

13.4.3 THE BODE PLOT: SKETCHING THE FREQUENCY RESPONSE OF GENERAL FUNCTIONS *

Sections 13.4.1 and 13.4.2 demonstrated the ease with which we can sketch the frequency response of simple circuits by observing their behavior at low frequencies and high frequencies. Things get more complicated for a network with several inductors or capacitors. This section discusses a simple and intuitive method called *Bode plots* for sketching the frequency response of more general circuits. The Bode method uses the insight gained from Sections 13.4.1 and 13.4.2 that the frequency response plots can be closely approximated by straight line segments derived from the asymptotic behavior of the transfer functions.

The method proceeds as follows: First, write the relationship (Equation 13.46, for example) in the form of a system function, the ratio of the complex amplitude of the response to the complex amplitude of the input:

$$H(s) = \frac{\text{Response}}{\text{Input}}. \quad (13.91)$$

In general, the system function $H(s)$ will be the ratio of two polynomials:

$$H(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0} \quad (13.92)$$

where the coefficients a_i and b_i are real numbers since our circuit parameters are real numbers. We saw one example of this in Equation 13.64. We can factor the numerator and denominator polynomials and write

$$H(s) = \frac{K_1(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \quad (13.93)$$

where K_1 is a constant and z_1, z_2, \dots, z_m are the roots of the numerator polynomial, and p_1, p_2, \dots, p_n are the roots of the denominator polynomial.⁹

9. Because the system function goes to zero when $s = z_i$, the roots of the numerator, z_1, z_2, \dots, z_m , are called the *zeros*, *definition* of the system function. Similarly, the roots of the denominator, p_1, p_2, \dots, p_n are called the *poles* of the system function. The system function goes to infinity when s takes on the value of one of the poles (in other words, when $s = p_i$). When one or more of the z_i 's or p_i 's is zero, the system is said to have zeros or poles at the origin. The poles and zeros of a system function are important system parameters because they characterize the general behavior

In general, some of the roots of the numerator or the denominator polynomials can be zero. Furthermore, the roots of the numerator and denominator polynomials can also be complex. If any of the roots are complex, then they must appear in complex conjugate pairs, so that the overall system function remains real. We will rewrite Equation 13.93 into the following standard form to reflect these facts:

$$H(s) = \frac{K_o s^l (s + a_1)(s + a_2) \cdots (s^2 + 2\alpha_1 s + \omega_1^2) \cdots}{(s + a_3)(s + a_4) \cdots (s^2 + 2\alpha_2 s + \omega_2^2) \cdots}. \quad (13.94)$$

In Equation 13.94, we have combined complex conjugate pairs into quadratic terms of the form $(s^2 + 2\alpha s + \omega^2)$. Thus all the remaining a_i values are real. The s^l term, where l can be positive or negative, reflects the case where the roots in Equation 13.93 are zero.

We will now show that it is possible to sketch without formal calculation the general shape of $H(s)$ as a function of frequency. More precisely, we can make an approximate sketch of the magnitude and phase of $H(s)$ as a function of the input frequency ω . The resulting pair of graphics representing an approximate sketch of the frequency response is called a *Bode plot*, in honor of the Bell Laboratories engineer who devised it to study stability in feedback amplifiers.¹⁰

The Bode plot is an approximation of the frequency response and accordingly has two parts: a sketch of the log magnitude of $H(j\omega)$ versus $\log \omega$ and a sketch of the angle of $H(j\omega)$ versus $\log \omega$. These coordinates are chosen because they facilitate straightforward construction of the frequency response graphs even for complicated functions without the use of a computer. Taking the magnitude and log on both sides of Equation 13.94,

$$\begin{aligned} \log|H(s)| &= \log K_o + \\ &\log |s| + \log |s| + \cdots \text{ (} l \text{ terms)} + \\ &\log |s + a_1| + \log |s + a_2| + \cdots - \log |s + a_3| - \log |s + a_4| + \cdots \\ &\log |s^2 + 2\alpha_1 s + \omega_1^2| + \cdots - \log |s^2 + 2\alpha_2 s + \omega_2^2| + \cdots \end{aligned} \quad (13.95)$$

of the system. A detailed discussion of system analysis using poles and zeros is beyond the scope of this book.

10. Bode, H.W., *Network Analysis and Feedback Amplifier Designs*, Van Nostrand, New York, 1945, Chapter 15.

and for the phase

$$\begin{aligned}
 \angle H(s) = & \angle K_o + \\
 & \angle s + \angle s + \dots \text{ (} l \text{ terms)} + \\
 & \angle(s + a_1) + \angle(s + a_2) + \dots - \angle(s + a_3) - \angle(s + a_4) - \dots \\
 & + \angle(s^2 + 2\alpha_1 s + \omega_1^2) + \dots - \angle(s^2 + 2\alpha_2 s + \omega_2^2) - \dots
 \end{aligned} \quad (13.96)$$

Notice that there are four types of terms in the magnitude and phase equations:

1. The K_o constant term,
2. the s terms,
3. terms of the form $(s + a)$, and
4. quadratic terms of the form $(s^2 + 2\alpha s + \omega^2)$, which have complex roots.

This gives us a simple way of approximating the magnitude and phase curves of the frequency response plot. First, draw the individual magnitude and angle curves for each of the four types of terms in the numerator and denominator of Equation 13.94. Then, construct the overall magnitude and phase plots by simply adding together the individual curves.

Let us now address each of the four terms:

1. The K_o constant term.
We saw how to draw the frequency response of constant terms in Section 13.4.1. Essentially constant terms result in horizontal lines on the magnitude plot and have a phase of zero.
2. The s terms.
Terms of the form s and $1/s$ (if l is negative) were also plotted in Section 13.4.1. We saw that each of these terms result in lines of $+1$ or -1 slope on the log magnitude plot and contribute to a phase of 90° or -90° , respectively.
3. Terms of the form $(s + a)$.
Section 13.4.2 addressed terms of the form $(s + a)$. We showed that the magnitude part of the frequency response of these terms is approximated by two straight lines corresponding to the low and high frequency asymptotes meeting at the break frequency a . Accordingly, Bode plots result in a series of straight line segments attached together at the break frequencies.

The phase plot also uses low- and high-frequency asymptotes and passes through 45° at the break frequency a . For more accuracy, the phase curve can be approximated by a straight line that passes through 45° at the

break frequency a , and meets the low- and high-frequency asymptotes at 0.1 times the break frequency ($0.1a$) and 10 times the break frequency ($10a$), respectively.

4. Quadratic terms of the form $(s^2 + 2\alpha s + \omega^2)$ with complex roots.
Although not as straightforward, it is possible to sketch frequency response plots for system functions of the form $(s^2 + 2\alpha s + \omega^2)$, where the roots are complex. However, we will defer a further discussion on plotting Bode plots for complex roots to Section 14.4. For now, we will focus on real roots.