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## Statistical Physics - Solutions

1. There are sixteen possible outcomes of flipping a coin four times. Using Eq. (7.3) to calculate the statistical weight for  $N = 4$ , with  $n$  equal to the number of heads obtained in each distribution:

$$HHHH : \binom{4}{4} = \frac{4!}{4!0!} = 1$$

$$HHHT : \binom{4}{3} = \frac{4!}{3!1!} = 4$$

$$HHTT : \binom{4}{2} = \frac{4!}{2!2!} = 6$$

$$HTTT : \binom{4}{1} = \frac{4!}{1!3!} = 4$$

$$TTTT : \binom{4}{0} = \frac{4!}{0!4!} = 1$$

2. Since there is no particular order to the grouping, use Eq. (7.3) with  $N = 4$  and  $n = 2$ :

$$\binom{4}{2} = \frac{24}{4} = 6$$

3. Using Eq. (7.7) for each energy level:

$$\frac{n_2}{g_2} = \frac{n_2}{2} = \frac{N}{Z} e^{\epsilon_2/k_B T}$$

$$\frac{n_1}{g_1} = \frac{n_1}{1} = \frac{N}{Z} e^{\epsilon_1/k_B T}$$

Divide the above two equations:

$$\frac{n_2}{n_1} = 2e^{-(\epsilon_2 - \epsilon_1)/k_B T}$$

We are given  $\epsilon_2 - \epsilon_1 = 0.025 \text{ eV}$ ,  $T = 298\text{K}$ , and  $k_B$  from Appendix A:

$$\frac{n_2}{n_1} = 2 \exp\left(\frac{-0.025 \text{ eV}}{(8.617 \times 10^{-5} \text{ eV/K})(298\text{K})}\right) = 0.755$$

So 75.5% of the electrons are in the first excited state and 24.5% are in the ground state.

4. Starting with Eq. (7.20), and letting  $a = 1/(2mk_B T)$ ,

$$\int_0^\infty P(p) dp = \frac{4a^{3/2}}{\sqrt{\pi}} \int_0^\infty p^2 e^{-ap^2} dp$$

The integral itself is of the form of Eq. (7.18) with  $u = p$  and  $n = 2$ :

$$\int_0^\infty u^2 e^{-au^2} du = I_2(a)$$

Using Appendix G:

$$I_2(a) = \frac{\sqrt{\pi}}{4a^{3/2}}$$

shows that the probability distribution is normalized as given in Eq. (7.21).

5.  $F_1(u)$  is defined as:

$$F_1(u) = \frac{4}{\sqrt{\pi}} u^2 e^{-u^2}$$

Maximizing  $F_1(u)$  by setting the derivative of  $F_1(u)$  equal to zero:

$$\begin{aligned} \frac{dF_1}{du} &= \frac{4}{\sqrt{\pi}} (2ue^{-u^2} - 2u^3 e^{-u^2}) = 0 \\ 2ue^{-u^2} (1 - u^2) &= 0 \\ u &= 0, \pm 1 \end{aligned}$$

The negative solution has no physical interpretation. Also,  $F_1(0) = 0$ , so  $F_1(u)$  has a maximum at  $u = 1$ . This agrees with the plot shown in Figure 7.2.

6. The average value of  $v^2$  is given by:

$$\begin{aligned} \langle v^2 \rangle &= \int_0^\infty v^2 P(v) dv \\ &= \int_0^\infty 4\pi \left(\frac{a}{\pi}\right)^{3/2} v^4 e^{-av^2} dv \end{aligned}$$

where  $a = m/(2k_B T)$ . Pulling out the constant terms, the integral is of the form given by Eq. (7.18)

$$\int_0^\infty v^4 e^{-av^2} dv = I_4(a) = \frac{3}{(2a)^2} \cdot \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

So the average value is:

$$\langle v^2 \rangle = \frac{4a^{3/2}}{\sqrt{\pi}} \left( \frac{3\sqrt{\pi}}{8a^{5/2}} \right) = \frac{3}{2a} = \frac{3k_B T}{m}$$

7. The root-mean-square (rms) speed is defined as:

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_B T}{m}}$$

From Problem 5, the most probably speed corresponds to  $u = 1$  or, using Eq. (7.23),

$$v_p = \sqrt{\frac{2k_B T}{m}}$$

Comparing the two,  $v_{rms} > v_p$  by a factor of  $\sqrt{3/2} = 1.22$ . In Figure 7.2, the rms speed would be plotted at  $u = 1.22$ .

8. The problem is to find the percentage of  $\text{CO}_2$  molecules that have a speed greater than the given escape velocity  $v_e = 5000 \text{ m/s}$ , assuming a temperature of  $T = 240 \text{ K}$ . Using the masses from Appendix B:

$$m = m_C + 2m_O \simeq (12.0 + 32.0) \frac{10^{-3} \text{ kg}}{\text{mole}} \left( \frac{1 \text{ mole}}{6.022 \times 10^{23}} \right) = 7.31 \times 10^{-26} \frac{\text{kg}}{\text{atom}}$$

Plugging this into Eq. (7.23), the dimensionless variable is:

$$u = v \sqrt{\frac{m}{2k_B T}} = v \sqrt{\frac{7.31 \times 10^{-26} \text{ kg}}{2(1.38 \times 10^{-23} \text{ J/K})(240 \text{ K})}} = v \left( 3.32 \times 10^{-3} \frac{\text{s}}{\text{m}} \right)$$

and using the value of  $v_e$  gives  $u_e = 16.6$ . To get the fraction of atoms with speeds above this value, integrate the probability distribution (given just below Eq. (7.23)):

$$\int_{u_e}^\infty P(u) du = \frac{4}{\sqrt{\pi}} \int_{16.6}^\infty u^2 e^{-u^2} du$$

This integral can be done numerically with MATLAB (see Appendix C) using the following commands:

```
P = @(u) u.*u.*exp(-u.*u);
int_value = integral(P,16.6,100.0)
```

where infinity has been approximated by 100. The result is about  $1.0 \times 10^{-117}$ , which means that essentially no  $\text{CO}_2$  atoms have a speed above the escape velocity.

**9.** The most probable speed  $v_p$  is given by Eq. (7.24), which corresponds to  $u_p = 1$  as shown in Problem 5. Calculating the fraction of atoms with speeds above  $u_p$  follows along the same lines as the previous problem by integrating the probability distribution:

$$\int_{u_p}^{\infty} P(u) du = \frac{4}{\sqrt{\pi}} \int_1^{\infty} u^2 e^{-u^2} du$$

In terms of MATLAB commands, the integral part is found by:

```
P = @(u) u.*u.*exp(-u.*u);
int_value = integral(P,1.0,100.0)
```

where again infinity has been approximated by 100. The integral has a value of about 0.2536. Multiplying by  $4/\sqrt{\pi}$  gives 0.572. So about 57.2% of atoms have speeds greater than the most probably one. Similarly, the integral from zero to 1.0 gives 0.1895, and multiplying by  $4/\sqrt{\pi}$  gives 0.428, or 42.8% have speeds less than  $v_p$ .

**10.** The average kinetic energy for each degree of freedom is (Example 7.4):

$$\epsilon_{av} = \frac{1}{2} k_B T$$

Plugging in for  $T = 298 \text{ K}$  and using  $k_B$  from Appendix A:

$$\epsilon_{av} = \frac{1}{2} (1.38 \times 10^{-23} \text{ J/K})(298 \text{ K}) = 2.06 \times 10^{-21} \text{ J}$$

In units of  $eV$ , this is  $\epsilon_{av} = 0.013 \text{ eV}$ .

**11.** As in the problem above:

$$\epsilon_{av} = \frac{3}{2} k_B T$$

The total kinetic energy of a mole of gas is (using  $N_A$  from Appendix A):

$$\epsilon = \frac{3}{2} (1.381 \times 10^{-23} \text{ J/K})(298 \text{ K})(6.022 \times 10^{23}) = 3717 \text{ J}$$

**12.** The specific heat is defined as the energy divided by the temperature:

$$\frac{\epsilon}{T} = \frac{1}{2} k_B$$

for each degree of freedom. Using the information provided:

$$\begin{aligned} \text{Atoms} &: \frac{3}{2}k_B N_A = 12.5 \text{ J/K} \\ \text{Diatomic} &: \frac{5}{2}k_B N_A = 20.8 \text{ J/K} \\ \text{Polyatomic} &: \frac{6}{2}k_B N_A = 24.9 \text{ J/K} \end{aligned}$$

13. Using Eq. (7.28):

$$u(f) = \frac{8\pi f^2}{c^3} \left( \frac{hf}{e^{hf/k_B T} - 1} \right)$$

Setting  $x = hf/(K_B T)$  and taking the differential:

$$\begin{aligned} u(x) &= 8\pi \left( \frac{k_B T}{hc} \right)^3 h x^3 (e^x - 1)^{-1} \\ \frac{du(x)}{dx} &= 8\pi \left( \frac{k_B T}{hc} \right)^3 h x^3 [3x^2(e^x - 1)^{-1} - x^3(e^x - 1)^{-2} \cdot e^x] \end{aligned}$$

Setting it equal to zero, the constant terms divide out. Multiplying by  $(e^x - 1)^2$ :

$$\frac{du}{dx} = 3x^2(e^x - 1) - x^3 e^x = 0$$

The maximum is not at  $x = 0$ , so dividing by  $x^2$ :

$$(3e^x - 3) - x e^x = 0$$

gives the transcendental equation  $(3 - x)e^x = 3$ . This can be solved by either plotting it or using the technique of successive approximation. The answer is  $x = 2.821$ .

14. Integrating the energy density for a frequency range, Eq. (7.28), over all frequencies:

$$\int_0^\infty u(f) df = \int_0^\infty \frac{8\pi f^2}{c^3} \left( \frac{hf}{e^{hf/k_B T} - 1} \right) df$$

Let  $\mu = hf/k_B T$  and pull out the constant terms:

$$\frac{8\pi k_B^4 T^4}{c^3 h^3} \int_0^\infty \frac{\mu^3}{e^\mu - 1} d\mu = \frac{8\pi k_B^4 T^4}{c^3 h^3} (3!) \left( \frac{\pi^4}{90} \right)$$

where the value of the integral is from Appendix G, Eqs. (G.20) and (G.22). Plugging in the numbers with  $T = 3000 \text{ K}$ :

$$\int_0^\infty u(f) df = \frac{48\pi^5}{90} \left( \frac{(1.38 \times 10^{-23} \cdot 3000)^4}{(3.00 \times 10^8 \cdot 6.626 \times 10^{-34})^3} \right) = 0.0611 \frac{J}{m^3}$$

15. Using Eq. (7.28):

$$u(f) = \frac{8\pi f^2}{c^3} \left( \frac{hf}{e^{hf/k_B T} - 1} \right)$$

Converting to wavelength using  $f = c/\lambda$ .

$$u(\lambda) = \frac{8\pi(c/\lambda)^2}{c^3} \left( \frac{h(c/\lambda)}{e^{hc/\lambda k_B T} - 1} \right) = \frac{8\pi}{\lambda^3} \left( \frac{h}{e^{hc/\lambda k_B T} - 1} \right)$$

16. Using Eq. (7.39) and given  $T = 5800 K$ :

$$I = \sigma T^4 = \left( 5.67 \times 10^{-8} \frac{J}{m^2 s K^4} \right) (5800 K)^4 = 6.42 \times 10^7 \frac{W}{m^2}$$

Note that this is the intensity of light at the Sun's surface. By the time it reaches the Earth, the intensity has dropped by  $1/r^2$  where  $r$  is approximately the Earth-Sun distance.

17. Using Eq. (7.37), the most probably frequency of light is:

$$\frac{hf}{k_B T} = 2.821$$

Since we want the energy of the photons, use  $E = hf$ , given  $T = 5800 K$ :

$$\begin{aligned} E = hf &= (2.821)(k_B T) \\ &= (2.821)(8.62 \times 10^{-5} eV/K)(5800 K) \\ &= 1.41 eV \end{aligned}$$

The probability of photons in a given energy range  $\Delta E$  is:

$$P = \frac{I(E)\Delta E}{I_{total}}$$

where Eq. (7.38) gives the numerator and Eq. (7.39) gives the denominator (found in the previous problem). We are given an energy range, from 1.75 to 1.80 eV. It is most convenient to work with the dimensionless variable  $u = hf/k_B T$  with  $E = hf$  and here  $T = 5800 K$  gives  $k_B T = 0.50 eV$ . So the range 1.75-1.80 eV corresponds to  $u$  in the range 3.5-3.6. Plugging the numbers into Eq. (7.38) in terms of  $u$ :

$$I(u) du = \frac{\hbar}{4\pi^2 c^2} \left( \frac{k_B T}{\hbar} \right)^4 \int_{3.5}^{3.6} \frac{u^3}{e^u - 1} du$$

For a small enough range in  $u$ , we can approximate  $u$  by its central value,  $u = 3.55$ , and  $du \simeq \Delta u = 0.1$ . For constant  $u$ :

$$\begin{aligned} I(u) \Delta u &\simeq \frac{c}{4\pi^2(\hbar c)^3} (k_B T)^4 \frac{u^3}{e^u - 1} \Delta u \\ &= \frac{3.0 \times 10^{17} \text{ nm/s}}{4\pi^2(197.3 \text{ eV} \cdot \text{nm})^3} (0.5 \text{ eV})^4 \frac{(3.55)^3}{e^{3.55} - 1} \quad (0.1) \\ &= 8.18 \times 10^6 \frac{\text{eV}}{\text{nm}^2 \cdot \text{s}} \end{aligned}$$

Converting this to standard units,  $I(E) \Delta E = 1.31 \times 10^6 \text{ J}/(\text{m}^2 \cdot \text{s})$ . Then:

$$P = \frac{1.31 \times 10^6}{6.42 \times 10^7} = 0.0204$$

so about 2% of all photons from the Sun are in the energy range 1.75-1.8 eV. Repeating the same procedure, but for the energy range 3.05-3.10 eV, the probability is only  $P = 0.0077$  or less than 0.8%.

**18.** From Eqs. (7.46) and (7.47):

$$dE = -P dV$$

and by conservation of energy for a closed system:

$$dE_1 + dE_2 = 0$$

Combining the above equations:

$$P_1 dV_1 = -P_2 dV_2$$

If the pressure of the two parts are equal, then  $dV_1 = -dV_2$ . In other words the volume of each partition will adjust, since the wall is moveable (keeping the total volume constant), until equilibrium is obtained.

**19.** Using Eq. (7.50), the work done on the gas is given by:

$$dW = -PdV$$

We are given the ideal gas law. Solving for  $P$ :

$$P = \frac{nRT}{V}$$

Using this  $P$  and integrating for constant  $T$ :

$$W = -nRT \int_{V_2}^{V_3} \frac{dV}{V} = -nRT \ln \left( \frac{V_3}{V_2} \right)$$

as desired. Next, using Eq. (7.48):

$$dE = dQ_{to} + dW_{on}$$

The energy remains constant between points 2 and 3, so  $dE = 0$  and using the above result for  $dW$ :

$$dQ_{to} = nRT \ln \left( \frac{V_3}{V_2} \right)$$

**20.** Using the first law of thermodynamics, Eq. (7.48) gives

$$dE = dQ_{to} + dW_{on}$$

Between points 1 and 2, no heat is exchanged, so  $dQ = 0$  and

$$dE = dW_{on} = PdV \quad 1 \rightarrow 2$$

where  $P$  depends on both  $T$  and  $V$ . In contrast, between points 2 and 3, the temperature remains constant, so  $P$  only depends on  $V$ . Since there is an added term for  $T$  in the first case, we expect a steeper change in  $P$  for an equivalent change in  $V$ .

**21.** Using Eq. (7.66):

$$\frac{1}{2.612} \left( \frac{N}{V} \right) = \left( \frac{2\pi mk_B T_c}{h^2} \right)^{3/2}$$

Solving for  $T_c$ :

$$T_c = \left( \frac{1}{2.612} \frac{N}{V} \right)^{2/3} \frac{h^2}{2\pi mk_B}$$

Substitute the given density:

$$\frac{N}{V} = 5.0 \times 10^{14} \frac{\text{atoms}}{\text{cm}^3} = 5.0 \times 10^{20} \frac{\text{atoms}}{\text{m}^3}$$

Using the value of  $h$  and  $k_B$  from Appendix A:

$$T_c = (3.32 \times 10^{13}) \frac{h^2}{2\pi mk_B} = \frac{1.68 \times 10^{-31} \text{ kg} \cdot \text{K}}{m}$$

Given the mass of an atom (or a molecule) the gas,  $m$  (units of  $kg$ ), the critical temperature is easily calculated.

**22.**



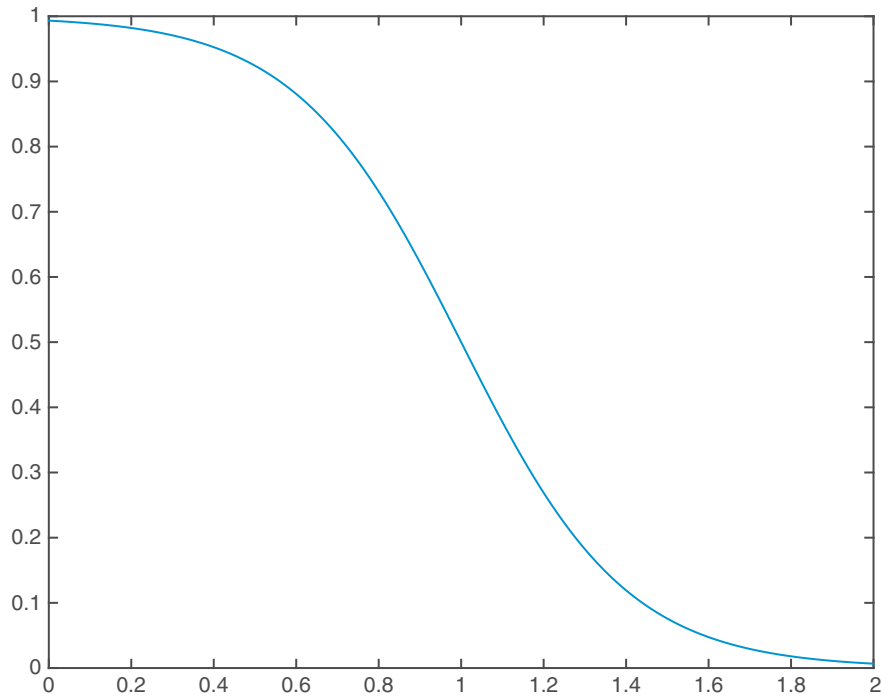
**23.** Following MATLAB Program 7.2, except using  $k_B T = 0.2\mu$ :

```

u = linspace(0, 2.0, 300);
f = @(u) 1./ ( exp( (u-1)*5) + 1);
plot(u,f(u))

```

The result looks very similar to Figure 7.12(a), except that the sloping region extends further in both directions around  $\epsilon_F/\mu = 1$ .



**24.** Using Eqs. (7.81) through (7.83):

$$Prob(u > 1) = \int_1^{\infty} f_n p(u) du$$

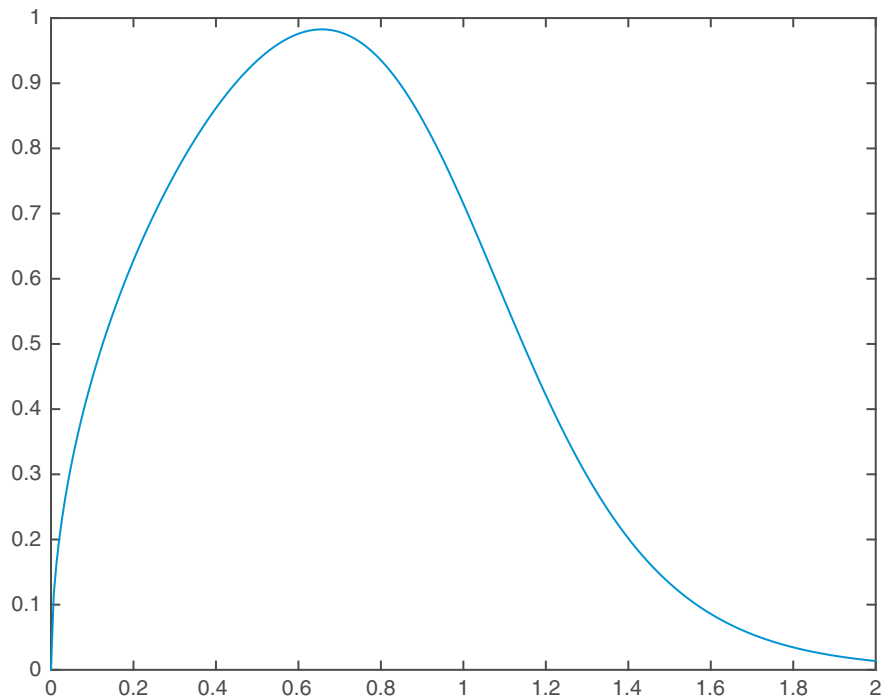
where  $f_n$  is the normalization constant and  $p(u) = \sqrt{u}f(u)$ . As for the above problem,  $k_B T = 0.2\mu$  so

$$p(u) = \sqrt{u} \frac{1}{e^{(u-1)/0.2} + 1}$$

Following MATLAB Program 7.3:

```
u = linspace(0, 2.0, 300);
f = @(u) 1./ ( exp( (u-1)*5 ) + 1 );
p = @(u) sqrt(u) .* f(u);
fn = 1/integral(p,0,2.0);
prob = @(u) fn*p(u);
plot( u, prob(u) )
int = integral( prob, 1.0, 2.0 )
```

The result of the integral is 0.217. So about 21.7% of electrons have an energy greater than the chemical potential when  $k_B T = 0.2\mu$ .



**25.** Let  $n_c$  be the number of conduction electrons per atom. Given that  $M$  is the molar mass, and  $\rho$  is the mass density, then  $\rho N_A / M$  is the number of atoms per volume. Multiplying this by the number of conduction electrons per atom

$$(n_c) \left( \frac{\rho N_A}{M} \right) = \left( \frac{\text{electrons}}{\text{atom}} \right) \left( \frac{\text{atoms}}{\text{volume}} \right)$$

gives the number of conduction electrons per volume,  $N/V$ .

**26.** Given  $n_c = 1$  since gold is monovalent, along with the other given values:

$$\begin{aligned} \frac{N}{V} &= \left( 1 \frac{\text{electron}}{\text{atom}} \right) \frac{(19.32 \frac{\text{g}}{\text{cm}^3})(6.022 \times 10^{23} \frac{\text{atoms}}{\text{mole}})}{197 \frac{\text{g}}{\text{mole}}} \\ &= 5.90 \times 10^{22} \frac{\text{electrons}}{\text{cm}^3} \end{aligned}$$

**27.** Using the result from the previous problem, and given  $V = 1 \text{ cm}^3$ :

$$N = \left( \frac{N}{V} \right) (V) = (5.90 \times 10^{22} \text{cm}^{-3})(1.0 \text{ cm}^3) = 5.90 \times 10^{22}$$

Obviously, there are a lot of conduction electron is a cubic centimeter of gold.

**28.(a)** Using the result from problem 24 with  $n_c = 1$  for sodium:

$$\begin{aligned} \frac{N}{V} &= \frac{n_c \rho N_A}{M} = \left( 1 \frac{\text{electron}}{\text{atom}} \right) \frac{(0.971 \frac{\text{g}}{\text{cm}^3})(6.022 \times 10^{23} \frac{\text{atoms}}{\text{mole}})}{23.0 \frac{\text{g}}{\text{mole}}} \\ &= 2.54 \times 10^{22} \frac{\text{electrons}}{\text{cm}^3} \end{aligned}$$

**(b)** Using Eq. (7.73) the Fermi energy, along with the above result and the electron mass for  $m$ :

$$\begin{aligned} \epsilon_F &= \frac{h^2}{2m} \left[ \frac{3}{8\pi} \frac{N}{V} \right]^{2/3} = \frac{(hc)^2}{2mc^2} \left[ \frac{3}{8\pi} \frac{N}{V} \right]^{2/3} \\ &= \frac{(1240 \text{ eV} \cdot \text{nm})^2}{2(5.11 \times 10^5 \text{ eV})} \left[ \frac{3}{8\pi} (2.54 \times 10^{22} \frac{1}{\text{cm}^3}) \left( \frac{10^{-21} \text{cm}^3}{\text{nm}^3} \right) \right]^{2/3} \\ &= 3.15 \text{ eV} = 5.05 \times 10^{-19} \text{ J} \end{aligned}$$

**(c)** Using the equation at the end of Section 7.6:

$$T_F = \frac{\epsilon_F}{k_B}$$

Plugging in the result from part (b):

$$T_F = \frac{3.15 \text{ eV}}{8.617 \times 10^{-5} \frac{\text{eV}}{\text{K}}} = 3.66 \times 10^4 \text{ K}$$

29. Following the same procedure as the previous problem:

$$\begin{aligned}\frac{N}{V} &= \frac{n_c \rho N_A}{M} = \left( 2 \frac{\text{electrons}}{\text{atom}} \right) \frac{(1.7 \frac{\text{g}}{\text{cm}^3})(6.022 \times 10^{23} \frac{\text{atoms}}{\text{mole}})}{24.3 \frac{\text{g}}{\text{mole}}} \\ &= 8.43 \times 10^{22} \frac{\text{electrons}}{\text{cm}^3}\end{aligned}$$

Then the Fermi energy is:

$$\begin{aligned}\epsilon_F &= \frac{(hc)^2}{2mc^2} \left[ \frac{3}{8\pi} \frac{N}{V} \right]^{2/3} \\ &= \frac{(1240 \text{ eV} \cdot \text{nm})^2}{2(5.11 \times 10^5 \text{ eV})} \left[ \frac{3}{8\pi} (8.43 \times 10^{22} \frac{1}{\text{nm}^3}) \right]^{2/3} \\ &= 7.01 \text{ eV} = 1.12 \times 10^{-19} \text{ J}\end{aligned}$$

giving the Fermi Temperature:

$$T_F = \frac{7.01 \text{ eV}}{8.617 \times 10^{-5} \frac{\text{eV}}{\text{K}}} = 8.14 \times 10^4 \text{ K}$$