Function Spaces

Prerequisite: Section 4.7, Coordinatization

In this section, we apply the techniques of Chapter 4 to vector spaces whose elements are functions. The vector spaces $\mathcal{P}_n$ and $\mathcal{P}$ are familiar examples of such spaces. Other important examples are $C^0(\mathbb{R}) = \{$all continuous real-valued functions on $\mathbb{R}\}$ and $C^1(\mathbb{R}) = \{$all continuously differentiable real-valued functions on $\mathbb{R}\}$.

▶ Linear Independence in Function Spaces

Proving that a finite subset S of a function space is linearly independent usually requires a modification of the strategy used in $\mathbb{R}^n$.

EXAMPLE 1 Consider the subset $S = \left\{x^3 - x,\ xe^{-x^2},\ \sin\left(\frac{\pi}{2}x\right)\right\}$ of $C^1(\mathbb{R})$. We will show that S is linearly independent using the definition of linear independence. Let a, b, and c be real numbers such that

$$a\left(x^3 - x\right) + b\left(xe^{-x^2}\right) + c\left(\sin\left(\frac{\pi}{2}x\right)\right) = 0$$

for every value of x. We must show that $a = b = c = 0$.

The above equation must be satisfied for every value of x. In particular, it is true for $x = 1$, $x = 2$, and $x = 3$. This yields the following system:

$$\begin{cases} (\text{Letting } x = 1 \Longrightarrow) & a(0) + b\left(\dfrac{1}{e}\right) + c\,(1) = 0 \\ (\text{Letting } x = 2 \Longrightarrow) & a(6) + b\left(\dfrac{2}{e^4}\right) + c\,(0) = 0 \\ (\text{Letting } x = 3 \Longrightarrow) & a(24) + b\left(\dfrac{3}{e^9}\right) + c\,(-1) = 0 \end{cases}.$$

Row reducing the matrix

$$\begin{array}{c} \begin{matrix} a & b & c & \end{matrix} \\ \left[\begin{array}{ccc|c} 0 & \frac{1}{e} & 1 & 0 \\ 6 & \frac{2}{e^4} & 0 & 0 \\ 24 & \frac{3}{e^9} & -1 & 0 \end{array}\right] \end{array} \quad \text{to} \quad \begin{array}{c} \begin{matrix} a & b & c & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

shows that the trivial solution $a = b = c = 0$ is the only solution to this homogeneous system. Hence, the set S is linearly independent by the definition of linear independence. ■

When proving linear independence using the technique of Example 1, we try to choose "nice" values of x to make computations easier. Even so, the use of a calculator or computer is often desirable when working with function spaces.

Other problems may occur because of the choice of x-values. Returning to Example 1, if instead we had plugged in $x = -1$, $x = 0$, and $x = 1$, we would have obtained the system

$$\begin{cases} (x = -1 \Longrightarrow) & a(0) + b\left(-\dfrac{1}{e}\right) + c\,(-1) = 0 \\ (x = \ \ 0 \Longrightarrow) & a(0) + b\,(0) + c\,(0) = 0 \\ (x = \ \ 1 \Longrightarrow) & a(0) + b\left(\dfrac{1}{e}\right) + c\,(1) = 0 \end{cases},$$

which has infinitely many nontrivial solutions. To prove linear independence, we must examine further values of x, generating more equations for the system, until the new system we obtain has only the trivial solution, as in Example 1.

Suppose, however, that after substituting many values for x and creating a huge homogeneous system, we still have nontrivial solutions. We cannot conclude that the set of functions is linearly dependent, although we may suspect that it is. In general, to *prove* that a set of functions $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is linearly dependent, we must find real numbers $a_1, \ldots, a_n$, not all zero, such that

$$a_1\mathbf{f}_1(x) + a_2\mathbf{f}_2(x) + \cdots + a_n\mathbf{f}_n(x) = 0$$

is a functional identity for every value of x, not just those we have tried.

EXAMPLE 2 Let $S = \{\sin 2x, \cos 2x, \sin^2 x, \cos^2 x\}$, a subset of $C^1(\mathbb{R})$. Suppose we attempt to show that S is linearly independent using the definition of linear independence. Let a, b, c, and d represent real numbers such that

$$a(\sin 2x) + b(\cos 2x) + c(\sin^2 x) + d(\cos^2 x) = 0.$$

Since we have four vectors in S, we substitute four different values for x into this equation to obtain the following system:

$$\begin{cases} (x = 0 \implies) & a(0) + b(1) + c(0) + d(1) = 0 \\ (x = \frac{\pi}{4} \implies) & a(1) + b(0) + c\left(\frac{1}{2}\right) + d\left(\frac{1}{2}\right) = 0 \\ (x = \frac{\pi}{2} \implies) & a(0) + b(-1) + c(1) + d(0) = 0 \\ (x = \frac{3\pi}{4} \implies) & a(-1) + b(0) + c\left(\frac{1}{2}\right) + d\left(\frac{1}{2}\right) = 0 \end{cases}.$$

Since the coefficient matrix for this homogeneous system row reduces to

$$\begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

there are nontrivial solutions to the system, such as $a = 0$, $b = -1$, $c = -1$, $d = 1$.

At this point, we cannot infer that S is linearly independent because we have nontrivial solutions. We also cannot conclude that S is linearly dependent because we have tested only a few values for x. We could try more values, such as $x = \frac{\pi}{6}$ and $x = \pi$, but we would still find that $a = 0$, $b = -1$, $c = -1$, $d = 1$ satisfies each equation we generate. This situation leads us to believe that the set S is linearly dependent. To be certain, we must check that the values $a = 0$, $b = -1$, $c = -1$, and $d = 1$ yield a functional identity when plugged into the original functional equation. Substituting these values yields

$$0(\sin 2x) + (-1)(\cos 2x) + (-1)(\sin^2 x) + (1)(\cos^2 x) = 0,$$

or $\cos 2x = \cos^2 x - \sin^2 x$, a well-known trigonometric identity. Thus, one vector in S can be expressed as a linear combination of the other vectors in S, and S is linearly dependent. ■

► New Vocabulary

$C^0(\mathbb{R})$ (continuous real-valued functions on $\mathbb{R}$)
$C^1(\mathbb{R})$ (real-valued functions on $\mathbb{R}$ having a continuous derivative)
function spaces
linearly dependent set (in a function space)
linearly independent set (in a function space)

▶ Highlights

- Function spaces are vector spaces whose elements are functions.
- Examples of function spaces are $\mathcal{P}_n$, $\mathcal{P}$, $C^0(\mathbb{R})$, and $C^1(\mathbb{R})$.
- A set of functions $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ (in a function space) is linearly independent if there are n different values of x so that the resulting n equations of the form $a_1\mathbf{f}_1(x) + a_2\mathbf{f}_2(x) + \cdots + a_n\mathbf{f}_n(x) = 0$ form a system having only the trivial solution $a_1 = a_2 = \cdots = a_n = 0$.
- A set of functions $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ (in a function space) is linearly dependent if the equation $a_1\mathbf{f}_1(x) + a_2\mathbf{f}_2(x) + \cdots + a_n\mathbf{f}_n(x) = 0$ has a nontrivial solution for $a_1, a_2, \ldots, a_n$ for every possible value of x.

▶ EXERCISES

1. In each part of this exercise, determine whether the given subset S of $C^1(\mathbb{R})$ is linearly independent. If S is linearly independent, prove that it is. If S is linearly dependent, solve for a functional identity that expresses one function in S as a linear combination of the others.

★ **a)** $S = \{e^x, e^{2x}, e^{3x}\}$

b) $S = \{\sin x, \sin 2x, \sin 3x, \sin 4x\}$

★ **c)** $S = \{(5x-1)/(1+x^2), (3x+1)/(2+x^2), (7x^3-3x^2+17x-5)/(x^4+3x^2+2)\}$

d) $S = \{\sin x, \ \sin(x+1), \ \sin(x+2), \ \sin(x+3)\}$

2. Recall that a function $\mathbf{f}(x) \in C^0(\mathbb{R})$ is **even** if $\mathbf{f}(x) = \mathbf{f}(-x)$ for all $x \in \mathbb{R}$ and is **odd** if $\mathbf{f}(x) = -\mathbf{f}(-x)$ for all $x \in \mathbb{R}$. Suppose we want to prove that a finite subset S of $C^0(\mathbb{R})$ is linearly independent by the method of Example 1.

a) Suppose that every element of S is an odd function of x (as in Example 1). Explain why we would not want to substitute both 1 and -1 for x into the appropriate functional equation. Also explain why $x = 0$ would be a poor choice.

b) Suppose that every element of S is an even function. Would we want to substitute both 1 and -1 for x into the appropriate functional equation? Why? How about $x = 0$?

3. Let S be the subset $\{\cos(x+1), \cos(x+2), \cos(x+3)\}$ of $C^1(\mathbb{R})$.

a) Show that span(S) has $\{\cos x, \sin x\}$ for a basis. (Hint: The identity $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ is useful.)

b) Use part (a) to prove that S is linearly dependent.

4. For each given subset S of $C^1(\mathbb{R})$, find a subset B of S that is a basis for $\mathcal{V} =$ span(S).

★ **a)** $S = \{\sin 2x, \cos 2x, \sin^2 x, \cos^2 x, \sin x \cos x, 1\}$

b) $S = \{e^x, 1, e^{-x}\}$

★ **c)** $S = \{\sin(x+1), \cos(x+1), \sin(x+2), \cos(x+2)\}$

5. In each part of this exercise, let B represent an ordered basis for a subspace $\mathcal{V}$ of $C^1(\mathbb{R})$ and find $[\mathbf{v}]_B$ for the given $\mathbf{v} \in \mathcal{V}$.

★ **a)** $B = (e^x,\, e^{2x},\, e^{3x})$, $\mathbf{v} = 5e^x - 7e^{3x}$

b) $B = (\sin 2x,\, \cos 2x,\, \sin^2 x)$, $\mathbf{v} = 1$

★ **c)** $B = (\sin(x+1),\, \sin(x+2))$, $\mathbf{v} = \cos x$

★ **6.** True or False:

a) A subset $\{\mathbf{f}_1, \mathbf{f}_2\}$ of nonzero functions in $C^0(\mathbb{R})$ is linearly dependent if and only if $\mathbf{f}_1$ is a nonzero constant multiple of $\mathbf{f}_2$.

b) The set $\{x^2, x^3, x^4, x^5\}$ is a linearly independent subset of $C^1(\mathbb{R})$.

c) Let $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \in C^0(\mathbb{R})$. If plugging values for x into $a\mathbf{f}_1(x) + b\mathbf{f}_2(x) + c\mathbf{f}_3(x) = 0$ leads to $a = b = c = 0$, then $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ are linearly dependent.

d) Let $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \in C^0(\mathbb{R})$. If plugging 3 different values for x into $a\mathbf{f}_1(x) + b\mathbf{f}_2(x) + c\mathbf{f}_3(x) = 0$ does not allow us to conclude that $a = b = c = 0$, then $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ are linearly dependent.

▶ Answers to Selected Exercises

(1) (a) Linearly independent; to prove that it is, substitute the values $x = 0$, $x = 1$, $x = 2$, and follow the method of Example 1

(c) Linearly dependent ($a = -2$, $b = 1$, $c = 1$)

(4) (a) $B = \{\sin(2x), \cos(2x), \sin^2 x\}$

(c) $B = \{\sin(x+1), \cos(x+1)\}$

(5) (a) $[\mathbf{v}]_B = [5, 0, -7]$

(c) $[\mathbf{v}]_B = [-\frac{\cos 2}{\sin 1}, \frac{\cos 1}{\sin 1}] \approx [0.4945, 0.6421]$. (If your answer is more complicated than this, compare numerical approximations.)

(6) (a) T

(b) T

(c) F

(d) F