6

The Emergence of Masers and Lasers - Solutions

1. From Figure 6.7, the ${}^4F_2$ band ranges from about 17, 000 cm^{-1} to 19, 000 cm^{-1}. The units tell us that these quantities are given in terms of $1/\lambda$, the inverse of the light's wavelength. Using Eq. (1.22):

$$f = \frac{c}{\lambda}$$

where c is the speed of light (in units of cm/s). Converting to frequency:

$$17 \times 10^3 cm^{-1} \left(3 \times 10^{10} \frac{cm}{s}\right) = 5.1 \times 10^{14} Hz$$

$$19 \times 10^3 cm^{-1} \left(3 \times 10^{10} \frac{cm}{s}\right) = 5.7 \times 10^{14} Hz$$

Similarly, for ${}^4F_1$:

$$23 \times 10^3 cm^{-1} \left(3 \times 10^{10} \frac{cm}{s}\right) = 6.9 \times 10^{14} Hz$$

$$27 \times 10^3 cm^{-1} \left(3 \times 10^{10} \frac{cm}{s}\right) = 8.1 \times 10^{14} Hz$$

2. From Figure 6.9, the $2p$ to $1s$ transitions are shown by dashed lines with wavelengths ranging from 5944 Å to 6678 Å. The level splittings are are shown in Figure 6.9 next to the $2p$ and $1s$ symbols (note that $2p$ is repeated twice in the Paschen notation used in this figure–we want the lower-energy one here). By carefully measuring the energy of the $2p_3$ and $1s_3$ states, the wavelength of this transition can be estimated at approximately 6250 Å = 6250×10^{-10} m.

3. In the JK coupling scheme, we couple the angular momentum of the core, J, to the angular momentum of the electron, L. For neon, with a configuration $2p^5({}^2P_{3/2})3p^1$, where the term in the parenthesis uses spectroscopy notation

${}^{2s+1}L_J$ (see Section 5.4.1), the core has $J = 3/2$ coupled with the excited electron in the $3p$ orbital with $l = 1$. Following the rules for addition of angular momentum from Section 4.3.4:

$$\mathbf{K} = \mathbf{J} + \mathbf{l}$$

then the possible values for K are:

$$|\mathbf{K}| = J + l,\ J + l - 1, \ldots |J - l| = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$$

The total angular momentum includes spin, which is $\mathbf{K} + \mathbf{S}$ where $|\mathbf{S}| = 1/2$. Hence, the possible values for the total angular momentum are: 3,2,1,0.

4. The hyperfine interaction couples the spin of the nucleus to the spin of the electron. Given that $I = 1/2$ and the electron's spin $S = 1/2$, and using the rule for addition of angular momentum from Section 4.3.4:

$$\mathbf{F} = \mathbf{I} + \mathbf{S}$$

giving the possible values of $|\mathbf{F}| = 0, 1$. In a magnetic field, these lines will be split, similar to the drawing shown in Fig. 4.21, with magnetic substate of $M_F = -1, 0, 1$. Of course, the state with $|\mathbf{F}| = 0$ has no splitting and is degenerate with the $M_F = 0$ substate.

5. Using Eq. (6.7) with $F = 1$, $I = 1/2$ and $S = 1/2$:

$$g_F = \frac{(1)(1+1) - \frac{1}{2}(\frac{1}{2}+1) + \frac{1}{2}(\frac{1}{2}+1)}{(1)(1+1)} = 1$$

The energy splitting in a magnetic field B is given by combining Eqs. (6.3) and (6.5):

$$E_{mag} = g_F \mu_B B M_F$$

where from the problem above the only non-zero M_F values are ± 1. Since $g_F > 0$, the states with positive M_F have a positive energy in a B-field. From Eq. (6.5),

$$\mu_z = -g_F \mu_B M_F$$

and so states with positive M_F have a negative μ_z and vice-versa. Hydrogen atoms with a positive μ_z value will be drawn to regions of high magnetic field outside the magnetic trap (since the trap has a local minimum of B, as described in Section 6.4). These atoms will be lost, decreasing the average kinetic energy and temperature of the remaining atoms caught in the trap.

6. We are given that $\mu_z = -g_S \mu_B S_z + g_N \mu_N I_z$ where S_z and I_z are the projections of the particle spin along the total angular momentum F:

$$S_z = \frac{(\mathbf{S} \cdot \mathbf{F}) F_z}{\mathbf{F} \cdot \mathbf{F}}$$
$$I_z = \frac{(\mathbf{I} \cdot \mathbf{F}) F_z}{\mathbf{F} \cdot \mathbf{F}}$$

Using the rules for addition of angular momentum from Section 4.3.4 with $\mathbf{F} = \mathbf{S} + \mathbf{I}$:

$$(\mathbf{S} \cdot \mathbf{F}) = \frac{1}{2}(F^2 + S^2 - I^2)$$
$$(\mathbf{I} \cdot \mathbf{F}) = \frac{1}{2}(F^2 + I^2 - S^2)$$

where the eigenvalue for any angular momentum operator J^2 is $J(J+1)$ and similarly for J_z the eigenvalue is M_J. Plugging this in gives:

$$S_z = \frac{1}{2}\left[\frac{F(F+1) + S(S+1) - I(I+1)}{F(F+1)}\right] M_F$$
$$I_z = \frac{1}{2}\left[\frac{F(F+1) + I(I+1) - S(S+1)}{F(F+1)}\right] M_F$$

The z-component of the magnetic field then becomes:

$$\mu_z = -\frac{g_s}{2}\mu_B \left[\frac{F(F+1) + S(S+1) - I(I+1)}{F(F+1)}\right] M_F + \frac{g_N}{2}\mu_N \left[\frac{F(F+1) + I(I+1) - S(S+1)}{F(F+1)}\right] M_F$$

Using the definition of g_F given by Eq. (6.6) and the approximate form (6.7), this simplifies to:

$$\mu_z = -g_F \mu_B M_F$$

which is the desired result.

7. Given $S = 1/2$ and $I = 5/2$, the possible values for F are $F = 2, 3$. These levels split as follows:

$$F = 3 : M_F = -3,\ -2,\ -1, 0, 1, 2, 3$$
$$F = 2 : M_F = -2,\ -1, 0, 1, 2$$

The $M_F = -3$ and $M_F = 3$ states are not mixed with any states, and will be "pure" states, with straight lines on a plot like Figure 6.11. The other states of $F = 3$ and all states where $F = 2$ will be mixed, and their energy values make curved lines on a plot like Figure 6.11.