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Relativity II - Solutions

1. The components of the four-velocity for velocity are found between Eqs. (12.3) and (12.4). Since the electron is traveling only in the x-direction:

$$v^2 = v^3 = 0$$

The relevant velocity components are:

$$v^0 = \gamma c, \quad v^1 = \gamma \frac{dx^1}{dt} = \gamma(0.2\ c)$$

Calculating γ:

$$\frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - (0.2)^2}} = 1.02$$

The results are:

$$v^0 = 1.02\ c$$
$$v^1 = 0.204\ c$$

b. Using Eq. (12.6) with the electron's mass ($m_e = 0.511\ MeV/c^2$):

$$p^0 = 1.02\ m_e c = 0.521\ MeV/c$$
$$p^1 = 0.204\ m_e c = 0.104\ MeV/c$$
$$p^2 = p^3 = 0$$

2. Using Eq. (12.13):

$$KE = (\gamma - 1)mc^2$$

Plug in the value of $\gamma = 1/\sqrt{1 - (0.2)^2} = 1.02$ and the electron mass:

$$KE = (1.02 - 1)(0.511\ MeV) = 0.011\ MeV$$

3. Using Eqs. (12.12) and (12.13) for the rest energy and kinetic energy, respectively:

$$(\gamma - 1)mc^2 = mc^2$$

gives $\gamma = 2.0$. Now using Eq. (12.3):

$$\frac{1}{\sqrt{1 - v^2/c^2}} = 2$$

$$\sqrt{(1 - v^2/c^2)} = \frac{1}{2}$$

$$\left(1 - \frac{v^2}{c^2}\right) = \frac{1}{4}$$

$$\frac{v^2}{c^2} = \frac{3}{4}$$

The particle must have a speed of $v = (\sqrt{3}/2)c$.

4. Using Eqs. (12.9) and (12.12) for the total energy and rest energy, respectively:

$$\gamma mc^2 = 2mc^2$$

gives $\gamma = 2.0$. This gives the same speed as the previous problem, $v = (\sqrt{3}/2)c$.

5. We first evaluate the derivatives of $f(x)$ called for in the Taylor series:

$$f'(x) = \frac{1}{2}\frac{1}{(1-x)^{3/2}}$$

$$f''(x) = \frac{3}{4}\frac{1}{(1-x)^{5/2}}$$

We now evaluate at x=0 and substitute into the given form of the Taylor series:

$$f(x) = 1 + \frac{1}{2}x = \frac{3}{8}x^2 + \ldots$$

Now, let $x = v^2/c^2$:

$$\frac{1}{\sqrt{1-u^2/c^2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} = \frac{3}{8}\frac{v^4}{c^4}$$

This result is identical to eq. 12.10.

6. From Eq. (12.13) with the mass of the electron:

$$KE = (\gamma - 1)(0.511\ MeV)$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$. This gives the following results:

$$\begin{aligned} v = 0\ &: \gamma = 1.00\ : \ KE = 0 \\ v = 0.99\ c\ &: \gamma = 7.09\ : \ KE = 3.11\ MeV \\ v = 0.999\ c\ &: \gamma = 22.37\ : \ KE = 10.9\ MeV \end{aligned}$$

7. Using Eq. (12.13) with m as the electron's mass:

$$KE = 100\ MeV = (\gamma - 1)(0.511\ MeV)$$

Solving for γ gives $\gamma = 196.7$. Now use the definition of γ in Eq. (12.3):

$$\begin{aligned} 196.7 &= \frac{1}{\sqrt{1-u^2/c^2}} \\ 38688 &= \frac{1}{1-\frac{v^2}{c^2}} \\ 1 - \frac{v^2}{c^2} &= 2.58 \times 10^{-5} \\ 0.999974 &= \frac{v^2}{c^2} \\ v &= 0.999987c \end{aligned}$$

8. Using Eq. (12.9) with the mass of the proton ($m_p = 938.3\ MeV$) and a total energy of $E = 1500\ MeV$:

$$1500\ MeV = \gamma(938.3\ MeV)$$

giving $\gamma = 1.60$, and plugging into Eq. (12.3) gives $v = 0.78\ c$.
b) Using Eq. (12.14),

$$\begin{aligned} pc &= \sqrt{E^2 - (mc^2)^2} = \sqrt{(1500)^2 - (938.3)^2}\ MeV \\ p &= 1170\ MeV/c \end{aligned}$$

9.a) Carrying out the four-vector addition (see Chapter 11) in the given expression for s:

$$\frac{|p_A + p_B|^2}{c^2} = \frac{[(E_A + E_B)^2 - (\mathbf{p}_A + \mathbf{p}_B)^2]}{c^2}$$

In the laboratory frame, E_B equals the rest energy of the particle and $\mathbf{p}_B = 0$:

$$s = \frac{[(E_A + m_B c^2)^2 - |\mathbf{p}_A|^2]}{c^2}$$

In the center of mass frame, we take the colliding particles to have equal and opposite momenta:

$$s = \frac{[(E_A + E_B)^2 - (\mathbf{p}_A + \mathbf{p}_B)^2]}{c^2} = \frac{(E_A + E_B)^2}{c^2}$$

b) Substituting the sums of the four-vectors into the sum of the Mandelstam variables:

$$\begin{aligned} s + t + u = \frac{1}{c^2}\Big[&\frac{(E_A + E_B)^2}{c^2} - (\mathbf{p}_A + \mathbf{p}_B)^2 + \frac{(E_A - E_C)^2}{c^2} - (\mathbf{p}_A - \mathbf{p}_C)^2 \\ &+ \frac{(E_A - E_D)^2}{c^2} - (\mathbf{p}_A - \mathbf{p}_D)^2\Big] \end{aligned}$$

Using conservation of energy and momentum:

$$\begin{aligned} E_A - E_D &= E_C - E_B \\ E_A - E_C &= E_D - E_B \\ \mathbf{p}_A - \mathbf{p}_D &= \mathbf{p}_C - \mathbf{p}_B \\ \mathbf{p}_A - \mathbf{p}_C &= \mathbf{p}_D - \mathbf{p}_B \end{aligned}$$

Substitute these quantities into the equation above:

$$\begin{aligned} s + t + u = \frac{1}{c^2}\Big[&\frac{(E_A + E_B)^2}{c^2} - (\mathbf{p}_A + \mathbf{p}_B)^2 + \frac{(E_D - E_B)^2}{c^2} - (\mathbf{p}_D - \mathbf{p}_B)^2 \\ &+ \frac{(E_C - E_B)^2}{c^2} - (\mathbf{p}_C - \mathbf{p}_B)^2\Big] \end{aligned}$$

In the laboratory frame, $\mathbf{p}_B = 0$ and $E_B = m_B c^2$ and so:

$$\begin{aligned} s + t + u &= \frac{2E_A m_B}{c^2} - \frac{2E_D m_B}{c^2} - \frac{2E_C m_B}{c^2} + m_A^2 + 3m_B^2 + m_c^2 + m_D^2 \\ &= \frac{2m_B}{c^2}(E_A - E_D - E_C) + m_A^2 + 3m_B^2 + m_c^2 + m_D^2 \end{aligned}$$

From the conservation of energy, $E_B = E_A - E_D - E_C$. Making this substitution and using $E_B = m_B c^2$:

$$\begin{aligned} s + t \ + u &= -2m_B^2 + m_A^2 + 3m_B^2 + m_c^2 + m_D^2 \\ &= m_A^2 + m_B^2 + m_c^2 + m_D^2 \end{aligned}$$

10. Starting from conservation of energy and momentum in the center of mass frame for the decay $\rho \to 2\pi^0$ (where the ρ meson is at rest):

$$E_\rho = E_{\pi 1} + E_{\pi 2}$$
$$\mathbf{p}_{\pi 1} = -\mathbf{p}_{\pi 2}$$

and E_ρ is just the given rest mass, $m_\rho c^2 = 775.5\ MeV$. Since both pions are symmetric, $E_{\pi 1} = E_{\pi 2}$. Using Eq. (12.14) for the RHS:

$$m_\rho c^2 = 2\sqrt{p_\pi^2 c^2 + m_\pi^2 c^4}$$

where the π subscript refers to either pion. Squaring both sides:

$$(775.5\ MeV)^2 = (4)[(p_\pi c)^2 + (135.0\ MeV)^2$$

gives $p_\pi = 365.5\ MeV/c$. To get the velocity of the pions, we need γ:

$$\gamma_\pi = \frac{E_\pi}{m_\pi c^2} = \frac{\sqrt{p_\pi^2 c^2 + m_\pi^2 c^4}}{m_\pi c^2}$$

and plugging in the above values gives $\gamma_\pi = 2.87$. So:

$$2.87 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

and solving for v gives $v = 0.937\ c$.

11. The decay here is $\Lambda^0 \to p + \pi^-$. Using four-vectors to conserve both momentum and energy:

$$p_\Lambda = p_p + p_\pi$$

First, isolate the proton momentum on the LHS:

$$p_p = p_\Lambda - p_\pi$$

Squaring both sides:

$$p_p^2 = p_\Lambda^2 + p_\pi^2 - 2p_\Lambda \cdot p_\pi$$

As shown above Eq. (12.14), the square of a momentum four-vector gives just its mass:

$$m_p^2 c^2 = m_\Lambda^2 c^2 + m_\pi^2 c^2 - 2p_\Lambda \cdot p_\pi$$

Carrying out the dot product:

$$m_p^2 c^2 = m_\Lambda^2 c^2 + m_\pi^2 c^2 - 2(\frac{E_\Lambda E_\pi}{c^2} - \mathbf{p}_\Lambda \cdot \mathbf{p}_\pi)$$

where the second term of the dot product is zero since $\mathbf{p}_\Lambda = 0$ in its rest frame. Collecting terms and noting $E_\Lambda = m_\Lambda$ in this frame:

$$E_\pi = \frac{m_\Lambda^2 c^4 + m_\pi^2 c^4 - m_p^2 c^4}{2 m_\Lambda c^2}$$
$$= \frac{(1115.7)^2 + (139.6)^2 - (938.3)^2}{2(1115.7)} \; MeV = 172.0 \; MeV$$

Using just energy conservation:

$$E_p = E_\lambda - E_\pi = 1115.7 \; MeV - 172.0 \; MeV = 943.7 \; MeV$$

To get the velocity of the each particle, find γ using Eq. (12.9):

$$\gamma = \frac{E}{mc^2}$$

giving $\gamma_\pi = 172.0/139.6 = 1.232$ and $\gamma_p = 943.7/938.3 = 1.0058$. The velocity is now found using Eq. (12.3):

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

which results in the pion's speed $v_\pi = 0.584 \; c$ and the proton's speed $v_p = 0.107 \; c$.

12. The particle collision here has two particles with equal mass and speed traveling in opposite directions. Using the conservation of momentum in the center of mass frame:

$$\mathbf{p}_A + \mathbf{p}_B = \mathbf{p}_C = 0$$

This says that the resulting particle is at rest. Using conservation of energy:

$$E_A + E_B = E_C$$
$$2\gamma mc^2 = m_C c^2$$

$$m_C = 2\gamma m$$

Using Eq. (12.3) with the given speed of $v = (2/3) \; c$:

$$m_C = \frac{2m}{\sqrt{1 - (2/3)^2}} = 2.68m$$

Since the mass of the resulting particle is greater than the sum of the colliding particles, some of their kinetic energy was converted into the resulting particle's rest-energy.

13. Using the result of the previous problem:

$$m_C = 2\gamma m_e$$

where in this case, we find γ using Eq. (12.9)

$$\gamma = \frac{E}{m_e c^2}$$

Plugging this in above gives $m_C = 20\ GeV/c^2$. Now we want to get the reaction $m_C \to \mu + \mu$ where μ is the symbol for a muon. Following the same steps as for Problem 10:

$$m_C^2 = 2\sqrt{p_\mu^2 c^2 + m_\mu^2 c^4}$$

where the μ subscript refers to either muon. Squaring both sides:

$$(20.0\ GeV)^2 = (4)[(p_\mu c)^2 + (0.1057\ GeV)^2$$

gives $p_\mu = 20\ GeV/c$ to high precision. To get the velocity of the muns, we need γ:

$$\gamma_\mu = \frac{E_\mu}{m_\mu c^2} = \frac{20.0\ GeV}{0.1057\ GeV} = 189.2$$

where E_μ is found using Eq. (12.9). So:

$$189.2 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

and solving for v gives $v = 0.999986\ c$. The kinetic energy is, from Eq. (12.13):

$$KE = (\gamma - 1)m_\mu c^2 = (189.2 - 1)(0.1057\ GeV) = 19.89\ GeV$$

14. In the center of mass frame where the proton and antiproton have equal and opposite momenta:

$$E_p + E_{\overline{p}} = E_X$$

where E_X is the energy of the resulting particle. As in Problem 12:

$$2\gamma m_p c^2 = m_X c^2$$
$$2\gamma(938\ MeV) = 9700\ MeV$$

giving $\gamma = 5.17$. Using Eq. (12.13) for the kinetic energies:

$$KE = (\gamma - 1)m_p c^2 = 3912\ MeV$$

Using Eq. (12.3) to get velocities:

$$\gamma = 5.17 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Solving for v (as in the problems above), the proton and antiproton will each have a speed of $v = 0.981\ c$ in the center of mass frame.

15. The reaction here is $\pi^+ \rightarrow \mu + \nu_{mu}$ where the neutrino is assumed to have zero mass. From conservation of energy in the center of mass frame:

$$E_\pi = m_\pi c^2 = E_\mu + E_\nu$$

where $E_\nu = |\mathbf{p}_\nu|c$ since $m_\nu = 0$. From conservation of momentum:

$$|\mathbf{p}_\nu| = |\mathbf{p}_\mu|$$

Using the above and Eq. (12.14) for E_μ:

$$m_\pi c^2 = \sqrt{p_\mu^2 c^2 + m_\mu^2 c^4} + p_\mu c$$

Isolating the square root and then squaring both sides:

$$(m_\pi c^2 - p_\mu c)^2 = p_\mu^2 c^2 + m_\mu^2 c^4$$
$$m_\pi^2 c^4 - 2m_\pi c^2 p_\mu c + p_\mu^2 c^2 = p_\mu^2 c^2 + m_\mu^2 c^4$$

Canceling terms and solving for p_μ:

$$p_\mu c = \frac{m_\pi^2 c^4 - m_\mu^2 c^4}{2m_\pi c^2}$$
$$= \frac{(139.6\ MeV)^2 - (105.7\ MeV)^2}{2(139.6\ MeV)} = 29.8\ MeV$$

To get the velocity, we need to get γ for the muon:

$$\gamma_\mu = \frac{E_\mu}{m_\mu c^2} = \frac{\sqrt{p_\mu^2 c^2 + m_\mu^2 c^4}}{m_\mu c^2}$$

giving $\gamma_\mu = 1.039$ Using Eq. (12.3):

$$\gamma_m u = 1.039 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

and solving for v results in $v = 0.271\ c$.
b) The mean distance traveled by the muon is given by $d = v\Delta t_M$ where

$\Delta t_R = 2.2 \times 10^{-6}\ s$ is the muon lifetime in the rest frame. Using Eq. (11.26) to relate Δt_M and Δt_R:

$$\begin{aligned} d &= v\Delta t_M = v\gamma_\mu \Delta t_R \\ &= (0.271)(3.0 \times 10^8\ m/s)(1.039)(2.2 \times 10^{-6}\ s) = 186\ m \end{aligned}$$

At this speed, the muon goes, on average, quite far before it decays.

16. First, test the relation:

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta_{ij}I$$

The Dirac-Pauli representation is given in Eq. (12.42):

$$\alpha^i\alpha^j = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{bmatrix} = \begin{bmatrix} \sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j \end{bmatrix}$$

$$\alpha^j\alpha^i = \begin{bmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} = \begin{bmatrix} \sigma^j\sigma^i & 0 \\ 0 & \sigma^j\sigma^i \end{bmatrix}$$

$$\alpha^i\alpha^j + \alpha^j\alpha^i = \begin{bmatrix} \sigma^i\sigma^j + \sigma^j\sigma^i & 0 \\ 0 & \sigma^i\sigma^j + \sigma^j\sigma^i \end{bmatrix} = (\sigma^i\sigma^j + \sigma^j\sigma^i)I$$

The Pauli matrices satisfy the following condition:

$$\sigma^i\sigma^j = \begin{cases} I & \text{if } \ i = j \\ -\sigma^j\sigma^i & \text{if } \ i \neq j \end{cases}$$

Therefore:

$$\sigma^i\sigma^j + \sigma^j\sigma^i = \begin{cases} 2 & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j \end{cases}$$

Or, more concisely, $\sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta_{ij}I$. Thus, $\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta_{ij}I$.

Next, check $\beta^2 = I$. From Eq. (12.42):

$$\beta = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

We carry out the multiplication:

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I^2 & 0 \\ 0 & I^2 \end{bmatrix}$$

Since $I^2 = I,\ \beta^2 = I$.

17. The definition of the co-variant γ^μ matrices are given in Eq. (12.47):

$$\gamma^i = \beta\alpha^i \quad , \quad \gamma^0 = \beta$$

where α^i and β matrices are defined in Eq. (12.42). From Example 12.6, it is shown by Eqs. (12.58)-(12.59) that:

$$\alpha \cdot \mathbf{p} = \begin{bmatrix} 0 & \sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & 0 \end{bmatrix} \quad , \quad \beta m = \begin{bmatrix} m\mathbf{I} & 0 \\ 0 & m\mathbf{I} \end{bmatrix}$$

where the Pauli spin matrices, σ^i are given by Eq. (12.43) and $\mathbf{I}$ is the 2 $\times$ 2 unit matrix. Since the zeroth component of the momenta is the energy, $p_0 = E$, and the zeroth component of the Dirac matrices is just $\gamma^0 = \beta$, then

$$\gamma^0 p_0 = \begin{bmatrix} E\mathbf{I} & 0 \\ 0 & E\mathbf{I} \end{bmatrix}$$

Similarly, we can use the above result for $\alpha \cdot \mathbf{p}$ and multiply by β giving:

$$\gamma^i p_i = -\beta\alpha \cdot \mathbf{p} = -\begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} 0 & \sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & 0 \end{bmatrix} = -\begin{bmatrix} 0 & \sigma \cdot \mathbf{p} \\ -\sigma \cdot \mathbf{p} & 0 \end{bmatrix}$$

where the minus sign comes from lowering the index, recall $p^\mu = (E, \mathbf{p})$ but $p_\mu = (E, -\mathbf{p})$. Adding the results together:

$$\gamma^\mu p_\mu + m = \begin{bmatrix} (E+m) & -\sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & (E+m) \end{bmatrix}$$

which is written in two-component form. Similarly,

$$\gamma^\mu p_\mu - m = \begin{bmatrix} (E-m) & -\sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & (E-m) \end{bmatrix}$$

18. Muons and electrons are both Dirac particles. The Feynman diagrams for muon-electron scattering are equivalent to the Feynman diagram shown in Fig. 12.4 with one of the incoming and outgoing lines corresponding to an electron and the other corresponding to a muon.