# Appendix G

# ANALYTICAL TECHNIQUES

## G.1 USEFUL INTEGRALS

$$\int_0^x \frac{\mathrm{d}x}{1-x} = \ln\frac{1}{1-x} \tag{G-1}$$

$$\int_0^x \frac{\mathrm{d}x}{(1-x)^2} = \frac{x}{1-x} \tag{G-2}$$

$$\int_0^x \frac{\mathrm{d}x}{1+\varepsilon x} = \frac{1}{\varepsilon}\ln(1+\varepsilon x) \tag{G-3}$$

$$\int_0^x \frac{1+\varepsilon x}{1-x}dx = (1+\varepsilon)\ln\frac{1}{1-x} - \varepsilon x \tag{G-4}$$

$$\int_0^x \frac{1+\varepsilon x}{(1-x)^2}\mathrm{d}x = \frac{(1-x)\,x}{1-x} - \varepsilon\ln\frac{1}{1-x} \tag{G-5}$$

$$\int_0^x \frac{(1+\varepsilon x)^2}{(1-x)^2}\mathrm{d}x = 2\varepsilon(1+\varepsilon)\ln(1-x) + \varepsilon^2 x + \frac{(1+\varepsilon)^2 x}{1-x} \tag{G-6}$$

$$\int_0^x \frac{\mathrm{d}x}{(1-x)(\Theta_B - x)} = \frac{1}{\Theta_B - 1}\ln\frac{\Theta_B - x}{\Theta_B(1-x)} \quad \Theta_B \neq 1 \tag{G-7}$$

$$\int_0^x \frac{\mathrm{d}x}{ax^2+bx+c} = \frac{-2}{2ax+b} + \frac{2}{b} \quad \text{for } b^2 = 4ac \tag{G-8}$$

$$\int_0^x \frac{\mathrm{d}x}{ax^2+bx+c} = \frac{1}{a(p-q)}\ln\left[\frac{q}{p}\cdot\frac{x-p}{x-q}\right] \quad \text{for } b^2 > 4ac \tag{G-9}$$

$$\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + C \tag{G-10}$$

$$\int \frac{1}{x}\mathrm{d}x = \ln x + C \tag{G-11}$$

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\ln(ax+b) + C \tag{G-12}$$

$$\int a^x \mathrm{d}x = \frac{1}{\ln a}a^x + C \tag{G-13}$$

$$\int e^x \mathrm{d}x = e^x + C \tag{G-14}$$

$$\int \sin x\,\mathrm{d}x = -\cos x + C \tag{G-15}$$

$$\int \cos x\,\mathrm{d}x = \sin x + C \tag{G-16}$$

$$\int \tan x\,\mathrm{d}x = -\ln(\cos x) + C = \ln(\sec x) + C \tag{G-17}$$

$$\int \sin ax\,\mathrm{d}x = -\frac{1}{a}\cos ax + C \tag{G-18}$$

$$\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + C \tag{G-19}$$

$$\int \cos ax\,\mathrm{d}x = \frac{1}{a}\sin ax + C \tag{G-20}$$

$$\int \cos(ax+b)\,\mathrm{d}x = \frac{1}{a}\sin(ax+b) + C \tag{G-21}$$

$$\int \tan a\,x\,\mathrm{d}x = \frac{1}{a}\ln(\sec\, ax) + C \tag{G-22}$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C \tag{G-23}$$

$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \tag{G-24}$$

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \tag{G-25}$$

$$\int \ln x \, \mathrm{d}x = x \ln(x) - x + C \quad \text{(G-26)}$$

$$\int \frac{x \, \mathrm{d}x}{a^2 + x^2} = \frac{1}{2} \ln\left(a^2 + x^2\right) + C \quad \text{(G-27)}$$

### USEFUL WEBSITES

- www.integrals.com
- http://integrals.wolfram.com

## G.2 LIEBNITZ'S RULE – HIGHER DERIVATIVES OF PRODUCTS

$$D^n(uv) = uD^n v + \binom{n}{1}(Du)\left(D^{n-1}v\right) + \binom{n}{2}\left(D^2 u\right)\left(D^{n-2}v\right) + \cdots + vD^n u \quad \text{(G-28)}$$

where

$D^n = \frac{\mathrm{d}^n}{\mathrm{d}x^n} =$ $n$th differential operator

$\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ are binomial coefficients.

### EXAMPLE

$$D^3(uv) = u\frac{\mathrm{d}^3 v}{\mathrm{d}x^3} + 3\frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + 3\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \cdot \frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \quad \text{(G-29)}$$

## G.3 DEFINITION OF A DERIVATIVE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{d}{\mathrm{d}x} f(x) = f'(x) = y' = \lim_{\Delta \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \quad y = f(x) \quad \text{(G-30)}$$

## G.4 PRODUCT RULE

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = V\frac{\mathrm{d}u}{\mathrm{d}x} + U\frac{\mathrm{d}v}{\mathrm{d}x} \quad \text{(G-31)}$$

## G.5 QUOTIENT RULE

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \quad \text{(G-32)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f(u) = \frac{\mathrm{d}f(u)}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{(G-33)}$$

## G.6 EXPONENTIAL/LOGARITHMIC FUNCTIONS

$$\frac{\mathrm{d}}{\mathrm{d}x} \log_a u = \frac{\log_a e}{u} \frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{(G-34)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} a^u = \frac{1}{u} \frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{(G-35)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} e^u = e^u \frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{(G-36)}$$

## G.7 TAYLOR'S AND MACLAURIN'S SERIES

### TAYLOR'S SERIES

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots + \frac{h^n}{n!} f^n(x) + \cdots \quad \text{(G-37)}$$

$$f(x+h) = (x+h)^n = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots \quad \text{(G-38)}$$

when $h = 0$

$$f(x) = x^n$$
$$f'(x) = nx^{n-1}$$
$$f''(x) = n(n-1)x^{n-2}$$
$$f'''(x) = n(n-1)(n-2)x^{n-3}$$

Substituting into the expansion for Taylor's expression

$$(x+h)^n = x^n + h\, nx^{n-1} + \frac{h^2}{2!} n(n-1)x^{n-2} + \frac{h^3}{3!} n(n-1)(n-2)x^{n-3} + \cdots \quad \text{(G-39)}$$

and re-arranging gives the following.

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2!} x^{n-2}h^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3}h^3 + \cdots \quad \text{(G-40)}$$

### MACLAURIN'S SERIES

Maclaurin's series is a special case of Taylor's series. If $x = 0$ and $h$ is replaced by $x$, then Taylor's expansion becomes

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^n}{n!} f^n(0) + \cdots \quad \text{(G-41)}$$

## G.8 DIFFERENTIAL EQUATIONS

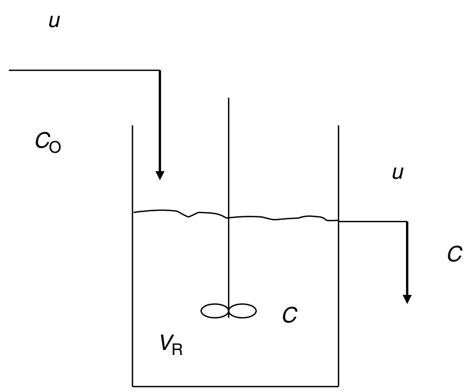
### SEPARATION OF VARIABLES

$$f(x)\, \mathrm{d}y + g(x)\, \mathrm{d}x = 0 \quad \text{(G-42)}$$

Solve by direct integration

$$\int f(y)\, \mathrm{d}y + \int g(x)\, \mathrm{d}x = C \quad \text{(G-43)}$$

## EXAMPLE CONTINUOUS FLOW STIRRED TANK REACTOR




$u$ = flow rate, L/s
$V_R$ = volume of tank, L
$C_O$ = inlet concentration, g/L
$C$ = tank and outlet concentration, g/L
$T$ = time, s.

Determine $C$ as a function of time.

*Step 1* A mass balance on tank

$$\text{Mass in} - \text{Mass out} = \text{Accumulation in tank}$$

$$u\,(C_O - C)\,dt = V_R\ dC \quad \text{(G-44)}$$

*Step 2* Separate $C$ and $t$ and integrate

$$\int \frac{dC}{C_O - C} = \frac{u}{V_R}\int dt \quad \text{(G-45)}$$

$$-\ln(C_O - C) = \frac{u\ t}{V_R} + \ln A \quad \text{(G-46)}$$

or

$$\ln(C - C_O) = \frac{-ut}{V_R} + \ln A \quad \text{(G-47)}$$

$$C - C_O = Ae^{\frac{-ut}{V_R}} \quad \text{(G-48)}$$

*Step 3* Determine $A$ using the boundary condition
$t = 0$, $C = C_i$, therefore $A = C_i - C_O$

*Step 4.* The final solution is

$$C - C_O = (C_i - C_O)\,e^{\frac{-ut}{V_R}} \quad \text{(G-49)}$$

or

$$C = C_O + (C_i - C_O)\,e^{\frac{-ut}{V_R}} \quad \text{(G-50)}$$

## VARIABLE SEPARABLE

$$f(y)\,\frac{dy}{dx} + F(x) = 0 \quad \text{(G-51)}$$

or

$$f(y)\ dy + F(x)\ dx = 0 \quad \text{(G-52)}$$

in which $F(x)$ is a function of $x$ only, and $f(y)$ a function of $y$ only.
The general solution is

$$\int f(y)\ dy + \int F(x)\,dx = C \quad \text{(G-53)}$$

## EXAMPLE

Solve the differential

$$xdy + ydx = 0 \quad \text{(G-54)}$$

dividing by $xy$,

$$\frac{dy}{y} + \frac{dx}{x} = 0 \quad \text{(G-55)}$$

$$\int \frac{dy}{y} + \int \frac{dx}{x} = 0 \quad \text{(G-56)}$$

$$\ln y + \ln x = C_1 \quad \text{(G-57)}$$

where the constant $C_1 = \ln C$

$$\ln y + \ln x = \ln C \quad \text{(G-58)}$$

giving

$$xy = c \quad \text{(G-59)}$$

The factor $\frac{1}{xy}$ used to multiply throughout to separate the variables is called an integrating factor.

## G.9 LINEAR EQUATIONS

An equation of the form

$$\frac{dy}{dx} + Py = Q \quad \text{(G-60)}$$

is called a linear differential equation where $P$ and $Q$ are constants or functions of $x$ only.

Multiplying such an equation throughout by the integrating factor $e^{\int Pdx}$ gives an equation that can be solved.

$$e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int Pdx} \quad \text{(G-61)}$$

Differentiating $ye^{\int Pdx}$ gives

$$e^{\int Pdx}\frac{dy}{dx} + e^{\int Pdx}Py \quad \text{(G-62)}$$

which is the left-hand side of the equation.

Therefore the solution is

$$ye^{\int Pdx} = \int \left(Qe^{\int Pdx}\right)dx \quad \text{(G-63)}$$

EXAMPLE

Solve the equation

$$\frac{dy}{dx} + 2y = 0 \quad \text{(G-64)}$$

$$\text{Integrating Factor IF} = e^{\int P dx} = e^{\int 2 dx} = e^{2x} \quad \text{(G-65)}$$

Multiplying the equation by this integrating factor $e^{2x}$ gives

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 0 \quad \text{(G-66)}$$

Integrating this equation gives

$$ye^{2x} = A \quad \text{(G-67)}$$

where $A$ is a constant.

Therefore, the solution is

$$y = Ae^{-2x} \quad \text{(G-68)}$$

## G.10 EXACT DIFFERENTIAL EQUATION

If $M$ and $N$ are functions of $(x, y)$ and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $M dx + N dy = 0$ is an exact differential equation, the solution of which is

$$\int M dx + \int \left(N - \int \frac{\partial M}{\partial y} dx\right) dy + C = 0 \quad \text{(G-69)}$$

or

$$\int N dy + \int \left(M - \int \frac{\partial N}{\partial x} dy\right) dx + C = 0 \quad \text{(G-70)}$$

If the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is not satisfied, there exists a function $\psi(x, y) = \psi$ such that $\frac{\partial(\psi M)}{\partial y} = \frac{\partial(\psi N)}{\partial x}$

## G.11 HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \quad \text{(G-71)}$$

First construct and solve the auxiliary equation

$$m^2 + am + b = 0 \quad \text{(G-72)}$$

If the auxiliary equation has distinct real roots $m_1$ and $m_2$, then the solution is

$$y = Ae^{m_1 x} + Be^{m_2 x} \quad \text{(G-73)}$$

If the auxiliary equation has complex roots, $m_1 + im_2$ and $m_1 - im_2$, then the solution is:

$$y = e^{m_1 x}(A\cos m_2 x + B\sin m_2 x) \quad \text{(G-74)}$$

If the auxiliary equation has two roots equal to $m$, then the solution is

$$y = (Ax + B)e^{mx} \quad \text{(G-75)}$$

## G.12 TABLE OF LAPLACE TRANSFORM

| $\bar{f}(s)$ | $f(t)$ |
|---|---|
| $\frac{1}{s}$ | $1$ |
| $\frac{1}{s^2}$ | $t$ |
| $\frac{1}{s^n}\ (n = 1, 2, \cdots)$ | $\frac{t^{n-1}}{(n-1)!}$ |
| $\frac{1}{s-a}$ | $e^{at}$ |
| $\frac{1}{(s-a)^2}$ | $te^{at}$ |
| $\frac{1}{(s-a)^n}\ (n = 1, 2, \cdots)$ | $\frac{1}{(n-1)!}t^{n-1}e^{at}$ |
| $\frac{1}{s^2+a^2}$ | $\frac{1}{a}\sin at$ |
| $\frac{s}{s^2+a^2}$ | $\cos at$ |
| $\frac{1}{(s-a)^2+b^2}$ | $\frac{1}{b}e^{at}\sin bt$ |
| $\frac{(s-a)}{(s-a)^2+b^2}$ | $e^{at}\cos bt$ |
| $\frac{e^{-ks}}{s^2}$ | $\begin{cases} 0 & \text{when } 0 < t < k \\ t-k & \text{when } t > k \end{cases}$ |
| $s\bar{f}(s) - f(0)$ | $\frac{df(t)}{dt}$ |
| $s^2\bar{f}(s) - sf(0) - f'(0)$ | $\frac{d^2f(t)}{dt^2}$ |
| $\frac{n!}{s^{n+1}}$ where $n$ is an integer | $t^n$ |
| $\frac{\Gamma(n)}{s^{n+1}}$ $n$ is not an integer | $t^n$ |
| $\frac{1}{(s-b)^2+a^2}$ | $\frac{e^{bt}\sin at}{a}$ |
| $\frac{s-b}{(s-b)^2+a^2}$ | $e^{bt}\cos at$ |
| $s^n\bar{f}(s) - \left[s^{n-1}f(0) + s^{n-2}f'(0) + \cdots + sf^{n-2}(0) + f^{n-1}(0)\right]$ | $\frac{d^nf(t)}{dt^n}$ |

where $f'(0), f''(0), \cdots f^{(n-1)}(0)$ are the values of the first, second, up to the (n−1)th derivative of the function when the independent variable is zero.

## G.13 CUBIC EQUATIONS

$$ax^3 + bx^2 + cx + d = 0 \quad \text{(G-76)}$$

This reduces by the substitution

$$x = y - \frac{b}{3a} \text{ to } y^3 + py + q = 0 \quad \text{(G-77)}$$

where

$$p = \frac{1}{3}\left[3\left(\frac{c}{a}\right) - \left(\frac{b}{a}\right)^2\right] \quad \text{(G-78)}$$

$$q = \frac{1}{27}\left[2\left(\frac{b}{a}\right)^3 - 9\left(\frac{b}{a}\right)\left(\frac{c}{a}\right) + 27\left(\frac{d}{a}\right)\right] \quad \text{(G-79)}$$

$$y_1 = u + v$$

$$y_2 = \frac{(-u+v)}{2} + \frac{(u-v)}{2} i\sqrt{3}$$

$$y_3 = \frac{(-u+v)}{2} - \frac{(u-v)}{2} i\sqrt{3}. \quad \text{(G-80)}$$

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \quad \text{(G-81)}$$

$$u = \sqrt[3]{\frac{-q}{2} + \sqrt{D}} \quad v = \sqrt[3]{\frac{-q}{2} - \sqrt{D}} \quad \text{(G-82)}$$

If $a$, $b$, $c$, and $d$ are real and if

- $D > 0$ there are one real and two conjugate imaginary roots
- $D = 0$ there are three real roots of which at least two are equal
- $D < 0$, trigonometric formulas of the roots are

$$y_1 = \sqrt[2]{\frac{|p|}{3}} \cos \frac{\phi}{3} \quad \text{(G-83)}$$

$$y_2 = -2\sqrt{\frac{|p|}{3}} \cos\left(\frac{\phi + \pi}{3}\right) \quad \text{(G-84)}$$

$$y_2 = -2\sqrt{\frac{|p|}{3}} \cos\left(\frac{\phi - \pi}{3}\right) \quad \text{(G-85)}$$

The value of $\phi$ is calculated from the expression

$$\cos\phi = \frac{-q/2}{\sqrt{\frac{|p^3|}{27}}} \quad \text{(G-86)}$$