega G_c'\right]\right\} d\tau = 0 \tag{CC-19-16}$$

and the total differential is

$$dG = G_t dt^* + G_x d\tau. \tag{CC-19-17}$$

We will break up the thickness of the slab, L, into N_x equally-spaced nodes of width $\Delta x = L/N_x$, or $\Delta\tau = \tau_L/N_x$. In t^*-τ-space the characteristics then are straight lines as shown in Fig. 1, with the lines going up to the right corresponding to the upper sign in equation (CC-19-16), and the lines going down to the right to the lower sign. As time step $\Delta\tau$ we take the time it takes to move along the characteristics from adjacent points (n, i) and $(n, i+1)$ to their intersection at $(n+1, i+{}^1\!/_2)$ as shown in the figure. During that time the signal moves a distance $\pm\Delta x/2$, so that

$$\Delta t^* = \Delta\tau/2\alpha. \tag{CC-19-18}$$

We can finite-difference equations (CC-19-16) and (CC-19-17) along the characteristics by using $d\phi = \phi^n_{i+1/2} - \phi^{n-1}_i$ for the left-to-right characteristics, and $d\phi = \phi^n_{i+1/2} - \phi^{n-1}_{i+1}$ for the right-to-left characteristics, where ϕ stands for any of the variables τ, G, G_t and G_x. In the finite differencing we distinguish between odd time steps (all nodes, such as $i+{}^1\!/_2$, are internal) and even time steps (all nodes are at integer locations, including two boundary nodes $i = 0$ and $i = N_x$).

Odd Time Steps (n odd) All new positions are at $i+{}^1\!/_2$ ($i = 0, 1 \dots N_x - 1$); all old positions are at i ($i = 0, N_x - 1$) for the left-to-right characteristics, and at $i+1$ ($i+1 = 1, N_x$) for the right-to-left characteristics. Thus,

$$\alpha(G_{t,i+1/2} - G_{t,i}) - \alpha^2(G_{x,i+1/2} - G_{x,i}) + \Big\{(1-\omega)(G_{t,i+1/2} + G_{t,i}) \\ + 3\alpha^2\left[G_{t,i+1/2} + G_{t,i} + (1-\omega)(G_{i+1/2} + G_i) - \omega(G'_{c,i+1/2} + G'_{c,i})\right]\Big\}\frac{\Delta\tau}{4} = 0, \tag{CC-19-19}$$

where we have used averaged values, $\phi = \frac{1}{2}(\phi^n_{i+1/2} + \phi^{n-1}_i)$ for the terms within braces, and have omitted the time superscripts, since the distinction between new and old is clear. Bringing all unknown quantities at the new time to the left-hand side we get

$$B_p G_{t,i+1/2} - C_4 G_{x,i+1/2} + C_2 G_{i+1/2} = -B_m G_{t,i} - C_4 G_{x,i} - C_2 G_i + C_3(G'_{c,i+1/2} + G'_{c,i}) = E_1, \\ i = 0, N_x - 1, \tag{CC-19-20}$$

where

$$B_p = \alpha + (1-\omega+3\alpha^2)\frac{\Delta\tau}{4}, \quad B_m = \alpha - (1-\omega+3\alpha^2)\frac{\Delta\tau}{4}, \\ C_2 = 3\alpha^2(1-\omega)\frac{\Delta\tau}{4}, \quad C_3 = 3\alpha^2\omega\frac{\Delta\tau}{4}, \quad C_4 = \alpha^2. \tag{CC-19-21}$$

Similarly, we obtain for the right-to-left characteristics, by switching the signs in equation (CC-19-16) and replacing i by $i+1$:

$$B_p G_{t,i+1/2} + C_4 G_{x,i+1/2} + C_2 G_{i+1/2} \\ = -B_m G_{t,i+1} + C_4 G_{x,i+1} - C_2 G_{i+1} + C_3(G'_{c,i+1/2} + G'_{c,i+1}) = E_2, \quad i = 0, N_x - 1. \tag{CC-19-22}$$

We now have two equations in the three unknowns $G_{t,i+1/2}$, $G_{x,i+1/2}$ and $G_{i+1/2}$: one more relation is needed and will come from equation (CC-19-17), which may be finite-differenced from the left or from the right as

$$G_{i+1/2} = G_i + \frac{1}{2}(G_{t,i+1/2} + G_{t,i})\Delta t^* + \frac{1}{2}(G_{x,i+1/2} + G_{x,i})\frac{\Delta\tau}{2}, \qquad l \to r \\ = G_{i+1} + \frac{1}{2}(G_{t,i+1/2} + G_{t,i+1})\Delta t^* - \frac{1}{2}(G_{x,i+1/2} + G_{x,i+1})\frac{\Delta\tau}{2}, \qquad r \to l. \tag{CC-19-23}$$

For better accuracy, we take the average, or

$$-\frac{\Delta t^*}{2}G_{t,i+1/2} + G_{i+1/2} = \frac{1}{2}(G_i + G_{i+1}) + \frac{\Delta t^*}{4}(G_{t,i} + G_{t,i+1}) + \frac{\Delta\tau}{8}(G_{x,i} - G_{x,i+1}) = D_2. \quad \text{(CC-19-24)}$$

Subtracting equation (CC-19-20) from (CC-19-22) leads to

$$G_{x,i+1/2} = (E_2 - E_1)/2C_4, \quad i = 0, N_x - 1, \quad \text{(CC-19-25)}$$

while adding them gives

$$B_p G_{t,i+1/2} + C_2 G_{i+1/2} = \frac{1}{2}(E_1 + E_2) = D_1, \quad \text{(CC-19-26)}$$

which, together with equation (CC-19-24) leads to

$$G_{i+1/2} = \frac{D_1\Delta t^*/2 + D_2 B_p}{C_2\Delta t^*/2 + B_p}, \quad G_{t,i+1/2} = \frac{D_1 - C_2 D_2}{C_2\Delta t^*/2 + B_p}, \quad i = 0, N_x - 1.$$

Even Time Steps (n even) Even time steps are a little more difficult to handle, because two of the nodes lie on the boundaries, and for them the boundary conditions must be invoked. Internal nodes, on the other hand, are the same as those for odd n, except that nodes are displaced by half a node. Replacing every i by $i - 1/2$ we obtain

$$\begin{aligned} G_{x,i} &= (E_2 - E_1)/2C_4, \quad G_{t,i} = \frac{D_1 - C_2 D_2}{C_2\Delta t^*/2 + B_p}, \\ G_i &= \frac{D_1\Delta t^*/2 + D_2 B_p}{C_2\Delta t^*/2 + B_p}; \qquad i = 1, N_x - 1, \end{aligned} \quad \text{(CC-19-27)}$$

where

$$\begin{aligned} E_1 &= -B_m G_{t,i-1/2} - C_4 G_{x,i-1/2} - C_2 G_{i-\frac{1}{2}} + C_3(G'_{c,i} + G'_{c,i-1/2}) \\ E_2 &= -B_m G_{t,i+1/2} + C_4 G_{x,i+1/2} - C_2 G_{i+\frac{1}{2}} + C_3(G'_{c,i} + G'_{c,i+1/2}) \\ D_1 &= \frac{1}{2}(E_1 + E_2) \\ D_2 &= (G_{i-1/2} + G_{i+1/2}) + \frac{\Delta t^*}{4}(G_{t,i-1/2} + G_{t,i+1/2}) + \frac{\Delta\tau}{8}(G_{x,i-1/2} - G_{x,i+1/2}) \end{aligned}$$

At the left boundary, $i = 0$, equation (CC-19-20) is not valid and must be replaced by the boundary condition, slightly rewritten as

$$G_{x,i} = \frac{3}{2}G_i + \frac{1}{2\alpha^2}\, G_{t,i}. \quad \text{(CC-19-28)}$$

Sticking this into equation (CC-19-22) (with $i + 1/2$ replaced by i) gives

$$f_1 G_{t,i} + f_2 G_i = E_2; \quad f_1 = B_p + \frac{C_4}{2\alpha^2} = B_p + \frac{1}{2}; \quad f_2 = C_2 + \frac{3}{2}C_4. \quad \text{(CC-19-29)}$$

Also, for the total derivative we can only use the $r \to l$ form, or

$$G_i = G_{i+1/2} + \frac{1}{2}(G_{t,i} + G_{t,i+1/2})\Delta t^* - \frac{1}{2}(G_{x,i} + G_{x,i+1/2})\frac{\Delta\tau}{2}, \quad \text{(CC-19-30)}$$

or, after eliminating $G_{x,i}$ through equation (CC-19-28)

$$f_3 G_{t,i} + f_4 G_i = D_2, \quad f_3 = \frac{\Delta\tau}{8\alpha^2} - \frac{\Delta t^*}{2}; \quad f_4 = 1 + \frac{3\Delta\tau}{8}, \quad \text{(CC-19-31)}$$

and, thus,

$$G_i = \frac{f_3 E_2 - f_1 D_2}{f_3 f_2 - f_1 f_4}; \quad G_{t,i} = \frac{f_2 D_2 - f_4 E_2}{f_3 f_2 - f_1 f_4}, \tag{CC-19-32}$$

and $G_{x,i}$ from equation (CC-19-28).

Similarly, for $i = N_x$ equation (CC-19-22) is not valid and must be replaced by the boundary at $\tau = \tau_L$, and for the total derivative the $l \to r$ version must be used, leading to very similar expressions.

Finally, transmissivity and reflectivity of the slab are simply the absolute value of the nondimensional fluxes at the boundaries, i.e.,

$$\begin{aligned} \text{Reflectivity} &= \left|q(t^*, 0)\right| = \frac{1}{2} G(t^*, 0) \\ \text{Transmissivity} &= q(t^*, \tau_L) + q_c(t^*, \tau_L) = \frac{1}{2} G(t^*, \tau_L) + G_c(t^*, \tau_L). \end{aligned} \tag{CC-19-33}$$

Input:

`Nx`	=	Number of equally-spaced nodes across slab,
`a`	=	P_a-approximation switch: `a` = 1 for P_1-approximation, `a` = $^1\!/_3$ for $P_{1/3}$-approximation,
`L`	=	Thickness of slab, in m,
`beta`	=	Extinction coefficient β, in m^{-1},
`omga`	=	single scattering albedo, ω,
`tmax`	=	Maximum $t^*_{\max}$ to be considered in calculation,
`tps`	=	Total nondimensional pulse energy,
`tme`	=	Starting time for calculation; `tme` = `0` will start top-hat pulse at $t^* = 0$, `tme` = `−tps/2` will center top-hat pulse at $t^* = 0$, etc.
`tc,tp`	=	Pulse parameters for clipped-Gaussian pulse; note that `tp` = `tps`/ $\sqrt{\pi}$ results in a total pulse energy of `tps` (i.e., the same as for the top-hat pulse).

Output:

For every even time step the program prints out the value for `tme` = t^*, Transmissivity and Reflectivity as defined in equation (CC-19-33). Total pulse energy, total time integrated reflectivity and transmissivity are also printed out, which — for $\omega = 1$ — gives a check of truncation error and the proper choice for `tmax` to simulate the entire pulse.

Example: As an example we will analyze a slab of 1 m width using the $P_{1/3}$-approximation ($a = {}^1\!/_3$), with an extinction coefficient of $\beta = 5\,\text{m}^{-1}$ (leading to an optical thickness of $\tau_L = 5$), and a scattering albedo of $\omega = 1$ (or 100%). Thus, we call the output file `transP3rd-5-100.dat`. We will use a top-hat laser pulse centered at $t = 0$, with a nondimensional pulse length of $t^*_p = 0.3$. Finally, we will use a spatial resolution of 200 nodes and, since it takes the signal 5 nondimensional time units to penetrate the slab and pure scattering will bounce around the beam for much longer, we choose a maximum t^* of 80. Thus, the beginning of the program looks as follows: (i) in the fifth line we have set `Nx=200`, (ii) under "pulse shape" we have uncommented the 4 'top-hat' lines, and (iii) we have fashioned the numbers below 'Input data' to fit our needs:

```
    program transPN
! Program to calculate energy transmitted as a function of time
! from a pulsed collimated laser source, through absorbing-scattering slab,
! using P1 and P1/3
    IMPLICIT NONE
    INTEGER, PARAMETER :: Nx=200
    INTEGER :: i,n
    DOUBLE PRECISION    :: L,tp,tps,beta,omga,tauL,dx,dt,trmsv,reflc,Bp,Bm,tme,tc
    DOUBLE PRECISION    :: G(0:Nx),Gx(0:Nx),Gt(0:Nx),G5(0:Nx),Gx5(0:Nx),Gt5(0:Nx)
    DOUBLE PRECISION    :: alf,c1,c2,c3,c4,Gc,Gc5,Gcp,Gcp5,Heav,y,E1,E2,D1,D2,f1,f2,f3,f4
    DOUBLE PRECISION    :: tmax,a,sumpls,sumtrn,sumref
```

```
    Heav(y)=FLOAT(INT(1.+.5*y/(abs(y)+1.d-15)))
! *******************   Pulse shape  ***********************************
! uncomment only one set of laser data below!!
! the following 4 lines simulate a top hat laser starting at n*dt=0
    Gc(n,i)=(Heav(n*dt-i*dx)-Heav(n*dt-i*dx-tps))*exp(-i*dx)
    Gc5(n,i)=(Heav(n*dt-(i+.5)*dx)-Heav(n*dt-(i+.5)*dx-tps))*exp(-(i+.5)*dx)
    Gcp(n,i)=Gc(n,i)
    Gcp5(n,i)=Gc5(n,i)
! the following 6 lines simulate a clipped Gaussian laser centered at n*dt=tc
!    Gc(n,i)=exp(-i*dx-((n*dt-i*dx-tc)/tp)**2) &
!            *(Heav(n*dt-i*dx)-Heav(n*dt-i*dx-2.*tc))
!    Gcp(n,i)=Gc(n,i)*(1.-2.*a*(n*dt-i*dx-tc)/tp**2)
!    Gc5(n,i)=exp(-(i+.5)*dx-((n*dt-(i+.5)*dx-tc)/tp)**2) &
!            *(Heav(n*dt-(i+.5)*dx)-Heav(n*dt-(i+.5)*dx-2.*tc))
!    Gcp5(n,i)=Gc5(n,i)*(1.-2.*a*(n*dt-(i+.5)*dx-tc)/tp**2)
!
! ********************  Output file  ***********************************
    open(unit=8,file='transP3rd-5-100.dat',status='unknown')
! ****************  Input data  *****************************************
    a=1.d0/3.d0 ! =1 for P1, =1/3 for P1/3 approximation
    L=1.        ! m
    beta=5.     ! 1/m
    omga=1
    tmax=80.    ! maximum t* to be considered
! pulse data: make sure to uncomment only 1 starting time "tme"
! pulse width for top-hat laser
    tps=0.3             ! total pulse duration = total pulse power
    tme=-tps/2.         ! non-zero value moves beginning of pulse; -tps/2 centers pulse at 0
! pulse shape for clipped Gaussian laser
    tc=0.5
    tp=tps/1.77245d0    ! total pulse power/sqrt(pi)
!   tme=-tc
! ****************  End of input data  *********************************
```

This leads to the following results stored in:

```
VARIABLES = tme,trmsv,reflc
zone
  -0.125   0.0000E+00   0.2536E-03
  -0.100   0.0000E+00   0.8391E-03
  -0.075   0.0000E+00   0.1675E-02
  -0.050   0.0000E+00   0.2744E-02
  -0.025   0.0000E+00   0.4027E-02
   0.000   0.0000E+00   0.5507E-02
   0.025   0.0000E+00   0.7167E-02
   0.050   0.0000E+00   0.8993E-02
   0.075   0.0000E+00   0.1097E-01
   0.100   0.0000E+00   0.1308E-01
   0.125   0.0000E+00   0.1533E-01
   0.150   0.0000E+00   0.1768E-01
   0.175   0.0000E+00   0.2012E-01
   0.200   0.0000E+00   0.2237E-01
   0.225   0.0000E+00   0.2444E-01
   0.250   0.0000E+00   0.2632E-01
   0.275   0.0000E+00   0.2804E-01
   0.300   0.0000E+00   0.2960E-01
   0.325   0.0000E+00   0.3103E-01
   0.350   0.0000E+00   0.3232E-01
   0.375   0.0000E+00   0.3348E-01
   0.400   0.0000E+00   0.3453E-01
```

0.425	0.0000E+00	0.3548E-01
0.450	0.0000E+00	0.3633E-01
0.475	0.0000E+00	0.3708E-01
0.500	0.0000E+00	0.3775E-01
0.525	0.0000E+00	0.3835E-01
0.550	0.0000E+00	0.3887E-01
0.575	0.0000E+00	0.3932E-01
0.600	0.0000E+00	0.3971E-01
0.625	0.0000E+00	0.4004E-01
0.650	0.0000E+00	0.4032E-01
0.675	0.0000E+00	0.4055E-01
0.700	0.0000E+00	0.4074E-01
0.725	0.0000E+00	0.4088E-01
0.750	0.0000E+00	0.4098E-01
0.775	0.0000E+00	0.4105E-01
0.800	0.0000E+00	0.4109E-01
0.825	0.0000E+00	0.4109E-01
0.850	0.0000E+00	0.4107E-01
0.875	0.0000E+00	0.4102E-01
0.900	0.0000E+00	0.4095E-01
0.925	0.0000E+00	0.4086E-01
0.950	0.0000E+00	0.4074E-01
0.975	0.0000E+00	0.4061E-01
1.000	0.0000E+00	0.4046E-01
.		
.		
4.500	0.0000E+00	0.1518E-01
4.525	0.0000E+00	0.1509E-01
4.550	0.0000E+00	0.1501E-01
4.575	0.0000E+00	0.1492E-01
4.600	0.0000E+00	0.1484E-01
4.625	0.0000E+00	0.1476E-01
4.650	0.0000E+00	0.1468E-01
4.675	0.0000E+00	0.1459E-01
4.700	0.0000E+00	0.1451E-01
4.725	0.0000E+00	0.1443E-01
4.750	0.0000E+00	0.1436E-01
4.775	0.0000E+00	0.1428E-01
4.800	0.0000E+00	0.1420E-01
4.825	0.0000E+00	0.1412E-01
4.850	0.6893E-02	0.1405E-01
4.875	0.7201E-02	0.1397E-01
4.900	0.7507E-02	0.1390E-01
4.925	0.7811E-02	0.1382E-01
4.950	0.8114E-02	0.1375E-01
4.975	0.8417E-02	0.1368E-01
5.000	0.8718E-02	0.1361E-01
5.025	0.9019E-02	0.1353E-01
5.050	0.9319E-02	0.1346E-01
5.075	0.9618E-02	0.1339E-01
5.100	0.9917E-02	0.1332E-01
5.125	0.1022E-01	0.1326E-01
5.150	0.1052E-01	0.1319E-01
5.175	0.3921E-02	0.1312E-01
5.200	0.3912E-02	0.1305E-01
5.225	0.3905E-02	0.1299E-01
5.250	0.3899E-02	0.1292E-01
5.275	0.3895E-02	0.1285E-01
5.300	0.3892E-02	0.1279E-01

5.325	0.3891E-02	0.1272E-01
5.350	0.3891E-02	0.1266E-01
5.375	0.3892E-02	0.1260E-01
5.400	0.3893E-02	0.1253E-01
5.425	0.3896E-02	0.1247E-01
5.450	0.3899E-02	0.1241E-01
5.475	0.3904E-02	0.1235E-01
5.500	0.3909E-02	0.1229E-01
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8.000	0.4665E-02	0.8036E-02
8.025	0.4667E-02	0.8006E-02
8.050	0.4669E-02	0.7977E-02
8.075	0.4671E-02	0.7948E-02
8.100	0.4673E-02	0.7919E-02
8.125	0.4674E-02	0.7890E-02
8.150	0.4676E-02	0.7861E-02
8.175	0.4677E-02	0.7833E-02
8.200	0.4679E-02	0.7804E-02
8.225	0.4680E-02	0.7776E-02
8.250	0.4681E-02	0.7748E-02
8.275	0.4682E-02	0.7720E-02
8.300	0.4683E-02	0.7693E-02
8.325	0.4684E-02	0.7665E-02
8.350	0.4685E-02	0.7638E-02
8.375	0.4685E-02	0.7611E-02
8.400	0.4686E-02	0.7584E-02
8.425	0.4686E-02	0.7557E-02
8.450	0.4687E-02	0.7530E-02
8.475	0.4687E-02	0.7504E-02
8.500	0.4687E-02	0.7477E-02
8.525	0.4687E-02	0.7451E-02
8.550	0.4687E-02	0.7425E-02
8.575	0.4687E-02	0.7399E-02
8.600	0.4687E-02	0.7373E-02
8.625	0.4687E-02	0.7348E-02
8.650	0.4686E-02	0.7322E-02
8.675	0.4686E-02	0.7297E-02
8.700	0.4685E-02	0.7272E-02
8.725	0.4685E-02	0.7247E-02
8.750	0.4684E-02	0.7222E-02
8.775	0.4683E-02	0.7197E-02
8.800	0.4682E-02	0.7173E-02
8.825	0.4681E-02	0.7148E-02
8.850	0.4680E-02	0.7124E-02
8.875	0.4679E-02	0.7100E-02
8.900	0.4678E-02	0.7076E-02
8.925	0.4677E-02	0.7052E-02
8.950	0.4675E-02	0.7028E-02
8.975	0.4674E-02	0.7004E-02
9.000	0.4673E-02	0.6981E-02
.		
.		
79.900	0.1443E-04	0.1431E-04
79.925	0.1440E-04	0.1428E-04
79.950	0.1437E-04	0.1425E-04
79.975	0.1434E-04	0.1422E-04
80.000	0.1431E-04	0.1419E-04
80.025	0.1428E-04	0.1416E-04

```
Total transmission:    8.525E-02
Total reflection:      2.394E-01
Total trans+reflec:    3.246E-01
Total pulse enrg:      3.063E-01
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Note that the transmissivity remains 0 until $t^* = 4.85$, when the beginning of the pulse has reached the opposite end by direct travel, and has its maximum at around $t^* \simeq 8.6$ (while the reflectivity peaks around $t^* \simeq 0.8$. Note that, for the present case of conservative scattering $\omega = 1$, the sum of transmissivity and reflectivity should equal the total pulse energy, or 0.3 ($= t_p^*$). The departures are due to the relatively coarse grid and the nonconservative nature of the P_a-approximation.

References

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