12.4 UNDRIVEN, PARALLEL RLC CIRCUIT*

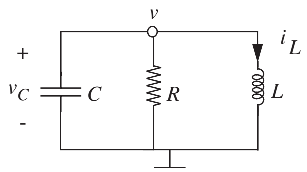

FIGURE 12.24 The parallel second-order RLC circuit shown in Figure 2.14a.

We will now analyze the undriven parallel RLC circuit shown in Figure 12.24, which is copied from Figure 2.14a. To analyze the behavior of this circuit we can again employ the node method, and this analysis closely parallels that of Section 12.1. As in Figure 12.6, a ground node is already selected in Figure 12.24, and the unknown node voltage v is already labeled. So, we may again proceed immediately to Step 3 of the node method. Here, we write KCL in terms of v for the node at which v is defined. This yields

$$C\frac{dv(t)}{dt} + \frac{1}{R}v(t) + \frac{1}{L}\int_{-\infty}^{t} v(\tilde{t})d\tilde{t} = 0. \qquad (12.81)$$

The first term in Equation 12.81 is the capacitor current, the second term is the resistor current, and the third term is the inductor current. Because the circuit contains an inductor, Equation 12.81 contains a time integral. To remove this integral, we differentiate Equation 12.81 with respect to time, and also divide by C, to obtain

$$\frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0, \qquad (12.82)$$

which is easier to work with.

To complete the node analysis, we complete Steps 4 and 5 by solving Equation 12.82 for v. Then, we use v to determine the other branch variables of interest, for example, i_L and v_C. Like Equation 12.4, Equation 12.82 is an ordinary second-order linear differential equation with constant coefficients. Since the circuit does not have a drive, its homogeneous solution is also the complete solution. Therefore, as with Equation 12.4, we expect its solution to be a superposition of two terms of the form

$$Ae^{st}.$$

The substitution of this candidate term into Equation 12.82 yields

$$A\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)e^{st} = 0 \qquad (12.83)$$

from which it follows that

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0. \qquad (12.84)$$

Equation 12.84 is the *characteristic equation* of the circuit. To simplify Equation 12.84, and to put it in a form that is more standard for the characteristic equation in second-order circuits, we write it as

$$s^2 + 2\alpha s + \omega_\circ^2 = 0 \tag{12.85}$$

where

$$\alpha \equiv \frac{1}{2RC} \tag{12.86}$$

$$\omega_\circ \equiv \sqrt{\frac{1}{LC}}; \tag{12.87}$$

note that Equation 12.87 is the same as Equation 12.9. Equation 12.85 is a quadratic equation having two roots. Those roots are

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_\circ^2} \tag{12.88}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_\circ^2}. \tag{12.89}$$

Therefore, the solution for v is a linear combination of the two functions $e^{s_1 t}$ and $e^{s_2 t}$, and takes the form

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{12.90}$$

where A_1 and A_2 are as yet unknown constants that are equivalent to the two constants of integration encountered when integrating Equation 12.82 twice to find v.

To complete the solution to Equation 12.82 we must again determine A_1 and A_2 from initial conditions v_C and i_L specified at $t = 0$, To do so, note that

$$v_C(t) = v(t). \tag{12.91}$$

Further, from KCL applied to either node, that is, from Equation 12.81,

$$i_L(t) = -\frac{1}{R}v(t) - C\frac{dv}{dt}. \tag{12.92}$$

Equations 12.91 and 12.92 can be solved to determine v and dv/dt in terms of i_L and v_C. Doing so, and evaluating the result at $t = 0$, then yields

$$v(0) = v_C(0) \tag{12.93}$$

$$\frac{dv}{dt}(0) = -\frac{1}{C}i_L(0) - \frac{1}{RC}v_C(0). \tag{12.94}$$

Now, to find A_1 and A_2, we evaluate Equation 12.90 and its derivative at $t = 0$, and set the results equal to Equations 12.93 and 12.94, respectively. This results in

$$v(0) = A_1 + A_2 = v_C(0) \tag{12.95}$$

$$\begin{aligned}\frac{dv}{dt}(0) &= s_1 A_1 + s_2 A_2 \\ &= -\frac{1}{C} i_L(0) - \frac{1}{RC} v_C(0).\end{aligned} \tag{12.96}$$

Equations 12.95 and 12.96 can be jointly solved for A_1 and A_2 to obtain

$$A_1 = \frac{(1 + RCs_2)v_C(0) + Ri_L(0)}{RC(s_2 - s_1)} = \frac{s_1 v_C(0) - i_L(0)/C}{(s_1 - s_2)} \tag{12.97}$$

$$A_2 = \frac{(1 + RCs_1)v_C(0) + Ri_L(0)}{RC(s_1 - s_2)} = \frac{s_2 v_C(0) - i_L(0)/C}{(s_2 - s_1)} \tag{12.98}$$

where we have used the fact that both s_1 and s_2 satisfy Equation 12.84, and the fact that $LCs_1 s_2 = 1$, to obtain the second equalities. Finally, Equations 12.97 and 12.98 can now be substituted into Equation 12.90 to obtain

$$v(t) = \frac{s_1 v_C(0) - i_L(0)/C}{(s_1 - s_2)} e^{s_1 t} + \frac{s_2 v_C(0) - i_L(0)/C}{(s_2 - s_1)} e^{s_2 t}. \tag{12.99}$$

Further, the substitution of Equation 12.99 into Equations 12.91 and 12.92 yields

$$v_C(t) = \frac{s_1 v_C(0) - i_L(0)/C}{(s_1 - s_2)} e^{s_1 t} + \frac{s_2 v_C(0) - i_L(0)/C}{(s_2 - s_1)} e^{s_2 t} \tag{12.100}$$

$$\begin{aligned} i_L(t) &= -\left(\frac{1 + RCs_1}{R}\right) \frac{s_1 v_C(0) - i_L(0)/C}{(s_1 - s_2)} e^{s_1 t} \\ &\quad - \left(\frac{1 + RCs_2}{R}\right) \frac{s_2 v_C(0) - i_L(0)/C}{(s_2 - s_1)} e^{s_2 t} \\ &= \frac{v_C(0)/L - s_2 i_L(0)}{(s_1 - s_2)} e^{s_1 t} + \frac{v_C(0)/L - s_1 i_L(0)}{(s_2 - s_1)} e^{s_2 t} \end{aligned} \tag{12.101}$$

as the states of the parallel circuit. To obtain the second equality in Equation 12.101 we have again used the fact that both s_1 and s_2 satisfy

Equation 12.84, and the fact that $LCs_1s_2 = 1$. This completes the formal node analysis of the circuit shown in Figure 12.24.

We will now close this subsection by examining the dynamic behavior of v_C and i_L for the same three cases defined in Section 12.2. Those are the cases of under-damped, critically-damped, and over-damped dynamics. As we shall see, the dynamics of the series circuit are essentially identical to those of the parallel circuit for all three cases, except for the details of the role of R. In the series circuit, small R caused light damping while large R caused heavy damping. This role reverses for the parallel circuit because it is in the limit of large R that Figure 12.24 reduces to Figure 12.6.

12.4.1 UNDER-DAMPED DYNAMICS

The case of under-damped dynamics is characterized by

$$\alpha < \omega_\circ$$

or, after substitution of Equations 12.86 and 12.87, by

$$2R > \sqrt{L/C}.$$

As R becomes large, the corresponding resistor approaches an open circuit, and so the circuit shown in Figure 12.24 approaches the LC circuit shown in Figure 12.6. Therefore, we should expect the under-damped dynamics to be oscillatory in nature. As we shall see shortly, this is indeed the case.

With $\alpha < \omega_\circ$, the quantity inside the radicals in Equations 12.88 and 12.89 is negative, and so the natural frequencies s_1 and s_2 are complex numbers. To simplify matters, let us again define ω_d according to

$$\omega_d \equiv \sqrt{\omega_o^2 - \alpha^2}. \tag{12.102}$$

With this definition, s_1 and s_2 from Equations 12.88 and 12.89 can be written as

$$s_1 = -\alpha + j\omega_d \tag{12.103}$$

$$s_2 = -\alpha - j\omega_d. \tag{12.104}$$

The real and imaginary parts of s_1 and s_2 are now more apparent.

Since s_1 and s_2 are now complex, the exponentials in Equations 12.100 and 12.101 are also complex. Thus, i_L and v_C will exhibit both oscillatory and decaying behavior. To see this, we substitute Equations 12.103 and 12.104 into

Equations 12.100 and 12.101, and use the Euler relation to obtain

$$\begin{aligned} v_C(t) &= v_C(0)e^{-\alpha t}\cos(\omega_d t) - \left(\frac{\alpha C v_C(0) + i_L(0)}{C\omega_d}\right) e^{-\alpha t}\sin(\omega_d t) \\ &= \sqrt{v_C^2(0) + \left(\frac{\alpha C v_C(0) + i_L(0)}{C\omega_d}\right)^2}\; e^{-\alpha t} \\ &\quad \times \cos\left(\omega_d t + \tan^{-1}\left(\frac{\alpha C v_C(0) + i_L(0)}{C\omega_d v_C(0)}\right)\right) \end{aligned} \tag{12.105}$$

$$\begin{aligned} i_L(t) &= i_L(0)e^{-\alpha t}\cos(\omega_d t) + \left(\frac{v_C(0) + \alpha L i_L(0)}{L\omega_d}\right) e^{-\alpha t}\sin(\omega_d t) \\ &= \sqrt{i_L^2(0) + \left(\frac{v_C(0) + \alpha L i_L(0)}{L\omega_d}\right)^2}\; e^{-\alpha t} \\ &\quad \times \sin\left(\omega_d t + \tan^{-1}\left(\frac{L\omega_d i_L(0)}{v_C(0) + \alpha L i_L(0)}\right)\right). \end{aligned} \tag{12.106}$$

Sketches of i_L and v_C are shown in Figure 12.25 for the special case of

$$i_L(0) = 0.$$

As was the case for the series circuit, the states in the parallel circuit display oscillatory and decaying behavior. It is also the case that Equations 12.105 and 12.106 reduce to Equations 12.21 and 12.22, respectively, as the circuit damping characterized by α vanishes. The difference here is that this occurs as $R \to \infty$ because it is in this limit that Figure 12.24 reduces to Figure 12.6.

A comparison of Equations 12.105 and 12.106 with Equations 12.63 and 12.64 shows that the under-damped dynamics of the parallel and series circuits are quite similar. This is to be expected because their characteristic equations are identical. It is for this reason that α, $\omega_\circ$, ω_d, and Q have the same interpretations for the two circuits. Our comments concerning stored energy also hold for both circuits. Therefore, we will not repeat the details here. Rather, we will identify three important differences. The first difference, which has been mentioned already, is the reversed role of R. A large R in the series circuit corresponds to a small R in the parallel circuit and vice versa. The second difference is the evaluation of the quality factor Q. While Equation 12.65 still holds for the parallel circuit, that is,

$$Q \equiv \frac{\omega_\circ}{2\alpha}, \tag{12.107}$$

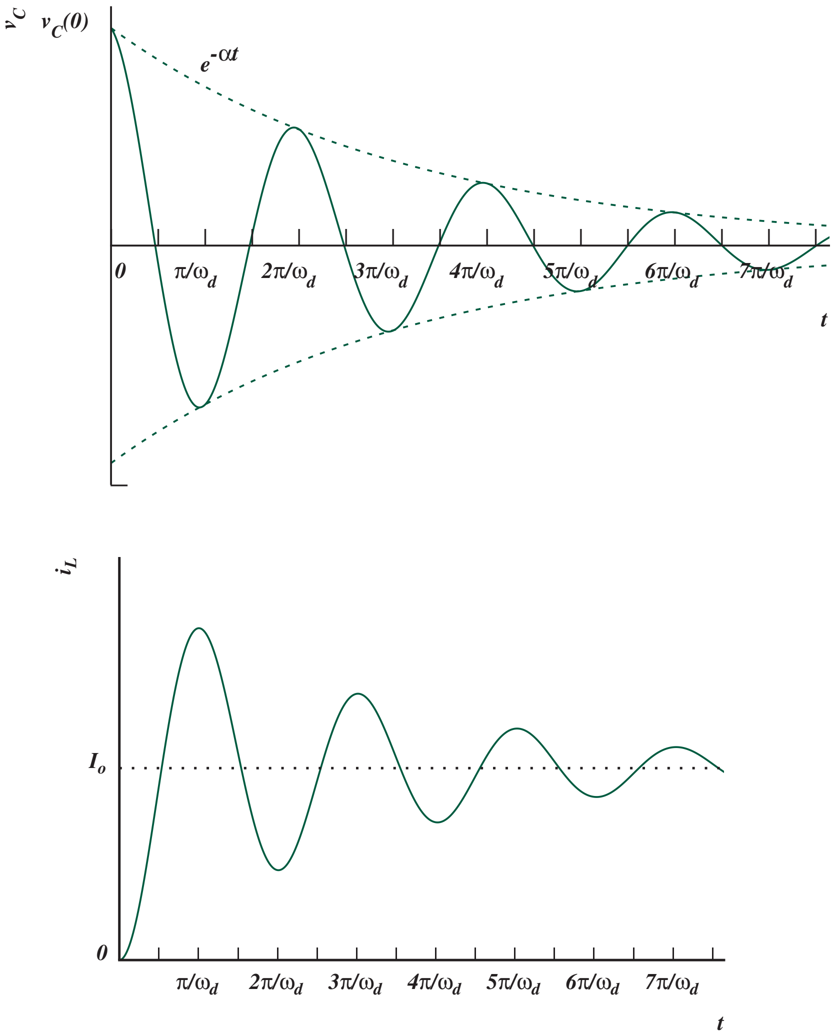

FIGURE 12.25 Waveforms of v_C and i_L in undriven, parallel RLC circuit for the case of $i_L(0) = 0$.

Equation 12.66 does not. Rather, for the parallel circuit shown in Figure 12.24, the substitution of Equations 12.86 and 12.87 into Equation 12.65 yields

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{12.108}$$

or

$$Q = \frac{\omega_o L}{R}. \tag{12.109}$$

The third difference is the role of ϕ in Figure 12.17. For the parallel circuit, assuming $i_L(0) = 0$ in Equations 12.105 and 12.106, v_C is advanced with respect to i_L by quadrature plus the additional angle ϕ, where $\phi = \tan^{-1}(\alpha/\omega_d)$.

12.4.2 OVER-DAMPED DYNAMICS

As with the case of the series circuit, the case of over-damped dynamics is characterized by

$$\alpha > \omega_\circ$$

or, after substitution of Equations 12.86 and 12.87, by

$$2R < \sqrt{L/C}.$$

In this case, the quantity inside the radicals in Equations 12.88 and 12.89 is positive, and so both s_1 and s_2 are real. For this reason, the dynamic behavior of i_L and v_C, as expressed by Equations 12.100 and 12.101, does not exhibit oscillation. Rather, it involves two real exponential functions that decay at different rates, as the two equations show. The expressions for v_C and i_L for the case of $i_L(0) = 0$ with over-damping are obtained from Equations 12.100 and 12.101, and are shown here:

$$v_C(t) = \frac{s_1 v_C(0)}{(s_1 - s_2)} e^{s_1 t} + \frac{s_2 v_C(0)}{(s_2 - s_1)} e^{s_2 t} \tag{12.110}$$

$$i_L(t) = \frac{v_C(0)}{L(s_1 - s_2)} e^{s_1 t} + \frac{v_C(0)}{L(s_2 - s_1)} e^{s_2 t}. \tag{12.111}$$

Since $\alpha > \omega_\circ$ for over-damped circuits, note that s_1 and s_2 are both real in these two equations.

As R becomes small, in particular smaller than $1/2\sqrt{L/C}$, it becomes a significant short circuit across the capacitor and inductor. In this way it diverts the oscillating current that the capacitor and inductor share for larger values of R. As a consequence, the energy exchange between the capacitor and inductor is interrupted, and the circuit ceases to oscillate. Instead, its behavior is more like that of an independent capacitor and an independent inductor discharging through the resistor. To see this, let us determine the asymptotic values of s_1 and s_2 as R becomes small and hence as α becomes large. They are

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_\circ^2} = \alpha \left(-1 + \sqrt{1 - \left(\frac{\omega_\circ}{\alpha}\right)^2} \right) \approx \alpha \frac{-\omega_\circ^2}{2\alpha^2} = \frac{-R}{L} \tag{12.112}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_\circ^2} = \alpha\left(-1 - \sqrt{1 - \left(\frac{\omega_\circ}{\alpha}\right)^2}\right) \approx -2\alpha = \frac{-1}{RC}. \quad (12.113)$$

As expected the corresponding time constants approach RC and L/R, the time constants of an independent capacitor-resistor circuit and an independent inductor-resistor circuit. Note that, for over-damped dynamics, $\alpha > \omega_\circ$ from which it follows that RC is the faster time constant and L/R is the slower time constant.

12.4.3 CRITICALLY-DAMPED DYNAMICS

The case of critically-damped dynamics is characterized by

$$\alpha = \omega_\circ.$$

In this case, it follows from Equations 12.88 and 12.89 that

$$s_1 = s_2 = -\alpha$$

and that the characteristic equation, Equation 12.85, has a repeated root. Because of this, $e^{s_1 t}$ and $e^{s_2 t}$ are no longer independent functions, and so the general solution for v is no longer the superposition of these two functions as given by Equation 12.90. Rather, it is again the superposition of the repeated exponential function

$$e^{s_1 t} = e^{s_2 t} = e^{-\alpha t} \quad \text{and} \quad te^{-\alpha t}.$$

From this observation, and Equations 12.91 and 12.92, it follows that v_C and i_L will exhibit similar behavior.

Perhaps the easiest way to determine v_C and i_L for the case of critical damping is to evaluate Equations 12.105 and 12.106 under the conditions of that case. To do so, observe from Equation 12.102 that, for critical damping $\omega_\circ = \alpha$, and so $\omega_d = 0$. Therefore, we can obtain v_C and i_L for the case of critical-damping by evaluating Equations 12.105 and 12.106 in the limit $\omega_d \to 0$. This results in

$$v_C(t) = v_C(0)e^{-\alpha t} - \frac{\alpha C v_C(0) + i_L(0)}{C} te^{-\alpha t} \quad (12.114)$$

$$i_L(t) = i_L(0)e^{-\alpha t} + \frac{v_C(0) + \alpha L i_L(0)}{L} te^{-\alpha t}. \quad (12.115)$$

From Equations 12.114 and 12.115 we see that v_C and i_L contain both the decaying exponential function $e^{-\alpha t}$ and the function $te^{-\alpha t}$, as expected.