14

Nuclear Physics - Solutions

1. From Section 14.2, the notation for isotopes is:

$$^A_Z X$$

where A is the atomic mass number, Z is the atomic number, and X is the element symbol. The number of neutrons is equal to A minus Z:

$$^3_7 Li : 3 \text{ protons}, \ 4 \text{ neutrons}$$
$$^{63}_{29} Cu : 29 \text{ protons}, \ 34 \text{ neutrons}$$
$$^{238}_{92} U : 92 \text{ protons}, \ 146 \text{ neutrons}$$

2. Using the notation explained above, the atomic numbers add up, $92 = 36 + 56$. So any loss of atomic mass number must be due to neutrons:

$$N_{neutrons} = 236 - (90 + 144) = 2$$

With 2 neutrons per fission, a chain reaction can develop if the neutrons are mostly contained within the volume of enriched ^{236}U.

3. Using Eq. (14.3), an estimate of the nuclear radius is:

$$R = 1.12 A^{1/3} \ fm$$

where A is the atomic mass number. Using this formula:

$$^4_2 He : R = 1.12(4)^{1/3} \ fm = 1.78 \ fm$$
$$^{16}_8 He : R = 1.12(16)^{1/3} \ fm = 2.82 \ fm$$
$$^{56}_{26} Fe : R = 1.12(56)^{1/3} \ fm = 4.28 \ fm$$
$$^{208}_{82} Pb : R = 1.12(208)^{1/3} \ fm = 6.64 \ fm$$
$$^{237}_{93} Np : R = 1.12(237)^{1/3} \ fm = 6.93 \ fm$$

4. Using Eq. (14.5) for the binding energy:

$$\begin{aligned}
{}^{4}_{2}He &: B = [2(1.007825) + 2(1.008665) - 4.002602]u \\
&\quad = 0.030378(931.5\ MeV) = 28.3\ MeV \\
{}^{16}_{8}O &: B = [8(1.007825) + 8(1.008665) - 15.994915]u \\
&\quad = 0.137005(931.5\ MeV) = 127.62\ MeV \\
{}^{56}_{26}Fe &: B = [26(1.007825) + 30(1.008665) - 55.934938]u \\
&\quad = 0.528462(931.5\ MeV) = 492.26\ MeV \\
{}^{208}_{82}Fe &: B = [82(1.007825) + 126(1.008665) - 207.97665]u \\
&\quad = 1.75679(931.5\ MeV) = 1636.45\ MeV \\
{}^{238}_{92}U &: B = [92(1.007825) + 146(1.008665) - 238.05079]u \\
&\quad = 1.9342(931.5\ MeV) = 1801.71\ MeV
\end{aligned}$$

To get the binding energy per nucleon, just divide by A:

$$\begin{aligned}
{}^{4}_{2}He &: B/A = (28.3/4)\ MeV = 7.08 MeV/\text{nucleon} \\
{}^{16}_{8}O &: B/A = (127.62/16)\ MeV = 7.98 MeV/\text{nucleon} \\
{}^{56}_{26}Fe &: B/A = (492.26/56)\ MeV = 8.79 MeV/\text{nucleon} \\
{}^{208}_{82}Fe &: B/A = (1636.45/208)\ MeV = 7.87 MeV/\text{nucleon} \\
{}^{238}_{92}U &: B/A = (1801.71/238)\ MeV = 7.57 MeV/\text{nucleon}
\end{aligned}$$

5. To find the binding energy of a proton, take the difference between the nuclear masses and subtract the proton mass:

$$B_p = m({}^{A}_{Z}X) - m({}^{A-1}_{Z-1}Y) - m(H)$$

where X and Y are the chemical symbols for the original and final nucleus, respectively, and $m(H)$ is the mass of hydrogen. The result is:

$$\begin{aligned}
{}^{4}_{2}He : B_p &= m({}^{4}_{2}He) - m({}^{3}_{1}H) - m(H) = [4.002603 - 3.016049 - 1.007825]u \\
&= -0.021271(931.5\ MeV) = -19.81\ MeV \\
{}^{56}_{26}Fe : B_p &= m({}^{56}_{26}Fe) - m({}^{55}_{25}Mn) - m(H) = [55.93494 - 54.93805 - 1.00783]u \\
&= -0.01094(931.5\ MeV) = -10.2\ MeV \\
{}^{208}_{82}Pb : B_p &= m({}^{208}_{82}Pb) - m({}^{207}_{81}Tl) - m(H) = [207.9767 - 206.9774 - 1.0078]u \\
&= -0.0085(931.5\ MeV) = -7.9\ MeV
\end{aligned}$$

The energy to remove the proton is just $E = -B_p$. Note that another way to do this is simply to subtract the binding energies of the parent and daughter nucleus. However, tables of binding energies are not as accessible as tables of masses.

6. To find the binding energy of a neutron, take the difference between the nuclear masses and subtract the neutron mass:

$$B_n = m({}^{A}_{Z}X) - m({}^{A-1}_{Z}X) - m_n$$

where X is the chemical symbol for the original nucleus and m_n is the neutron mass. The result is:

$$\begin{aligned}
{}^{4}_{2}He : B_n &= m({}^{4}_{2}He) - m({}^{3}_{2}He) - m_n = [4.002603 - 3.016029 - 1.008665]u \\
&= -0.022091(931.5\ MeV) = -20.58\ MeV \\
{}^{56}_{26}Fe : B_n &= m({}^{56}_{26}Fe) - m({}^{55}_{26}Fe) - m_n = [55.93494 - 54.93829 - 1.00867]u \\
&= -0.01202(931.5\ MeV) = -11.2\ MeV \\
{}^{208}_{82}Pb : B_n &= m({}^{208}_{82}Pb) - m({}^{207}_{82}Pb) - m_n = [207.9767 - 206.9759 - 1.0087]u \\
&= -0.0079(931.5\ MeV) = -7.4\ MeV
\end{aligned}$$

The energy to remove the neutron is just $E = -B_n$.

7. Using Eq. (14.6) to calculate the binding energies, with the parameters of Eq. (14.7)-(14.8):

$$\begin{aligned}
{}^{4}_{2}He &: B = 9.75\ MeV,\ B/A = 2.44\ MeV \\
{}^{16}_{8}O &: B = 116.04\ MeV,\ B/A = 7.25\ MeV \\
{}^{56}_{26}Fe &: B = 484.18\ MeV,\ B/A = 8.65\ MeV \\
{}^{208}_{82}Fe &: B = 1623.2\ MeV,\ B/A = 7.80\ MeV \\
{}^{238}_{92}U &: B = 1804.6\ MeV,\ B/A = 7.58\ MeV
\end{aligned}$$

Except for the first case of ^{4}He, the agreement with B/A of problem 4 is within about 1%.

8. The semi-empirical formula without the pairing term is as follows:

$$B(N,\ Z) = aA - bA^{2/3} - \frac{dZ^2}{A^{1/3}} - s\frac{(N-Z)^2}{A}$$

When N=Z, the fourth is zero. To get the binding energy per nucleon, divide by A:

$$\frac{B(N,Z)}{A} = a - bA^{-1/3} - \frac{dZ^2}{A^{4/3}}$$

Now let Z=A/2:

$$\text{Binding energy per nucleon} = a - bA^{-1/3} - \frac{d}{4}A^{2/3}$$

To find the maximum, differentiate with respect to A and set equal to zero:

$$\frac{1}{3}A^{-4/3} - \frac{1}{2}dA^{-1/3} = 0$$

Solving for A (assuming $A \neq 0$):

$$A = \frac{2b}{d}$$

Plugging in the values for b and d from Eq. (14.7), the maximum is at $A = 51.3$. Rounding up, this gives $A/2 \approx 26$ as given in the problem.

9. Eq. (14.13) relates the half-life to the proportionality constant λ:

$$t_{1/2} = 2\ min = 120\ s = \frac{ln(2)}{\lambda}$$
$$\lambda = \frac{0.693}{120\ s} = 0.006\ s^{-1}$$

The decay rate, given just below Eq. (14.12), and the given initial rate $R = 1200\ s^{-1}$ at $t = 0$:

$$R = 1200\ s^{-1} = (0.006\ s^{-1})N_0(1)$$

giving $N_0 = 2.0 \times 10^5$.The same equation gives the decay rates at later times:

$$t = 4\ min = 240\ s : R = (0.006s^{-1})(2 \times 10^5)e^{-0.006(240)} = 284\ s^{-1}$$
$$t = 6\ min = 360\ s : R = (0.006s^{-1})(2 \times 10^5)e^{-0.006(360)} = 138\ s^{-1}$$
$$t = 8\ min = 480\ s : R = (0.006s^{-1})(2 \times 10^5)e^{-0.006(480)} = 67\ s^{-1}$$

10. Using Eq. (14.13) to calculate the decay constant:

$$\lambda = \frac{ln(2)}{t_{1/2}} = \frac{0.693}{12.3\ yr} = 0.0563\ yr^{-1}$$

Using Eq. (14.12) to calculate the fraction left after 40 years:

$$\frac{N}{N_0} = e^{-(0.0563)(40)} = 0.105$$

About 10% of the original amount is left after waiting 40 years.

11.a) Using Eq. (14.13) to get the decay constant:

$$\lambda = \frac{ln(2)}{5730\ yr} = \frac{0.693}{3.012 \times 10^9\ min} = 2.30 \times 10^{-10}\ min^{-1}$$

Using this in the rate equation to get the initial amount:

$$15.3\ min^{-1} = (2.30 \times 10^{-10}\ min^{-1})N_0\ e^{-(2.30\times10^{-10})(1)}$$
$$N_0 = 6.65 \times 10^{10} \text{ atoms of } {}^{14}_{6}C$$

The total number of carbon atoms in the sample is:

$$1\ g \cdot \frac{6.022 \times 10^{23}\ atoms}{12\ g/mol} = 5.018 \times 10^{22} \text{ atoms of } C$$

The proportion is:

$$\frac{6.65 \times 10^{10}}{5.018 \times 10^{22}} = \frac{1.32 \times 10^{-12} \text{ atoms } {}^{14}_{6}C}{\text{total atoms C}}$$

This indicates the rarity of ^{14}C for isolated objects and thus its reliability for dating ancient relics since these nuclear decays can be easily measured.

b) Using the above values for λ and N_0 when the organism was alive and the given time period:

$$20,000\ years = 1.051 \times 10^{10}\ minutes$$

then the rate of decays is:

$$\begin{aligned} R &= (2.30 \times 10^{-10}\ min^{-1})(6.65 \times 10^{10}\ decays)\ e^{-(2.30\times10^{-10})(1051\times10^{10})} \\ &= 1.36 \text{ decays per minute} \end{aligned}$$

12. After calculating A and Z for the missing nucleus, then use the semi-empirical mass formula, Eq. (14.6), to calculate the mass of each nucleus, with $u = 931.5\ MeV$ from Appendix A.
a)

$$\begin{aligned} &{}^{209}_{83}\text{Bi} \rightarrow {}^{205}_{81}\text{Tl} + {}^{4}_{2}\text{He} \\ \Delta E &= (208.98037 - 204.97440 - 4.00260)\ u \\ &= 0.00337u = 3.14\ MeV \end{aligned}$$

b)

$$\begin{aligned} &{}^{238}_{92}\text{U} \rightarrow {}^{234}_{90}\text{Th} + {}^{4}_{2}\text{He} \\ \Delta E &= (238.05078 - 234.04360 - 4.00260)\ u \\ &= 0.00458u = 4.27\ MeV \end{aligned}$$

c) For the next two reactions, the Q-value of beta-decay is given, see Example 14.6. The Q-value is equal to the energy release (including the positron mass of 0.511 MeV).

$$\begin{aligned} &{}^{77}_{36}\text{Kr} \rightarrow {}^{77}_{35}\text{Br} + e^{+} + \nu_e \\ Q &= (71655.330 - 71652.265)\ MeV \\ &= 3.065\ MeV \end{aligned}$$

d)

$$\begin{aligned}{}^{77}_{35}\mathrm{Br} \rightarrow\ {}^{77}_{34}\mathrm{Se}\ +\ e^{+}\ +\ \nu_e \\ Q = (71652.265 - 71650.900)\ MeV \\ = 1.365\ MeV\end{aligned}$$

13. The energy release can be found by taking the difference in masses between ${}^{7}_{4}$Be and ${}^{7}_{3}$Li. Note that the mass of the electron is irrelevant, since the masses are tabulated for neutral atoms, and Li has one fewer electron than Be. (The mass of the electron has been converted to energy via the weak decay mechanism.) Using Table 14.2 for the ${}^{7}_{3}$Li mass and looking up the mass of ${}^{7}_{4}$Be:

$$\Delta E = (7.016930 - 7.016004)\ u = 0.000926\ u = 0.86 MeV$$

14. The most likely decay of ${}^{6}_{2}$He is beta-decay, since it is a light nucleus and neutron-rich:

$$\begin{aligned}{}^{6}_{2}He\ \rightarrow {}^{6}_{3}\ Li\ + e^{-}\ + \bar{\nu}_e \\ Q = (6.01889 - 6.01512)u = 3.512\ MeV\end{aligned}$$

The case of ${}^{8}_{4}$Be is unusual, since it can split exactly into two ${}^{4}_{2}$He nuclei:

$$\begin{aligned}{}^{8}_{4}Be\ \rightarrow {}^{4}_{2}\ He\ + {}^{4}_{2}\ He \\ Q = (8.005305 - 2 \cdot 4.002603)u = 0.092\ MeV\end{aligned}$$

With 12 nucleons, ${}^{12}_{4}$Be quickly decays via beta-decay:

$$\begin{aligned}{}^{12}_{4}Be\ \rightarrow {}^{12}_{5}\ B\ + e^{-}\ + \bar{\nu}_e \\ Q = (12.02692 - 12.01435)u = 11.71\ MeV\end{aligned}$$

The nucleus ${}^{15}_{8}$O has more protons than neutrons, so it will emit a positron:

$$\begin{aligned}{}^{15}_{8}O\ \rightarrow {}^{15}_{7}\ N\ +\ e^{+}\ + \nu_e \\ Q = (15.00307 - 15.00011)u = 2.76\ MeV\end{aligned}$$

Although a heavy nucleus like ${}^{240}_{95}$Am could α-decay, this nucleus prefers to emit a positron:

$$\begin{aligned}{}^{240}_{95}Am\ \rightarrow {}^{240}_{94}\ Pu\ + e^{+}\ + \nu_e \\ Q = (240.05529 - 240.05381)u = 1.38\ MeV\end{aligned}$$

The energy releases for beta decay here are the Q-value (see Example 14.6) which includes the electron or positron mass.

15.a) Start by looking up the masses of each nucleus:

$$m({}^{234}_{92}U) = 234.04095\ u$$
$$m({}^{230}_{90}Th) = 230.03313\ u$$
$$m({}^{4}_{2}He) = 4.00260\ u$$

Using $u = 931.5\ MeV$, the energy release of the decay is:

$$\Delta E = (234.04095 - 230.03313 - 4.00260)u = 4.86\ MeV$$

b) The kinetic energy of the α-particle comes from conservation of momentum:

$$m_{Th}v_{Th} = m_{He}v_{He}$$

and the sum of the kinetic energies must add to the total energy release:

$$\frac{1}{2}m_{Th}v_{Th}^2 + \frac{1}{2}m_{He}v_{He}^2 = 4.86\ MeV$$

Solving these equations for v_{Th} in the first equation gives:

$$\frac{1}{2}\left(\frac{m_{He}}{m_{Th}} + 1\right)m_{He}v_{He}^2 = 4.86\ MeV$$

so the kinetic energy of the α-particle is:

$$KE_\alpha = \frac{4.86\ MeV}{1 + m_{He}/m_{Th}} = 4.78\ MeV$$

16.a Using the information in Table 14.5, along with the example of mirror nuclei in Fig. 14.16, and the text below it. The angular momentum, j and parity, π, of the ground states are:

$$\begin{aligned}
{}^{17}_{8}O\ &: j^\pi = \frac{5}{2}^+ \ \text{reason: one extra nucleon in } 1\mathrm{d}_{5/2} \text{ orbit} \\
{}^{17}_{9}F\ &: j^\pi = \frac{5}{2}^+ \ \text{reason: mirror nucleus} \\
{}^{31}_{15}P\ &: j^\pi = \frac{1}{2}^+ \ \text{reason: mirror nucleus} \\
{}^{31}_{16}S\ &: j^\pi = \frac{1}{2}^+ \ \text{reason: one nucleon missing in } 2\mathrm{s}_{1/2} \text{ orbit} \\
{}^{32}_{16}S\ &: j^\pi = 0^+ \ \text{reason: even} - \text{even nucleus} \\
{}^{40}_{20}Ca\ &: j^\pi = 0^+ \ \text{reason: even} - \text{even nucleus} \\
{}^{45}_{21}Sc\ &: j^\pi = \frac{7}{2}^- \ \text{reason: one nucleon in } 1\mathrm{f}_{7/2} \text{ orbit}
\end{aligned}$$

b) The single-particle excited states have one nucleon from the closed shell moving up into the next shell. The angular momentum and parity of the excited states are not easy to surmise, but here they are:

$$\begin{aligned}
{}^{17}_{8}O &: excited\ j^\pi = \frac{1}{2}^+ \ \text{reason : one extra nucleon in } 2\text{s}_{1/2} \text{ shell} \\
{}^{17}_{9}F &: excited\ j^\pi = \frac{1}{2}^+ \ \text{reason : mirror nucleus} \\
{}^{31}_{15}P &: excited\ j^\pi = \frac{3}{2}^+ \\
{}^{31}_{16}S &: excited\ j^\pi = \frac{3}{2}^+ \\
{}^{32}_{16}S &: excited\ j^\pi = 2^+ \ \text{reason : even} - \text{even nucleus} \\
{}^{40}_{20}Ca &: excited\ j^\pi = 2^+ \ \text{reason : even} - \text{even nucleus} \\
{}^{45}_{21}Sc &: excited\ j^\pi = \frac{3}{2}^+
\end{aligned}$$

For the case of ${}^{17}_{8}$O, one nucleon gets promoted from the $1d_{5/2}$ shell to the $2s_{1/2}$ shell. This single nucleon determines the spin-parity of the nucleus. The other 16 nucleons in the core are tightly bound, and not easy to excite. One could imagine that for ${}^{31}_{16}$S, the unpaired nucleon in the $2s_{1/2}$ shell of the ground stategets promoted to the $1d_{5/2}$ shell, which would result in $j^\pi = (5/2)^+$, but this is not what actually happens in nature. The reason for the $(3/2)^+$ excited state can be explained as the spin-coupling between two excited nucleons, which is not a simple case.

17. Following section 14.5, but now with subscripts where D represents the deuteron, X represents the excited nucleus, and p represents the proton. Starting with conservation of momentum:

$$p_D = p_p\ cos\theta + P_x$$
$$0 = p_p\ sin\theta + P_y$$

Solving for P_x and P_y:

$$P_x = p_D - p_p\ cos\theta$$
$$P_x = -p_p\ sin\theta$$

The total kinetic energy before the collision is just from the deuteron:

$$E_i = \frac{p_D^2}{2m_D}$$

The total kinetic energy after the collision is:

$$E_f = \frac{p_p^2}{2m_p} + \frac{1}{2m_X}\left((p_D - p_p cos\theta)^2 + (p_p\ sin\theta)^2\right)$$

The energy absorbed by the nucleus will be equal to the loss of kinetic energy:

$$E = E_i - E_f = \frac{p_D^2}{2m_D} - \frac{p_p^2}{2m_p} - \frac{1}{2m_X}(p_D^2 + p_p^2 - 2p_D p_p\ cos\theta)$$

Substitute the corresponding energies for the first two terms:

$$\frac{p_D^2}{2m_D} = E_D$$
$$\frac{p_p^2}{2m_p} = E_p$$
$$p_D p_p = \sqrt{4E_D E_p m_D m_p}$$

The energy of the excited nucleus due to deuteron stripping is:

$$E = E_D\left(1 - \frac{m_D}{m_X}\right) - E_p\left(1 - \frac{m_p}{m_X}\right) - \frac{2\sqrt{E_D E_p m_D m_p}}{m_X}\ cos\theta$$

18. Using conservation of momentum:

$$p_\alpha = -p_d$$

Substituting this into the given energy:

$$E = \frac{1}{2m_\alpha}\mathbf{p}_\alpha^2 + \frac{1}{2m_d}\mathbf{p}_\alpha^2$$

Collecting terms:

$$E = \frac{1}{2}\left(\frac{1}{m_\alpha} + \frac{1}{m_d}\right)\mathbf{p}_\alpha^2$$

Adding the fractions to get a common denominator:

$$E = \frac{1}{2}\left(\frac{m_d + m_\alpha}{m_\alpha m_d}\right)\mathbf{p}_\alpha^2$$

which is the desired result.

19. The Q-value for this reaction has already been done in problem 14 above. The result is:

$$Q = 0.092MeV$$

The mean lifetime can be estimated using the uncertainty principle:

$$\Delta E \cdot \Delta t \sim \hbar$$

where $\Delta E \simeq Q$. Using $\hbar c = 197.3\ MeV \cdot fm$ and $c = 3.0 \times 10^8\ m/s = 3.0 \times 10^{23}\ fm/s$:

$$\Delta t \sim \frac{197.3}{0.092(3.0 \times 10^{23})}\ s = 7 \times 10^{-21}\ s$$

This is shorter than the observed lifetime, but this is a very crude estimate. A better calculation of the lifetime can be obtained using WKB approximation, which is presented in several advanced textbooks on quantum theory.

20. Start by looking up the masses of each nucleus:

$$\begin{aligned} m({}^{238}_{94}Pu) &= 238.04955\ u \\ m({}^{234}_{92}U) &= 234.04095\ u \\ m({}^{4}_{2}He) &= 4.00260\ u \end{aligned}$$

Using $u = 931.5\ MeV$, the Q-value of the decay is:

$$Q = (238.04955 - 234.04095 - 4.00260)u = 5.6\ MeV$$

The half-life can be estimated using the uncertainty principle:

$$\Delta E \cdot \Delta t \sim \hbar$$

where $\Delta E \simeq Q$. Using $\hbar c = 197.3\ MeV \cdot fm$ and $c = 3.0 \times 10^{23}\ fm/s$:

$$\Delta t \sim \frac{197.3}{5.6(3.0 \times 10^{23})}\ s = 1 \times 10^{-22}\ s$$

In this case, the lifetime is off (compared with the observed lifetime) by many orders of magnitude. The reason is that the α must penetrate the Coulomb barrier, which is substantial in a high-Z nucleus like ${}^{238}_{94}$Pu but much smaller in a light nucleus (as in the previous problem). See also the comments at the end of section 3.3.2.

21. Start by looking up the masses of each nucleus:

$$\begin{aligned} m({}^{194}_{79}Au) &= 193.96534\ u \\ m({}^{194}_{78}Pt) &= 193.96266\ u \\ m({}^{190}_{77}Ir) &= 189.96059\ um({}^{4}_{2}He) \qquad &= 4.00260\ u \end{aligned}$$

Using $u = 931.5\ MeV$, the Q-value of the β-decay is:

$$Q = (193.96534 - 193.96266)u = 2.5\ MeV$$

The Q-value for the α-decay is:

$$Q = (193.96534 - 189.96059 - 4.0026)u = 2.0\ MeV$$

The lifetime for the β-decay can be estimated as in Problem 19, and is on the order of 10^{-22} s, but again this is a very crude estimate. As shown in Problem 20, it is not possible to get a good estimate on the lifetime of α-decay (see the comments at the end of section 3.3.2) since the lifetime is very sensitive to the height of the Coulomb barrier.