11

Relativity I - Solutions

1. Einstein's second postulate on the special theory of relativity states that the speed of light is independent of the motion of the source, therefore the velocity of the light in the reference frame of the spaceship is c.

2.a) Starting with Eq. (11.16), calculate the gamma factor:

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-(0.2)^2}} = 1.0206$$

Using the Lorentz transformation from Eq. (11.15):

$$\begin{aligned} x' =& \gamma(x-ut) = (1.0206)(2\ m - (0.2\ c)(4\ m/c)) = 1.2247\ m \\ y' =& 0\ m \\ z' =& 0\ m \\ t' =& \gamma(t - \frac{ux}{c^2}) = (1.0206)\left(4\frac{m}{c} - (0.2\ c)\frac{2\ m}{c^2}\right) = 3.6742\ \frac{m}{c} \end{aligned}$$

b) Using Eq. (11.17) for the inverse transformation:

$$\begin{aligned} x =& \gamma(x'+ut') = (1.0206)(1.2247\ m - (0.2\ c)(3.6742\ m/c)) = 2.000\ m \\ y =& 0\ m \\ z =& 0\ m \\ t =& \gamma(t' + \frac{ux'}{c^2}) = (1.0206)\left(3.6742\frac{m}{c} - (0.2\ c)\frac{1.2247\ m}{c^2}\right) = 4.000\ \frac{m}{c} \end{aligned}$$

3.a) Starting with Eq. (11.16), calculate the gamma factor:

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-(0.25)^2}} = 1.033$$

Transforming from S to S' using Eq. (11.17):

$$\begin{aligned} x =& \gamma(x' + ut') = (1.033)(2\ m + (0.25\ c)(4\ m/c)) = 3.10\ m \\ y =& 0\ m \\ z =& 0\ m \\ t =& \gamma(t' + \frac{ux'}{c^2}) = (1.033)\left(4\frac{m}{c} + (0.25)\frac{2\ m}{c^2}\right) = 4.65\ \frac{m}{c} \end{aligned}$$

b) To transform back to S', use Eq. (11.15):

$$\begin{aligned} x' =& \gamma(x - ut) = (1.033)(3.10\ m - (0.25\ c)(4.65\ m/c)) = 2.00\ m \\ y' =& 0\ m \\ z' =& 0\ m \\ t' =& \gamma(t - \frac{ux}{c^2}) = (1.033)\left(4.65\frac{m}{c} - (0.25\ c)\frac{3.10\ m}{c^2}\right) = 4.00\ \frac{m}{c} \end{aligned}$$

4. Using the length contraction formula, Eq. (11.23):

$$\begin{aligned} L_M =& \sqrt{1 - \frac{u^2}{c^2}}(L_R) \\ \frac{L_M}{L_R} =& \sqrt{1 - \frac{u^2}{c^2}} \end{aligned}$$

Setting the length ratio equal to 1/2 and squaring both sides:

$$\frac{1}{4} = 1 - \frac{u^2}{c^2}$$

Solving for the speed u:

$$u = \frac{\sqrt{3}}{2}c$$

5.a) To calculate the length as perceived in your reference frame, use Eq. (11.23) with $L_R = 1\ m$:

$$L_M = \sqrt{1 - \frac{(0.5\ c)^2}{c^2}}\ (1\ m) = 0.866\ m$$

b) First, calculate the time it would take in the relative frame:

$$\Delta t_R = \frac{d}{v} = \frac{1.0\ m}{(0.5)(3.0 \times 10^8\ m/s)} = 6.667 \times 10^{-9}\ s$$

Transforming to your frame of reference:

$$\Delta t_M = \gamma \Delta t_R = \frac{6.667 \times 10^{-9}\ s}{\sqrt{1 - \frac{(0.5\ c)^2}{c^2}}} = 7.70 \times 10^{-9}\ s$$

6. Length contraction occurs only along the axis of motion. So the x-component will contract but the y-component is not changed. The x-component in the rest frame is:

$$(L_R)_x = (1.0\ m)\cos 30^\circ = 0.866\ m$$

Converting to the moving frame:

$$(L_M)_x = (0.866\ m)\sqrt{1 - (0.8\ c)^2/c^2} = 0.520\ m$$

The y-component (in both frames) is just $L_y = (1.0\ m)\sin 30^\circ = 0.5\ m$. Using the Pythagorean formula:

$$L_M = \sqrt{(L_x)^2 + (L_y)^2} = \sqrt{(0.52)^2 + (0.5)^2} = 0.721\ m$$

7. The percentage of length contraction is:

$$\left(1 - \frac{L_M}{L_R}\right) \times 100\%$$

Using Eq. (11.23):

$$\frac{L_M}{L_R} = \sqrt{1 - \frac{u^2}{c^2}} = \sqrt{1 - (3 \times 10^{-6})}$$

Even at this high speed, the ratio of length contraction is still unity to about one part in 10^{11}. In comparison, the time dilation, using Eq. (11.26), is:

$$\Delta t_M = \gamma(\Delta t_R) = \frac{1\ yr}{\sqrt{1 - (3 \times 10^{-6})^2}}$$

which is a similarly small effect.

8. Using Eq. (11.39), with v as the velocity of rocket A and u as the velocity of rocket B:

$$v' = \frac{(0.8\ c) - (0.6\ c)}{1 - (0.8)(0.6)} = 0.385\ c$$

9.a) We are given that the position of the center of mass, X, is:

$$2mX = (mx_e + mx_p)$$

and both particles have the same mass m. Taking the differential:

$$\frac{dX}{dt} = \frac{1}{2}(u_e + u_p)$$

where u_e and u_p are the velocities of the electron and positron, respectively. Substituting in the given speeds for the electron and positron:

$$u_X = \frac{1}{2}(0.95\ c + 0.2\ c) = 0.575\ c$$

b) Given a lifetime of $2.0 \times 10^{-8}\ s$ in the rest frame, Eq. (11.26) gives:

$$\Delta t_M = \frac{2.0 \times 10^{-8}\ s}{\sqrt{1 - (.575c)^2}} = 2.445 \times 10^{-8}\ s$$

10.a) An observer on Earth measures the time to be:

$$\Delta t_M = \frac{d}{v} = \frac{8.6\ c \cdot yr}{0.99\ c} = 8.687\ yr$$

b) An observer on the spaceship measures using proper time:

$$\Delta t_R = \Delta t_M \sqrt{1 - (0.99)^2} = 1.225\ yr$$

11.a) First, change the wavelength to frequency using the equation from Chapter 1:

$$f_0 = \frac{c}{\lambda_0} = \frac{3.0 \times 10^8\ m/s}{589.0 \times 10^{-9}\ m} = 5.09 \times 10^{14}\ Hz$$

Using Eq. (11.41) for an approaching light source:

$$f = \sqrt{\frac{1+\beta}{1-\beta}}\ f_0$$

where $\beta = v/c = 0.3$. Then:

$$f = \sqrt{\frac{1.3}{0.7}}(5.09 \times 10^{14}\ Hz) = 6.94 \times 10^{14}\ Hz$$

The correspond wavelength is:

$$\lambda = \frac{c}{f} = \frac{3.0 \times 10^8\ m/s}{6.94 \times 10^{14}\ Hz} = 432.2\ nm$$

and the shift in wavelength is:

$$\lambda - \lambda_0 = 432.2\ nm - 589.0\ nm = -156.8\ nm$$

b) Using Eq. (11.41) for a receding light source:

$$f = \sqrt{\frac{1-\beta}{1+\beta}} f_0 = \sqrt{\frac{0.7}{1.3}}(5.09 \times 10^{14}\ Hz) = 3.74 \times 10^{14}\ Hz$$

$$\lambda = \frac{c}{f} = 802.1\ nm$$

$$\lambda - \lambda_0 = 802.1\ nm - 589.0\ nm = 213.1\ nm$$

c) Figure 11.12 shows how to calculate the Doppler shift due to transverse motion:

$$f = \frac{f_0}{\gamma}$$

where the gamma factor is:

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - (0.3)^2}} = 1.05$$

Substituting:

$$f = \frac{5.09 \times 10^{14}\ Hz}{1.05} = 4.86 \times 10^{14}\ Hz$$

Now the wavelength and the wavelength shift are:

$$\lambda = \frac{c}{f} = \frac{3.0 \times 10^8\ m/s}{4.86 \times 10^{14}\ Hz} = 617.2\ nm$$

$$\lambda - \lambda_0 = 617.2\ nm - 589.0\ nm = 28.2\ nm$$

12. Change the wavelength to frequency using the equation from Chapter 1:

$$f_0 = \frac{c}{\lambda_0} = \frac{3.0 \times 10^8\ m/s}{656.5 \times 10^{-9}\ m} = 4.57 \times 10^{14}\ Hz$$

Using Eq. (11.41) for an receding light source:

$$f = \sqrt{\frac{1-\beta}{1+\beta}}\ f_0$$

where $\beta = v/c = (1.2 \times 10^8)/(3.0 \times 10^8) = 0.4$. Then:

$$f = \sqrt{\frac{0.6}{1.4}}(4.57 \times 10^{14}\ Hz) = 2.99 \times 10^{14}\ Hz$$

The correspond wavelength is:

$$\lambda = \frac{c}{f} = \frac{3.0 \times 10^8\ m/s}{2.99 \times 10^{14}\ Hz} = 1003\ nm$$

and the relative shift in wavelength is:

$$\frac{\lambda - \lambda_0}{\lambda_0} = \frac{1003\ nm - 656.5\ nm}{656.5\ nm} = 0.528$$

13. Using Eq. (11.41) for the receding source with $\beta = 0.8$:

$$f = \sqrt{\frac{1 - .80}{1 + .80}} \times 100\ MHz = 33\ MHz$$

14. Plugging the relation $f = c/\lambda$ into Eq. (11.41) for a receding source:

$$\frac{c}{\lambda} = \frac{c}{\lambda_0}\sqrt{\frac{1-\beta}{1+\beta}}$$

Canceling the common factor c and multiplying through by λ_0:

$$R = \frac{\lambda_0}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}}$$

Solving for β in terms of the ratio R:

$$\beta = \frac{1-R^2}{1+R^2}$$

We want to find β when $\lambda = 650\ nm$. The results are:

$$\begin{aligned} \lambda_0 &= 590\ nm: \ \ \beta = 0.0965 \\ \lambda_0 &= 525\ nm: \ \ \beta = 0.210 \\ \lambda_0 &= 460\ nm: \ \ \beta = 0.333 \end{aligned}$$

15. Start with Eq. (11.16) for γ:

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

Then solve Eq. (11.38) for v:

$$v' = \gamma(v - u) \cdot \gamma(1 + uv'/c^2)$$

Simplify the right-hand side, then bring the terms containing v' to the left:

$$\begin{aligned} v' &= \gamma^2 \left(v - u + \frac{vuv'}{c^2} - \frac{u^2 v'}{c^2} \right) \\ \frac{v'}{\gamma^2} - \frac{v'vu}{c^2} + \frac{u^2 v'}{c^2} &= v - u \\ v' \left(\frac{1}{\gamma^2} - \frac{uv}{c^2} + \frac{u^2}{c^2} \right) &= v - u \\ v' \left(1 - \frac{u^2}{c^2} - \frac{uv}{c^2} + \frac{u^2}{c^2} \right) &= v - u \\ v' &= \frac{v - u}{1 - \frac{uv}{c^2}} \end{aligned}$$

This is exactly Eq. (11.39) as desired.

16. The proper time is just the invariant interval, in the equations below Fig. 11.15:

$$(\text{proper time})^2 = (ct)^2 - x^2$$

where here the distance is $x = 3\ m$ and the lab time is $(ct) = 8\ m$. Hence,

$$\text{proper time} = \sqrt{(ct)^2 - x^2} = \sqrt{(8\ m)^2 - (3\ m)^2} = 7.4\ m$$

17. The world line for this traveler can be plotted in a similar way as done in Fig. 11.16. For the first leg, the traveler remains stationary for 1 m of time:

$$(\text{proper time})^2 = 1^2 - 0^2$$
$$\text{proper time} = 1\ m$$

Next, the traveler goes a distance of 4 m in a time of 5 m:

$$(\text{proper time})^2 = 5^2 - 4^2$$
$$\text{proper time} = 3\ m$$

Finally the traveler goes 1 m in a time of 2 m:

$$(\text{proper time})^2 = 2^2 - 1^2$$
$$\text{proper time} = \sqrt{3}\ m$$

The total proper time is $4 + \sqrt{3}\ m$.

18. The world line is drawn similar way to Fig. 11.16. Since $u = 0.2\ c$, the slope of the x'-axis is 0.2 and the slope of the ct'-axis is 5.

19. The world line is drawn similar way to Fig. 11.16. Since $u = 0.25\ c$, the slope of the x'-axis is 0.25 and the slope of the ct'-axis is 4.

20. Using Eq. (11.44) with the usual convention $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ and a velocity of $\beta = 0.2$, giving $\gamma = 1.02$:

$$\begin{aligned} x'^0 =& \gamma(x^0 - \beta x^1) = (1.02)(4\ m - (0.2)(2\ m)) = 3.672\ m \\ x'^1 =& \gamma(x^1 - \beta x^0) = (1.02)(2\ m - (0.2)(4\ m)) = 1.224\ m \\ x'^2 =& x^2 = 0 \\ x'^3 =& x^3 = 0 \end{aligned}$$

21. Using Eqs. (11.15), solve for x:

$$x = \frac{x'}{\gamma} + ut$$

Again using Eqs. (11.15), solve for t:

$$t = \frac{t'}{\gamma} + \frac{ux}{c^2}$$

Substitute in for t:

$$x = \frac{x'}{\gamma} + \frac{ut'}{\gamma} + \frac{u^2 x}{c^2}$$

Collecting terms of x

$$x\left(1 - \frac{u^2}{c^2}\right) = \frac{1}{\gamma}(x' + ut')$$

and using the definition of $\gamma^2 = (1 - u^2\ c^2)$:

$$x = \gamma(x' + ct')$$

as desired. Now substitute the first equation for x into the second one:

$$t = \frac{t'}{\gamma}\frac{ux'}{\gamma c^2} + \frac{u^2 t}{c^2}$$

Collecting terms for t and using the definition of γ:

$$t = \gamma(t' + \frac{ux'}{c^2})$$

22. Given Eq. (11.57), $t_\mu = g_{\mu\nu} t^\nu$, and Eq. (11.59), $t^\sigma = g^{\sigma\mu} t_\mu$, we see thatthe g matrices will serve to raise or lower the indices of a vector. Using $\Lambda_\mu^\nu = g_{\mu\rho}\Lambda_\sigma^\rho g^{\sigma\nu}$ to operate on v_ν:

$$g^{\sigma\nu} v_\nu = v^\sigma$$

The action of Λ_σ^ρ is to transform to the ρ' indices:

$$\Lambda_\sigma^\rho g^{\sigma\nu} v_\nu = \Lambda_\sigma^\rho v^\sigma = v'^\rho$$

Now use $g_{\mu\rho}$ to lower the index:

$$g_{\mu\rho}\Lambda_\sigma^\rho g^{\sigma\nu} v_\nu = g_{\mu\rho} v'^\rho = v'_\rho$$

This shows that $v'_\mu = \Lambda_\mu^\nu v_\nu$, as desired. This gives us a way to transform the covariant components of a vector.

23. First recall that $g_{\mu\rho}$ and $g^{\sigma\nu}$ will have the form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The first operation, $g_{\mu\rho}\Lambda^{\rho}_{\sigma}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ \beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Now, $g_{\mu\rho}\Lambda^{\rho}_{\sigma}g^{\sigma\nu}$:

$$\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ \beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

24.