CHAPTER 2

Elasticity and Viscoelasticity
CHAPTER 2.1

Introduction to Elasticity and Viscoelasticity

JEAN LEMAITRE
Université Paris 6, LMT-Cachan, 61 avenue du Président Wilson, 94235 Cachan Cedex, France

For all solid materials there is a domain in stress space in which strains are reversible due to small relative movements of atoms. For many materials like metals, ceramics, concrete, wood and polymers, in a small range of strains, the hypotheses of isotropy and linearity are good enough for many engineering purposes. Then the classical Hooke’s law of elasticity applies. It can be derived from a quadratic form of the state potential, depending on two parameters characteristics of each material: the Young’s modulus $E$ and the Poisson’s ratio $\nu$.

$$\psi^* = \frac{1}{2\rho} A_{ijkl(E,\nu)} \sigma_{ij} \sigma_{kl}$$  \hfill (1)

$$\epsilon_{ij} = \rho \frac{\partial \psi^*}{\partial \sigma_{ij}} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}$$  \hfill (2)

$E$ and $\nu$ are identified from tensile tests either in statics or dynamics. A great deal of accuracy is needed in the measurement of the longitudinal and transverse strains ($\delta\epsilon \approx \pm10^{-6}$ in absolute value).

When structural calculations are performed under the approximation of plane stress (thin sheets) or plane strain (thick sheets), it is convenient to write these conditions in the constitutive equation.

- **Plane stress** ($\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$):

$$\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{12}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & 0 \\
\frac{1}{E} & \frac{1 + \nu}{E} & 0 \\
\text{Sym} & \frac{1 + \nu}{E}
\end{bmatrix} \begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix}$$  \hfill (3)
- **Plane strain** ($\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$):

$$\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} = \begin{bmatrix}
\lambda + 2\mu & \lambda & 0 \\
\lambda & \lambda + 2\mu & 0 \\
\text{Sym} & 2\mu & 
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{12}
\end{bmatrix}$$

with

$$\begin{cases}
\lambda = \frac{vE}{(1+v)(1-2v)} \\
\mu = \frac{E}{2(1+v)}
\end{cases}$$

(4)

For orthotropic materials having three planes of symmetry, nine independent parameters are needed: three tension moduli $E_1$, $E_2$, $E_3$ in the orthotropic directions, three shear moduli $G_{12}$, $G_{23}$, $G_{31}$, and three contraction ratios $\nu_{12}, \nu_{23}, \nu_{31}$. In the frame of orthotropy:

$$\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{23} \\
\varepsilon_{31} \\
\varepsilon_{12}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\
1 & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\
1 & \frac{1}{E_3} & 0 & 0 & 0 & 0 \\
\text{Sym} & 1 & \frac{1}{2G_{23}} & 0 & 0 & 0 \\
\frac{1}{2G_{31}} & 0 & 1 & \frac{1}{2G_{12}} & 0 & 0 \\
\frac{1}{2G_{12}} & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{31} \\
\sigma_{12}
\end{bmatrix}$$

(5)

Nonlinear elasticity in large deformations is described in Section 2.2, with applications for porous materials in Section 2.3 and for elastomers in Section 2.4.

Thermoelasticity takes into account the stresses and strains induced by thermal expansion with dilatation coefficient $\alpha$. For small variations of temperature $\theta$ for which the elasticity parameters may be considered as constant:

$$\varepsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} + \alpha \theta \delta_{ij}$$

(6)

For large variations of temperature, $E$, $\nu$, and $\alpha$ will vary. In rate formulations, such as are needed in elastoviscoplasticity, for example, the
derivative of $E$, $v$, and $\alpha$ must be considered.

$$\dot{e}_{ij} = \frac{1 + v}{E} \sigma_{ij} - \frac{v}{E} \sigma_{kk} \delta_{ij} + 2 \dot{\theta} \delta_{ij} + \left[ \frac{\partial}{\partial \theta} \left( \frac{1 + v}{E} \right) \sigma_{ij} - \frac{\partial}{\partial \theta} \left( \frac{v}{E} \right) \sigma_{kk} \delta_{ij} + 2 \frac{\partial \theta}{\partial \theta} \delta_{ij} \right] \dot{\theta}$$  

(7)

Viscoelasticity considers in addition a dissipative phenomenon due to “internal friction,” such as between molecules in polymers or between cells in wood. Here again, isotropy, linearity, and small strains allow for simple models. Quadratic functions for the state potential and the dissipative potential lead to either Kelvin-Voigt or Maxwell's models, depending upon the partition of stress or strains in a reversible part and in an irreversible part. They are described in detail for the one-dimensional case in Section 2.5 and recalled here in three dimensions.

- **Kelvin-Voigt model:**
  \[
  \sigma_{ij} = \lambda (\varepsilon_{kk} + \theta_2 \dot{\varepsilon}_{kk}) \delta_{ij} + 2\mu (\varepsilon_{ij} + \theta_1 \dot{\varepsilon}_{ij})
  \]
  (8)

  Here $\lambda$ and $\mu$ are Lame's coefficients at steady state, and $\theta_2$ and $\theta_1$ are two time parameters responsible for viscosity. These four coefficients may be identified from creep tests in tension and shear.

- **Maxwell model:**
  \[
  \dot{e}_{ij} = \frac{1 + v}{E} \left( \sigma_{ij} + \frac{\sigma}{\tau_1} \right) - \frac{v}{E} \left( \sigma_{kk} + \frac{\sigma}{\tau_2} \right) \delta_{ij}
  \]
  (9)

  Here $E$ and $v$ are Young's modulus and Poisson's ratio at steady state, and $\tau_1$ and $\tau_2$ are two other time parameters. It is a fluidlike model: equilibrium at constant stress does not exist.

In fact, a more general way to write linear viscoelastic constitutive models is through the *functional formulation* by the convolution product as any linear system. The hereditary integral is described in detail for the one-dimensional case, together with its use by the Laplace transform, in Section 2.5.

$$\varepsilon_{ij(t)} = \int_0^t J_{ijkl}(t - \tau) \frac{d\sigma_{kl}}{d\tau} d\tau + \sum_{p=1}^n J_{ijkl}(t - \tau) \Delta \sigma_{kl}^p$$

(10)

$[J(t)]$ is the *creep functions* matrix, and $\Delta \sigma_{kl}^p$ are the eventual stress steps.

The dual formulation introduces the *relaxation functions* matrix $[R(t)]$

$$\sigma_{ij(t)} = \int_0^t R_{ijkl}(t - \tau) \frac{d\varepsilon_{kl}}{d\tau} d\tau + \sum_{p=1}^n R_{ijkl} \Delta \varepsilon_{kl}^p$$

(11)

When isotropy is considered the matrix, $[J]$ and $[R]$ each reduce to two functions: either $J(t)$, the creep function in tension, is identified from a creep
test at constant stress; \( J(t) = \varepsilon(t)/\sigma \) and \( K \), the second function, from the creep function in shear. This leads to

\[
\varepsilon_{ij} = (J + K) \otimes \frac{D\sigma_{ij}}{D\tau} - K \otimes \frac{D\sigma_{kk}}{D\tau} \delta_{ij}
\]

where \( \otimes \) stands for the convolution product and \( D \) for the distribution derivative, taking into account the stress steps.

Or \( M(t) \), the relaxation function in shear, and \( L(t) \), a function deduced from \( M \) and from a relaxation test in tension \( R(t) = \sigma(t)/\varepsilon; \ L(t) = M(R - 2M)/(3M - R) \)

\[
\sigma_{ij} = L \otimes \frac{D(\varepsilon_{kk})}{D\tau} \delta_{ij} + 2M \otimes \frac{D\varepsilon_{ij}}{D\tau}
\]

All of this is for linear behavior. A nonlinear model is described in Section 2.6, and interaction with damage is described in Section 2.7.
2.2.1 VALIDITY

The theory is applicable to materials, such as rubberlike solids and certain soft biological tissues, which are capable of undergoing large elastic deformations. More details of the theory and its applications can be found in Beatty [1] and Ogden [3].

2.2.2 DEFORMATION

For a continuous body, a reference configuration, denoted by $B_r$, is identified and $\partial B_r$ denotes the boundary of $B_r$. Points in $B_r$ are labeled by their position vectors $X$ relative to some origin. The body is deformed quasi-statically from $B_r$ so that it occupies a new configuration, denoted $B$, with
This is the current or deformed configuration of the body. The deformation is represented by the mapping \( \chi : B_r \rightarrow B \), so that
\[
x = \chi(X) \quad X \in B_r,
\]
where \( x \) is the position vector of the point \( X \) in \( B \). The mapping \( \chi \) is called the deformation from \( B_r \) to \( B \), and \( \chi \) is required to be one-to-one and to satisfy appropriate regularity conditions. For simplicity, we consider only Cartesian coordinate systems and let \( X \) and \( x \), respectively, have coordinates \( X_i \) and \( x_i \), where \( i \in \{1, 2, 3\} \), so that \( x_i = \chi_i(X) \). Greek and Roman indices refer, respectively, to \( B_r \) and \( B \), and the usual summation convention for repeated indices is used.

The deformation gradient tensor, denoted \( F \), is given by
\[
F = \text{Grad } x \quad F_{ix} = \frac{\partial x_i}{\partial X_x}
\]
Grad being the gradient operator in \( B_r \). Local invertibility of \( \chi \) and its inverse requires that
\[
0 < J \equiv \det F < \infty
\]
wherein the notation \( J \) is defined.

The deformation gradient has the (unique) polar decompositions
\[
F = RU = VR
\]
where \( R \) is a proper orthogonal tensor and \( U \), \( V \) are positive definite and symmetric tensors. Respectively, \( U \) and \( V \) are called the right and left stretch tensors. They may be put in the spectral forms
\[
U = \sum_{i=1}^{3} \lambda_i u^{(i)} \otimes u^{(i)} \quad V = \sum_{i=1}^{3} \lambda_i v^{(i)} \otimes v^{(i)}
\]
where \( v^{(i)} = Ru^{(i)} \), \( i \in \{1, 2, 3\} \), \( \lambda_i \) are the principal stretches, \( u^{(i)} \) the unit eigenvectors of \( U \) (the Lagrangian principal axes), \( v^{(i)} \) those of \( V \) (the Eulerian principal axes), and \( \otimes \) denotes the tensor product. It follows from Eq. 3 that
\[
J = \lambda_1 \lambda_2 \lambda_3
\]
where \( J \) is the determinant of the deformation gradient.

The right and left Cauchy-Green deformation tensors, denoted \( C \) and \( B \), respectively, are defined by
\[
C = F^T F = U^2 \quad B = F F^T = V^2
\]

2.2.3 STRESS AND EQUILIBRIUM

Let \( \rho_r \) and \( \rho \) be the mass densities in \( B_r \) and \( B \), respectively. The mass conservation equation has the form
\[
\rho_r = \rho J
\]
The Cauchy stress tensor, denoted $\sigma$, and the nominal stress tensor, denoted $S$, are related by

$$S = JF^{-1}\sigma$$

The equation of equilibrium may be written in the equivalent forms

$$\text{div}\sigma + \rho b = 0 \quad \text{Div} S + \rho_{r} b = 0$$

where \text{div} and \text{Div} denote the divergence operators in $\mathcal{B}$ and $\mathcal{B}_r$, respectively, and $b$ denotes the body force per unit mass. In components, the second equation in Eq. 9 is

$$\frac{\partial S_{zi}}{\partial X_x} + \rho_{r} b_i = 0$$

Balance of the moments of the forces acting on the body yields simply $\sigma^T = \sigma$, equivalently $S^TF^T = FS$. The Lagrangian formulation based on the use of $S$ and Eq. 10, with $X$ as the independent variable, is used henceforth.

### 2.2.4 ELASTICITY

The constitutive equation of an elastic material is given in the equivalent forms

$$S = H(F) = \frac{\partial W}{\partial F}(F) \quad \sigma = G(F) \equiv J^{-1}FH(F)$$

where $H$ is a tensor-valued function, defined on the space of deformation gradients $F$, $W$ is a scalar function of $F$ and the symmetric tensor-valued function $G$ is defined by the latter equation in Eq. 11. In general, the form of $H$ depends on the choice of reference configuration and it is referred to as the response function of the material relative to $\mathcal{B}_r$ associated with $S$. For a given $\mathcal{B}_r$, therefore, the stress in $\mathcal{B}$ at a (material) point $X$ depends only on the deformation gradient at $X$. A material whose constitutive law has the form of Eq. 11 is generally referred to as a hyperelastic material and $W$ is called a strain-energy function (or stored-energy function). In components, (11)$_1$ has the form $S_{zi} = \partial W/\partial F_{iz}$, which provides the convention for ordering of the indices in the partial derivative with respect to $F$.

If $W$ and the stress vanish in $\mathcal{B}_r$, so that

$$W(I) = 0 \quad \frac{\partial W}{\partial F}(I) = 0$$

where $I$ is the identity and $O$ the zero tensor, then $\mathcal{B}_r$ is called a natural configuration.
Suppose that a rigid-body deformation \( x^* = Qx + c \) is superimposed on the deformation \( x = \chi(X) \), where \( Q \) and \( c \) are constants, \( Q \) being a rotation tensor and \( c \) a translation vector. The resulting deformation gradient, \( F^* \), is given by \( F^* = QF \). The elastic stored energy is required to be independent of superimposed rigid deformations, and it follows that

\[
W(QF) = W(F)
\]

for all rotations \( Q \). A strain-energy function satisfying this requirement is said to be \textit{objective}.

Use of the polar decomposition (Eq. 4) and the choice \( Q = R^T \) in Eq. 13 shows that \( W(F) = W(U) \). Thus, \( W \) depends on \( F \) only through the stretch tensor \( U \) and may therefore be defined on the class of positive definite symmetric tensors. We write

\[
T = \frac{\partial W}{\partial U}
\]

for the (symmetric) \textit{Biot stress tensor}, which is related to \( S \) by

\[
T = (SR + R^T S^T)/2.
\]

### 2.2.5 MATERIAL SYMMETRY

Let \( F \) and \( F' \) be the deformation gradients in \( \mathcal{B} \) relative to two different reference configurations, \( \mathcal{B}_r \) and \( \mathcal{B}'_r \) respectively. In general, the response of the material relative to \( \mathcal{B}'_r \) differs from that relative to \( \mathcal{B}_r \), and we denote by \( W \) and \( W' \) the strain-energy functions relative to \( \mathcal{B}_r \) and \( \mathcal{B}'_r \). Now let \( P = \text{Grad} X' \) be the deformation gradient of \( \mathcal{B}'_r \) relative to \( \mathcal{B}_r \), where \( X' \) is the position vector of a point in \( \mathcal{B}'_r \). Then \( F = F'P \). For specific \( P \) we may have \( W' = W \), and then

\[
W(F'P) = W(F')
\]

for all deformation gradients \( F' \). The set of tensors \( P \) for which Eq. 15 holds forms a multiplicative group, called the \textit{symmetry group of the material relative to \( \mathcal{B}_r \)}. This group characterizes the physical symmetry properties of the material.

For \textit{isotropic elastic materials}, for which the symmetry group is the \textit{proper orthogonal group}, we have

\[
W(FQ) = W(F)
\]

for all rotations \( Q \). Since the \( Q \)'s appearing in Eqs. 13 and 16 are independent, the combination of these two equations yields

\[
W(QUQ^T) = W(U)
\]
for all rotations $Q$. Equation 17 states that $W$ is an isotropic function of $U$. It follows from the spectral decomposition (Eq. 5) that $W$ depends on $U$ only through the principal stretches $\lambda_1$, $\lambda_2$, and $\lambda_3$ and is symmetric in these stretches.

For an isotropic elastic material, $\sigma$ is coaxial with $V$, and we may write

$$\sigma = \sigma_0 I + \sigma_1 B + \sigma_2 B^2$$  \hspace{1cm} (18)

where $\sigma_0$, $\sigma_1$, and $\sigma_2$ are scalar invariants of $B$ (and hence of $V$) given by

$$\sigma_0 = 2I_3^{1/2} \frac{\partial W}{\partial I_3}, \quad \sigma_1 = 2I_3^{-1/2} \left( \frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right), \quad \sigma_2 = -2I_3^{-1/2} \frac{\partial W}{\partial I_2}$$  \hspace{1cm} (19)

and $W$ is now regarded as a function of $I_1$, $I_2$, and $I_3$, the principal invariants of $B$ defined by

$$I_1 = \text{tr}(B) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$  \hspace{1cm} (20)

$$I_2 = \frac{1}{2} [I_1^2 - \text{tr}(B^2)] = \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 + \lambda_1^2 \lambda_2^2,$$  \hspace{1cm} (21)

$$I_3 = \det B = \lambda_1^2 \lambda_2^2 \lambda_3^2.$$  \hspace{1cm} (22)

Another consequence of isotropy is that $S$ and $\sigma$ have the decompositions

$$S = \sum_{i=1}^{3} t_i u^{(i)} \otimes v^{(i)}, \quad \sigma = \sum_{i=1}^{3} \sigma_i v^{(i)} \otimes v^{(i)}$$  \hspace{1cm} (23)

where $\sigma_i$, $i \in \{1, 2, 3\}$ are the principal Cauchy stresses and $t_i$ the principal Biot stresses, connected by

$$t_i = \frac{\partial W}{\partial \lambda_i} = J \lambda_i^{-1} \sigma_i$$  \hspace{1cm} (24)

Let the unit vector $M$ be a preferred direction in the reference configuration of the material, i.e., a direction for which the material response is indifferent to arbitrary rotations about the direction and to replacement of $M$ by $-M$. Such a material can be characterized by a strain energy which depends on $F$ and the tensor $M \otimes M$ [2, 4, 5]. Thus, we write $W(F, M \otimes M)$. The required symmetry (transverse isotropy) reduces $W$ to dependence on the five invariants

$$I_1, I_2, I_3, I_4 = M \cdot (CM) \quad I_5 = M \cdot (C^2M)$$  \hspace{1cm} (25)

where $I_1$, $I_2$, and $I_3$ are defined in Eqs. (20)–(22). The resulting nominal stress tensor is given by

$$S = 2W_1 F^T + 2W_2 (I_1 I - C) F^T + 2I_3 W_3 F^{-1} + 2W_4 M \otimes FM + 2W_5 (M \otimes FCM + CM \otimes FM)$$  \hspace{1cm} (26)

where $W_i = \partial W/\partial I_i$, $i = 1, \ldots, 5$. 

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2.2 Background on Nonlinear Elasticity
When there are two families of fibers corresponding to two preferred directions in the reference configuration, \( M \) and \( M' \) say, then, in addition to Eq. 25, the strain energy depends on the invariants

\[
I_6 = M' \cdot (CM') \quad I_7 = M' \cdot (C^2M') \quad I_8 = M \cdot (CM')
\] (27)

and also on \( M \cdot M' \) (which does not depend on the deformation); see Spencer [4, 5] for details. The nominal stress tensor can be calculated in a similar way to Eq. 26.

### 2.2.6 CONSTRAINED MATERIALS

An internal constraint, given in the form \( C(F) = 0 \), must be satisfied for all possible deformation gradients \( F \), where \( C \) is a scalar function. Two commonly used constraints are *incompressibility* and *inextensibility*, for which, respectively,

\[
C(F) = \text{det}F - 1 \quad C(F) = M \cdot (F^T FM) - 1
\] (28)

where the unit vector \( M \) is the direction of inextensibility in \( \mathcal{B}_r \). Since any constraint is unaffected by a superimposed rigid deformation, \( C \) must be an objective scalar function, so that \( C(QF) = C(F) \) for all rotations \( Q \).

Any stress normal to the hypersurface \( C(F) = 0 \) in the (nine-dimensional) space of deformation gradients does no work in any (virtual) incremental deformation compatible with the constraint. The stress is therefore determined by the constitutive law (11)$_1$ only to within an additive contribution parallel to the normal. Thus, for a constrained material, the stress-deformation relation (11)$_1$ is replaced by

\[
\sigma = H(F) + q \frac{\partial C}{\partial F} = \frac{\partial W}{\partial F} + q \frac{\partial C}{\partial F}
\] (29)

where \( q \) is an arbitrary (Lagrange) multiplier. The term in \( q \) is referred to as the *constraint stress* since it arises from the constraint and is not otherwise derivable from the material properties.

For incompressibility and inextensibility we have

\[
\sigma = \frac{\partial W}{\partial F} + qF^{-1} \quad \sigma = \frac{\partial W}{\partial F} + 2qM \otimes FM
\] (30)

respectively. For an incompressible material the Biot and Cauchy stresses are given by

\[
T = \frac{\partial W}{\partial U} - pU^{-1} \quad \det U = 1
\] (31)
and

\[ \sigma = F \frac{\partial W}{\partial F} - pI \quad \det F = 1 \]  

(32)

where \( q \) has been replaced by \(-p\), which is called an arbitrary hydrostatic pressure. The term in \( z_0 \) in Eq. 18 is absorbed into \( p \), and \( I_3 = 1 \) in the remaining terms in Eq. 18. For an incompressible isotropic material the principal components of Eqs. 31 and 32 yield

\[ t_i = \frac{\partial W}{\partial \lambda_i} - p\lambda_i^{-1} \quad \sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \]  

(33)

respectively, subject to \( \lambda_1\lambda_2\lambda_3 = 1 \).

For an incompressible transversely isotropic material with preferred direction \( M \), the dependence on \( I_3 \) is omitted and the Cauchy stress tensor is given by

\[ \sigma = -pI + 2W_1B + 2W_2(I_1B - B^2) + 2W_4FM \otimes FM \]  

\[ + 2W_5(FM \otimes BFM + BFM \otimes FM) \]  

(34)

For a material with two preferred directions, \( M \) and \( M' \), the Cauchy stress tensor for an incompressible material is

\[ \sigma = -pI + 2W_1B + 2W_2(I_1B - B^2) + 2W_4FM \otimes FM \]  

\[ + 2W_5(FM \otimes BFM + BFM \otimes FM) + 2W_6FM' \otimes FM' \]  

\[ + 2W_7(FM' \otimes BFM' + BFM' \otimes FM') + W_8(FM \otimes FM' + FM' \otimes FM) \]  

(35)

where the notation \( W_i = \partial W/\partial I_i \) now applies for \( i = 1, 2, 4, \ldots, 8 \).

### 2.2.7 BOUNDARY-VALUE PROBLEMS

The equilibrium equation (second part of Eq. 9), the stress-deformation relation (Eq. 11), and the deformation gradient (Eq. 2) coupled with Eq. 1 are combined to give

\[ \text{Div} \left( \frac{\partial W}{\partial F} \right) + \rho_f b = 0 \quad F = \text{Grad } x \quad x = \chi(X) \quad X \in \mathcal{B}_r \]  

(36)

Typical boundary conditions in nonlinear elasticity are

\[ x = \xi(X) \text{ on } \partial \mathcal{B}_r^e \]  

(37)

\[ S^T N = \tau(F, X) \text{ on } \partial \mathcal{B}_r^e \]  

(38)

where \( \xi \) and \( \tau \) are specified functions, \( N \) is the unit outward normal to \( \partial \mathcal{B}_r \),
and $\partial B^c_r$ and $\partial B^r_r$ are complementary parts of $\partial B$. In general, $\tau$ may depend on the deformation through $F$. For a dead-load traction $\tau$ is independent of $F$.

For a hydrostatic pressure boundary condition, Eq. 38 has the form

$$\tau = -JPF^{-T}N \text{ on } \partial B^r_r$$

Equations 36–38 constitute the basic boundary-value problem in non-linear elasticity.

In components, the equilibrium equation in Eq. 36 is written

$$\mathcal{A}_{\alpha\beta} \frac{\partial^2 x_j}{\partial x_i \partial x_j} + \rho r b_i = 0$$

for $i \in \{1, 2, 3\}$, where the coefficients $\mathcal{A}_{\alpha\beta}$ are defined by

$$\mathcal{A}_{\alpha\beta} = \mathcal{A}_{\beta\alpha} = \frac{\partial^2 W}{\partial F_{ij} \partial F_{\beta j}}$$

When coupled with suitable boundary conditions, Eq. 41 forms a system of quasi-linear partial differential equations for $x_i = x_i(X_a)$. The coefficients $\mathcal{A}_{\alpha\beta}$ are, in general, nonlinear functions of the components of the deformation gradient.

For incompressible materials the corresponding equations are obtained by substituting the first part of Eq. 30 into the second part of Eq. 9 to give

$$\mathcal{A}_{\alpha\beta} \frac{\partial^2 x_j}{\partial x_i \partial x_j} - \frac{\partial p}{\partial x_i} + \rho r b_i = 0 \quad \det(\partial x_i / \partial x_j) = 1$$

where the coefficients are again given by Eq. 41.

In order to solve a boundary-value problem, a specific form of $W$ needs to be given. The form of $W$ chosen will depend on the particular material considered and on mathematical requirements relating to the properties of the equations, an example of which is the strong ellipticity condition. Equations 40 are said to be strongly elliptic if the inequality

$$\mathcal{A}_{\alpha\beta} m_i m_j N_\alpha N_\beta > 0$$

holds for all nonzero vectors $m$ and $N$. Note that Eq. 43 is independent of any boundary conditions. For an incompressible material, the strong ellipticity condition associated with Eq. 42 again has the form of Eq. 43, but the incompressibility constraint now imposes the restriction $m \cdot (F^{-T}N) = 0$ on $m$ and $N$.

REFERENCES


CHAPTER 2.3

Elasticity of Porous Materials

N. D. Cristescu
231 Aerospace Building, University of Florida, Gainesville, Florida

Contents
2.3.1 Validity ........................................ 84
2.3.2 Formulation ...................................... 85
2.3.3 Identification of the Parameters ................. 85
2.3.4 Examples ........................................ 88
References ........................................... 90

2.3.1 VALIDITY

The methods used to determine the elasticity of porous materials and/or particulate materials as geomaterials or powderlike materials are distinct from those used with, say, metals. The reason is that such materials possess pores and/or microcracks. For various stress states these may either open or closed, thus influencing the values of the elastic parameters. Also, the stress-strain curves for such materials are strongly loading-rate-dependent, starting from the smallest applied stresses, and creep (generally any time-dependent phenomena) is exhibited from the smallest applied stresses (see Fig. 2.3.1 for schist, showing three uniaxial stress-strain curves for three loading rates and a creep curve [1]). Thus information concerning the magnitude of the elastic parameters cannot be obtained:

- from the initial slope of the stress-strain curves, since these are loading-rate-dependent;
- by the often used “chord” procedure, obviously;
- from the unloading slopes, since significant hysteresis loops are generally present.
2.3.2 FORMULATION

The elasticity of such materials can be expressed as “instantaneous response” by

\[ D = \frac{\dot{T}}{2G} + \left( \frac{1}{3K} - \frac{1}{2G} \right)\frac{1}{3}(\text{tr}\dot{T})I \]

where \( G \) and \( K \) are the elastic parameters that are not constant, \( D \) is the strain rate tensor, \( T \) is the stress tensor, \( \text{tr}(\ ) \) is the trace operator, and \( I \) is the unit tensor. Besides the elastic properties described by Eq. 1, some other mechanical properties can be described by additional terms to be added to Eq. 1. For isotropic geomaterials the elastic parameters are expected to depend on stress invariants and, perhaps, on some damage parameters, since during loading some pores and microcracks may close or open, thus influencing the elastic parameters.

2.3.3 IDENTIFICATION OF THE PARAMETERS

The elastic parameters can be determined experimentally by two procedures. With the *dynamic procedure*, one is determining the travel time of the two
elastic (seismic) extended longitudinal and transverse waves, which are traveling in the body. If both these waves are recorded, then the instantaneous response is of the form of Eq. 1. The elastic parameters are obtained from

\[
K = \rho \left( v_p^2 - \frac{4}{3} v_S^2 \right) \\
G = \rho v_S^2
\]

where \( v_S \) is the velocity of propagation of the shearing waves, \( v_p \) the velocity of the longitudinal waves, and \( \rho \) the density.

The static procedure takes into account that the constitutive equations for geomaterials are strongly time-dependent. Thus, in triaxial tests performed under constant confining pressure \( \sigma \), after loading up to a desired stress state \( \tau \) (octahedral shearing stress), one is keeping the stress constant for a certain time period \( t_c \) [2, 3]. During this time period the rock is creeping. When the strain rates recorded during creep become small enough, one is performing an unloading–reloading cycle (see Fig. 2.3.2). From the slopes

\[
\left( \frac{1}{3G} + \frac{1}{9K} \right)^{-1} \left( -\frac{1}{6G} + \frac{1}{9K} \right)^{-1}
\]

of these unloading–reloading curves one can determine the elastic parameters. For each geomaterial, if the time \( t_c \) is chosen so that the subsequent unloading is performed in a comparatively much shorter time interval, no significant interference between creep and unloading phenomena will take place. An example for schist is shown in Figure 2.3.3, obtained in a triaxial test with five unloading–reloading cycles.

![Figure 2.3.2](image_url)  
**FIGURE 2.3.2** Static procedures to determine the elastic parameters from partial unloading processes preceded by short-term creep.
If only a partial unloading is performed (one third or even one quarter of the total stress, and sometimes even less), the unloading and reloading follow quite closely straight lines that practically coincide. If a hysteresis loop is still recorded, it means that the time $t_c$ was chosen too short. The reason for performing only a partial unloading is that the specimen is quite "thick" and as such the stress state in the specimen is not really uniaxial. During complete unloading, additional phenomena due to the "thickness" of the specimen will be involved, including, e.g., kinematic hardening in the opposite direction, etc.

Similar results can be obtained if, instead of keeping the stress constant, one is keeping the axial strain constant for some time period during which the axial stress is relaxing. Afterwards, when the stress rate becomes relatively small, an unloading–reloading is applied to determine of the elastic parameters. This procedure is easy to apply mainly for particulate materials (sand, soils, etc.) when standard (Karman) three-axial testing devices are used and the elastic parameters follow from

$$K = \frac{1}{3} \frac{\Delta \tau}{\Delta \varepsilon_1 + 2 \Delta \varepsilon_2} \quad G = \frac{1}{2} \frac{\Delta \tau}{\Delta \varepsilon_1 - \Delta \varepsilon_2}$$

(4)

where $\Delta$ is the variation of stress and elastic strains during the unloading–reloading cycle. The same method is used to determine the bulk modulus $K$ in hydrostatic tests when the formula to be used is

$$K = \frac{\Delta \sigma}{\Delta \varepsilon_v}$$

(5)

with $\sigma$ the mean stress and $\varepsilon_v$ the volumetric strain.

Generally, $K$ is increasing with $\sigma$ and reaching an asymptotic constant value when $\sigma$ is increasing very much and all pores and microcracks are closed.
under this high pressure. The variation of the elastic parameters with \( \tau \) is more involved: when \( \tau \) increases but is still under the compressibility–dilatancy boundary, the elastic parameters are increasing. For higher values, above this boundary, the elastic parameters are decreasing. Thus their variation is related to the variation of irreversible volumetric strain, which, in turn, is describing the evolution of the pores and microcracks existing in the geomaterial. That is why the compressibility–dilatancy boundary plays the role of reference configuration for the values of the elastic parameters so long as the loading path (increasing \( \sigma \) and/or \( \tau \)) is in the compressibility domain, the elastic parameters are increasing, whereas if the loading path is in the dilatancy domain (increasing under constant \( \sigma \)), the elastic parameters are decreasing. If stress is kept constant and strain is varying by creep, in the compressibility domain volumetric creep produces a closing of pores and microcracks and thus the elastic parameters increase, and vice versa in the dilatancy domain. Thus, for each value of \( \sigma \) the maximum values of the elastic parameters are reached on the compressibility–dilatancy boundary.

### 2.3.4 EXAMPLES

As an example, for rock salt in uniaxial stress tests, the variation of the elastic moduli \( G \) and \( K \) with the axial stress \( \sigma_1 \) is shown in Figure 2.3.4 [4]. The variation of \( G \) and \( K \) is very similar to that of the irreversible volumetric strain.

![Figure 2.3.4 Variation of the elastic parameters K and G and of irreversible volumetric strain in monotonic uniaxial tests.](image)
strain $e_{IV}$. If stress is increased in steps, and if after each increase the stress is kept constant for two days, the elastic parameters are varying during volumetric creep, as shown in Figure 2.3.5. Here $\Delta$ is the ratio of the applied stress and the strength in uniaxial compression $\sigma_c = 17.88 \text{ MPa}$. Again, a similarity with the variation of $e_{IV}$ is quite evident. Figure 2.3.6 shows for a different kind of rock salt the variation of the elastic velocities $v_P$ and $v_S$ in true triaxial tests under confining pressure $p_c = 5 \text{ MPa}$ (data by Popp, Schultze, and Kern [5]). Again, these velocities increase in the compressibility domain, reach their maxima on the compressibility–dilatancy boundary, and then decrease in the dilatancy domain.

For shale, and the conventional (Karman) triaxial tests shown in Figure 2.3.3, the values of $E$ and $G$ for the five unloading–reloading cycles shown are: $E = 9.9$, 24.7, 29.0, 26.3, and 22.3 GPa, respectively, while $G = 4.4$, 10.7, 12.1, 10.4, and 8.5 GPa.

For granite, the variation of $K$ with $\sigma$ is given as [2]

$$K(\sigma) := \begin{cases} K_0 - K_1 \left(1 - \frac{\sigma}{\sigma_0}\right), & \text{if } \sigma \leq \sigma_0 \\ K_0, & \text{if } \sigma \geq \sigma_0 \end{cases}$$

with $K_0 = 59 \text{ GPa}$, $K_1 = 48 \text{ GPa}$, and $\sigma_0 = 0.344 \text{ GPa}$, the limit pressure when all pores are expected to be closed.
The same formula for alumina powder is

\[ K(\sigma) := K^\infty - p_a \exp\left(-b \frac{\sigma}{p_a}\right) \]  

(7)

with \( K^\infty = 1 \times 10^7 \) kPa the constant value toward which the bulk modulus tends at high pressures, \( a = 10^7 \), \( b = -1.2 \times 10^{-4} \), and \( p_a = 1 \) kPa. Also for alumina powder we have

\[ E(\sigma) := E^\infty - p_a \beta \exp(-d\sigma) \]  

(8)

with \( E^\infty = 7 \times 10^3 \) kPa, \( \beta = 6.95 \times 10^5 \), and \( d = 0.002 \).

For the shale shown in Figure 2.3.3, the variation of \( K \) with \( \sigma \) for \( 0 \leq \sigma \leq 45 \) MPa is

\[ K(\sigma) := -0.78\sigma^2 + 65.32\sigma - 369 \]  

(9)

REFERENCES

2.4.1 VALIDITY

Many rubberlike solids can be treated as isotropic and incompressible elastic materials to a high degree of approximation. Here describe the mechanical properties of such solids through the use of an isotropic elastic strain-energy function in the context of finite deformations. For general background on finite elasticity, we refer to Ogden [5].

2.4.2 BACKGROUND

Locally, the finite deformation of a material can be described in terms of the three principal stretches, denoted by \( \lambda_1, \lambda_2, \) and \( \lambda_3 \). For an incompressible material these satisfy the constraint

\[
\lambda_1 \lambda_2 \lambda_3 = 1
\]

The material is isotropic relative to an unstressed undeformed (natural) configuration, and its elastic properties are characterized in terms of a
strain-energy function \( W(\lambda_1, \lambda_2, \lambda_3) \) per unit volume, where \( W \) depends symmetrically on the stretches subject to Eq. 1.

The principal Cauchy stresses associated with this deformation are given by

\[
\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad i \in \{1, 2, 3\}
\]

where \( p \) is an arbitrary hydrostatic pressure (Lagrange multiplier). By regarding two of the stretches as independent and treating the strain energy as a function of these through the definition \( \hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1} \lambda_2^{-1}) \), we obtain

\[
\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1} \quad \sigma_2 - \sigma_3 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}
\]

For consistency with the classical theory, we must have

\[
\hat{W}(1, 1) = 0, \quad \frac{\partial^2 \hat{W}}{\partial \lambda_1 \partial \lambda_2}(1, 1) = 2\mu, \quad \frac{\partial \hat{W}}{\partial \lambda_\alpha}(1, 1) = 0, \quad \frac{\partial^2 \hat{W}}{\partial \lambda_\alpha^2}(1, 1) = 4\mu, \quad \alpha \in \{1, 2\}
\]

where \( \mu \) is the shear modulus in the natural configuration. The equations in Eq. 3 are unaffected by superposition of an arbitrary hydrostatic stress. Thus, in determining the characteristics of \( \hat{W} \), and hence those of \( W \), it suffices to set \( \sigma_3 = 0 \) in Eq. 3, so that

\[
\sigma_1 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1} \quad \sigma_2 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}
\]

Biaxial experiments in which \( \lambda_1, \lambda_2 \) and \( \sigma_1, \sigma_2 \) are measured then provide data for the determination of \( \hat{W} \). Biaxial deformation of a thin sheet where the deformation corresponds effectively to a state of plane stress, or the combined extension and inflation of a thin-walled (membranelike) tube with closed ends provide suitable tests. In the latter case the governing equations are written

\[
P^* = \lambda_1^{-1} \lambda_2^{-1} \frac{\partial \hat{W}}{\partial \lambda_2} \quad F^* = \frac{\partial \hat{W}}{\partial \lambda_1} - \frac{1}{2} \lambda_2 \lambda_1^{-1} \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}
\]

where \( P^* = PR/H, P \) is the inflating pressure, \( H \) the undeformed membrane thickness, and \( R \) the corresponding radius of the tube, while \( F^* = F/2\pi RH \), with \( F \) the axial force on the membrane (note that the pressure contributes to the total load on the ends of the tube). Here \( \lambda_1 \) is the axial stretch and \( \lambda_2 \) the azimuthal stretch in the membrane.
2.4.3 DESCRIPTION OF THE MODEL

A specific model which fits very well the available data on various rubbers is that defined by

\[ W = \sum_{n=1}^{N} \mu_n (\dot{\lambda}_1^{2n} + \dot{\lambda}_2^{2n} + \dot{\lambda}_3^{2n} - 3)/\alpha_n \]  

(7)

where \( \mu_n \) and \( \alpha_n \) are material constants and \( N \) is a positive integer, which for many practical purposes may be taken as 2 or 3 [3]. For consistency with Eq. 4 we must have

\[ \sum_{n=1}^{N} \mu_n \alpha_n = 2\mu \]  

(8)

and in practice it is usual to take \( \mu_n \alpha_n > 0 \) for each \( n = 1, \ldots, N \).

In respect of Eq. 7, the equations in Eq. 3 become

\[ \sigma_1 - \sigma_3 = \sum_{n=1}^{N} \mu_n (\dot{\lambda}_1^{2n} - \dot{\lambda}_3^{2n}) \quad \sigma_2 - \sigma_3 = \sum_{n=1}^{N} \mu_n (\dot{\lambda}_2^{2n} - \dot{\lambda}_3^{2n}) \]  

(9)

2.4.4 IDENTIFICATION OF PARAMETERS

Biaxial experiments with \( \sigma_3 = 0 \) indicate that the shapes of the curves of \( \sigma_1 - \sigma_2 \) plotted against \( \lambda_1 \) are essentially independent of \( \lambda_2 \) for many rubbers. Thus the shape may be determined by the pure shear test with \( \lambda_2 = 1 \), so that

\[ \sigma_1 - \sigma_2 = \sum_{n=1}^{N} \mu_n (\dot{\lambda}_1^{2n} - 1) \quad -\sigma_2 = \sum_{n=1}^{N} \mu_n (\dot{\lambda}_3^{2n} - 1) \]  

(10)

for \( \lambda_1 \geq 1, \lambda_3 \leq 1 \). The shift factor to be added to the first equation in Eq. 10 when \( \lambda_2 \) differs from 1 is

\[ \sum_{n=1}^{N} \mu_n (1 - \dot{\lambda}_2^{2n}) \]  

(11)

Information on both the shape and shift obtained from experiments at fixed \( \lambda_2 \) then suffice to determine the material parameters, as described in detail in References [3] or [4].

Data from the extension and inflation of a tube can be studied on this basis by considering the combination of equations in Eq. 6 in the form

\[ \sigma_1 - \sigma_2 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial W}{\partial \lambda_2} = \lambda_1 F^* - \frac{1}{2} \lambda_2^2 \lambda_1 P^* \]  

(12)
2.4.5 HOW TO USE IT

The strain-energy function is incorporated in many commercial Finite Element (FE) software packages, such as ABAQUS and MARC, and can be used in terms of principal stretches and principal stresses in the FE solution of boundary-value problems.

2.4.6 TABLE OF PARAMETERS

Values of the parameters corresponding to a three-term form of Eq. 7 are now given in respect of two different but representative vulcanized natural rubbers. The first is the material used by Jones and Treloar [2]:

\[ \alpha_1 = 1.3, \quad \alpha_2 = 4.0, \quad \alpha_3 = -2.0, \]
\[ \mu_1 = 0.69, \quad \mu_2 = 0.01, \quad \mu_3 = -0.0122 \text{ Nmm}^{-2} \]

The second is the material used by James et al. [1], the material constants having been obtained by Treloar and Riding [6]:

\[ \alpha_1 = 0.707, \quad \alpha_2 = 2.9, \quad \alpha_3 = -2.62, \]
\[ \mu_1 = 0.941, \quad \mu_2 = 0.093, \quad \mu_3 = -0.0029 \text{ Nmm}^{-2} \]

For detailed descriptions of the rubbers concerned, reference should be made to these papers.

REFERENCES

CHAPTER 2.5

Background on Viscoelasticity

KOZO IKEGAMI
Tokyo Denki University, Kanda-Nishikicho 2-2, Chiyodaku, Tokyo 101-8457, Japan

Contents
2.5.1 Validity ........................................ 95
2.5.2 Mechanical Models ....................... 95
2.5.3 Static Viscoelastic Deformation......... 98
2.5.4 Dynamic Viscoelastic Deformation .... 100
2.5.5 Hereditary Integral ....................... 102
2.5.6 Viscoelastic Constitutive Equation by the Laplace Transformation ................... 103
2.5.7 Correspondence Principle ................. 104
References ....................................... 106

2.5.1 VALIDITY

Fundamental deformation of materials is classified into three types: elastic, plastic, and viscous deformations. Polymeric material shows time-dependent properties even at room temperature. Deformation of metallic materials is also time-dependent at high temperature. The theory of viscoelasticity can be applied to represent elastic and viscous deformations exhibiting time-dependent properties. This paper offers an outline of the linear theory of viscoelasticity.

2.5.2 MECHANICAL MODELS

Spring and dashpot elements as shown in Figure 2.5.1 are used to represent elastic and viscous deformation, respectively, within the framework of the
linear theory of viscoelasticity. The constitutive equations between stress $\sigma$ and stress $\varepsilon$ of the spring and dashpot are, respectively, as follows:

$$
\sigma = k\varepsilon \quad \sigma = \eta \frac{d\varepsilon}{dt}
$$

where the notations $k$ and $\eta$ are elastic and viscous constants, respectively. Stress of spring elements is linearly related with strain. Stress of dashpot elements is related with strain differentiated by time $t$, and the constitutive relation is time-dependent.

Linear viscoelastic deformation is represented by the constitutive equations combining spring and dashpot elements. For example, the constitutive equations of series model of spring and dashpot shown in Figure 2.5.2 is as follows:

$$
\sigma + \frac{\eta}{k} \frac{d\sigma}{dt} = \eta \frac{d\varepsilon}{dt}
$$

This is called the Maxwell model. The constitutive equation of the parallel model of spring and dashpot elements shown in Figure 2.5.3 is as follows:

$$
\sigma = k\varepsilon + \eta \frac{d\varepsilon}{dt}
$$

This is called the Voigt or Kelvin model.
There are many variations of constitutive equations giving linear viscoelastic deformation by using different numbers of spring and dashpot elements. Their constitutive equations are generally represented by the following ordinary differential equation:

\[
p_0 \sigma + p_1 \frac{d\sigma}{dt} + p_2 \frac{d^2\sigma}{dt^2} + \ldots + p_n \frac{d^n\sigma}{dt^n} = q_0 \varepsilon + q_1 \frac{d\varepsilon}{dt} + q_2 \frac{d^2\varepsilon}{dt^2} + \ldots + q_n \frac{d^n\varepsilon}{dt^n}
\]  

The coefficients \( p \) and \( q \) of Eq. 4 give the characteristic properties of linear viscoelastic deformation and take different values according to the number of spring and dashpot elements of the viscoelastic mechanical model.
2.5.3 STATIC VISCOELASTIC DEFORMATION

There are two functions representing static viscoelastic deformation; one is creep compliance, and another is the relaxation modulus. Creep compliance is defined by strain variations under constant unit stress. This is obtained by solving Eqs. 2 or 3 for step input of unit stress. For the Maxwell model and the Voigt model, their creep compliances are represented, respectively, by the following expressions. For the Maxwell model, the creep compliance is

$\epsilon(t) = \frac{1}{\eta} + \frac{1}{k} = \frac{1}{k} \left( \frac{t}{\tau_M} + 1 \right)$

where $\tau_M = \eta/k$, and this is denoted as relaxation time. For the Voigt model, the creep compliance is

$\epsilon(t) = \frac{1}{k} \left[ 1 - \exp \left( -\frac{kt}{\eta} \right) \right] = \frac{1}{k} \left[ 1 - \exp \left( -\frac{t}{\tau_K} \right) \right]

where $\tau_K = \eta/k$, and this is denoted as retardation time.

Creep deformations of the Maxwell and Voigt models are illustrated in Figures 2.5.4 and 2.5.5, respectively. Creep strain of the Maxwell model
2.5 Background on Viscoelasticity

**FIGURE 2.5.4** Creep compliance of the Maxwell model.

**FIGURE 2.5.5** Creep compliance of the Voigt model.
increases linearly with respect to time duration. The Voigt model exhibits saturated creep strain for a long time.

The relaxation modulus is defined by stress variations under constant unit strain. This is obtained by solving Eqs. 2 or 3 for step input of unit strain. For the Maxwell and Voigt models, their relaxation moduli are represented by the following expressions, respectively. For the Maxwell model,

$$\sigma = k \exp \left( -\frac{kt}{\eta} \right) = k \exp \left( -\frac{t}{\tau_M} \right)$$ (7)

For the Voigt model,

$$\sigma = k$$ (8)

Relaxation behaviors of the Maxwell and Voigt models are illustrated in Figures 2.5.6 and 2.5.7, respectively. Applied stress is relaxed by Maxwell model, but stress relaxation dose not appear in Voigt model.

### 2.5.4 DYNAMIC VISCOELASTIC DEFORMATION

The characteristic properties of dynamic viscoelastic deformation are represented by the dynamic response for cyclically changing stress or strain.

![Diagram of Relaxation Modulus](image)
The viscoelastic effect causes delayed phase phenomena between input and output responses. Viscoelastic responses for changing stress or strain are defined by complex compliance or modulus, respectively. The dynamic viscoelastic responses are represented by a complex function due to the phase difference between input and output.

Complex compliance $J^*_{/C3}$ of the Maxwell model is obtained by calculating changing strain for cyclically changing stress with unit amplitude. Substituting changing complex stress $\sigma = \exp(i\omega t)$, where $i$ is an imaginary unit and $\omega$ is the frequency of changing stress, into Eq. 2, complex compliance $J^*$ is obtained as follows:

$$J^* = \frac{1}{k} - i\frac{1}{\omega\eta} = \frac{1}{k} - i\frac{1}{k\omega\tau_M} = J' - ij''$$  \hspace{1cm} (9)

where the real part $J' = 1/k$ is denoted as storage compliance, and the imaginary part $J'' = 1/k\omega\tau_M$ is denoted as loss compliance.

The complex modulus $Y^*$ of the Maxwell model is similarly obtained by calculating the complex changing strain for the complex changing strain

---

**FIGURE 2.5.7** Relaxation modulus of the Voigt model.
\( \varepsilon = \exp(i\omega t) \) as follows:
\[
Y^* = k \frac{(\omega \tau_M)^2}{1 + (\omega \tau_M)^2} + ik \frac{\omega \tau_M}{1 + (\omega \tau_M)^2} = Y' + iY'' \quad (10)
\]
where \( Y' = k((\omega \tau_M)^2/(1 + (\omega \tau_M)^2)) \) and \( Y'' = k(\omega \tau_M/(1 + (\omega \tau_M)^2)) \). The notations \( Y' \) and \( Y'' \) are denoted as dynamic modulus and dynamic loss, respectively. The phase difference \( \delta \) between input strain and output stress is given by
\[
\tan \delta = \frac{Y''}{Y'} = \frac{1}{\omega \tau_M} \quad (11)
\]
This is called mechanical loss.

Similarly, the complex compliance and the modulus of the Voigt model are able to be obtained. The complex compliance is
\[
J^* = \frac{1}{k} \left[ \frac{1}{1 + (\omega \tau_K)^2} \right] - i \frac{1}{k} \left[ \frac{\omega \tau_K}{1 + (\omega \tau_K)^2} \right] = J' - iJ'' \quad (12)
\]
where \( J' = \frac{1}{k} \left[ \frac{1}{1 + (\omega \tau_K)^2} \right] \) and \( J'' = \frac{1}{k} \left[ \frac{\omega \tau_K}{1 + (\omega \tau_K)^2} \right] \).

The complex modulus is
\[
Y^* = k + i\omega \tau_K = Y' + iY'' \quad (13)
\]
where \( Y' = k \) and \( Y'' = k\omega \tau_K \).

### 2.5.5 HEREDITARY INTEGRAL

The hereditary integral offers a method of calculating strain or stress variation for arbitrary input of stress or strain. The method of calculating strain for stress history is explained by using creep compliance as illustrated in Figure 2.5.8. An arbitrary stress history is divided into incremental constant stress history \( d\sigma' \). Strain variation induced by each incremental stress history is obtained by creep compliance with the constant stress values. In Figure 2.5.8 the strain induced by stress history for \( t' < t \) is represented by the following integral:
\[
\varepsilon(t) = \sigma_0 J(t) + \int_0^t J(t - t') \frac{d\sigma'}{dt'} dt' \quad (14)
\]
This equation is transformed by partially integrating as follows:

$$\varepsilon(t) = \sigma(t) J(0) + \int_0^t \sigma(t') \frac{dJ(t - t')}{d(t - t')} dt'$$  \hspace{1cm} (15)

Similarly, stress variation for arbitrary strain history becomes

$$\sigma(t) = \varepsilon_0 Y(t) + \int_0^t Y(t - t') \frac{d\sigma'}{dt'} dt'$$  \hspace{1cm} (16)

Partial integration of Eq. 15 gives the following equation:

$$\sigma(t) = \varepsilon(t) Y(0) + \int_0^t \sigma(t') \frac{dY(t - t')}{d(t - t')} dt'$$  \hspace{1cm} (17)

Integrals in Eqs. 14 to 17 are called hereditary integrals.

### 2.5.6 VISCOELASTIC CONSTITUTIVE EQUATION BY THE LAPLACE TRANSFORMATION

The constitutive equation of viscoelastic deformation is the ordinary differential equation as given by Eq. 4. That is,

$$\sum_{k=0}^{n} p_k \frac{d^k \sigma}{dt^k} = \sum_{k=0}^{m} q_k \frac{d^k \varepsilon}{dt^k}$$  \hspace{1cm} (18)
This equation is written by using differential operators \( P \) and \( Q \),

\[
P\sigma = Q\varepsilon
\]  

(19)

where \( P = \sum_{k=0}^{n} p_k \frac{d^k}{dt^k} \) and \( Q = \sum_{k=0}^{m} q_k \frac{d^k}{dt^k} \).

Equation (19) is represented by the Laplace transformation as follows.

\[
\sum_{k=0}^{n} p_k s^k \hat{\sigma} = \sum_{k=0}^{m} q_k s^k \hat{\varepsilon}
\]  

(20)

where \( \hat{\sigma} \) and \( \hat{\varepsilon} \) are transformed stress and strain, and \( s \) is the variable of the Laplace transformation. Equation 20 is written by using the Laplace transformed operators of time derivatives \( \hat{P} \) and \( \hat{Q} \) as follows:

\[
\hat{\sigma} = \frac{\hat{Q}}{\hat{P}} \hat{\varepsilon}
\]  

(21)

where \( \hat{P} = \sum_{k=0}^{n} p_k s^k \) and \( \hat{Q} = \sum_{k=0}^{m} q_k s^k \).

Comparing Eq. 21 with Hooke’s law in one dimension, the coefficient \( \frac{\hat{Q}}{\hat{P}} \) corresponds to Young’s modulus of linear elastic deformation. This fact implies that linear viscoelastic deformation is transformed into elastic deformation in the Laplace transformed state.

### 2.5.7 CORRESPONDENCE PRINCIPLE

In the previous section, viscoelastic deformation in the one-dimensional state was able to be represented by elastic deformation through the Laplace transformation. This can apply to three-dimensional viscoelastic deformation. The constitutive relations of linear viscoelastic deformation are divided into the relations between hydrostatic pressure and dilatation, and between deviatoric stress and strain.

The relation between hydrostatic pressure and dilatation is represented by

\[
\sum_{k=0}^{m} p'_k \frac{d^k}{dt^k} \sigma'_{ij} = \sum_{k=0}^{n} q''_k \frac{d^k}{dt^k} \varepsilon_{ij}
\]  

(22)

\[
P'' \sigma_{ii} = Q'' \varepsilon_{ii}
\]  

(23)

where \( P'' \sum_{k=0}^{m} p''_k \frac{d^k}{dt^k} \) and \( Q'' = \sum_{k=0}^{n} q''_k \frac{d^k}{dt^k} \). In Eq. 22 hydrostatic pressure is \((1/3)\sigma_{ii}\) and dilatation is \(\varepsilon_{ii}\).
The relation between deviatoric stress and strain is represented by

\[
\sum_{k=0}^{m} p'_k \frac{d^k \sigma'_{ij}}{dt^k} = \sum_{k=0}^{n} q'_k \frac{d^k \epsilon'_{ij}}{dt^k}
\]

(24)

where \( P' = \sum_{k=0}^{m} p'_k \frac{d^k}{dt^k} \) and \( Q' = \sum_{k=0}^{n} q'_k \frac{d^k}{dt^k} \). In Eq. 24 deviatoric stress and strain are \( \sigma'_{ij} \) and \( \epsilon'_{ij} \), respectively.

The Laplace transformations of Eqs. 22 and 24 are written, respectively, as follows:

\[
\bar{P}'' \bar{\sigma}_{ii} = \bar{Q}'' \bar{\epsilon}_{ii}
\]

(26)

where \( \bar{P}'' = \bar{P}''(s) \) and \( \bar{Q}'' = \bar{Q}''(s) \), and

\[
\bar{P}' \bar{\sigma}'_{ij} = \bar{Q}' \bar{\epsilon}'_{ij}
\]

(27)

The linear elastic constitutive relations between hydrostatic pressure and dilatation and between deviatoric stress and strain are represented as follows:

\[
\sigma_{ii} = 3K \epsilon_{ii}
\]

(28)

\[
\sigma'_{ij} = 2G \epsilon'_{ij}
\]

(29)

Comparing Eq. 17 with Eq. 19, and Eq. 18 with Eq. 20, the transformed viscoelastic operators correspond to elastic constants as follows:

\[
3K = \frac{\bar{Q}''}{\bar{P}''}
\]

(30)

\[
2G = \frac{\bar{Q}'}{\bar{P}'}
\]

(31)

where \( K \) and \( G \) are volumetric coefficient and shear modulus, respectively.

For isotropic elastic deformation, volumetric coefficient \( K \) and shear modulus \( G \) are connected with Young’s modulus \( E \) and Poisson’s ratio \( v \) as follows:

\[
G = \frac{E}{2(1 + v)}
\]

(32)

\[
K = \frac{E}{3(1 - 2v)}
\]

(33)
Using Eqs. 30–33, Young’s modulus $E$ and Poisson’s ratio are connected with the Laplace transformed coefficient of linear viscoelastic deformation as follows:

$$E = \frac{3\ddot{Q}' \dddot{Q}''}{2\dddot{P}' \dddot{Q}'' + \dddot{P}'' \dddot{Q}'}$$  \hspace{1cm} (34)

$$v = \frac{\dddot{P}' \dddot{Q}'' - \dddot{P}'' \dddot{Q}'}{2\dddot{P}' \dddot{Q}'' + \dddot{P}'' \dddot{Q}'}$$  \hspace{1cm} (35)

Linear viscoelastic deformation corresponds to linear elastic deformation through Eqs. 30–31 and Eqs. 34–35. This is called the correspondence principle between linear viscoelastic deformation and linear elastic deformation. The linear viscoelastic problem is the transformed linear elastic problem in the Laplace transformed state. Therefore, the linear viscoelastic problem is able to be solved as a linear elastic problem in the Laplace transformed state, and then the elastic constants of solved solutions are replaced with the Laplace transformed operator of Eqs. 30–31 and Eqs. 34–35 by using the correspondence principle. The solutions replaced the elastic constants become the solution of the linear viscoelastic problem by inverting the Laplace transformation.

REFERENCES

A Nonlinear Viscoelastic Model Based on Fluctuating Modes

RACHID RAHOUADJ AND CHRISTIAN CUNAT
LEMTA, UMR CNRS 7563, ENSEM INPL 2, avenue de la Forêt-de-Haye, 54500 Vandoeuvre-lès-Nancy, France

Contents
2.6.1 Validity .................................. 107
2.6.2 Background of the DNLR ............. 108
   2.6.2.1 Thermodynamics of Irreversible
           Processes and Constitutive Laws . . . 108
   2.6.2.2 Kinetics and Complementary
           Laws ..................................... 110
   2.6.2.3 Constitutive Equations of
           the DNLR ............................. 112
2.6.3 Description of the Model in the Case
       of Mechanical Solicitations .......... 113
2.6.4 Identification of the Parameters ...... 113
2.6.5 How to Use It .......................... 115
2.6.6 Table of Parameters ................... 115
References ................................ 116

2.6.1 VALIDITY

We will formulate a viscoelastic modeling for polymers in the temperature range of glass transition. This physical modeling may be applied using integral or differential forms. Its fundamental basis comes from a generalization of the Gibbs relation, and leads to a formulation of constitutive laws involving control and internal thermodynamic variables. The latter must traduce
different microstructural rearrangements. In practice, both modal analysis and fluctuation theory are well adapted to the study of the irreversible transformations.

Such a general formulation also permits us to consider various nonlinearities as functions of material specificities and applied perturbations.

To clarify the present modeling, called “the distribution of nonlinear relaxations” (DNLR), we will consider the viscoelastic behavior in the simple case of small applied perturbations near the thermodynamic equilibrium. In addition, we will focus our attention upon the nonlinearities induced by temperature and frequency perturbations.

2.6.2 BACKGROUND OF THE DNLR

2.6.2.1 THERMODYNAMICS OF IRREVERSIBLE PROCESSES AND CONSTITUTIVE LAWS

As mentioned, the present irreversible thermodynamics are based on a generalization of the fundamental Gibbs equation to systems evolving outside equilibrium. Note that Coleman and Gurtin [1], have also applied this postulate in the framework of rational thermodynamics. At first, a set of internal variables (generalized vector denoted \( z \)) is introduced to describe the microstructural state. The generalized Gibbs relation combines the two laws of thermodynamics into a single one, i.e., the internal energy potential:

\[
e = e(s, e, n, \ldots ; z)
\]

which depends on overall state variables, including the specific entropy, \( s \). Furthermore, with the positivity of the entropy production being always respected, one obtains for open systems:

\[
T \frac{d\Delta s}{dt} = T \sigma_s = -J^s \cdot \nabla T - \sum_{k=1}^n J^k \cdot \nabla \mu^k + A \cdot \dot{z} \geq 0
\]

where the nonequilibrium thermodynamic forces may be separated into two groups: (i) the gradient ones, such as the gradient of temperature gradient \( \nabla T \), and the gradient of generalized chemical potential \( \nabla \mu^k \); and (ii) The generalized forces \( A \), or affinities as defined by De Donder [2] for chemical reactions, which characterize the nonequilibrium state of a uniform medium.

The vectors \( J^s \), \( J^k \), and \( \dot{z} \) correspond to the dual, fluxes, or rate-type variables.

To simplify the formulation of the constitutive laws, we will now consider the behavior of a uniform representative volume element (RVE without any
2.6 A Nonlinear Viscoelastic Model Based on Fluctuating Modes

gradient, thus:

\[ T \sigma_s = A \cdot \dot{z} \geq 0 \]  

(3)

The equilibrium or relaxed state (denoted by the index \( r \)) is currently described by a suitable thermodynamic potential \( \psi' \) obtained via the Legendre transformation of Eq. 1 with respect to the control or state variable \( (\gamma) \). In this particular state, the set of internal variables is completely governed by \( (\gamma) \):

\[ \psi' = \psi'(\gamma; z^r(\gamma)) = \psi'(\gamma) \]  

(4)

Our first hypothesis [3] states that it is always possible to define a thermodynamic potential \( \psi \) only as a function of \( \gamma \) and \( z \), even for systems outside equilibrium:

\[ \psi = \psi(\gamma; z) \]  

(5)

Then, we assume that the constitutive equations may be obtained as functions of the first partial derivatives of this potential with respect to the dual variables, and depend consequently on both control and internal variables; i.e., \( \beta = \beta(\gamma; z) \) and \( A = A(\gamma; z) \). In fact, this description is consistent with the principle of equipresence, as postulated in rational thermodynamics. Therefore, the thermodynamic potential becomes in a differential form:

\[ d\psi_k = \sum_{m=1}^{q} \beta_m d\gamma_m - \sum_{j=1}^{r} A_j dz_j \]  

(6)

Thus the time evolution of the global response, \( \beta \), obeys a nonlinear differential equation involving both the applied perturbation \( \gamma \) and the internal variable \( z \) (generalized vector):

\[ \dot{\beta} = a^u \cdot \dot{\gamma} + b \cdot \dot{z} \]  

(7a)

\[ \dot{A} = -b \cdot \dot{\gamma} - g \cdot \dot{z} \]  

(7b)

This differential system resumes in a general and condensed form the announced constitutive relationships. The symmetrical matrix \( a^u = \partial^2 \psi / \partial \gamma \partial \gamma \) is the matrix of Tisza, and the symmetrical matrix \( g = \partial^2 \psi / \partial z \partial z \) traduces the interaction between the dissipation processes [3]. The rectangular matrix \( b = \partial^2 \psi / \partial z \partial \gamma \) expresses the coupling effect between the state variables and the dissipation variables.

In other respects, the equilibrium state classically imposes the thermodynamic forces and their rate to be zero; i.e., \( A = 0 \) and \( \dot{A} = 0 \). From Eq. 7b we find, for any equilibrium state, that the internal variables’ evolution results directly from the variation of the control variables:

\[ \dot{z}^r = -g^{-1} \cdot b \cdot \dot{\gamma} \]  

(8)
According to Eqs. 7b and 8, the evolution of the generalized force becomes

\[ \dot{A} = -g.(\dot{z} - \dot{z}') \]  

(9)

and its time integration for transformation near equilibrium leads to the simple linear relationship

\[ A = -g(z - z') \]  

(10)

where \( g \) is assumed to be constant.

### 2.6.2.2 Kinetics and Complementary Laws

To solve the preceding three equations (7a–b, 10), with the unknown variables being \( \beta, z, z', \) and \( A \), one has to get further information about the kinetic relations between the nonequilibrium driving forces \( A \) and their fluxes \( \dot{z} \).

#### 2.6.2.2.1 First-Order Nonlinear Kinetics and Relaxation Times

We know that the kinetic relations are not submitted to the same thermodynamic constraints as the constitutive ones. Thus we shall consider for simplicity an affine relation between fluxes and forces. Note that this well-known modeling, early established by Onsager, Casimir, Meixner, de Donder, De Groot, and Mazur, is only valid in the vicinity of equilibrium:

\[ \dot{z} = L.A \]  

(11)

and hence, with Eq. 10:

\[ \dot{z} = -L.g.(z - z') = -\tau^{-1}.(z - z') \]  

(12)

According to this nonlinear kinetics, Meixner [4] has judiciously suggested a base change in which the relaxation time operator \( \tau \) is diagonal. Here, we consider this base, which also represents a normal base for the dissipation modes. In what follows, the relaxation spectrum will be explicitly defined on this normal base. To extend this kinetic modeling to nonequilibrium transformations, which is the object of the nonlinear TIP, we also suggest referring to Eq. 12 but with variable relaxation times. Indeed, each relaxation time is inversely proportional to the jump frequency, \( \nu \), and to the probability \( p_j = \exp(-\Delta F_{j\rightarrow r}/RT) \) of overcoming a free energy barrier, \( \Delta F_{j\rightarrow r} \). It follows that the relaxation time of the process \( j \) may be written:

\[ \tau_j = 1/\nu \exp(\Delta F_{j\rightarrow r}/RT) \]  

(13)

where the symbol (+) denotes the activated state, and the index (r) refers to the activation barrier of the REV near the equilibrium.
The reference jump frequency, \( v_0 = k_B T / h \), has been estimated from Guggenheim's theory, which considers elementary movements of translation at the atomic level. The parameters \( h \), \( k_B \), and \( r \) represent the constants of Plank, Boltzmann, and of the perfect gas, respectively, and \( T \) is the absolute temperature. It seems natural to assume that the frequency of the microscopic rearrangements is mainly governed by the applied perturbation rate, \( \gamma \), through a shift function \( a(\gamma) \):

\[
v = v_0 / a(\gamma)
\]

Assuming now that the variation of the activation energy for each process is governed by the evolution of the overall set of internal variables leads us to the following approximation of first order:

\[
\Delta F_j^+ = \Delta E_j^{+,r} + K_z (z - z')
\]

In the particular case of a viscoelastic behavior, this variation of the free energy becomes negligible. The temperature dependence obviously intervenes into the basic definition of the activation energy as

\[
\Delta F_j^{+,r} = \Delta E^{+,r} - T \Delta S_j^{+,r}
\]

where the internal energy \( \Delta E^{+,r} \) is supposed to be the same for all processes. It follows that we may define another important shift function, noted \( a(T) \), which accounts for the effect of temperature. According to the Arrhenius approximation, \( \Delta E^{+,r} \) being quasi-constant, this shift function verifies the following relation:

\[
\ln a(T, T_{\text{ref}}) = \Delta E^{+,r} (1/T - 1/T_{\text{ref}})
\]

where \( T_{\text{ref}} \) is a reference temperature. For many polymers near the glass transition, this last shift function obeys the WLF empiric law developed by William, Landel, and Ferry [5]:

\[
\ln(a_T) = c_1 (T - T_{\text{ref}}) / [c_2 + (T - T_{\text{ref}})]
\]

In summary, the relaxation times can be generally expressed as

\[
\tau_j(T) = \tau_j'(T_{\text{ref}}) a(T, T_{\text{ref}}) a(\gamma) a(z, z')
\]

and the shift function \( a(z, z') \) becomes negligible in viscoelasticity.

### 2.6.2.2.2 Form of the Relaxation Spectrum near the Equilibrium

We now examine the distribution of the relaxation modes evolving during the solicitation. In fact, this applied solicitation, \( \gamma \), induces a state of fluctuations which may be approximately compared to the corresponding equilibrium one. According to prigogine [6], these fluctuations obey the equipartition of the entropy production. Therefore, we can deduce the expected distribution in
the vicinity of equilibrium as

\[ p_j^0 = B \sqrt{\tau_j^r} \quad \text{with} \quad \sum_{j=1}^{n} p_j^0 = 1 \quad \text{and} \quad B = 1 \left/ \sum_{j=1}^{n} \sqrt{\tau_j^r} \right. \] (20)

where \( \tau_j^r \) is the relaxation time of the process \( j \), \( p_j^0 \) its relative weight in the overall spectrum, and \( n \) the number of dissipation processes [3].

As a first approximation, the continuous spectrum defined by Eq. 20 may be described with only two parameters: the longest relaxation time corresponding to the fundamental mode, and the spectrum width. Note that a regular numerical discretization of the relaxation time scale using a sufficiently high number \( n \) of dissipation modes, e.g., 30, gives a sufficient accuracy.

### 2.6.2.3 Constitutive Equations of the DNLR

Combining Eqs. 7a and 12 gives, whatever the chosen kinetics,

\[ \dot{\beta} = a^u \dot{\gamma} - b \cdot (z - z^r) \cdot \tau_z^{-1} = a^u \dot{\gamma} - (\hat{a} - \hat{a}') \cdot \tau_{\beta}^{-1} \] (21a)

To simplify the notation, \( \tau_{\beta} \) will be denoted \( \tau \). In a similar form and after introducing each process contribution in the base defined above, one has

\[ \dot{\beta}_m = \sum_{p=1}^{n} a^u_{mp} \dot{\gamma}_p - \sum_{j=1}^{n} \frac{\beta_{jm} - p_j^0 \beta_m^r}{\tau_j} \] (21b)

where the indices \( u \) and \( r \) denote the instantaneous and the relaxed values, respectively.

Now we shall examine the dynamic response due to sinusoidally varying perturbations \( \gamma_n = \gamma^0 \exp(i\omega t) \), where \( \omega \) is the applied frequency, and \( i^2 = -1 \), i.e., \( \dot{\gamma}_n = i\omega \dot{\gamma}_n \). The response is obtained by integrating the above differential relationship. Evidently, the main problem encountered in the numerical integration consists in using a time step that must be consistent with the applied frequency and the shortest time of relaxation. Furthermore, a convenient possibility for very small perturbations is to assume that the corresponding response is periodic and out of phase:

\[ \beta_n = \beta^0 \exp(i\omega t + \varphi) \quad \text{and} \quad \dot{\beta}_n = i\omega \beta_n \] (22)

where \( \varphi \) is the phase angle. In fact, such relations are representative of various physical properties as shown by Kramers [7] and Kronig [8].

The coefficients of the matrices of Tisza, \( a^u \) and \( a^r \), and the relaxation times, \( \tau_j \), may be dependent on temperature and/or frequency. In uniaxial
tests of mechanical damping, these Tisza’s coefficients correspond to the storage and loss modulus $E'$ (or $G'$) and $E''$ (or $G''$), respectively.

2.6.3 DESCRIPTION OF THE MODEL IN THE CASE OF MECHANICAL SOLICITATIONS

We consider a mechanical solicitation under an imposed strain $\varepsilon$. Here, the perturbation $\gamma$ and the response $\beta$ are respectively denoted $\varepsilon$ and $\sigma$. According to Eqs. 19 and 21b, the stress rate response, $\dot{\sigma}$, may be finally written

$$\dot{\sigma} = \sum_{j=1}^{n} p_j^0 a^u : \dot{\varepsilon} - \sum_{j=1}^{n} \frac{\sigma_j - p_j^0 a^r : \varepsilon}{a(\dot{\varepsilon}) a(\varepsilon, \varepsilon') a(T, T_{ref}) \tau_j(T_{ref})} \quad (23)$$

As an example, for a pure shear stress this becomes

$$\dot{\sigma}_{12} = \sum_{j=1}^{n} p_j^0 G^u \dot{\varepsilon}_{12} - \sum_{j=1}^{n} \frac{\sigma_{j 12} - p_j^0 G^r \dot{\varepsilon}_{12}}{a(\dot{\varepsilon}) a(\varepsilon, \varepsilon') a(T, T_{ref}) \tau_j(T_{ref})} \quad (24)$$

In the case of sinusoidally varying deformation, the complex modulus is given by

$$G^*(\omega) = G^u + (G^r - G^u) \sum_{j=1}^{n} p_j^0 \frac{1}{1 + i\omega \tau_j^G} \quad (25)$$

It follows that its real and imaginary components are, respectively,

$$G'(\omega) = G^u + (G^r - G^u) \sum_{j=1}^{n} p_j^0 \frac{1}{1 + \omega^2 (\tau_j^G)^2} \quad (26)$$

$$G''(\omega) = (G^r - G^u) \sum_{j=1}^{n} p_j^0 \frac{\omega \tau_j^G}{1 + \omega^2 (\tau_j^G)^2} \quad (27)$$

2.6.4 IDENTIFICATION OF THE PARAMETERS

The crucial problem in vibration experiments concerns the accurate determination of the viscoelastic parameters over a broad range of frequency. Generally, to avoid this difficulty one has recourse to the appropriate principle of equivalence between temperature and frequency, assuming implicitly identical microstructural states. A detailed analysis of the literature has brought us to a narrow comparison of the empirical model of Havriliak and
Negami (HN) [9] with the DNLR. The HN approach appears to be successful for a wide variety of polymers; it combines the advantages of the previous modeling of Cole and Cole [10] and of Davidson and Cole [11]. For pure shear stress the response given by this HN approach is

$$G^* = G^u_{HN} + (G^r_{HN} - G^u_{HN}) \frac{1}{[1 + (i\omega \tau_{HN})^2]^\beta}$$  \hspace{1cm} (28)

where $G^u_{HN}$, $G^r_{HN}$, $\alpha$, and $\beta$ are empirical parameters. Thus the real and imaginary components are, respectively,

$$G' = G^u_{HN} + (G^r_{HN} - G^u_{HN}) \frac{\cos(\beta \theta)}{[1 + 2\omega^2 \tau^2_{HN}\cos(\alpha \pi/2) + \omega^2 \tau^2_{HN}]^{\beta/2}}$$  \hspace{1cm} (29)

$$G'' = (G^r_{HN} - G^u_{HN}) \frac{\sin(\beta \theta)}{[1 + 2\omega^2 \tau^2_{HN}\cos(\alpha \pi/2) + \omega^2 \tau^2_{HN}]^{\beta/2}}$$  \hspace{1cm} (30)

The function $\theta$ is defined by

$$\theta = \tan^{-1}\left(\frac{\omega^2 \tau^2_{HN}\sin(\alpha \pi/2)}{1 + \omega^2 \tau^2_{HN}\cos(\alpha \pi/2)}\right)$$  \hspace{1cm} (31)

Eqs. 28 to 30 are respectively compared to Eqs. 25 to 27 in order to establish a correspondence between the relaxation times of the two models:

$$\log(\tau^G_j) = \log(\tau_{HN}) + j\Lambda/n + \Theta$$  \hspace{1cm} (32)

where $\Theta$, $\Lambda$, and $n$ are a scale parameter, the number of decades of the spectrum, and the number of processes, respectively. A precise empirical connection is obtained by identifying the shift function for the time scale with the relation

$$\tau^G_j = a(j)\tau^G_j = a(\omega)\tau^G_j = \left(\frac{\tan(\beta \theta)}{\omega \tau_{HN}}\right)\tau^G_j$$  \hspace{1cm} (33)

This involves a progressive evolution of the difference of modulus as a function of the applied frequency:

$$(G' - G^u) = (G^r_{HN} - G^u_{HN})f_G$$  \hspace{1cm} (34)

The function $f_G$ is given by

$$f_G = \cos(\beta \theta) \frac{1 + \tan^2(\beta \theta)}{[1 + 2\omega^2 \tau^2_{HN}\cos(\alpha \pi/2) + \omega^2 \tau^2_{HN}]^{\beta/2}}$$  \hspace{1cm} (35)
2.6.5 HOW TO USE IT

In practice, knowledge of the only empirical parameters of HN’s modeling (and/or Cole and Cole’s and Davidson and Cole’s) permits us, in the framework of the DNLR, to account for a large variety of loading histories.

2.6.6 TABLE OF PARAMETERS

As a typical example given by Hartmann et al. [12], we consider the case of a polymer whose chemical composition is 1PTMG2000/3MIDI/2DMPD*. The master curve is plotted at 298 K in Figure 2.6.1. The spectrum is discretized

* PTMG: poly (tetramethylene ether) glycol; MIDI: 4,4’-diphenylmethane diisocyanate; DMPD: 2,2-dimethyl-1, 3-propanediol with a density of 1.074 g/cm³, and glass transition $T_g = -40^\circ$C.
with $\Lambda = 6$, a scale parameter $\Theta$ equal to $-5.6$, and 50 relaxation times. The parameters $G_{HN}^{L} = 2.14 \text{ MPa}$, $G_{HN}^{u} = G_{u} = 1859 \text{ MPa}$, $\tau_{HN} = 1.649 \times 10^{-7} \text{ s}$, $\alpha = 0.5709$ and $\beta = 0.0363$ allow us to calculate the shift function $a(\omega)$ and the function $f_G$ which is necessary to estimate the difference between the relaxed and nonrelaxed modulus, taking into account the experimental conditions. Figure 2.6.1 illustrates the calculated viscoelastic response, which is superposed to HN’s one. The function $f_G$ and the shift function $a(\omega)$ illustrate the nonlinearities introduced in the DNLR modeling (Fig. 2.6.2).

REFERENCES

Linear Viscoelasticity with Damage

R. A. Schapery
Department of Aerospace Engineering and Engineering Mechanics, University of Texas, Austin, Texas

Contents
2.7.1 Validity .................................. 117
2.7.2 Background ............................ 118
2.7.3 Description of the Model ............... 119
2.7.4 Identification of the Material Functions and Parameters .......................... 121
2.7.5 How to Use It ........................... 123
References .................................... 123

2.7.1 VALIDITY

This paper describes a homogenized constitutive model for viscoelastic materials with constant or growing distributed damage. Included are three-dimensional constitutive equations and equations of evolution for damage parameters (internal state variables, ISVs) which are measures of damage. Anisotropy may exist without damage or may develop as a result of damage. For time-independent damage, the specific model covered here is that for a linearly viscoelastic, thermorheologically simple material in which all hereditary effects are expressed through a convolution integral with one creep or relaxation function of reduced time; nonlinear effects of transient crack face contact and friction are excluded. More general cases that account for intrinsic nonlinear viscoelastic and viscoplastic effects as well as thermorheologically complex behavior and multiple relaxation functions are published elsewhere [10].
2.7.2 BACKGROUND

As background to the model with time-dependent damage, consider first the constitutive equation with constant damage, in which $\varepsilon$ and $\sigma$ represent the strain and stress tensors, respectively,

$$
\varepsilon = \{ Sd\sigma \} + \varepsilon_T
$$

(1)

where $S$ is a fully symmetric, fourth order order creep compliance tensor and $\varepsilon_T$ is the strain tensor due to temperature and moisture (and other absorbed substances which affect the strains). The braces are abbreviated notation for a linear hereditary integral. Although the most general form could be used, allowing for general aging effects, for notational simplicity we shall use the familiar form for thermorheologically simple materials,

$$
\{ fdg \} = \int_0^t f(\xi - \xi') \frac{dg}{dt'} dt' = \int_0^\xi f(\xi - \xi') \frac{dg}{d\xi'} d\xi'
$$

(2)

where it is assumed $f = g = o$ for $t < 0$ and

$$
\xi = \int_0^t dt''/a_T[T(t'')] \quad \xi' = \xi(t')
$$

(3)

Also, $a_T(T)$ is the temperature-dependent shift factor. If the temperature is constant in time, then $\xi - \xi' = (t - t')/a_T$. Physical aging [12] may be taken into account by introducing explicit time dependence in $a_T$; i.e., use $a_T = a_T(T, t'')$ in Eq. 3. The effect of plasticizers, such as moisture, may also be included in $a_T$. When Eq. 2 is used with Eq. 1, $f$ and $g$ are components of the creep compliance and stress tensors, respectively.

In certain important cases, the creep compliance components are proportional to one function of time,

$$
S = kD
$$

(4)

where $k$ is a constant, dimensionless tensor and $D = D(\xi)$ is a creep compliance (taken here to be that obtained under a uniaxial stress state). Isotropic materials with a constant Poisson’s ratio satisfy Eq. 4. If such a material has mechanically rigid reinforcements and/or holes (of any shape), it is easily shown by dimensional analysis that its homogenized constitutive equation satisfies Eq. 4; in this case the stress and strain tensors in Eq. 1 should be interpreted as volume-averaged quantities [2]. The Poisson’s ratio for polymers at temperatures which are not close to their glass-transition temperature, $T_g$, is nearly constant; except at time or rate extremes, somewhat above $T_g$ Poisson’s ratio is essentially one half, while below $T_g$ it is commonly in the range 0.35–0.40 [5].
Equations 1 and 4 give
\[ \varepsilon = \{Dd(k\sigma)\} + \varepsilon_T \] (5)

The inverse is
\[ \sigma = k_I\{Ede\} - k_I\{Ede_T \} \] (6)

where \( k_I = k^{-1} \) and \( E = E(\xi) \) is the uniaxial relaxation modulus in which, for \( t > 0 \),
\[ \{DdE\} = \{EdD\} = 1 \] (7)

In relating solutions of elastic and viscoelastic boundary value problems, and for later use with growing damage, it is helpful to introduce the dimensionless quantities
\[ \varepsilon_R \equiv \frac{1}{E_R}\{Ede\} \quad \varepsilon_T^R \equiv \frac{1}{E_R}\{Ede_T \} \quad u^R \equiv \frac{1}{E_R}\{Edu\} \] (8)

where \( E_R \) is an arbitrary constant with dimensions of modulus, called the reference modulus; also, \( \varepsilon_R \) and \( \varepsilon_T^R \) are so-called pseudo-strains and \( u^R \) is the pseudo-displacement. Equation 6 becomes
\[ \sigma = C\varepsilon_R - C\varepsilon_T^R \] (9)

where \( C \equiv E_R k_I \) is like an elastic modulus tensor; its elements are called pseudo-moduli. Equation 9 reduces to that for an elastic material by taking \( E = E_R \); it reduces to the constitutive equation for a viscous material if \( E \) is proportional to a Dirac delta function of \( \xi \). The inverse of Eq. 9 gives the pseudo-strain \( \varepsilon_R \) in terms of stress,
\[ \varepsilon_R = \hat{S}\sigma + \varepsilon_T^R \] (10)

where \( \hat{S} = C^{-1} = k/E_R \). The physical strain is given in Eq. 5.

2.7.3 DESCRIPTION OF THE MODEL

The correspondence principle (CPII in Schapery [4, 8]) that relates elastic and viscoelastic solutions shows that Eqs. 1–10 remain valid, under assumption Eq. 4, with damage growth when the damage consists of cracks whose faces are either unloaded or have loading that is proportional to the external loads. With growing damage \( k, C, \) and \( \hat{S} \) are time-dependent because they are functions of one or more damage-related ISVs; the strain \( \varepsilon_T \) may also depend on damage. The fourth-order tensor \( k \) must remain inside the convolution integral in Eq. 5, just as shown. This position is required by the correspondence principle. The elastic-like Eqs. 9 and 10 come from Eq. 5, and thus have the appropriate form with growing damage. However, with
healing of cracks, pseudo-stresses replace pseudo-strains because \( k \) must appear outside the convolution integral in Eq. 5 [8].

The damage evolution equations are based on viscoelastic crack growth equations or, in a more general context, on nonequilibrium thermodynamic equations. Specifically, let \( W^R \) and \( W^C \) denote pseudo-strain energy density and pseudo-complementary strain energy density, respectively,

\[
W^R = \frac{1}{2} C (\varepsilon^R - \varepsilon_T^R) (\varepsilon^R - \varepsilon_T^R) - F \tag{11}
\]

\[
W^C = \frac{1}{2} \dot{S} \sigma \sigma + \varepsilon_T^R \sigma + F \tag{12}
\]

so that

\[
W^C = -W^R + \sigma \varepsilon^R \tag{13}
\]

and

\[
\sigma = \frac{\partial W^R}{\partial \varepsilon^R} \quad \varepsilon^R = \frac{\partial W^C}{\partial S} \tag{14}
\]

The function \( F \) is a function of damage and physical variables that cause residual stresses such as temperature and moisture.

For later use in Section 2.7.4, assume the damage is fully defined by a set of scalar ISVs, \( S_p (\ p = 1, 2, \ldots P) \) instead of tensor ISVs. Thermodynamic forces, which are like energy release rates, are introduced,

\[
f_p \equiv -\frac{\partial W^R}{\partial S_p} \tag{15}
\]

or

\[
f_p \equiv \frac{\partial W^C}{\partial S_p} \tag{16}
\]

where the equality of these derivatives follows directly from the total differential of Eq. 13.

Although more general forms could be used, the evolution equations for \( \dot{S}_p \equiv dS_p/d\xi \) are assumed in the form

\[
\dot{S}_p = \dot{S}_p (S_q, f_p) \tag{17}
\]

in which \( \dot{S}_p \) may depend on one or more \( S_q (q = 1, \ldots P) \), but on only one force \( f_p \). The entropy production rate due to damage is non-negative if

\[
\sum_p f_p \dot{S}_p \geq 0 \tag{18}
\]

thus satisfying the Second Law of Thermodynamics. It is assumed that when \( |f_p| \) is less than some threshold value, then \( \dot{S}_p = 0 \).
Observe that even when the stress vanishes, there may be damage growth due to $F$. According to Eqs. 12 and 16,

$$f_p = \frac{\partial W_C^R}{\partial S_p} = \frac{1}{2} \frac{\partial \hat{S}}{\partial S_p} \sigma \sigma + \frac{\partial e_{T}^{R}}{\partial S_p} \sigma + \frac{\partial F}{\partial S_p}$$

which does not vanish when $\sigma = 0$, unless $\partial F/\partial S_p = 0$.

The use of tensor ISVs is discussed and compared with scalar ISVs by Schapery [10]. The equations in this section are equally valid for tensor and scalar ISVs.

### 2.7.4 IDENTIFICATION OF THE MATERIAL FUNCTIONS AND PARAMETERS

The model outlined above is based on thermorheologically simple behavior in that reduced time is used throughout, including damage evolution (Eq. 17). In studies of particle-reinforced rubber [4], this simplicity was found, implying that even the microcrack growth rate behavior was affected by temperature only through viscoelastic behavior of the rubber. If the damage growth is affected differently by temperature (and plasticizers), then explicit dependence may be introduced in the rate (Eq. 17). In the discussion that follows, complete thermorheological simplicity is assumed.

The behavior of particle-reinforced rubber and asphalt concrete has been characterized using a power law when $f_p > o$,

$$\hat{S}_p = (f_p)^{z_p}$$

where $z_p$ is a positive constant. (For the rubber composite two ISVs, with $z_1 = 4.5$ and $z_2 = 6$, were used for uniaxial and multiaxial behavior, whereas for asphalt one ISV, with $z = 2.5$, was used for uniaxial behavior.) A coefficient depending on $S_p$ may be included in Eq. 20; but it does not really generalize the equation because a simple change of the variable $S_p$ may be used to eliminate the coefficient.

Only an outline of the identification process is given here, but details are provided by Park et al. [3] for uniaxial behavior and by Park and Schapery [4] and Ha and Schapery [1] for multiaxial behavior. Schapery and Sicking [11] and Schapery [9] discuss the model’s use for fiber composites. The effects of \( e_T \) and \( F \) are neglected here.

(a) The first step is to obtain the linear viscoelastic relaxation modulus $E(\xi)$ and shift factor $a_T$ for the undamaged state. This may be done
using any standard method, such as uniaxial constant strain rate tests at a series of rates and temperatures. Alternatively, for example, uniaxial creep tests may be used to find \( D(\xi) \), after which \( E(\xi) \) is derived from Eq. 7.

(b) Constant strain rate (or stress rate) tests to failure at a series of rates or temperatures may be conveniently used to obtain the additional data needed for identification of the model. (However, depending on the complexity of the material and intended use of the model, unloading and reloading tests may be needed [7].) Constant strain rate tests often are preferred over constant stress rate tests because meaningful post-stress peak behavior (prior to significant strain localization) may be found from the former tests.

For isothermal, constant strain rate, \( R \), tests, the input is \( Rt = \hat{R} \xi \), where \( \hat{R} = RaT \) and \( \xi = t/aT \). Inasmuch as the model does not depend on temperature when reduced time is used, all stress vs. reduced time response curves depend on only one input parameter \( \hat{R} \), regardless of temperature. Thus, one may obtain a complete identification of the model from a series of tests over a range of \( \hat{R} \) using one temperature and different rates or one rate and different temperatures; both types of tests may be needed in practice for \( \hat{R} \) to cover a sufficiently broad range. One should, however, conduct at least a small number of both types of tests to check the thermorheologically simple assumption.

(c) Convert all experimental values of displacements and strains from step (b) tests to pseudo-quantities using Eq. 8. This removes intrinsic viscoelastic effects, thus enabling all subsequent identification steps to be those for a linear elastic material with rate-dependent damage. If controlled strain (stress) tests are used, then one would employ \( W^R(W^C_\xi) \) in the identification. However, mixed variables may be input test parameters, such as constant strain rate tests of specimens in a test chamber at a series of specified pressures [4]. In this case it is convenient to use mixed pseudo-energy functions in terms of strain and stress variables. Appropriate energy functions may be easily constructed using methods based on linear elasticity theory.

(d) The procedure for finding the exponent \( a \) and pseudo Young’s modulus in terms of one damage parameter is given by Park et al. [3]. After this, the remaining pseudo-moduli or compliances may be found in terms of one or more ISVs, as described by Park and Schapery [4] using constant strain rate tests of bar specimens under several confining pressures. The material employed by them was initially isotropic, but it became transversely isotropic as a result of damage. Identification of the full set of five pseudo-moduli and the pseudo-strain energy function, as functions of two ISVs, is detailed by Ha and Schapery [1].
2.7.5 HOW TO USE IT

Implementation of user-defined constitutive relations based on this model in a finite element analysis is described by Ha and Schapery [1]. Included are comparisons between theory and experiment for overall load-displacement behavior and for local strain distributions. The model employed assumes the material is locally transversely isotropic with the axis of isotropy assumed parallel to the local maximum principal stress direction, accounting for prior stress history at each point. A procedure is proposed by Schapery [10] that enables use of the same model when transverse isotropy is lost due to rotation of the local maximum principal stress direction.

REFERENCES