12.6 DRIVEN, PARALLEL RLC CIRCUIT*

Consider now the circuit shown in Figure 12.50. As in previous sections of this chapter, we will analyze the behavior of this circuit using the node method beginning at Step 3. In doing so, we will follow the analysis presented in Section 12.4 very closely.

We begin by completing Step 3, of the node method. To do so, we write KCL in terms of v_C for the node at which v_C is defined to obtain

$$C\frac{dv_C(t)}{dt} + \frac{1}{R}v_C(t) + \frac{1}{L}\int_{-\infty}^{t} v_C(\tilde{t})d\tilde{t} = i_{IN}(t), \tag{12.198}$$

which upon differentiation and division by C becomes

$$\frac{d^2v_C(t)}{dt^2} + \frac{1}{RC}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{1}{C}\frac{di_{IN}(t)}{dt}. \tag{12.199}$$

Unlike Equations 12.4, 12.82 and 12.40, Equation 12.199 is an inhomogeneous differential equation because it is driven by the external signal i_{IN}. Unfortunately, i_{IN} enters Equation 12.199 through a derivative, which poses an unnecessary complication. To eliminate this complication, we substitute the constitutive law for the inductor,

$$v_C(t) = L\frac{di_L(t)}{dt}, \tag{12.200}$$

into Equation 12.198 and divide by LC. This yields

$$\frac{d^2i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC}i_L(t) = \frac{1}{LC}i_{IN}(t), \tag{12.201}$$

which is easier to work with.

Equation 12.201 is an inhomogeneous differential equation, which unlike our previous undriven examples (for example, Equation 12.4 for the undriven

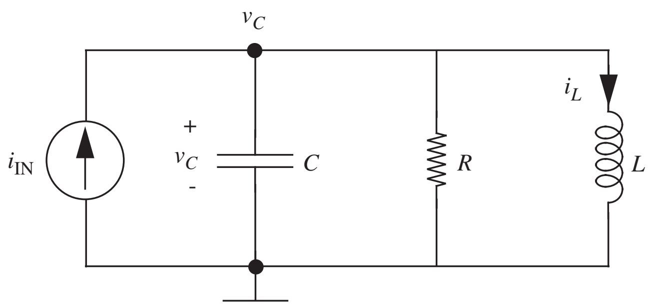

FIGURE 12.50 The parallel second-order circuit with a resistor, capacitor, inductor, and current source.

LC circuit) has an additional term for the input drive. Furthermore, notice the term proportional to di_L/dt. As we saw in Section 12.5, this term modifies the homogeneous response to include damping. Therefore we now expect the oscillations in the step and impulse responses to decay in time.

To complete the node analysis, we complete Steps 4 and 5 by solving Equation 12.201 for i_L, and using it to determine v_C and any other variables of interest. To do so we employ our usual method of solving differential equations:

1. Find the homogeneous solution $i_{LH}(t)$.
2. Find the particular solution $i_{LP}(t)$.
3. The total solution is then the sum of the homogeneous solution and the particular solution as follows:

$$i_L(t) = i_{LH}(t) + i_{LP}(t).$$

Use the initial conditions to solve for the remaining constants.

The homogeneous solution $i_{LH}(t)$ to Equation 12.201 is obtained by solving the differential equation with the drive $i_{IN} \equiv 0$. With $i_{IN} \equiv 0$, the circuit shown in Figure 12.50 is identical to the parallel, undriven RLC circuit shown in Figure 12.24, and so the two circuits have the same homogeneous equation. The homogeneous equation in terms of the current is given by

$$\frac{d^2 i_{LH}(t)}{dt^2} + \frac{1}{RC}\frac{di_{LH}(t)}{dt} + \frac{1}{LC}i_{LH}(t) = 0. \tag{12.202}$$

Note the similarity between this homogeneous equation and Equation 12.4 for the undriven, parallel RLC circuit. Following the solution (Equation 12.10) of the homogeneous equation for the undriven, parallel RLC circuit, we can write the form of the homogeneous solution for our driven, parallel RLC circuit as

$$i_{LH}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{12.203}$$

where K_1 and K_2 are as yet unknown constants that will be determined from the initial conditions after the total solution has been formed. s_1 and s_2, the roots of the characteristic equation,

$$s^2 + 2\alpha s + \omega_\circ^2 = 0 \tag{12.204}$$

$$\alpha \equiv \frac{1}{2RC} \tag{12.205}$$

$$\omega_\circ \equiv \sqrt{\frac{1}{LC}}. \tag{12.206}$$

The roots are given by

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_\circ^2} \tag{12.207}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_\circ^2}. \tag{12.208}$$

As observed with other second-order circuits, the circuit exhibits under-damped, over-damped, or critically-damped behavior depending on the relative values of α and $\omega_\circ$:

$$\begin{array}{lll} \alpha < \omega_\circ & \Rightarrow & \text{under-damped dynamics;} \\ \alpha = \omega_\circ & \Rightarrow & \text{critically-damped dynamics;} \\ \alpha > \omega_\circ & \Rightarrow & \text{over-damped dynamics.} \end{array}$$

For brevity, the rest of the section will assume that

$$\alpha < \omega_\circ$$

so that the circuit displays under-damped dynamics. For the under-damped case, since s_1 and s_2 are now complex, they can be written explicitly in complex form as

$$s_1 = -\alpha + j\omega_d$$

$$s_2 = -\alpha - j\omega_d \tag{12.209}$$

where

$$\omega_d \equiv \sqrt{\omega_\circ^2 - \alpha^2}. \tag{12.210}$$

As we did with the series RLC circuit, we shall rewrite the homogeneous solution in Equation 12.121 into a more intuitive form using the Euler relation as follows:

$$i_{LH}(t) = A_1 e^{-\alpha t} \cos(\omega_d t) + A_2 e^{-\alpha t} \sin(\omega_d t) \tag{12.211}$$

where A_1 and A_2 are unknown constants we will evaluate later depending on the initial conditions of the circuit.

Next, we need to find $i_{LP}(t)$. Knowing it, we can write the total solution as

$$i_L(t) = i_{LP}(t) + i_{LH}(t) = i_{LP}(t) + A_1 e^{-\alpha t} \cos(\omega_d t) + A_2 e^{-\alpha t} \sin(\omega_d t). \tag{12.212}$$

At this point, only i_{LP}, and A_1 and A_2, remain as unknowns.

We will now proceed to find the i_{LP}, and then A_1 and A_2. i_{LP} depends on the input drive. We will find i_{LP} for two cases of i_{IN}, namely a step and an impulse. That is, we will proceed to find the step response and the impulse response of the circuit. To simplify matters, we will assume that the circuit is under-damped, that both the step and the impulse occur at $t = 0$, and that the circuit is initially at rest prior to that time. The latter assumption implies that we are seeking the zero-state response for which

$$i_L(0) = 0 \tag{12.213}$$

and

$$v_C(0) = 0. \tag{12.214}$$

The zero-state response is the response of the circuit for zero initial state. Equations 12.213 and 12.214 provide the initial conditions for the solution of Equation 12.201 after the step and impulse occur, that is, for $t > 0$.

12.6.1 STEP RESPONSE

Let i_{IN} be the current step given by

$$i_{\mathrm{IN}}(t) = I_\circ u(t) \tag{12.215}$$

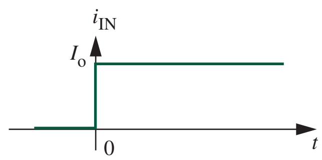

FIGURE 12.51 A current step input.

and shown in Figure 12.51. With the substitution of Equation 12.215, Equation 12.201 becomes

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC}i_L(t) = \frac{1}{LC}I_\circ \tag{12.216}$$

for $t > 0$. Any function that satisfies Equation 12.216 for $t > 0$ is an acceptable i_{LP}. One such function is

$$i_{LP}(t) = I_\circ. \tag{12.217}$$

Thus, we have the particular solution for a step input.
The total solution is given by summing the homogeneous solution (Equation 12.211) and the particular solution (Equation 12.217) as

$$i_L(t) = I_\circ + A_1 e^{-\alpha t}\cos(\omega_d t) + A_2 e^{-\alpha t}\sin(\omega_d t), \tag{12.218}$$

again for $t > 0$. Additionally, the substitution of Equation 12.218 into Equation 12.200 yields

$$v_C(t) = (\omega_d L A_2 - \alpha L A_1)e^{-\alpha t}\cos(\omega_d t) - (\omega_d L A_1 + \alpha L A_2)e^{-\alpha t}\sin(\omega_d t), \tag{12.219}$$

also for $t > 0$. Now only A_1 and A_2 remain as unknowns.

In Chapter 9, we saw that the voltage across a capacitor is continuous unless the current through it contains an impulse. We also saw that the current through an inductor is continuous unless the voltage across it contains an impulse. Since i_{IN} contains no impulses, we can therefore assume that both v_C and i_L are continuous across the step at $t = 0$. Consequently, since both states are zero for $t \leq 0$, Equations 12.218 and 12.219 must both evaluate to zero as $t \to 0$. This observation allows us to use the initial conditions to determine A_1 and A_2. Evaluation of both equations as $t \to 0$, followed by the substitution of the initial conditions, yields

$$i_L(0) = I_\circ + A_1 = 0 \tag{12.220}$$

$$v_C(0) = \omega_d L A_2 - \alpha L A_1 = 0. \tag{12.221}$$

Equations 12.220 and 12.221 can be solved to yield

$$A_1 = -I_\circ \tag{12.222}$$

$$A_2 = -\frac{\alpha}{\omega_d} I_\circ. \tag{12.223}$$

Finally, the substitution of Equations 12.222 and 12.223 into Equations 12.218 and 12.219 yields

$$i_L(t) = I_\circ \left(1 - \frac{\omega_\circ}{\omega_d} e^{-\alpha t} \cos\left(\omega_d t - \tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right)\right) u(t) \tag{12.224}$$

$$v_C(t) = \frac{I_\circ}{\omega_d C} e^{-\alpha t} \sin(\omega_d t) u(t); \tag{12.225}$$

Equations 12.210 and 12.206 have also been used to simplify the results. Note that the unit step function u has been introduced into Equations 12.224 and 12.225 so that they are valid for all time. The validity of Equations 12.224 and 12.225 can be demonstrated by observing that they satisfy the initial conditions, and Equations 12.201 and 12.200, respectively, for all time. Because they do, our assumption that the states are continuous at $t = 0$ is justified.

Figure 12.52 shows i_L and v_C as given by Equations 12.224 and 12.225. As expected, the ringing in both states now decays as $t \to \infty$. This decay is well characterized by the quality factor Q, as defined in Equation 12.66 and discussed shortly thereafter. In fact, because the circuits shown in Figures 12.24 and 12.50 have the same homogeneous response, the entire discussion of α, ω_d, and $\omega_\circ$ given in Section 12.4 applies here as well. In fact, the series and parallel circuits are duals. This can be observed by comparing the evolution of their branch variables. For example, like the capacitor voltage v_C in the series circuit, i_L undergoes nearly a two-fold overshoot during the initial transient.

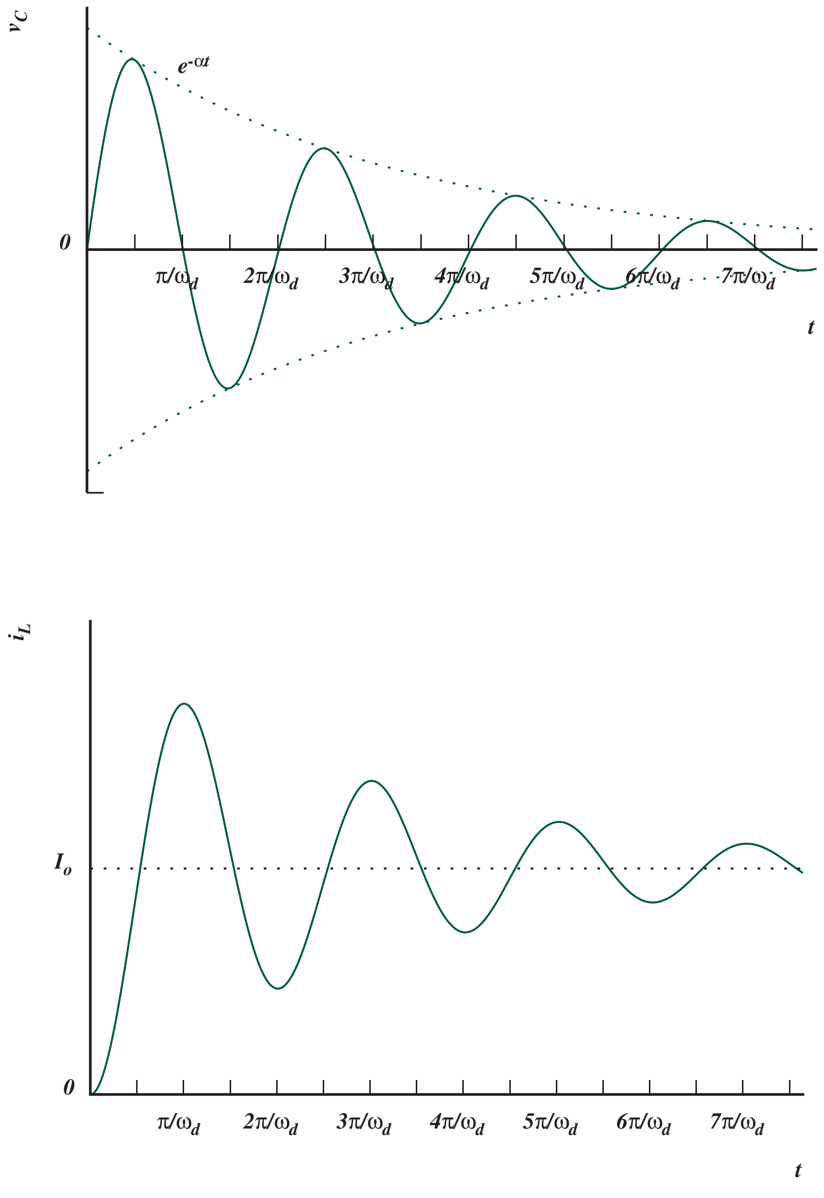

FIGURE 12.52 i_L and v_C for the parallel RLC circuit shown in Figure 12.50 for the case of a step input through i_{IN}.

Another observation concerns the short-time behavior of the circuit. We have seen in Chapter 10 that the transient behavior of an uncharged capacitor is to act as a short circuit during the early part of a transient, while the corresponding transient behavior of an uncharged inductor is to act as an open circuit. This behavior is observed in Figure 12.52 since i_{IN} is carried entirely by the capacitor (and i_L is 0) at the start of the transient, and v_C ramps up correspondingly.

A related observation concerns the long-time behavior of the circuit. We have also seen in Chapter 10 that the transient behavior of a capacitor is to act as an open circuit as $t \to \infty$, while the corresponding transient behavior of an inductor is to act as a short circuit. This behavior is also observed in Figure 12.52, since i_{IN} is carried entirely by i_L as $t \to \infty$.

We also note the overshoot of i_L above the input current step of $I_{\circ}$ during the transient. Although the average value of i_L is close to $I_{\circ}$ during the transient, the peak value is closer to $2I_{\circ}$.

Finally, note that as $t \to \infty$, i_{IN} is carried entirely by the inductor since $i_L \to I_{\circ}$. This is consistent with the relative long-time transient behavior of the inductor, resistor, and capacitor.

12.6.2 IMPULSE RESPONSE

Let i_{IN} be the impulse given by

$$i_{\mathrm{IN}} = Q_{\circ}\delta(t) \tag{12.226}$$

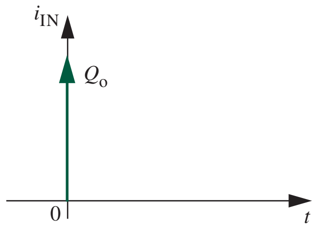

FIGURE 12.53 The current impulse i_{IN}.

as shown in Figure 12.53. Because i_{IN} is an impulse, it vanishes for $t > 0$. Therefore, Equation 12.201 reduces to a homogeneous equation for $t > 0$, and so the simplest acceptable particular solution is

$$i_{LP}(t) = 0. \tag{12.227}$$

The substitution of Equation 12.227 into Equation 12.212 now yields

$$i_L(t) = A_1 e^{-\alpha t}\cos(\omega_d t) + A_2 e^{-\alpha t}\sin(\omega_d t) \tag{12.228}$$

again for $t > 0$. Additionally, we can obtain $v_C(t)$ by using

$$v_C(t) = L\frac{di_L(t)}{dt}$$

as

$$v_C(t) = (LA_2\omega_d - L\alpha A_1)e^{-\alpha t}\cos(\omega_d t) - (LA_1\omega_d + L\alpha A_2)e^{-\alpha t}\sin(\omega_d t) \tag{12.229}$$

also for $t > 0$. Now only A_1 and A_2 remain as unknowns.

From this discussion, it is apparent that the role of the impulse in i_{IN} is to establish the initial conditions for a subsequent homogeneous response. This, incidentally, might explain how the circuit shown in Figure 12.24 began its operation.

As mentioned earlier during our discussion of the step response, the transient behavior of an uncharged capacitor is to act as a short circuit during the early part of a transient, while the corresponding transient behavior of an uncharged inductor is to act as an open circuit. Because of this the impulse in i_{IN} passes entirely through the capacitor while i_L remains zero at $t = 0$. An important consequence of this is that the charge $Q_\circ$ delivered by i_{IN} is delivered entirely to the capacitor, and so v_C steps to $Q_\circ/C$ at $t = 0$. This establishes the initial conditions after the impulse needed to determine A_1 and A_2. The evaluation of Equations 12.228 and 12.229 as $t \to 0$, followed by the substitution of these initial conditions ($v_C(0) = Q_\circ/C$ and $i_L(0) = 0$), yields

$$i_L(0) = A_1 = 0 \tag{12.230}$$

$$v_C(0) = \omega_\circ L A_2 = \frac{Q_\circ}{C}. \tag{12.231}$$

Equations 12.230 and 12.231 can be rearranged to yield

$$A_1 = 0 \tag{12.232}$$

$$A_2 = \frac{Q_\circ}{LC\omega_d}. \tag{12.233}$$

Finally, the substitution of Equations 12.232 and 12.233 into Equations 12.228 and 12.229 yields

$$i_L(t) = \frac{Q_\circ \omega_\circ^2}{\omega_d} e^{-\alpha t} \sin(\omega_d t) \tag{12.234}$$

$$v_C(t) = \frac{Q_\circ}{C}\frac{\omega_\circ}{\omega_d} e^{-\alpha t} \cos\left(\omega_d t + \tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right) u(t), \tag{12.235}$$

where the unit step function u has been introduced into Equations 12.234 and 12.235 so they are valid for all time.

Note that our solution in Equation 12.234 satisfies the initial conditions established by the impulse, and that it satisfies Equation 12.201. Because it does, it justifies our interpretation of the circuit behavior at $t = 0$. The waveforms for v_C and i_L are as shown in Figure 12.54.

It is interesting to note that the impulse response of the circuit can also be obtained from the step response. The circuit shown in Figure 12.50 is a linear circuit. Therefore, since the impulse i_{IN} given in Equation 12.226 is a scaled derivative of the step i_{IN} given in Equation 12.215, it follows that the impulse response is the same scaled derivative of the step response. (See Section 10.6.2

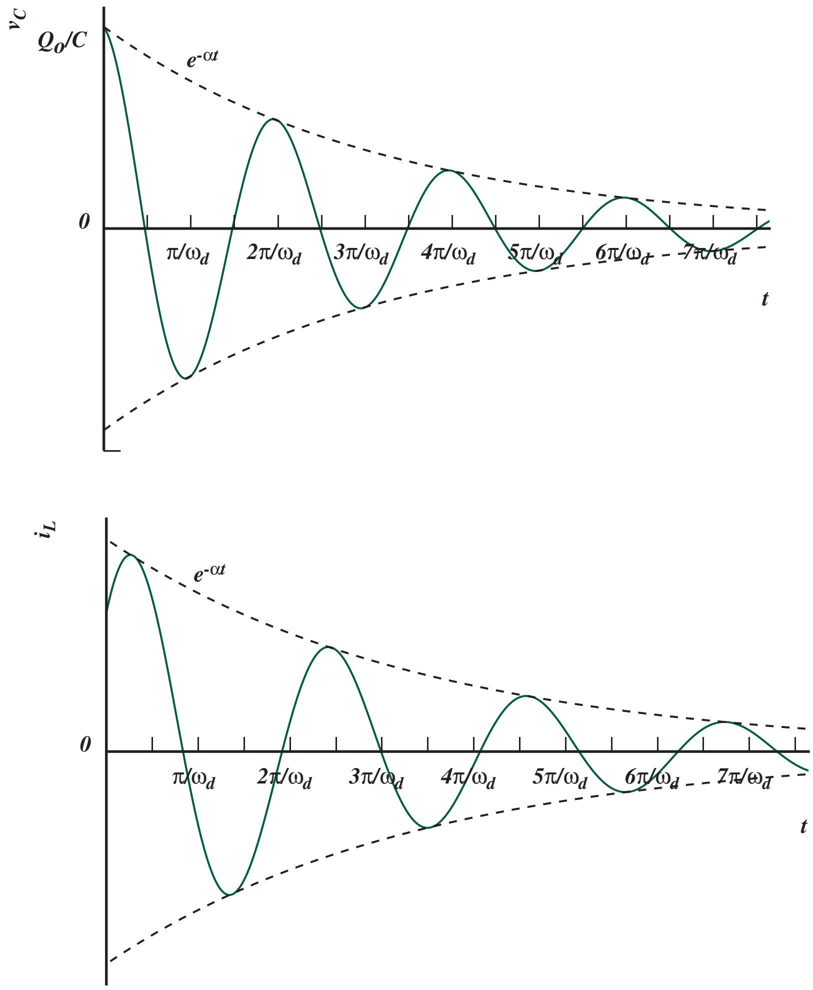

FIGURE 12.54 Waveforms of i_L and v_C for the parallel RLC circuit for a short pulse input.

for a more detailed discussion on the use of linearity to obtain responses to the derivative or the integral of an input, once the response to the input is known.)

To be more specific, i_{IN} as given in Equation 12.226, can be constructed by applying $(Q_\circ/I_\circ)d/dt$ to i_{IN} as given in Equation 12.215. In other words,

$$Q_\circ \delta(t) = (Q_\circ/I_\circ)\frac{d}{dt}I_\circ u(t).$$

Therefore, the same operator may be applied to Equations 12.234 and 12.235 to determine i_L and v_C respectively, for the impulse response. Thus, we can obtain the impulse response by differentiating the step response. Thus, applying the operator $(Q_\circ/I_\circ)d/dt$ to the step response, we obtain

$$\begin{aligned}
i_L(t) &= \frac{Q_\circ}{I_\circ}\frac{d}{dt}\left(I_\circ\left(1-\frac{\omega_\circ}{\omega_d}e^{-\alpha t}\cos\left(\omega_d t-\tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right)\right)u(t)\right)\\
&= \omega_\circ Q_\circ e^{-\alpha t}\sin\left(\omega_d t-\tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right)u(t)\\
&\quad+\frac{\alpha\omega_\circ Q_\circ}{\omega_d}e^{-\alpha t}\cos\left(\omega_d t-\tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right)u(t)\\
&\quad+Q_\circ\left(1-\frac{\omega_\circ}{\omega_d}e^{-\alpha t}\cos\left(\omega_d t-\tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right)\right)\delta(t)\\
&= Q_\circ\frac{\omega_\circ^2}{\omega_d}e^{-\alpha t}\sin(\omega_d t)u(t) \qquad (12.236)
\end{aligned}$$

$$\begin{aligned}
v_C(t) &= \frac{Q_\circ}{I_\circ}\frac{d}{dt}\left(\frac{I_\circ}{\omega_d C}e^{-\alpha t}\sin(\omega_d t)u(t)\right)\\
&= \frac{Q_\circ}{C}e^{-\alpha t}\cos(\omega_d t)u(t)-\frac{\alpha Q_\circ}{\omega_d C}e^{-\alpha t}\sin(\omega_d t)u(t)+\frac{Q_\circ}{\omega_d C}e^{-\alpha t}\sin(\omega_d t)\delta(t)\\
&= \frac{Q_\circ}{C}\frac{\omega_\circ}{\omega_d}e^{-\alpha t}\cos\left(\omega_d t+\tan^{-1}\left(\frac{\alpha}{\omega_d}\right)\right)u(t) \qquad (12.237)
\end{aligned}$$

as the impulse response of the circuit. Note that terms involving the impulse δ vanish in Equations 12.236 and 12.237 because δ is itself zero everywhere except $t = 0$, and the coefficients of the impulse are both zero at $t = 0$.

From this experience with the impulse, we can see that the impulse response of the circuit is essentially a homogeneous response. Thus this response is identical to that studied in Section 12.4. In fact, the role of the impulse is to establish initial conditions for the subsequent homogeneous response. As we argued in Section 12.6, the current impulse passes entirely through the capacitor delivering its charge in the process. Therefore, v_C steps to $Q_\circ/C$ as i_L remains zero. As a result, for $t > 0$, the impulse response described by Equations 12.236 and 12.237 are identical to Equations 12.105 and 12.106, respectively, with $v_C(0)$ replaced by $Q_\circ/C$, and $i_L(0)$ replaced by zero. Therefore, the entire discussion of the circuit shown in Figure 12.24 is applicable. Not surprisingly, note that v_C and i_L for the impulse as shown in Figure 12.54 are the same as those in Figure 12.25 with $v_C(0)$ replaced by $Q_\circ/C$.