## Contents

Preface to the Seventh Edition ................................................................. xxvi
Acknowledgments ................................................................................. xxviii
The Order of Presentation of the Formulas ........................................... xxix
Use of the Tables ....................................................................................... xxx
Index of Special Functions ...................................................................... xxxi
Notation ..................................................................................................... xliii
Note on the Bibliographic References .................................................. xlviii

### 0 Introduction

0.1 Finite Sums .......................................................................................... 1
0.11 Progressions ....................................................................................... 1
0.12 Sums of powers of natural numbers .................................................. 1
0.13 Sums of reciprocals of natural numbers .......................................... 3
0.14 Sums of products of reciprocals of natural numbers ....................... 3
0.15 Sums of the binomial coefficients .................................................... 3
0.2 Numerical Series and Infinite Products .............................................. 6
0.21 The convergence of numerical series ............................................... 6
0.22 Convergence tests ............................................................................. 6
0.23–0.24 Examples of numerical series ................................................ 8
0.25 Infinite products ............................................................................... 14
0.26 Examples of infinite products .......................................................... 14
0.3 Functional Series ............................................................................... 15
0.30 Definitions and theorems ................................................................ 15
0.31 Power series ..................................................................................... 16
0.32 Fourier series ................................................................................... 19
0.33 Asymptotic series ............................................................................. 21
0.4 Certain Formulas from Differential Calculus ..................................... 21
0.41 Differentiation of a definite integral with respect to a parameter ....... 21
0.42 The \( n \)th derivative of a product (Leibniz's rule) ........................... 22
0.43 The \( n \)th derivative of a composite function .................................... 22
0.44 Integration by substitution ............................................................... 23

### 1 Elementary Functions

1.1 Power of Binomials ............................................................................ 25
1.11 Power series .................................................................................... 25
1.12 Series of rational fractions ............................................................. 26
1.2 The Exponential Function .................................................................. 26
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.21</td>
<td>Series representation</td>
<td>26</td>
</tr>
<tr>
<td>1.22</td>
<td>Functional relations</td>
<td>27</td>
</tr>
<tr>
<td>1.23</td>
<td>Series of exponentials</td>
<td>27</td>
</tr>
<tr>
<td>1.3–1.4</td>
<td>Trigonometric and Hyperbolic Functions</td>
<td>28</td>
</tr>
<tr>
<td>1.30</td>
<td>Introduction</td>
<td>28</td>
</tr>
<tr>
<td>1.31</td>
<td>The basic functional relations</td>
<td>28</td>
</tr>
<tr>
<td>1.32</td>
<td>The representation of powers of trigonometric and hyperbolic functions in terms of functions of multiples of the argument (angle)</td>
<td>31</td>
</tr>
<tr>
<td>1.33</td>
<td>The representation of trigonometric and hyperbolic functions of multiples of the argument (angle) in terms of powers of these functions</td>
<td>33</td>
</tr>
<tr>
<td>1.34</td>
<td>Certain sums of trigonometric and hyperbolic functions</td>
<td>36</td>
</tr>
<tr>
<td>1.35</td>
<td>Sums of powers of trigonometric functions of multiple angles</td>
<td>37</td>
</tr>
<tr>
<td>1.36</td>
<td>Sums of products of trigonometric functions of multiple angles</td>
<td>38</td>
</tr>
<tr>
<td>1.37</td>
<td>Sums of tangents of multiple angles</td>
<td>39</td>
</tr>
<tr>
<td>1.38</td>
<td>Sums leading to hyperbolic tangents and cotangents</td>
<td>39</td>
</tr>
<tr>
<td>1.39</td>
<td>The representation of cosines and sines of multiples of the argument as finite products</td>
<td>41</td>
</tr>
<tr>
<td>1.41</td>
<td>The expansion of trigonometric and hyperbolic functions in power series</td>
<td>42</td>
</tr>
<tr>
<td>1.42</td>
<td>Expansion in series of simple fractions</td>
<td>44</td>
</tr>
<tr>
<td>1.43</td>
<td>Representation in the form of an infinite product</td>
<td>45</td>
</tr>
<tr>
<td>1.44–1.45</td>
<td>Trigonometric (Fourier) series</td>
<td>46</td>
</tr>
<tr>
<td>1.46</td>
<td>Series of products of exponential and trigonometric functions</td>
<td>51</td>
</tr>
<tr>
<td>1.47</td>
<td>Series of hyperbolic functions</td>
<td>51</td>
</tr>
<tr>
<td>1.48</td>
<td>Lobachevskiy’s “Angle of Parallelism” II((x))</td>
<td>51</td>
</tr>
<tr>
<td>1.49</td>
<td>The hyperbolic amplitude (the Gudermannian) (gd\ x)</td>
<td>52</td>
</tr>
<tr>
<td>1.5</td>
<td>The Logarithm</td>
<td>53</td>
</tr>
<tr>
<td>1.51</td>
<td>Series representation</td>
<td>53</td>
</tr>
<tr>
<td>1.52</td>
<td>Series of logarithms (cf. 1.431)</td>
<td>55</td>
</tr>
<tr>
<td>1.6</td>
<td>The Inverse Trigonometric and Hyperbolic Functions</td>
<td>56</td>
</tr>
<tr>
<td>1.61</td>
<td>The domain of definition</td>
<td>56</td>
</tr>
<tr>
<td>1.62–1.63</td>
<td>Functional relations</td>
<td>56</td>
</tr>
<tr>
<td>1.64</td>
<td>Series representations</td>
<td>60</td>
</tr>
</tbody>
</table>

2 Indefinite Integrals of Elementary Functions

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>Introduction</td>
<td>63</td>
</tr>
<tr>
<td>2.00</td>
<td>General remarks</td>
<td>63</td>
</tr>
<tr>
<td>2.01</td>
<td>The basic integrals</td>
<td>64</td>
</tr>
<tr>
<td>2.02</td>
<td>General formulas</td>
<td>65</td>
</tr>
<tr>
<td>2.1</td>
<td>Rational Functions</td>
<td>66</td>
</tr>
<tr>
<td>2.10</td>
<td>General integration rules</td>
<td>66</td>
</tr>
<tr>
<td>2.11–2.13</td>
<td>Forms containing the binomial (a + bx^k)</td>
<td>68</td>
</tr>
<tr>
<td>2.14</td>
<td>Forms containing the binomial (1 \pm x^n)</td>
<td>74</td>
</tr>
<tr>
<td>2.15</td>
<td>Forms containing pairs of binomials: (a + bx) and (\alpha + \beta x)</td>
<td>78</td>
</tr>
<tr>
<td>2.16</td>
<td>Forms containing the trinomial (a + bx^k + cx^{2k})</td>
<td>78</td>
</tr>
<tr>
<td>2.17</td>
<td>Forms containing the quadratic trinomial (a + bx + cx^2) and powers of (x)</td>
<td>79</td>
</tr>
<tr>
<td>2.18</td>
<td>Forms containing the quadratic trinomial (a + bx + cx^2) and the binomial (\alpha + \beta x)</td>
<td>81</td>
</tr>
<tr>
<td>2.2</td>
<td>Algebraic Functions</td>
<td>82</td>
</tr>
<tr>
<td>2.20</td>
<td>Introduction</td>
<td>82</td>
</tr>
<tr>
<td>2.21</td>
<td>Forms containing the binomial (a + bx^k) and (\sqrt{x})</td>
<td>83</td>
</tr>
<tr>
<td>Section</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>2.22–2.23</td>
<td>Forms containing $\sqrt{(a + bx)^k}$</td>
<td>84</td>
</tr>
<tr>
<td>2.24</td>
<td>Forms containing $\sqrt{a + bx}$ and the binomial $\alpha + \beta x$</td>
<td>88</td>
</tr>
<tr>
<td>2.25</td>
<td>Forms containing $\sqrt{a + bx + cx^2}$</td>
<td>92</td>
</tr>
<tr>
<td>2.26</td>
<td>Forms containing $\sqrt{a + bx + cx^2}$ and integral powers of $x$</td>
<td>94</td>
</tr>
<tr>
<td>2.27</td>
<td>Forms containing $\sqrt{a + cx^2}$ and integral powers of $x$</td>
<td>99</td>
</tr>
<tr>
<td>2.28</td>
<td>Forms containing $\sqrt{a + bx + cx^2}$ and first- and second-degree polynomials</td>
<td>103</td>
</tr>
<tr>
<td>2.29</td>
<td>Integrals that can be reduced to elliptic or pseudo-elliptic integrals</td>
<td>104</td>
</tr>
<tr>
<td>2.3</td>
<td>The Exponential Function</td>
<td>106</td>
</tr>
<tr>
<td>2.31</td>
<td>Forms containing $e^{ax}$</td>
<td>106</td>
</tr>
<tr>
<td>2.32</td>
<td>The exponential combined with rational functions of $x$</td>
<td>106</td>
</tr>
<tr>
<td>2.4</td>
<td>Hyperbolic Functions</td>
<td>110</td>
</tr>
<tr>
<td>2.41–2.43</td>
<td>Powers of $\sinh x$, $\cosh x$, $\tanh x$, and $\coth x$</td>
<td>110</td>
</tr>
<tr>
<td>2.44–2.45</td>
<td>Rational functions of hyperbolic functions</td>
<td>125</td>
</tr>
<tr>
<td>2.46</td>
<td>Algebraic functions of hyperbolic functions</td>
<td>132</td>
</tr>
<tr>
<td>2.47</td>
<td>Combinations of hyperbolic functions and powers</td>
<td>139</td>
</tr>
<tr>
<td>2.48</td>
<td>Combinations of hyperbolic functions, exponentials, and powers</td>
<td>148</td>
</tr>
<tr>
<td>2.5–2.6</td>
<td>Trigonometric Functions</td>
<td>151</td>
</tr>
<tr>
<td>2.50</td>
<td>Introduction</td>
<td>151</td>
</tr>
<tr>
<td>2.51–2.52</td>
<td>Powers of trigonometric functions</td>
<td>151</td>
</tr>
<tr>
<td>2.53–2.54</td>
<td>Sines and cosines of multiple angles and of linear and more complicated fun-</td>
<td>161</td>
</tr>
<tr>
<td></td>
<td>tions of the argument</td>
<td></td>
</tr>
<tr>
<td>2.55–2.56</td>
<td>Rational functions of the sine and cosine</td>
<td>171</td>
</tr>
<tr>
<td>2.57</td>
<td>Integrals containing $\sqrt{a \pm b \sin x}$ or $\sqrt{a \pm b \cos x}$</td>
<td>179</td>
</tr>
<tr>
<td>2.58–2.62</td>
<td>Integrals reducible to elliptic and pseudo-elliptic integrals</td>
<td>184</td>
</tr>
<tr>
<td>2.63–2.65</td>
<td>Products of trigonometric functions and powers</td>
<td>214</td>
</tr>
<tr>
<td>2.66</td>
<td>Combinations of trigonometric functions and exponentials</td>
<td>227</td>
</tr>
<tr>
<td>2.67</td>
<td>Combinations of trigonometric and hyperbolic functions</td>
<td>231</td>
</tr>
<tr>
<td>2.7</td>
<td>Logarithms and Inverse-Hyperbolic Functions</td>
<td>237</td>
</tr>
<tr>
<td>2.71</td>
<td>The logarithm</td>
<td>237</td>
</tr>
<tr>
<td>2.72–2.73</td>
<td>Combinations of logarithms and algebraic functions</td>
<td>238</td>
</tr>
<tr>
<td>2.74</td>
<td>Inverse hyperbolic functions</td>
<td>240</td>
</tr>
<tr>
<td>2.8</td>
<td>Inverse Trigonometric Functions</td>
<td>241</td>
</tr>
<tr>
<td>2.81</td>
<td>Arcsines and arccosines</td>
<td>241</td>
</tr>
<tr>
<td>2.82</td>
<td>The arcsecant, the arccosecant, the arctangent, and the arccotangent</td>
<td>242</td>
</tr>
<tr>
<td>2.83</td>
<td>Combinations of arcsine or arccosine and algebraic functions</td>
<td>242</td>
</tr>
<tr>
<td>2.84</td>
<td>Combinations of the arcsecant and arccosecant with powers of $x$</td>
<td>244</td>
</tr>
<tr>
<td>2.85</td>
<td>Combinations of the arctangent and arccotangent with algebraic functions</td>
<td>244</td>
</tr>
</tbody>
</table>

### Definite Integrals of Elementary Functions

<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>Introduction</td>
<td>247</td>
</tr>
<tr>
<td>3.01</td>
<td>Theorems of a general nature</td>
<td>247</td>
</tr>
<tr>
<td>3.02</td>
<td>Change of variable in a definite integral</td>
<td>248</td>
</tr>
<tr>
<td>3.03</td>
<td>General formulas</td>
<td>249</td>
</tr>
<tr>
<td>3.04</td>
<td>Improper integrals</td>
<td>251</td>
</tr>
<tr>
<td>3.05</td>
<td>The principal values of improper integrals</td>
<td>252</td>
</tr>
<tr>
<td>3.1–3.2</td>
<td>Power and Algebraic Functions</td>
<td>253</td>
</tr>
<tr>
<td>3.11</td>
<td>Rational functions</td>
<td>253</td>
</tr>
<tr>
<td>Section</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>---------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>3.12</td>
<td>Products of rational functions and expressions that can be reduced to square roots of first- and second-degree polynomials</td>
<td>254</td>
</tr>
<tr>
<td>3.13–3.17</td>
<td>Expressions that can be reduced to square roots of third- and fourth-degree polynomials and their products with rational functions</td>
<td>254</td>
</tr>
<tr>
<td>3.18</td>
<td>Expressions that can be reduced to fourth roots of second-degree polynomials and their products with rational functions</td>
<td>313</td>
</tr>
<tr>
<td>3.19–3.23</td>
<td>Combinations of powers of ( x ) and powers of binomials of the form ( (\alpha + \beta x) )</td>
<td>315</td>
</tr>
<tr>
<td>3.24–3.27</td>
<td>Powers of ( x ), of binomials of the form ( \alpha + \beta x^p ) and of polynomials in ( x )</td>
<td>322</td>
</tr>
<tr>
<td>3.3–3.4</td>
<td>Exponential Functions</td>
<td>334</td>
</tr>
<tr>
<td>3.31</td>
<td>Exponential functions</td>
<td>334</td>
</tr>
<tr>
<td>3.32–3.34</td>
<td>Combinations of exponentials and rational functions</td>
<td>340</td>
</tr>
<tr>
<td>3.35</td>
<td>Combinations of exponentials and algebraic functions</td>
<td>344</td>
</tr>
<tr>
<td>3.36–3.37</td>
<td>Combinations of exponential and algebraic functions</td>
<td>346</td>
</tr>
<tr>
<td>3.38–3.39</td>
<td>Combinations of exponentials and arbitrary powers</td>
<td>346</td>
</tr>
<tr>
<td>3.41–3.44</td>
<td>Combinations of rational functions of powers and exponentials</td>
<td>353</td>
</tr>
<tr>
<td>3.45</td>
<td>Combinations of powers and algebraic functions of exponentials</td>
<td>363</td>
</tr>
<tr>
<td>3.46–3.48</td>
<td>Combinations of exponentials of more complicated arguments and powers</td>
<td>364</td>
</tr>
<tr>
<td>3.5</td>
<td>Hyperbolic Functions</td>
<td>371</td>
</tr>
<tr>
<td>3.51</td>
<td>Hyperbolic functions</td>
<td>371</td>
</tr>
<tr>
<td>3.52–3.53</td>
<td>Combinations of hyperbolic functions and algebraic functions</td>
<td>375</td>
</tr>
<tr>
<td>3.54</td>
<td>Combinations of hyperbolic functions and exponentials</td>
<td>382</td>
</tr>
<tr>
<td>3.55–3.56</td>
<td>Combinations of hyperbolic functions, exponentials, and powers</td>
<td>386</td>
</tr>
<tr>
<td>3.6–4.1</td>
<td>Trigonometric Functions</td>
<td>390</td>
</tr>
<tr>
<td>3.61</td>
<td>Rational functions of sines and cosines and trigonometric functions of multiple angles</td>
<td>390</td>
</tr>
<tr>
<td>3.62</td>
<td>Powers of trigonometric functions</td>
<td>395</td>
</tr>
<tr>
<td>3.63</td>
<td>Powers of trigonometric functions and trigonometric functions of linear functions</td>
<td>397</td>
</tr>
<tr>
<td>3.64–3.65</td>
<td>Powers and rational functions of trigonometric functions</td>
<td>401</td>
</tr>
<tr>
<td>3.66</td>
<td>Forms containing powers of linear functions of trigonometric functions</td>
<td>405</td>
</tr>
<tr>
<td>3.67</td>
<td>Square roots of expressions containing trigonometric functions</td>
<td>408</td>
</tr>
<tr>
<td>3.68</td>
<td>Various forms of powers of trigonometric functions</td>
<td>411</td>
</tr>
<tr>
<td>3.69–3.71</td>
<td>Trigonometric functions of more complicated arguments</td>
<td>415</td>
</tr>
<tr>
<td>3.72–3.74</td>
<td>Combinations of trigonometric and rational functions</td>
<td>423</td>
</tr>
<tr>
<td>3.75</td>
<td>Combinations of trigonometric and algebraic functions</td>
<td>434</td>
</tr>
<tr>
<td>3.76–3.77</td>
<td>Combinations of trigonometric functions and powers</td>
<td>436</td>
</tr>
<tr>
<td>3.78–3.81</td>
<td>Rational functions of ( x ) and of trigonometric functions</td>
<td>447</td>
</tr>
<tr>
<td>3.82–3.83</td>
<td>Powers of trigonometric functions combined with other powers</td>
<td>459</td>
</tr>
<tr>
<td>3.84</td>
<td>Integrals containing ( \sqrt{1 - k^2 \sin^2 x} ), ( \sqrt{1 - k^2 \cos^2 x} ), and similar expressions</td>
<td>472</td>
</tr>
<tr>
<td>3.85–3.88</td>
<td>Trigonometric functions of more complicated arguments combined with powers</td>
<td>475</td>
</tr>
<tr>
<td>3.89–3.91</td>
<td>Trigonometric functions and exponentials</td>
<td>485</td>
</tr>
<tr>
<td>3.92</td>
<td>Trigonometric functions of more complicated arguments combined with exponentials</td>
<td>493</td>
</tr>
<tr>
<td>3.93</td>
<td>Trigonometric and exponential functions of trigonometric functions</td>
<td>495</td>
</tr>
<tr>
<td>3.94–3.97</td>
<td>Combinations involving trigonometric functions, exponentials, and powers</td>
<td>497</td>
</tr>
<tr>
<td>3.98–3.99</td>
<td>Combinations of trigonometric and hyperbolic functions</td>
<td>509</td>
</tr>
<tr>
<td>4.11–4.12</td>
<td>Combinations involving trigonometric and hyperbolic functions and powers</td>
<td>516</td>
</tr>
<tr>
<td>4.13</td>
<td>Combinations of trigonometric and hyperbolic functions and exponentials</td>
<td>522</td>
</tr>
</tbody>
</table>
4.14 Combinations of trigonometric and hyperbolic functions, exponentials, and powers 525
4.2–4.4 Logarithmic Functions ........................................ 527
4.21 Logarithmic functions ........................................... 527
4.22 Logarithms of more complicated arguments .................. 529
4.23 Combinations of logarithms and rational functions ........... 535
4.24 Combinations of logarithms and algebraic functions .......... 538
4.25 Combinations of logarithms and powers ...................... 540
4.26–4.27 Combinations involving powers of the logarithm and other powers 542
4.28 Combinations of rational functions of $\ln x$ and powers .......... 553
4.29–4.32 Combinations of logarithmic functions of more complicated arguments and powers 555
4.33–4.34 Combinations of logarithms and exponentials ............ 571
4.35–4.36 Combinations of logarithms, exponentials, and powers ...... 573
4.37 Combinations of logarithms and hyperbolic functions .......... 578
4.38–4.41 Logarithms and trigonometric functions .................. 581
4.42–4.43 Combinations of logarithms, trigonometric functions, and powers ...... 594
4.44 Combinations of logarithms, trigonometric functions, and exponentials .... 599
4.5 Inverse Trigonometric Functions ................................... 599
4.51 Inverse trigonometric functions ................................... 599
4.52 Combinations of arcsines, arccosines, and powers ............. 600
4.53–4.54 Combinations of arctangents, arccotangents, and powers .... 601
4.55 Combinations of inverse trigonometric functions and exponentials .... 605
4.56 A combination of the arctangent and a hyperbolic function ...... 605
4.57 Combinations of inverse and direct trigonometric functions ....... 605
4.58 A combination involving an inverse and a direct trigonometric function and a power ................. 607
4.59 Combinations of inverse trigonometric functions and logarithms .... 607
4.6 Multiple Integrals .................................................. 607
4.60 Change of variables in multiple integrals ...................... 607
4.61 Change of the order of integration and change of variables ...... 608
4.62 Double and triple integrals with constant limits ............... 610
4.63–4.64 Multiple integrals ........................................... 612
5 Indefinite Integrals of Special Functions 619
5.1 Elliptic Integrals and Functions ................................... 619
5.11 Complete elliptic integrals ..................................... 619
5.12 Elliptic integrals ................................................. 621
5.13 Jacobian elliptic functions ...................................... 623
5.14 Weierstrass elliptic functions ................................... 626
5.2 The Exponential Integral Function .................................. 627
5.21 The exponential integral function ................................ 627
5.22 Combinations of the exponential integral function and powers .... 627
5.23 Combinations of the exponential integral and the exponential .... 628
5.3 The Sine Integral and the Cosine Integral .......................... 628
5.4 The Probability Integral and Fresnel Integrals .................... 629
5.5 Bessel Functions .................................................. 629
6–7  Definite Integrals of Special Functions

6.1  Elliptic Integrals and Functions ........................................ 631
    6.11  Forms containing $F(x, k)$ ........................................... 631
    6.12  Forms containing $E(x, k)$ ........................................... 632
    6.13  Integration of elliptic integrals with respect to the modulus .... 632
    6.14–6.15  Complete elliptic integrals ................................... 632
    6.16  The theta function ................................................... 633
    6.17  Generalized elliptic integrals ....................................... 635

6.2–6.3  The Exponential Integral Function and Functions Generated by It .... 636
    6.21  The logarithm integral ................................................ 636
    6.22–6.23  The exponential integral function ............................ 638
    6.24–6.26  The sine integral and cosine integral functions .............. 639
    6.27  The hyperbolic sine integral and hyperbolic cosine integral functions 644
    6.28–6.31  The probability integral ....................................... 645

6.32  Fresnel integrals .......................................................... 649

6.4  The Gamma Function and Functions Generated by It ...................... 650
    6.41  The gamma function ................................................... 650
    6.42  Combinations of the gamma function, the exponential, and powers 652
    6.43  Combinations of the gamma function and trigonometric functions . 655
    6.44  The logarithm of the gamma function* ................................ 656
    6.45  The incomplete gamma function .................................... 657
    6.46–6.47  The function $\psi(x)$ ............................................. 658

6.5–6.7  Bessel Functions ....................................................... 659
    6.51  Bessel functions ....................................................... 659
    6.52  Bessel functions combined with $x$ and $x^2$ ......................... 664
    6.53–6.54  Combinations of Bessel functions and rational functions .... 670
    6.55  Combinations of Bessel functions and algebraic functions ........ 674
    6.56–6.58  Combinations of Bessel functions and powers ................ 675
    6.59  Combinations of powers and Bessel functions of more complicated arguments 689
    6.61  Combinations of Bessel functions and exponentials ................ 694
    6.62–6.63  Combinations of Bessel functions, exponentials, and powers 699
    6.64  Combinations of Bessel functions of more complicated arguments, exponentials, and powers 708
    6.65  Combinations of Bessel and exponential functions of more complicated arguments and powers 711
    6.66  Combinations of Bessel, hyperbolic, and exponential functions .... 713
    6.67–6.68  Combinations of Bessel and trigonometric functions .......... 717
    6.69–6.74  Combinations of Bessel and trigonometric functions and powers 727
    6.75  Combinations of Bessel, trigonometric, and exponential functions and powers 742
    6.76  Combinations of Bessel, trigonometric, and hyperbolic functions .. 747
    6.77  Combinations of Bessel functions and the logarithm, or arctangent 747
    6.78  Combinations of Bessel and other special functions ................ 748
    6.79  Integration of Bessel functions with respect to the order .......... 749

6.8  Functions Generated by Bessel Functions ................................ 753
    6.81  Struve functions ..................................................... 753
    6.82  Combinations of Struve functions, exponentials, and powers ........ 754
    6.83  Combinations of Struve and trigonometric functions ............... 755
6.84–6.85 Combinations of Struve and Bessel functions ................................................. 756
6.86 Lommel functions ........................................................................................................... 760
6.87 Thomson functions ......................................................................................................... 761
6.9 Mathieu Functions ............................................................................................................. 763
  6.91 Mathieu functions ............................................................................................................ 763
  6.92 Combinations of Mathieu, hyperbolic, and trigonometric functions .......................... 763
  6.93 Combinations of Mathieu and Bessel functions ............................................................ 767
  6.94 Relationships between eigenfunctions of the Helmholtz equation in different coordinate systems ......................................................... 767
7.1–7.2 Associated Legendre Functions ............................................................................... 769
  7.11 Associated Legendre functions ..................................................................................... 769
  7.12–7.13 Combinations of associated Legendre functions and powers ............................ 770
  7.14 Combinations of associated Legendre functions, exponentials, and powers ............ 776
  7.15 Combinations of associated Legendre and hyperbolic functions ............................... 778
  7.16 Combinations of associated Legendre functions, powers, and trigonometric functions ................................................................................................................. 779
  7.17 A combination of an associated Legendre function and the probability integral .......... 781
  7.18 Combinations of associated Legendre and Bessel functions ......................................... 782
  7.19 Combinations of associated Legendre functions and functions generated by Bessel functions ........................................................................................................................................................................... 787
  7.21 Integration of associated Legendre functions with respect to the order ....................... 788
  7.22 Combinations of Legendre polynomials, rational functions, and algebraic functions .................................................................................................................. 789
  7.23 Combinations of Legendre polynomials and powers .................................................... 791
  7.24 Combinations of Legendre polynomials and other elementary functions .................... 792
  7.25 Combinations of Legendre polynomials and Bessel functions .................................... 794
7.3–7.4 Orthogonal Polynomials ......................................................................................... 795
  7.31 Combinations of Gegenbauer polynomials $C_\nu^\nu_n(x)$ and powers ......................... 795
  7.32 Combinations of Gegenbauer polynomials $C_\nu^\nu_n(x)$ and elementary functions ........ 797
  7.325* Complete System of Orthogonal Step Functions .................................................... 798
  7.33 Combinations of the polynomials $C_\nu^\nu_n(x)$ and Bessel functions; Integration of Gegenbauer functions with respect to the index ................................................................................................................. 798
  7.34 Combinations of Chebyshev polynomials and powers .................................................. 800
  7.35 Combinations of Chebyshev polynomials and elementary functions ........................... 802
  7.36 Combinations of Chebyshev polynomials and Bessel functions ................................... 803
  7.37–7.38 Hermite polynomials ............................................................................................ 803
  7.39 Jacobi polynomials ........................................................................................................ 806
  7.41–7.42 Laguerre polynomials ........................................................................................... 808
7.5 Hypergeometric Functions ............................................................................................... 812
  7.51 Combinations of hypergeometric functions and powers ............................................ 812
  7.52 Combinations of hypergeometric functions and exponentials .................................... 814
  7.53 Hypergeometric and trigonometric functions ............................................................... 817
  7.54 Combinations of hypergeometric and Bessel functions .............................................. 817
7.6 Confluent Hypergeometric Functions ........................................................................... 820
  7.61 Combinations of confluent hypergeometric functions and powers ........................... 820
  7.62–7.63 Combinations of confluent hypergeometric functions and exponentials .......... 822
  7.64 Combinations of confluent hypergeometric and trigonometric functions ................. 829
  7.65 Combinations of confluent hypergeometric functions and Bessel functions ............ 830
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.66</td>
<td>Combinations of confluent hypergeometric functions, Bessel functions, and powers</td>
<td>831</td>
</tr>
<tr>
<td>7.67</td>
<td>Combinations of confluent hypergeometric functions, Bessel functions, exponentials, and powers</td>
<td>834</td>
</tr>
<tr>
<td>7.68</td>
<td>Combinations of confluent hypergeometric functions and other special functions</td>
<td>839</td>
</tr>
<tr>
<td>7.69</td>
<td>Integration of confluent hypergeometric functions with respect to the index</td>
<td>841</td>
</tr>
<tr>
<td>7.7</td>
<td>Parabolic Cylinder Functions</td>
<td>841</td>
</tr>
<tr>
<td>7.71</td>
<td>Parabolic cylinder functions</td>
<td>841</td>
</tr>
<tr>
<td>7.72</td>
<td>Combinations of parabolic cylinder functions, powers, and exponentials</td>
<td>842</td>
</tr>
<tr>
<td>7.73</td>
<td>Combinations of parabolic cylinder and hyperbolic functions</td>
<td>843</td>
</tr>
<tr>
<td>7.74</td>
<td>Combinations of parabolic cylinder and trigonometric functions</td>
<td>844</td>
</tr>
<tr>
<td>7.75</td>
<td>Combinations of parabolic cylinder and Bessel functions</td>
<td>845</td>
</tr>
<tr>
<td>7.76</td>
<td>Combinations of parabolic cylinder functions and confluent hypergeometric functions</td>
<td>849</td>
</tr>
<tr>
<td>7.77</td>
<td>Integration of a parabolic cylinder function with respect to the index</td>
<td>849</td>
</tr>
<tr>
<td>7.8</td>
<td>Meijer’s and MacRobert’s Functions (G and E)</td>
<td>850</td>
</tr>
<tr>
<td>7.81</td>
<td>Combinations of the functions G and E and the elementary functions</td>
<td>850</td>
</tr>
<tr>
<td>7.82</td>
<td>Combinations of the functions G and E and Bessel functions</td>
<td>854</td>
</tr>
<tr>
<td>7.83</td>
<td>Combinations of the functions G and E and other special functions</td>
<td>856</td>
</tr>
</tbody>
</table>

### 8–9 Special Functions

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.1</td>
<td>Elliptic Integrals and Functions</td>
<td>859</td>
</tr>
<tr>
<td>8.11</td>
<td>Elliptic integrals</td>
<td>859</td>
</tr>
<tr>
<td>8.12</td>
<td>Functional relations between elliptic integrals</td>
<td>863</td>
</tr>
<tr>
<td>8.13</td>
<td>Elliptic functions</td>
<td>865</td>
</tr>
<tr>
<td>8.14</td>
<td>Jacobian elliptic functions</td>
<td>866</td>
</tr>
<tr>
<td>8.15</td>
<td>Properties of Jacobian elliptic functions and functional relationships between them</td>
<td>870</td>
</tr>
<tr>
<td>8.16</td>
<td>The Weierstrass function ( \wp(u) )</td>
<td>873</td>
</tr>
<tr>
<td>8.17</td>
<td>The functions ( \zeta(u) ) and ( \sigma(u) )</td>
<td>876</td>
</tr>
<tr>
<td>8.18–8.19</td>
<td>Theta functions</td>
<td>877</td>
</tr>
<tr>
<td>8.2</td>
<td>The Exponential Integral Function and Functions Generated by It</td>
<td>883</td>
</tr>
<tr>
<td>8.21</td>
<td>The exponential integral function ( \text{Ei}(x) )</td>
<td>883</td>
</tr>
<tr>
<td>8.22</td>
<td>The hyperbolic sine integral ( \text{shi} x ) and the hyperbolic cosine integral ( \text{chi} x )</td>
<td>886</td>
</tr>
<tr>
<td>8.23</td>
<td>The sine integral and the cosine integral: ( \text{si} x ) and ( \text{ci} x )</td>
<td>886</td>
</tr>
<tr>
<td>8.24</td>
<td>The logarithmic integral ( \text{li}(x) )</td>
<td>887</td>
</tr>
<tr>
<td>8.25</td>
<td>The probability integral ( \Phi(x) ), the Fresnel integrals ( S(x) ) and ( C(x) ), the error function ( \text{erf}(x) ), and the complementary error function ( \text{erfc}(x) )</td>
<td>887</td>
</tr>
<tr>
<td>8.26</td>
<td>Lobachevsky’s function ( L(x) )</td>
<td>891</td>
</tr>
<tr>
<td>8.3</td>
<td>Euler’s Integrals of the First and Second Kinds</td>
<td>892</td>
</tr>
<tr>
<td>8.31</td>
<td>The gamma function (Euler’s integral of the second kind): ( \Gamma(z) )</td>
<td>892</td>
</tr>
<tr>
<td>8.32</td>
<td>Representation of the gamma function as series and products</td>
<td>894</td>
</tr>
<tr>
<td>8.33</td>
<td>Functional relations involving the gamma function</td>
<td>895</td>
</tr>
<tr>
<td>8.34</td>
<td>The logarithm of the gamma function</td>
<td>898</td>
</tr>
<tr>
<td>8.35</td>
<td>The incomplete gamma function</td>
<td>899</td>
</tr>
<tr>
<td>8.36</td>
<td>The psi function ( \psi(x) )</td>
<td>902</td>
</tr>
<tr>
<td>8.37</td>
<td>The function ( \beta(x) )</td>
<td>906</td>
</tr>
<tr>
<td>8.38</td>
<td>The beta function (Euler’s integral of the first kind): ( B(x,y) )</td>
<td>908</td>
</tr>
<tr>
<td>8.39</td>
<td>The incomplete beta function ( B_x(p,q) )</td>
<td>910</td>
</tr>
<tr>
<td>8.4–8.5</td>
<td>Bessel Functions and Functions Associated with Them</td>
<td>910</td>
</tr>
</tbody>
</table>
8.40 Definitions ............................................. 910
8.41 Integral representations of the functions \(J_\nu(z)\) and \(N_\nu(z)\) ............... 912
8.42 Integral representations of the functions \(H^{(1)}_\nu(z)\) and \(H^{(2)}_\nu(z)\) ............. 914
8.43 Integral representations of the functions \(I_\nu(z)\) and \(K_\nu(z)\) ..................... 916
8.44 Series representation ...................................... 918
8.45 Asymptotic expansions of Bessel functions ............................................. 920
8.46 Bessel functions of order equal to an integer plus one-half ...................... 924
8.47–8.48 Functional relations .................................... 926
8.49 Differential equations leading to Bessel functions .................................... 931
8.50 Series of Bessel functions .................................... 933
8.51–8.52 Expansion in products of Bessel functions .................................... 940
8.53 The zeros of Bessel functions ............................................. 941
8.54 Struve functions ............................................. 942
8.55 Thomson functions and their generalizations .................................... 944
8.56 Lommel functions ............................................. 945
8.57 Anger and Weber functions \(J_\nu(z)\) and \(E_\nu(z)\) .................................... 948
8.58 Neumann’s and Schlafli’s polynomials: \(O_n(z)\) and \(S_n(z)\) ..................... 949
8.6 Mathieu Functions ............................................. 950
8.60 Mathieu’s equation ............................................. 950
8.61 Periodic Mathieu functions ............................................. 951
8.62 Recursion relations for the coefficients \(A_{2r}^{(2n)}\), \(A_{2r+1}^{(2n+1)}\), \(B_{2r+1}^{(2n+1)}\), \(B_{2r+2}^{(2n+2)}\) .................................... 951
8.63 Mathieu functions with a purely imaginary argument .................................... 952
8.64 Non-periodic solutions of Mathieu’s equation .................................... 953
8.65 Mathieu functions for negative \(q\) .................................... 953
8.66 Representation of Mathieu functions as series of Bessel functions ................ 954
8.67 The general theory ............................................. 957
8.7–8.8 Associated Legendre Functions ............................................. 958
8.70 Introduction ............................................. 958
8.71 Integral representations ............................................. 960
8.72 Asymptotic series for large values of \(|\nu|\) .................................... 962
8.73–8.74 Functional relations ............................................. 964
8.75 Special cases and particular values ............................................. 968
8.76 Derivatives with respect to the order ............................................. 969
8.77 Series representation ............................................. 970
8.78 The zeros of associated Legendre functions ............................................. 972
8.79 Series of associated Legendre functions ............................................. 972
8.81 Associated Legendre functions with integer indices .................................... 974
8.82–8.83 Legendre functions ............................................. 975
8.84 Conical functions ............................................. 980
8.85 Toroidal functions ............................................. 981
8.9 Orthogonal Polynomials ............................................. 982
8.90 Introduction ............................................. 982
8.91 Legendre polynomials ............................................. 983
8.92 Series of Legendre polynomials ............................................. 988
8.93 Gegenbauer polynomials \(C_n^\lambda(t)\) .................................... 990
8.94 The Chebyshev polynomials \(T_n(x)\) and \(U_n(x)\) .................................... 993
8.95 The Hermite polynomials $H_n(x)$ ................................................. 996
8.96 Jacobi’s polynomials ................................................................. 998
8.97 The Laguerre polynomials .......................................................... 1000
9.1 Hypergeometric Functions ......................................................... 1005
  9.10 Definition ................................................................. 1005
  9.11 Integral representations .......................................................... 1005
  9.12 Representation of elementary functions in terms of a hypergeometric functions 1006
  9.13 Transformation formulas and the analytic continuation of functions defined by hypergeometric series ........................................ 1008
  9.14 A generalized hypergeometric series ........................................ 1010
  9.15 The hypergeometric differential equation .................................. 1010
  9.16 Riemann’s differential equation ................................................ 1014
  9.17 Representing the solutions to certain second-order differential equations using a Riemann scheme ................................................. 1017
  9.18 Hypergeometric functions of two variables .................................. 1018
  9.19 A hypergeometric function of several variables .............................. 1022
9.2 Confluent Hypergeometric Functions ......................................... 1022
  9.20 Introduction ................................................................. 1022
  9.21 The functions $\Phi(\alpha, \gamma; z)$ and $\Psi(\alpha, \gamma; z)$ ..................... 1023
  9.22-9.23 The Whittaker functions $M_{\lambda, \mu}(z)$ and $W_{\lambda, \mu}(z)$ .... 1024
  9.24-9.25 Parabolic cylinder functions $D_p(z)$ ........................................ 1028
  9.26 Confluent hypergeometric series of two variables ......................... 1031
9.3 Meijer’s $G$-Function ............................................................. 1032
  9.30 Definition ................................................................. 1032
  9.31 Functional relations ............................................................. 1033
  9.32 A differential equation for the $G$-function ................................ 1034
  9.33 Series of $G$-functions .......................................................... 1034
  9.34 Connections with other special functions .................................... 1034
9.4 MacRobert’s $E$-Function ........................................................ 1035
  9.41 Representation by means of multiple integrals ................................ 1035
  9.42 Functional relations ............................................................. 1035
  9.5 Riemann’s Zeta Functions $\zeta(z, q)$ and $\zeta(z)$, and the Functions $\Phi(z, s, v)$ and $\xi(s)$ 1036
    9.51 Definition and integral representations ..................................... 1036
    9.52 Representation as a series or as an infinite product .................... 1037
    9.53 Functional relations .......................................................... 1037
    9.54 Singular points and zeros .................................................... 1038
    9.55 The Lerch function $\Phi(z, s, v)$ .............................................. 1039
    9.56 The function $\xi(s)$ ............................................................. 1040
9.6 Bernoulli Numbers and Polynomials, Euler Numbers ...................... 1040
    9.61 Bernoulli numbers .............................................................. 1040
    9.62 Bernoulli polynomials .......................................................... 1041
    9.63 Euler numbers ................................................................. 1043
    9.64 The functions $\nu(x)$, $\nu(x, \alpha)$, $\mu(x, \beta)$, $\mu(x, \beta, \alpha)$, and $\lambda(x, y)$ 1043
    9.65 Euler polynomials ............................................................. 1044
9.7 Constants ................................................................................. 1045
  9.71 Bernoulli numbers .............................................................. 1045
  9.72 Euler numbers ................................................................. 1045
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.73</td>
<td>Euler’s and Catalan’s constants</td>
<td>1046</td>
</tr>
<tr>
<td>9.74</td>
<td>Stirling numbers</td>
<td>1046</td>
</tr>
<tr>
<td>10.1–10.8</td>
<td>Vectors, Vector Operators, and Integral Theorems</td>
<td>1049</td>
</tr>
<tr>
<td>10.1</td>
<td>Products of vectors</td>
<td>1049</td>
</tr>
<tr>
<td>10.12</td>
<td>Properties of scalar product</td>
<td>1049</td>
</tr>
<tr>
<td>10.13</td>
<td>Properties of vector product</td>
<td>1049</td>
</tr>
<tr>
<td>10.14</td>
<td>Differentiation of vectors</td>
<td>1050</td>
</tr>
<tr>
<td>10.21</td>
<td>Operators grad, div, and curl</td>
<td>1050</td>
</tr>
<tr>
<td>10.31</td>
<td>Properties of the operator ( \nabla )</td>
<td>1051</td>
</tr>
<tr>
<td>10.41</td>
<td>Solenoidal fields</td>
<td>1052</td>
</tr>
<tr>
<td>10.51–10.61</td>
<td>Orthogonal curvilinear coordinates</td>
<td>1052</td>
</tr>
<tr>
<td>10.71–10.72</td>
<td>Vector integral theorems</td>
<td>1055</td>
</tr>
<tr>
<td>10.81</td>
<td>Integral rate of change theorems</td>
<td>1057</td>
</tr>
<tr>
<td>11.1–11.3</td>
<td>General Algebraic Inequalities</td>
<td>1059</td>
</tr>
<tr>
<td>11.11</td>
<td>Algebraic inequalities involving real numbers</td>
<td>1059</td>
</tr>
<tr>
<td>11.21</td>
<td>Algebraic inequalities involving complex numbers</td>
<td>1060</td>
</tr>
<tr>
<td>11.31</td>
<td>Inequalities for sets of complex numbers</td>
<td>1061</td>
</tr>
<tr>
<td>12.11</td>
<td>Mean Value Theorems</td>
<td>1063</td>
</tr>
<tr>
<td>12.111</td>
<td>First mean value theorem</td>
<td>1063</td>
</tr>
<tr>
<td>12.112</td>
<td>Second mean value theorem</td>
<td>1063</td>
</tr>
<tr>
<td>12.113</td>
<td>First mean value theorem for infinite integrals</td>
<td>1063</td>
</tr>
<tr>
<td>12.114</td>
<td>Second mean value theorem for infinite integrals</td>
<td>1064</td>
</tr>
<tr>
<td>12.21</td>
<td>Differentiation of Definite Integral Containing a Parameter</td>
<td>1064</td>
</tr>
<tr>
<td>12.211</td>
<td>Differentiation when limits are finite</td>
<td>1064</td>
</tr>
<tr>
<td>12.212</td>
<td>Differentiation when a limit is infinite</td>
<td>1064</td>
</tr>
<tr>
<td>12.31</td>
<td>Integral Inequalities</td>
<td>1064</td>
</tr>
<tr>
<td>12.311</td>
<td>Cauchy-Schwarz-Buniakowsky inequality for integrals</td>
<td>1064</td>
</tr>
<tr>
<td>12.312</td>
<td>Hölder’s inequality for integrals</td>
<td>1064</td>
</tr>
<tr>
<td>12.313</td>
<td>Minkowski’s inequality for integrals</td>
<td>1065</td>
</tr>
<tr>
<td>12.314</td>
<td>Chebyshev’s inequality for integrals</td>
<td>1065</td>
</tr>
<tr>
<td>12.315</td>
<td>Young’s inequality for integrals</td>
<td>1065</td>
</tr>
<tr>
<td>12.316</td>
<td>Steffensen’s inequality for integrals</td>
<td>1065</td>
</tr>
<tr>
<td>12.317</td>
<td>Gram’s inequality for integrals</td>
<td>1065</td>
</tr>
<tr>
<td>12.318</td>
<td>Ostrowski’s inequality for integrals</td>
<td>1066</td>
</tr>
<tr>
<td>12.41</td>
<td>Convexity and Jensen’s Inequality</td>
<td>1066</td>
</tr>
<tr>
<td>12.411</td>
<td>Jensen’s inequality</td>
<td>1066</td>
</tr>
<tr>
<td>12.412</td>
<td>Carleman’s inequality for integrals</td>
<td>1066</td>
</tr>
<tr>
<td>12.51</td>
<td>Fourier Series and Related Inequalities</td>
<td>1066</td>
</tr>
<tr>
<td>12.511</td>
<td>Riemann-Lebesgue lemma</td>
<td>1067</td>
</tr>
<tr>
<td>12.512</td>
<td>Dirichlet lemma</td>
<td>1067</td>
</tr>
<tr>
<td>12.513</td>
<td>Parseval’s theorem for trigonometric Fourier series</td>
<td>1067</td>
</tr>
<tr>
<td>12.514</td>
<td>Integral representation of the ( n )th partial sum</td>
<td>1067</td>
</tr>
</tbody>
</table>
12.515 Generalized Fourier series .................................................. 1067
12.516 Bessel’s inequality for generalized Fourier series .................... 1068
12.517 Parseval’s theorem for generalized Fourier series ................. 1068

13 Matrices and Related Results .................................................. 1069
13.11–13.12 Special Matrices ....................................................... 1069
13.111 Diagonal matrix ............................................................... 1069
13.112 Identity matrix and null matrix .......................................... 1069
13.113 Reducible and irreducible matrices ..................................... 1069
13.114 Equivalent matrices ........................................................ 1069
13.115 Transpose of a matrix ......................................................... 1069
13.116 Adjoint matrix ............................................................... 1070
13.117 Inverse matrix ................................................................. 1070
13.118 Trace of a matrix ............................................................ 1070
13.119 Symmetric matrix ............................................................ 1070
13.120 Skew-symmetric matrix ..................................................... 1070
13.121 Triangular matrices .......................................................... 1070
13.122 Orthogonal matrices ........................................................ 1070
13.123 Hermitian transpose of a matrix ......................................... 1070
13.124 Hermitian matrix ............................................................. 1070
13.125 Unitary matrix ............................................................... 1071
13.126 Eigenvalues and eigenvectors .......................................... 1071
13.127 Nilpotent matrix .............................................................. 1071
13.128 Idempotent matrix .......................................................... 1071
13.129 Positive definite ............................................................. 1071
13.130 Non-negative definite ...................................................... 1071
13.131 Diagonally dominant ....................................................... 1071
13.21 Quadratic Forms ............................................................. 1071
13.211 Sylvester’s law of inertia ................................................. 1072
13.212 Rank ....................................................................... 1072
13.213 Signature ................................................................. 1072
13.214 Positive definite and semidefinite quadratic form ............... 1072
13.215 Basic theorems on quadratic forms .................................. 1072
13.31 Differentiation of Matrices ................................................ 1073
13.41 The Matrix Exponential .................................................. 1074

14 Determinants ................................................................. 1075
14.11 Expansion of Second- and Third-Order Determinants ............. 1075
14.12 Basic Properties ............................................................. 1075
14.13 Minors and Cofactors of a Determinant ................................ 1075
14.14 Principal Minors ............................................................ 1076
14.15* Laplace Expansion of a Determinant .................................. 1076
14.16 Jacobi’s Theorem ........................................................... 1076
14.17 Hadamard’s Theorem ....................................................... 1077
14.18 Hadamard’s Inequality .................................................... 1077
14.21 Cramer’s Rule .............................................................. 1077
14.31 Some Special Determinants .............................................. 1078
## CONTENTS

14.311 Vandermonde's determinant (alternant) ........................................... 1078
14.312 Circulants ......................................................................................... 1078
14.313 Jacobian determinant ........................................................................ 1078
14.314 Hessian determinants ........................................................................ 1079
14.315 Wronskian determinants .................................................................... 1079
14.316 Properties ......................................................................................... 1079
14.317 Gram-Kowalewski theorem on linear dependence ............................. 1080

### 15 Norms

15.1–15.9 Vector Norms .............................................................................. 1081
15.11 General Properties .............................................................................. 1081
15.21 Principal Vector Norms ...................................................................... 1082
15.211 The norm $||x||_1$ ............................................................................. 1082
15.212 The norm $||x||_2$ (Euclidean or $L_2$ norm) ..................................... 1082
15.213 The norm $||x||_\infty$ ........................................................................ 1082
15.31 Matrix Norms .................................................................................... 1082
15.311 General properties ............................................................................ 1082
15.312 Induced norms .................................................................................. 1082
15.313 Natural norm of unit matrix ............................................................... 1082
15.41 Principal Natural Norms .................................................................... 1082
15.411 Maximum absolute column sum norm ............................................ 1082
15.412 Spectral norm ................................................................................... 1082
15.413 Maximum absolute row sum norm .................................................. 1083
15.51 Spectral Radius of a Square Matrix .................................................... 1083
15.511 Inequalities concerning matrix norms and the spectral radius ............ 1083
15.512 Deductions from Gerschgorin's theorem (see 15.814) ...................... 1083
15.61 Inequalities Involving Eigenvalues of Matrices ................................... 1084
15.611 Cayley-Hamilton theorem ................................................................. 1084
15.612 Corollaries ......................................................................................... 1084
15.71 Inequalities for the Characteristic Polynomial ..................................... 1084
15.711 Named and unnamed inequalities ...................................................... 1085
15.712 Parodi's theorem .............................................................................. 1086
15.713 Corollary of Brauer's theorem ........................................................... 1086
15.714 Ballieu's theorem ............................................................................. 1086
15.715 Routh-Hurwitz theorem .................................................................... 1086
15.81–15.82 Named Theorems on Eigenvalues ............................................ 1087
15.811 Schur's inequalities .......................................................................... 1087
15.812 Sturmian separation theorem ............................................................. 1087
15.813 Poincare's separation theorem ........................................................... 1087
15.814 Gerschgorin's theorem .................................................................... 1088
15.815 Brauer's theorem ............................................................................. 1088
15.816 Perron's theorem ............................................................................. 1088
15.817 Frobenius theorem ........................................................................... 1088
15.818 Perron–Frobenius theorem ................................................................. 1088
15.819 Wielandt's theorem .......................................................................... 1088
15.820 Ostrowski's theorem ......................................................................... 1089
15.821 First theorem due to Lyapunov .......................................................... 1089
15.822 Second theorem due to Lyapunov ...................................................... 1089
15.823 Hermitian matrices and diophantine relations involving circular functions of rational angles due to Calogero and Perelomov ........................................... 1089
15.91 Variational Principles ................................................................. 1091
15.911 Rayleigh quotient ........................................................................ 1091
15.912 Basic theorems ........................................................................... 1091

16 Ordinary Differential Equations ......................................................... 1093
16.1–16.9 Results Relating to the Solution of Ordinary Differential Equations .................................................. 1093
16.11 First-Order Equations .................................................................... 1093
16.111 Solution of a first-order equation .................................................. 1093
16.112 Cauchy problem .......................................................................... 1093
16.113 Approximate solution to an equation ............................................. 1093
16.114 Lipschitz continuity of a function .................................................. 1094
16.21 Fundamental Inequalities and Related Results .................................. 1094
16.211 Gronwall’s lemma ........................................................................ 1094
16.212 Comparison of approximate solutions of a differential equation ........ 1094
16.31 First-Order Systems ....................................................................... 1094
16.311 Solution of a system of equations .................................................. 1094
16.312 Cauchy problem for a system ......................................................... 1095
16.313 Approximate solution to a system ................................................ 1095
16.314 Lipschitz continuity of a vector ....................................................... 1095
16.315 Comparison of approximate solutions of a system ......................... 1096
16.316 First-order linear differential equation .......................................... 1096
16.317 Linear systems of differential equations ...................................... 1096
16.41 Some Special Types of Elementary Differential Equations ............... 1097
16.411 Variables separable .................................................................... 1097
16.412 Exact differential equations .......................................................... 1097
16.413 Conditions for an exact equation .................................................. 1097
16.414 Homogeneous differential equations .......................................... 1097
16.51 Second-Order Equations ................................................................. 1098
16.511 Adjoint and self-adjoint equations ................................................ 1098
16.512 Abel’s identity ............................................................................. 1098
16.513 Lagrange identity ...................................................................... 1099
16.514 The Riccati equation ................................................................... 1099
16.515 Solutions of the Riccati equation .................................................. 1099
16.516 Solution of a second-order linear differential equation .................... 1100
16.61–16.62 Oscillation and Non-Oscillation Theorems for Second-Order Equations ................................................ 1100
16.611 First basic comparison theorem ................................................... 1100
16.622 Second basic comparison theorem ................................................. 1101
16.623 Interlacing of zeros .................................................................... 1101
16.624 Sturm separation theorem ............................................................ 1101
16.625 Sturm comparison theorem .......................................................... 1101
16.626 Szegö’s comparison theorem ........................................................ 1101
16.627 Picone’s identity ....................................................................... 1110
16.628 Sturm-Picone theorem ................................................................. 1110
16.629 Oscillation on the half line ............................................................. 1110
16.71 Two Related Comparison Theorems .............................................. 1113
16.711 Theorem 1 ............................................................................... 1113
### 16.712 Theorem 2

Theorem 2

1103

### 16.81–16.82 Non-Oscillatory Solutions

Non-Oscillatory Solutions

1103

### 16.811 Kneser’s non-oscillation theorem

Kneser’s non-oscillation theorem

1103

### 16.822 Comparison theorem for non-oscillation

Comparison theorem for non-oscillation

1104

### 16.823 Necessary and sufficient conditions for non-oscillation

Necessary and sufficient conditions for non-oscillation

1104

### 16.91 Some Growth Estimates for Solutions of Second-Order Equations

Some Growth Estimates for Solutions of Second-Order Equations

1104

### 16.911 Strictly increasing and decreasing solutions

Strictly increasing and decreasing solutions

1104

### 16.912 General result on dominant and subdominant solutions

General result on dominant and subdominant solutions

1104

### 16.913 Estimate of dominant solution

Estimate of dominant solution

1105

### 16.914 A theorem due to Lyapunov

A theorem due to Lyapunov

1105

### 16.92 Boundedness Theorems

Boundedness Theorems

1106

### 16.921 All solutions of the equation

All solutions of the equation

1106

### 16.922 If all solutions of the equation

If all solutions of the equation

1106

### 16.923 If $a(x) \to \infty$ monotonically as $x \to \infty$, then all solutions of

If $a(x) \to \infty$ monotonically as $x \to \infty$, then all solutions of

1106

### 16.924 Consider the equation

Consider the equation

1106

### 16.93 Growth of maxima of $|y|$.

Growth of maxima of $|y|$.

1106

### 17 Fourier, Laplace, and Mellin Transforms

17.1–17.4 Integral Transforms

Integral Transforms

1107

#### 17.11 Laplace transform

Laplace transform

1107

#### 17.12 Basic properties of the Laplace transform

Basic properties of the Laplace transform

1107

#### 17.13 Table of Laplace transform pairs

Table of Laplace transform pairs

1108

#### 17.21 Fourier transform

Fourier transform

1117

#### 17.22 Basic properties of the Fourier transform

Basic properties of the Fourier transform

1118

#### 17.23 Table of Fourier transform pairs

Table of Fourier transform pairs

1118

#### 17.24 Table of Fourier transform pairs for spherically symmetric functions

Table of Fourier transform pairs for spherically symmetric functions

1120

#### 17.31 Fourier sine and cosine transforms

Fourier sine and cosine transforms

1121

#### 17.32 Basic properties of the Fourier sine and cosine transforms

Basic properties of the Fourier sine and cosine transforms

1121

#### 17.33 Table of Fourier sine transforms

Table of Fourier sine transforms

1122

#### 17.34 Table of Fourier cosine transforms

Table of Fourier cosine transforms

1126

#### 17.35 Relationships between transforms

Relationships between transforms

1129

#### 17.41 Mellin transform

Mellin transform

1129

#### 17.42 Basic properties of the Mellin transform

Basic properties of the Mellin transform

1130

#### 17.43 Table of Mellin transforms

Table of Mellin transforms

1131

### 18 The z-Transform

18.1–18.3 Definition, Bilateral, and Unilateral $z$-Transforms

Definition, Bilateral, and Unilateral $z$-Transforms

1135

#### 18.1 Definitions

Definitions

1135

#### 18.2 Bilateral $z$-transform

Bilateral $z$-transform

1136

#### 18.3 Unilateral $z$-transform

Unilateral $z$-transform

1138

References

1141

Supplemental references

1145

Index of Functions and Constants

1151

General Index of Concepts

1161
Preface to the Seventh Edition

Since the publication in 2000 of the completely reset sixth edition of Gradshteyn and Ryzhik, users of the reference work have continued to submit corrections, new results that extend the work, and suggestions for changes that improve the presentation of existing entries. It is a matter of regret to us that the structure of the book makes it impossible to acknowledge these individual contributions, so, as usual, the names of the many new contributors have been added to the acknowledgment list at the front of the book.

This seventh edition contains the corrections received since the publication of the sixth edition in 2000, together with a considerable amount of new material acquired from isolated sources. Following our previous conventions, an amended entry has a superscript “11” added to its entry reference number, where the equivalent superscript number for the sixth edition was “10.” Similarly, an asterisk on an entry’s reference number indicates a new result. When, for technical reasons, an entry in a previous edition has been removed, to preserve the continuity of numbering between the new and older editions the subsequent entries have not been renumbered, so the numbering will jump.

We wish to express our gratitude to all who have been in contact with us with the object of improving and extending the book, and we want to give special thanks to Dr. Victor H. Moll for his interest in the book and for the many contributions he has made over an extended period of time. We also wish to acknowledge the contributions made by Dr. Francis J. O’Brien Jr. of the Naval Station in Newport, in particular for results involving integrands where exponentials are combined with algebraic functions.

Experience over many years has shown that each new edition of Gradshteyn and Ryzhik generates a fresh supply of suggestions for new entries, and for the improvement of the presentation of existing entries and errata. In view of this, we do not expect this new edition to be free from errors, so all users of this reference work who identify errors, or who wish to propose new entries, are invited to contact the authors, whose email addresses are listed below. Corrections will be posted on the web site www.az-tec.com/gr/errata.

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The Order of Presentation of the Formulas

The question of the most expedient order in which to give the formulas, in particular, in what division to include particular formulas such as the definite integrals, turned out to be quite complicated. The thought naturally occurs to set up an order analogous to that of a dictionary. However, it is almost impossible to create such a system for the formulas of integral calculus. Indeed, in an arbitrary formula of the form

\[ \int_{a}^{b} f(x) \, dx = A \]

one may make a large number of substitutions of the form \( x = \varphi(t) \) and thus obtain a number of “synonyms” of the given formula. We must point out that the table of definite integrals by Bierens de Haan and the earlier editions of the present reference both sin in the plethora of such “synonyms” and formulas of complicated form. In the present edition, we have tried to keep only the simplest of the “synonym” formulas. Basically, we judged the simplicity of a formula from the standpoint of the simplicity of the arguments of the “outer” functions that appear in the integrand. Where possible, we have replaced a complicated formula with a simpler one. Sometimes, several complicated formulas were thereby reduced to a single, simpler one. We then kept only the simplest formula. As a result of such substitutions, we sometimes obtained an integral that could be evaluated by use of the formulas of Chapter Two and the Newton–Leibniz formula, or to an integral of the form

\[ \int_{-a}^{a} f(x) \, dx, \]

where \( f(x) \) is an odd function. In such cases, the complicated integrals have been omitted.

Let us give an example using the expression

\[ \int_{0}^{\pi/4} (\cot x - 1)^{p-1} \ln \tan x \, dx = -\frac{\pi}{p} \csc p\pi. \]  

(0.1)

By making the natural substitution \( u = \cot x - 1 \), we obtain

\[ \int_{0}^{\infty} u^{p-1} \ln(1 + u) \, du = \frac{\pi}{p} \csc p\pi. \]  

(0.2)

Integrals similar to formula (0.1) are omitted in this new edition. Instead, we have formula (0.2).
As a second example, let us take

\[ I = \int_{0}^{\pi/2} \ln (\tan^p x + \cot^p x) \ln \tan x \, dx = 0. \]

The substitution \( u = \tan x \) yields

\[ I = \int_{0}^{\infty} \frac{\ln (u^p + u^{-p}) \ln u}{1 + u^2} \, du. \]

If we now set \( v = \ln u \), we obtain

\[ I = \int_{-\infty}^{\infty} \frac{ve^v \ln (e^{pv} + e^{-pv})}{2 \cosh v} \, dv = \int_{-\infty}^{\infty} v \frac{\ln (2 \cosh pv)}{2 \cosh v} \, dv. \]

The integrand is odd, and, consequently, the integral is equal to 0.

Thus, before looking for an integral in the tables, the user should simplify as much as possible the arguments (the “inner” functions) of the functions in the integrand.

The functions are ordered as follows: First we have the elementary functions:

1. The function \( f(x) = x \).
2. The exponential function.
3. The hyperbolic functions.
4. The trigonometric functions.
5. The logarithmic function.
6. The inverse hyperbolic functions. (These are replaced with the corresponding logarithms in the formulas containing definite integrals.)
7. The inverse trigonometric functions.

Then follow the special functions:

8. Elliptic integrals.
10. The logarithm integral, the exponential integral, the sine integral, and the cosine integral functions.
11. Probability integrals and Fresnel’s integrals.
12. The gamma function and related functions.
13. Bessel functions.
15. Legendre functions.
17. Hypergeometric functions.
18. Degenerate hypergeometric functions.
19. Parabolic cylinder functions.
20. Meijer’s and MacRobert’s functions.
21. Riemann’s zeta function.

The integrals are arranged in order of outer function according to the above scheme: the farther down in the list a function occurs, (i.e., the more complex it is) the later will the corresponding formula appear
in the tables. Suppose that several expressions have the same outer function. For example, consider \( \sin e^x, \sin x, \sin \ln x \). Here, the outer function is the sine function in all three cases. Such expressions are then arranged in order of the inner function. In the present work, these functions are therefore arranged in the following order: \( \sin x, \sin e^x, \sin \ln x \).

Our list does not include polynomials, rational functions, powers, or other algebraic functions. An algebraic function that is included in tables of definite integrals can usually be reduced to a finite combination of roots of rational power. Therefore, for classifying our formulas, we can conditionally treat a power function as a generalization of an algebraic and, consequently, of a rational function.\(^*\) We shall distinguish between all these functions and those listed above, and we shall treat them as operators. Thus, in the expression \( \sin^2 e^x \), we shall think of the squaring operator as applied to the outer function, namely, the sine. In the expression \( \sin x + \cos x \sin x - \cos x \), we shall think of the rational operator as applied to the trigonometric functions sine and cosine. We shall arrange the operators according to the following order:

1. Polynomials (listed in order of their degree).
2. Rational operators.
3. Algebraic operators (expressions of the form \( A^{p/q} \), where \( q \) and \( p \) are rational, and \( q > 0 \); these are listed according to the size of \( q \)).
4. Power operators.

Expressions with the same outer and inner functions are arranged in the order of complexity of the operators. For example, the following functions [whose outer functions are all trigonometric, and whose inner functions are all \( f(x) = x \)] are arranged in the order shown:

\[
\sin x, \quad \sin x \cos x, \quad \frac{1}{\sin x} = \csc x, \quad \frac{\sin x}{\cos x} = \tan x, \quad \frac{\sin x + \cos x}{\sin x - \cos x} = \sec x, \quad \frac{\sin^m x}{\sin x} - \frac{\cos x}{\sin x}, \quad \sin^m x, \quad \sin^m x \cos x.
\]

Furthermore, if two outer functions \( \varphi_1(x) \) and \( \varphi_2(x) \), where \( \varphi_1(x) \) is more complex than \( \varphi_2(x) \), appear in an integrand and if any of the operations mentioned are performed on them, the corresponding integral will appear [in the order determined by the position of \( \varphi_2(x) \) in the list] after all integrals containing only the function \( \varphi_1(x) \). Thus, following the trigonometric functions are the trigonometric and power functions [that is, \( \varphi_2(x) = x \)]. Then come

- combinations of trigonometric and exponential functions,
- combinations of trigonometric functions, exponential functions, and powers, etc.,
- combinations of trigonometric and hyperbolic functions, etc.

Integrals containing two functions \( \varphi_1(x) \) and \( \varphi_2(x) \) are located in the division and order corresponding to the more complicated function of the two. However, if the positions of several integrals coincide because they contain the same complicated function, these integrals are put in the position defined by the complexity of the second function.

To these rules of a general nature, we need to add certain particular considerations that will be easily understood from the tables. For example, according to the above remarks, the function \( e^x \) comes after \( e^{1/x} \) as regards complexity, but \( \ln x \) and \( \ln \frac{1}{x} \) are equally complex since \( \ln \frac{1}{x} = -\ln x \). In the section on “powers and algebraic functions,” polynomials, rational functions, and powers of powers are formed from power functions of the form \( (a + bx)^n \) and \( (\alpha + \beta x)^\nu \).

\(^*\)For any natural number \( n \), the involution \( (a + bx)^n \) of the binomial \( a + bx \) is a polynomial. If \( n \) is a negative integer, \( (a + bx)^n \) is a rational function. If \( n \) is irrational, the function \( (a + bx)^n \) is not even an algebraic function.
Use of the Tables*

For the effective use of the tables contained in this book, it is necessary that the user should first become familiar with the classification system for integrals devised by the authors Ryzhik and Gradshteyn. This classification is described in detail in the section entitled The Order of Presentation of the Formulas (see page xxvii) and essentially involves the separation of the integrand into inner and outer functions. The principal function involved in the integrand is called the outer function, and its argument, which is itself usually another function, is called the inner function. Thus, if the integrand comprised the expression \( \ln \sin x \), the outer function would be the logarithmic function while its argument, the inner function, would be the trigonometric function \( \sin x \). The desired integral would then be found in the section dealing with logarithmic functions, its position within that section being determined by the position of the inner function (here a trigonometric function) in Gradshteyn and Ryzhik’s list of functional forms.

It is inevitable that some duplication of symbols will occur within such a large collection of integrals, and this happens most frequently in the first part of the book dealing with algebraic and trigonometric integrands. The symbols most frequently involved are \( \alpha, \beta, \gamma, \delta, t, u, z, z_k \), and \( \Delta \). The expressions associated with these symbols are used consistently within each section and are defined at the start of each new section in which they occur. Consequently, reference should be made to the beginning of the section being used in order to verify the meaning of the substitutions involved.

Integrals of algebraic functions are expressed as combinations of roots with rational power indices, and definite integrals of such functions are frequently expressed in terms of the Legendre elliptic integrals \( F(\phi, k), E(\phi, k) \) and \( \Pi(\phi, n, k) \), respectively, of the first, second, and third kinds.

The four inverse hyperbolic functions \( \text{arcsinh} z, \text{arccosh} z, \text{arctanh} z, \text{and arccoth} z \) are introduced through the definitions

\[
\begin{align*}
\text{arcsin} z &= \frac{1}{i} \text{arcsinh}(iz) \\
\text{arccos} z &= \frac{1}{i} \text{arccosh}(z) \\
\text{arctan} z &= \frac{1}{i} \text{arctanh}(iz) \\
\text{arccot} z &= i \text{arccoth}(iz)
\end{align*}
\]

or
\[
\begin{align*}
\text{arcsinh } z &= \frac{1}{i} \arcsin (iz) \\
\text{arccosh } z &= i \arccos z \\
\text{arctanh } z &= \frac{1}{i} \arctan (iz) \\
\text{arccoth } z &= \frac{1}{i} \arccot (-iz)
\end{align*}
\]

The numerical constants \( C \) and \( G \) which often appear in the definite integrals denote Euler’s constant and Catalan’s constant, respectively. Euler’s constant \( C \) is defined by the limit
\[
C = \lim_{s \to \infty} \left( \sum_{m=1}^{s} \frac{1}{m} - \ln s \right) = 0.577215 \ldots
\]

On occasion, other writers denote Euler’s constant by the symbol \( \gamma \), but this is also often used instead to denote the constant
\[
\gamma = e^C = 1.781072 \ldots.
\]

Catalan’s constant \( G \) is related to the complete elliptic integral
\[
K \equiv K(k) \equiv \int_{0}^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}
\]
by the expression
\[
G = \frac{1}{2} \int_{0}^{1} K(k) \, dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} = 0.915965 \ldots.
\]

Since the notations and definitions for higher transcendental functions that are used by different authors are by no means uniform, it is advisable to check the definitions of the functions that occur in these tables. This can be done by identifying the required function by symbol and name in the Index of Special Functions and Notation on page xxxix, and by then referring to the defining formula or section number listed there. We now present a brief discussion of some of the most commonly used alternative notations and definitions for higher transcendental functions.

**Bernoulli and Euler Polynomials and Numbers**

Extensive use is made throughout the book of the Bernoulli and Euler numbers \( B_n \) and \( E_n \) that are defined in terms of the Bernoulli and Euler polynomials of order \( n \), \( B_n(x) \) and \( E_n(x) \), respectively. These polynomials are defined by the generating functions
\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{for } |t| < 2\pi
\]
and
\[
\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \text{for } |t| < \pi.
\]

The Bernoulli numbers are always denoted by \( B_n \) and are defined by the relation
\[
B_n = B_n(0) \quad \text{for } n = 0, 1, \ldots,
\]
when
\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \ldots
\]
The Euler numbers $E_n$ are defined by setting

$$E_n = 2^n E_n \left( \frac{1}{2} \right)$$

for $n = 0, 1, \ldots$

The $E_n$ are all integral, and $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, ... .

An alternative definition of Bernoulli numbers, which we shall denote by the symbol $B_n^*$, uses the same generating function but identifies the $B_n^*$ differently in the following manner:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2} t + B_1^* \frac{t^2}{2!} - B_2^* \frac{t^4}{4!} + \ldots$$

This definition then gives rise to the alternative set of Bernoulli numbers

$$B_1^* = 1/6, \quad B_2^* = 1/30, \quad B_3^* = 1/42, \quad B_4^* = 1/30, \quad B_5^* = 5/66,$$

$$B_6^* = 691/2730, \quad B_7^* = 7/6, \quad B_8^* = 3617/510, \quad \ldots$$

These differences in notation must also be taken into account when using the following relationships that exist between the Bernoulli and Euler polynomials:

$$B_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k} E_k(2x) \quad n = 0, 1, \ldots$$

$$E_{n-1}(x) = \frac{2^n}{n} \left\{ B_n \left( \frac{x + 1}{2} \right) - B_n \left( \frac{x}{2} \right) \right\}$$

or

$$E_{n-1}(x) = \frac{2}{n} \left\{ B_n(x) - 2^n B_n \left( \frac{x}{2} \right) \right\} \quad n = 1, 2, \ldots$$

and

$$E_{n-2}(x) = 2 \binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} (2^{n-k} - 1) B_{n-k} B_n(x) \quad n = 2, 3, \ldots$$

There are also alternative definitions of the Euler polynomial of order $n$, and it should be noted that some authors, using a modification of the third expression above, call

$$\left( \frac{2}{n+1} \right) \left\{ B_n(x) - 2^n B_n \left( \frac{x}{2} \right) \right\}$$

the Euler polynomial of order $n$.

### Elliptic Functions and Elliptic Integrals

The following notations are often used in connection with the inverse elliptic functions $sn u$, $cn u$, and $dn u$:

$$ns u = \frac{1}{sn u} \quad nc u = \frac{1}{cn u} \quad nd u = \frac{1}{dn u}$$

$$sc u = \frac{1}{cn u} \quad cs u = \frac{1}{sn u} \quad ds u = \frac{1}{sn u}$$

$$sd u = \frac{1}{dn u} \quad cd u = \frac{1}{dn u} \quad dc u = \frac{1}{cn u}$$
The elliptic integral of the third kind is defined by Gradshteyn and Ryzhik to be

\[ \Pi (\varphi, n^2, k) = \int_0^{\varphi} \frac{da}{(1 - n^2 \sin^2 a) \sqrt{1 - k^2 \sin^2 a}} \quad (-\infty < n^2 < \infty) \]

\[ = \int_0^{\sin \varphi} \frac{dx}{(1 - n^2 x^2) \sqrt{(1 - x^2)(1 - k^2 x^2)}} \]

**The Jacobi Zeta Function and Theta Functions**

The Jacobi zeta function \( z_n(u, k) \), frequently written \( Z(u) \), is defined by the relation

\[ z_n(u, k) = Z(u) = \int_0^u \left\{ \frac{dn}{\sqrt{1 - n^2}} - \frac{E}{K} \right\} dv = E(u) - \frac{E}{K} u. \]

This is related to the theta functions by the relationship

\[ z_n(u, k) = \frac{\partial}{\partial u} \ln \Theta(u) \]

giving

(i). \( z_n(u, k) = \pi \frac{\vartheta'_1}{2K} \left( \frac{\pi u}{2K} \right) - \frac{\text{cn} u \text{dn} u}{\text{sn} u} \)

(ii). \( z_n(u, k) = \pi \frac{\vartheta'_2}{2K} \left( \frac{\pi u}{2K} \right) - \frac{\text{dn} u \text{sn} u}{\text{cn} u} \)

(iii). \( z_n(u, k) = \pi \frac{\vartheta'_3}{2K} \left( \frac{\pi u}{2K} \right) - k^2 \frac{\text{sn} u \text{cn} u}{\text{dn} u} \)

(iv). \( z_n(u, k) = \pi \frac{\vartheta'_4}{2K} \left( \frac{\pi u}{2K} \right) \)

Many different notations for the theta function are in current use. The most common variants are the replacement of the argument \( u \) by the argument \( u/\pi \) and, occasionally, a permutation of the identification of the functions \( \vartheta_1 \) to \( \vartheta_4 \) with the function \( \vartheta_4 \) replaced by \( \vartheta \).

**The Factorial (Gamma) Function**

In older reference texts, the gamma function \( \Gamma(z) \), defined by the Euler integral

\[ \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \]

is sometimes expressed in the alternative notation

\[ \Gamma(1 + z) = z! = \Pi(z). \]

On occasions, the related derivative of the logarithmic factorial function \( \Psi(z) \) is used where

\[ \frac{d}{dz} \ln(z!) = \frac{(z!)'}{z!} = \Psi(z). \]
This function satisfies the recurrence relation

\[ \Psi(z) = \Psi(z - 1) + \frac{1}{z - 1} \]

and is defined by the series

\[ \Psi(z) = -C + \sum_{n=0}^{\infty} \left( \frac{1}{n + 1} - \frac{1}{z + n} \right). \]

The derivative \( \Psi'(z) \) satisfies the recurrence relation

\[ \Psi'(z + 1) = \Psi'(z) - \frac{1}{z^2} \]

and is defined by the series

\[ \Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z + n)^2}. \]

### Exponential and Related Integrals

The exponential integrals \( E_n(z) \) have been defined by Schloemilch using the integral

\[ E_n(z) = \int_1^{\infty} e^{-zt} t^{-n} \, dt \quad (n = 0, 1, \ldots, \ Re \ z > 0). \]

They should not be confused with the Euler polynomials already mentioned. The function \( E_1(z) \) is related to the exponential integral \( Ei(z) \) through the expressions

\[ E_1(z) = -Ei(-z) = \int_z^{\infty} e^{-t} t^{-1} \, dt \]

and

\[ \text{li}(z) = \int_0^z \frac{dt}{\ln t} = Ei(\ln z) \quad [z > 1]. \]

The functions \( E_n(z) \) satisfy the recurrence relations

\[ E_n(z) = -e^{-z} - z E_{n-1}(z) \quad [n > 1] \]

and

\[ E_n'(z) = -E_{n-1}(z) \]

with

\[ E_0(z) = e^{-z}/z. \]

The function \( E_n(z) \) has the asymptotic expansion

\[ E_n(z) \sim \frac{e^{-z}}{z} \left\{ 1 - \frac{n \cdot (n + 1)}{z^2} - \frac{n(n + 1)(n + 2)}{z^3} + \cdots \right\} \quad \left[ \arg z < \frac{3\pi}{2} \right] \]

while for large \( n \),

\[ E_n(x) = \frac{e^{-x}}{x + n} \left\{ 1 + \frac{n}{(x + n)^2} + \frac{n(n - 2)}{(x + n)^4} + \frac{n(6x^2 - 8nx + n^2)}{(x + n)^6} + R(n, x) \right\}, \]

where

\[ -0.36n^{-4} \leq R(n, x) \leq \left(1 + \frac{1}{x + n - 1}\right) n^{-4} \quad [x > 0]. \]

The sine and cosine integrals \( \text{si}(x) \) and \( \text{ci}(x) \) are related to the functions \( \text{Si}(x) \) and \( \text{Ci}(x) \) by the integrals

\[ \text{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt = \text{si}(x) + \frac{\pi}{2} \]

and
\[
\text{Ci}(x) = C + \ln x + \int_0^x \frac{\cos t - 1}{t} \, dt.
\]

The hyperbolic sine and cosine integrals \(\text{shi}(x)\) and \(\text{chi}(x)\) are defined by the relations

\[
\text{shi}(x) = \int_0^x \frac{\sinh t}{t} \, dt
\]

and

\[
\text{chi}(x) = C + \ln x + \int_0^x \frac{\cosh t - 1}{t} \, dt.
\]

Some authors write

\[
\text{Cin}(x) = \int_0^x \frac{1 - \cos t}{t} \, dt
\]

so that

\[
\text{Cin}(x) = -\text{Ci}(x) + \ln x + C.
\]

The error function \(\text{erf}(x)\) is defined by the relation

\[
\text{erf}(x) = \Phi(x) = 2 \sqrt{\frac{\pi}{2}} \int_0^x e^{-t^2} \, dt,
\]

and the complementary error function \(\text{erfc}(x)\) is related to the error function \(\text{erfc}(x)\) and to \(\Phi(x)\) by the expression

\[
\text{erfc}(x) = 1 - \text{erf}(x).
\]

The Fresnel integrals \(S(x)\) and \(C(x)\) are defined by Gradshteyn and Ryzhik as

\[
S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin \frac{\pi t^2}{2} \, dt
\]

and

\[
C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos \frac{\pi t^2}{2} \, dt.
\]

Other definitions that are in use are

\[
S_1(x) = \int_0^x \sin \frac{\pi t^2}{2} \, dt, \quad C_1(x) = \int_0^x \cos \frac{\pi t^2}{2} \, dt,
\]

and

\[
S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} \, dt, \quad C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} \, dt.
\]

These are related by the expressions

\[
S(x) = S_1 \left( x \sqrt{\frac{2}{\pi}} \right) = S_2 \left( x^2 \right)
\]

and

\[
C(x) = C_1 \left( x \sqrt{\frac{2}{\pi}} \right) = C_2 \left( x^2 \right)
\]

Hermite and Chebyshev Orthogonal Polynomials

The Hermite polynomials \(H_n(x)\) are related to the Hermite polynomials \(He_n(x)\) by the relations

\[
He_n(x) = 2^{-n/2} H_n \left( \frac{x}{\sqrt{2}} \right)
\]

and

\[
H_n(x) = 2^{n/2} He_n \left( x \sqrt{2} \right).
\]
These functions satisfy the differential equations

\[
\frac{d^2 H_n}{dx^2} - 2x \frac{d H_n}{dx} + 2n H_n = 0
\]

and

\[
\frac{d^2 He_n}{dx^2} - x \frac{d He_n}{dx} + n He_n = 0.
\]

They obey the recurrence relations

\[
H_{n+1} = 2x H_n - 2n H_{n-1}
\]

and

\[
He_{n+1} = x He_n - n He_{n-1}.
\]

The first six orthogonal polynomials \(He_n\) are

\[
He_0 = 1, \quad He_1 = x, \quad He_2 = x^2 - 1, \quad He_3 = x^3 - 3x, \quad He_4 = x^4 - 6x^2 + 3, \quad He_5 = x^5 - 10x^3 + 15x.
\]

Sometimes the Chebyshev polynomial \(U_n(x)\) of the second kind is defined as a solution of the equation

\[
\left(1 - x^2\right) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + n(n + 2)y = 0.
\]

**Bessel Functions**

A variety of different notations for Bessel functions are in use. Some common ones involve the replacement of \(Y_n(z)\) by \(N_n(z)\) and the introduction of the symbol

\[
\Lambda_n(z) = \left(\frac{1}{2}z\right)^{-n} \Gamma(n + 1) J_n(z).
\]

In the book by Gray, Mathews, and MacRobert, the symbol \(Y_n(z)\) is used to denote \(\frac{1}{2\pi} Y_n(z) + (\ln 2 - C) J_n(z)\) while Neumann uses the symbol \(Y^{(n)}(z)\) for the identical quantity.

The Hankel functions \(H_{\nu}^{(1)}(z)\) and \(H_{\nu}^{(2)}(z)\) are sometimes denoted by \(Hs_{\nu}(z)\) and \(Hi_{\nu}(z)\), and some authors write \(G_{\nu}(z) = \left(\frac{1}{2}\right) \pi i H_{\nu}^{(1)}(z)\).

The Neumann polynomial \(O_n(t)\) is a polynomial of degree \(n + 1\) in \(1/t\), with \(O_0(t) = 1/t\). The polynomials \(O_n(t)\) are defined by the generating function

\[
\frac{1}{t - z} = J_0(z) O_0(t) + 2 \sum_{k=1}^{\infty} J_k(z) O_k(t),
\]

giving

\[
O_n(t) = \frac{1}{4} \sum_{k=0}^{[n/2]} \frac{n(n - k - 1)!}{k!} \left(\frac{2}{t}\right)^{n-2k+1} \text{ for } n = 1, 2, \ldots,
\]

where \([\frac{1}{2}n]\) signifies the integral part of \(\frac{1}{2}n\). The following relationship holds between three successive polynomials:

\[
(n - 1) O_{n+1}(t) + (n + 1) O_{n-1}(t) - \frac{2(n^2 - 1)}{t} O_n(t) = \frac{2n}{t} \sin^2 \frac{n\pi}{2}.
\]
The Airy functions $\text{Ai}(z)$ and $\text{Bi}(z)$ are independent solutions of the equation
\[
\frac{d^2u}{dz^2} - zu = 0.
\]
The solutions can be represented in terms of Bessel functions by the expressions
\[
\text{Ai}(z) = \frac{1}{3} \sqrt{z} \left\{ I_{-1/3} \left( \frac{2}{3} z^{3/2} \right) - I_{1/3} \left( \frac{2}{3} z^{3/2} \right) \right\} = \frac{1}{\pi} \sqrt{\frac{z}{3}} K_{1/3} \left( \frac{2}{3} z^{3/2} \right)
\]
\[
\text{Ai}(-z) = \frac{1}{3} \sqrt{z} \left\{ J_{1/3} \left( \frac{2}{3} z^{3/2} \right) + J_{-1/3} \left( \frac{2}{3} z^{3/2} \right) \right\}
\]
and by
\[
\text{Bi}(z) = \sqrt{\frac{z}{3}} \left\{ I_{-1/3} \left( \frac{2}{3} z^{3/2} \right) + I_{1/3} \left( \frac{2}{3} z^{3/2} \right) \right\},
\]
\[
\text{Bi}(-z) = \sqrt{\frac{z}{3}} \left\{ J_{1/3} \left( \frac{2}{3} z^{3/2} \right) - J_{-1/3} \left( \frac{2}{3} z^{3/2} \right) \right\}.
\]

**Parabolic Cylinder Functions and Whittaker Functions**

The differential equation
\[
\frac{d^2y}{dz^2} + \left( az^2 + bz + c \right) y = 0
\]
has associated with it the two equations
\[
\frac{d^2y}{dz^2} + \left( \frac{1}{4} z^2 + a \right) y = 0 \quad \text{and} \quad \frac{d^2y}{dz^2} - \left( \frac{1}{4} z^2 + a \right) y = 0,
\]
the solutions of which are parabolic cylinder functions. The first equation can be derived from the second by replacing $z$ by $ze^{i\pi/4}$ and $a$ by $-ia$.

The solutions of the equation
\[
\frac{d^2y}{dz^2} - \left( \frac{1}{4} z^2 + a \right) y = 0
\]
are sometimes written $U(a, z)$ and $V(a, z)$. These solutions are related to Whittaker’s function $D_p(z)$ by the expressions
\[
U(a, z) = D_{-a-\frac{1}{2}}(z)
\]
and
\[
V(a, z) = \frac{1}{\pi} \Gamma \left( \frac{1}{2} + a \right) \left\{ D_{-a-\frac{1}{2}}(-z) + (\sin \pi a) D_{-a-\frac{1}{2}}(z) \right\}.
\]

**Mathieu Functions**

There are several accepted notations for Mathieu functions and for their associated parameters. The defining equation used by Gradshteyn and Ryzhik is
\[
\frac{d^2y}{dz^2} + \left( a - 2k^2 \cos 2z \right) y = 0 \quad \text{with} \quad k^2 = q.
\]
Different notations involve the replacement of $a$ and $q$ in this equation by $h$ and $\theta$, $\lambda$ and $h^2$, and $b$ and $c = 2\sqrt{q}$, respectively. The periodic solutions $\text{se}_n(z, q)$ and $\text{ce}_n(z, q)$ and the modified periodic solutions $\text{Se}_n(z, q)$ and $\text{Ce}_n(z, q)$ are suitably altered and, sometimes, re-normalized. A description of these relationships together with the normalizing factors is contained in: *Tables Relating to Mathieu Functions*. National Bureau of Standards, Columbia University Press, New York, 1951.
## Index of Special Functions

<table>
<thead>
<tr>
<th>Notation</th>
<th>Name of the function and the number of the formula containing its definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta(x) )</td>
<td>Gamma function</td>
</tr>
<tr>
<td>( \Gamma(z) )</td>
<td></td>
</tr>
<tr>
<td>( \gamma(a, x), \Gamma(a, x) )</td>
<td>Incomplete gamma functions</td>
</tr>
<tr>
<td>( \Delta(n - k) )</td>
<td>Unit integer pulse function</td>
</tr>
<tr>
<td>( \xi(s) )</td>
<td></td>
</tr>
<tr>
<td>( \lambda(x, y) )</td>
<td></td>
</tr>
<tr>
<td>( \mu(x, \beta) ), ( \mu(x, \beta, \alpha) )</td>
<td></td>
</tr>
<tr>
<td>( \nu(x), \nu(x, \alpha) )</td>
<td></td>
</tr>
<tr>
<td>( \Pi(x) )</td>
<td>Lobachevskiy’s angle of parallelism</td>
</tr>
<tr>
<td>( \Pi(\varphi, n, k) )</td>
<td>Elliptic integral of the third kind</td>
</tr>
<tr>
<td>( \zeta(s) )</td>
<td>Riemann’s zeta functions</td>
</tr>
<tr>
<td>( \Theta(u) = \vartheta_4 \left( \frac{\pi u}{2K} \right), \Theta_1(u) = \vartheta_3 \left( \frac{\pi u}{2K} \right) )</td>
<td>Jacobian theta function</td>
</tr>
<tr>
<td>( \vartheta_0(v</td>
<td>\tau), \vartheta_4(v</td>
</tr>
<tr>
<td>( \sigma(u) )</td>
<td></td>
</tr>
<tr>
<td>( \Phi(x) )</td>
<td></td>
</tr>
<tr>
<td>( \Phi(z, s, v) )</td>
<td></td>
</tr>
<tr>
<td>( \Phi(a, c; x) = {}_1F_1(\alpha; \gamma; x) )</td>
<td></td>
</tr>
<tr>
<td>( \Phi_1(\alpha, \beta, \gamma, x, y) )</td>
<td></td>
</tr>
<tr>
<td>( \Phi_2(\beta, \beta', \gamma, x, y) )</td>
<td></td>
</tr>
<tr>
<td>( \Phi_3(\beta, \gamma, x, y) )</td>
<td></td>
</tr>
<tr>
<td>( \psi(x) )</td>
<td></td>
</tr>
<tr>
<td>( \varphi(u) )</td>
<td></td>
</tr>
<tr>
<td>( \am(u, k) )</td>
<td>Amplitude (of an elliptic function)</td>
</tr>
<tr>
<td>( B_n )</td>
<td>Bernoulli numbers</td>
</tr>
<tr>
<td>( B_n(x) )</td>
<td>Bernoulli polynomials</td>
</tr>
<tr>
<td>( B(x, y) )</td>
<td>Beta functions</td>
</tr>
<tr>
<td>( B_x(p, q) )</td>
<td>Incomplete beta functions</td>
</tr>
<tr>
<td>( \text{bei}(z), \text{ber}(z) )</td>
<td>Thomson functions</td>
</tr>
</tbody>
</table>

*continued on next page*
<table>
<thead>
<tr>
<th>Notation</th>
<th>Name of the function and the number of the formula containing its definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C )</td>
<td>Euler constant ( 9.73, 8.367 )</td>
</tr>
<tr>
<td>( C(x) )</td>
<td>Fresnel cosine integral ( 8.25 )</td>
</tr>
<tr>
<td>( C_\nu(a) )</td>
<td>Young functions ( 3.76 )</td>
</tr>
<tr>
<td>( C_\lambda^A(t) )</td>
<td>Gegenbauer polynomials ( 8.93 )</td>
</tr>
<tr>
<td>( C_\lambda^A(x) )</td>
<td>Gegenbauer functions ( 8.932 )</td>
</tr>
<tr>
<td>( ce_{2n}(z,q), ce_{2n+1}(z,q) )</td>
<td>Periodic Mathieu functions (Mathieu functions of the first kind) ( 8.61 )</td>
</tr>
<tr>
<td>( Ce_{2n}(z,q), Ce_{2n+1}(z,q) )</td>
<td>Associated (modified) Mathieu functions of the first kind ( 8.63 )</td>
</tr>
<tr>
<td>( \text{chi}(x) )</td>
<td>Hyperbolic cosine integral function ( 8.22 )</td>
</tr>
<tr>
<td>( \text{ci}(x) )</td>
<td>Cosine integral ( 8.23 )</td>
</tr>
<tr>
<td>( \text{cn}(u) )</td>
<td>Cosine amplitude ( 8.14 )</td>
</tr>
<tr>
<td>( D(k) \equiv D )</td>
<td>Elliptic integral ( 8.11 )</td>
</tr>
<tr>
<td>( D(\varphi,k) )</td>
<td>Elliptic integral ( 8.11 )</td>
</tr>
<tr>
<td>( D_n(z), D_p(z) )</td>
<td>Parabolic cylinder functions ( 9.24-9.25 )</td>
</tr>
<tr>
<td>( \text{dn}(u) )</td>
<td>Delta amplitude ( 8.14 )</td>
</tr>
<tr>
<td>( e_1, e_2, e_3 )</td>
<td>(used with the Weierstrass function) ( 8.162 )</td>
</tr>
<tr>
<td>( E_n )</td>
<td>Euler numbers ( 9.63, 9.72 )</td>
</tr>
<tr>
<td>( E(\varphi,k) )</td>
<td>Elliptic integral of the second kind ( 8.11-8.12 )</td>
</tr>
<tr>
<td>( { E(k) = E } )</td>
<td>Complete elliptic integral of the second kind ( 8.11-8.12 )</td>
</tr>
<tr>
<td>( pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) )</td>
<td>MacRobert’s function ( 9.4 )</td>
</tr>
<tr>
<td>( 2F_1(\alpha, \beta; \gamma; z) = \Phi(\alpha, \beta; \gamma; z) )</td>
<td>Weber function ( 8.58 )</td>
</tr>
<tr>
<td>( F(\varphi,k) )</td>
<td>Exponential integral function ( 8.21 )</td>
</tr>
<tr>
<td>( \text{erf}(x) )</td>
<td>Error function ( 8.25 )</td>
</tr>
<tr>
<td>( \text{erfc}(x) = 1 - \text{erf}(x) )</td>
<td>Complementary error function ( 8.25 )</td>
</tr>
<tr>
<td>( \text{Fey}_n(z,q) )</td>
<td>Elliptic integral of the first kind ( 8.11-8.12 )</td>
</tr>
<tr>
<td>( \gamma_1, \ldots, \gamma_n : z_1, \ldots, z_n )</td>
<td>Generalized hypergeometric series ( 9.14 )</td>
</tr>
<tr>
<td>( F_1(z,q) )</td>
<td>Gauss hypergeometric function ( 9.10-9.13 )</td>
</tr>
<tr>
<td>( F_2(z,q) )</td>
<td>Degenerate hypergeometric function ( 9.21 )</td>
</tr>
<tr>
<td>( G )</td>
<td>Hypergeometric function of several variables ( 9.19 )</td>
</tr>
<tr>
<td>( g_2, g_3 )</td>
<td>Hypergeometric functions of two variables ( 9.18 )</td>
</tr>
<tr>
<td>( gd )</td>
<td>Other nonperiodic solutions of Mathieu’s equation ( 8.64, 8.663 )</td>
</tr>
<tr>
<td>( \gamma_1, \ldots, \gamma_n : z_1, \ldots, z_n )</td>
<td>Catalan constant ( 9.73 )</td>
</tr>
<tr>
<td>( { fe_n(z,q), Fe_n(z,q) } )</td>
<td>Invariants of the ( \varphi(u) )-function ( 8.161 )</td>
</tr>
<tr>
<td>( { Fey_n(z,q), Fey_n(z,q) } )</td>
<td>Gudermannian ( 1.49 )</td>
</tr>
<tr>
<td>( G_{p,q}^{m,n} )</td>
<td>Other nonperiodic solutions of Mathieu’s equation ( 8.64, 8.663 )</td>
</tr>
<tr>
<td>( g_e_n(z,q), Ge_n(z,q) )</td>
<td>Meijer’s functions ( 9.3 )</td>
</tr>
</tbody>
</table>

\[ \text{continued on next page} \]
<table>
<thead>
<tr>
<th>Notation</th>
<th>Name of the function and the number of the formula containing its definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(n)$</td>
<td>Unit integer function 18.1</td>
</tr>
<tr>
<td>$\text{hei}<em>{\nu}(z)$, $\text{her}</em>{\nu}(z)$</td>
<td>Hankel functions of the first and second kinds 8.405, 8.42</td>
</tr>
<tr>
<td>$H^{(1)}<em>{\nu}(z)$, $H^{(2)}</em>{\nu}(z)$</td>
<td>Theta function 8.192</td>
</tr>
<tr>
<td>$H(u) = \vartheta_1 \left( \frac{\pi u}{2} \right)$</td>
<td>Theta function 8.192</td>
</tr>
<tr>
<td>$I_\nu(z)$</td>
<td>Bessel functions of an imaginary argument 8.406, 8.43</td>
</tr>
<tr>
<td>$I_x(p, q)$</td>
<td>Normalized incomplete beta function 8.39</td>
</tr>
<tr>
<td>$J_\nu(z)$</td>
<td>Bessel function 8.407, 8.43</td>
</tr>
<tr>
<td>$L(x)$</td>
<td>Modified Struve function 8.55</td>
</tr>
<tr>
<td>$L_\nu(z)$</td>
<td>Struve functions 8.55</td>
</tr>
<tr>
<td>$K_\nu(z)$</td>
<td>Bessel functions of imaginary argument 8.407, 8.43</td>
</tr>
<tr>
<td>$K_\nu(z)$</td>
<td>Thomson functions 8.56</td>
</tr>
<tr>
<td>$\text{kei}(z)$, $\text{ker}(z)$</td>
<td>Thomson functions 8.56</td>
</tr>
<tr>
<td>$L(x)$</td>
<td>Lobachevskiy’s function 8.26</td>
</tr>
<tr>
<td>$L_\nu(z)$</td>
<td>Modified Struve function 8.55</td>
</tr>
<tr>
<td>$L^\alpha_\nu(z)$</td>
<td>Laguerre polynomials 8.97</td>
</tr>
<tr>
<td>$\text{hi}(z)$</td>
<td>Logarithm integral 8.24</td>
</tr>
<tr>
<td>$M_\lambda,\mu(z)$</td>
<td>Whittaker functions 9.22, 9.23</td>
</tr>
<tr>
<td>$O_n(x)$</td>
<td>Neumann’s polynomials 8.59</td>
</tr>
<tr>
<td>$P_{\alpha\beta\gamma\delta}^{a bc}(x)$</td>
<td>Riemann’s differential equation 9.160</td>
</tr>
<tr>
<td>$P_{\nu}(z)$, $P_{\nu}^\mu(x)$</td>
<td>Associated Legendre functions of the first kind 8.7, 8.8</td>
</tr>
<tr>
<td>$P_{\nu}(z)$, $P_{\nu}(x)$</td>
<td>Legendre functions and polynomials 8.82, 8.83, 8.91</td>
</tr>
<tr>
<td>$Q_{\alpha\beta\gamma\delta}^{a bc}(x)$</td>
<td>Jacobi’s polynomials 8.96</td>
</tr>
<tr>
<td>$Q_{\nu}(z)$, $Q_{\nu}^\mu(x)$</td>
<td>Associated Legendre functions of the second kind 8.7, 8.8</td>
</tr>
<tr>
<td>$S(x)$</td>
<td>Legendre functions of the second kind 8.82, 8.83</td>
</tr>
<tr>
<td>$S(z)$</td>
<td>Fresnel sine integral 8.25</td>
</tr>
<tr>
<td>$S_n(x)$</td>
<td>Schläflí’s polynomials 8.59</td>
</tr>
<tr>
<td>$s_{\mu,\nu}(z)$, $S_{\mu,\nu}(z)$</td>
<td>Lommel functions 8.57</td>
</tr>
<tr>
<td>$\text{se}<em>{2n+1}(z, q)$, $\text{se}</em>{2n+2}(z, q)$</td>
<td>Periodic Mathieu functions 8.61</td>
</tr>
<tr>
<td>$\text{Se}<em>{2n+1}(z, q)$, $\text{Se}</em>{2n+2}(z, q)$</td>
<td>Mathieu functions of an imaginary argument 8.63</td>
</tr>
<tr>
<td>$\text{sh}(x)$</td>
<td>Hyperbolic sine integral 8.22</td>
</tr>
<tr>
<td>$\text{si}(x)$</td>
<td>Sine integral 8.23</td>
</tr>
<tr>
<td>$\text{sn}u$</td>
<td>Sine amplitude 8.14</td>
</tr>
<tr>
<td>$T_n(x)$</td>
<td>Chebyshev polynomial of the 1st kind 8.94</td>
</tr>
<tr>
<td>$U_n(x)$</td>
<td>Chebyshev polynomials of the 2nd kind 8.94</td>
</tr>
</tbody>
</table>

continued on next page
<table>
<thead>
<tr>
<th>Notation</th>
<th>Name of the function and the number of the formula containing its definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_\nu(w, z) ), ( V_\nu(w, z) )</td>
<td>Lommel functions of two variables ( 8.578 )</td>
</tr>
<tr>
<td>( W_{\lambda, \mu}(z) )</td>
<td>Whittaker functions ( 9.22, 9.23 )</td>
</tr>
<tr>
<td>( Y_\nu(z) )</td>
<td>Neumann functions ( 8.403, 8.41 )</td>
</tr>
<tr>
<td>( Z_\nu(z) )</td>
<td>Bessel functions ( 8.401 )</td>
</tr>
<tr>
<td>( \mathfrak{z}_\nu(z) )</td>
<td>Bessel functions</td>
</tr>
</tbody>
</table>
### Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lfloor x \rfloor$</td>
<td>The integral part of the real number $x$ (also denoted by $[x]$)</td>
</tr>
<tr>
<td>$\int_a^{(b+)} \int_a^{(b-)}$</td>
<td>Contour integrals; the path of integration starting at the point $a$ extends to the point $b$ (along a straight line unless there is an indication to the contrary), encircles the point $b$ along a small circle in the positive (negative) direction, and returns to the point $a$, proceeding along the original path in the opposite direction.</td>
</tr>
<tr>
<td>$\int_C$</td>
<td>Line integral along the curve $C$</td>
</tr>
<tr>
<td>$\text{PV} \int$</td>
<td>Principal value integral</td>
</tr>
<tr>
<td>$\overline{z} = x - iy$</td>
<td>The complex conjugate of $z = x + iy$</td>
</tr>
<tr>
<td>$n!$</td>
<td>$= 1 \cdot 2 \cdot 3 \ldots n$, $0! = 1$</td>
</tr>
<tr>
<td>$(2n + 1)!!$</td>
<td>$= 1 \cdot 3 \ldots (2n + 1)$. (double factorial notation)</td>
</tr>
<tr>
<td>$(2n)!!$</td>
<td>$= 2 \cdot 4 \ldots (2n)$. (double factorial notation)</td>
</tr>
<tr>
<td>$0!! = 1$ and $(-1)!! = 1$</td>
<td>(cf. 3.372 for $n = 0$)</td>
</tr>
<tr>
<td>$0^0 = 1$</td>
<td>(cf. 0.112 and 0.113 for $q = 0$)</td>
</tr>
<tr>
<td>$\binom{p}{n}$</td>
<td>$= \frac{p(p - 1) \ldots (p - n + 1)}{1 \cdot 2 \cdot n} = \frac{p!}{n!(p - n)!}$, $\binom{p}{0} = 1$, $\binom{p}{n} = \frac{p!}{n!(p - n)!}$</td>
</tr>
<tr>
<td>$\binom{x}{n}$</td>
<td>$= x(x - 1) \ldots (x - n + 1)/n!$ [$n = 0, 1, \ldots$]</td>
</tr>
<tr>
<td>$(a)_n$</td>
<td>$= a(a + 1) \ldots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}$ (Pochhammer symbol)</td>
</tr>
<tr>
<td>$\sum_{k=m}^n u_k$</td>
<td>$= u_m + u_{m+1} + \ldots + u_n$. If $n &lt; m$, we define $\sum_{k=m}^n u_k = 0$</td>
</tr>
<tr>
<td>$\sum_{n}^\prime$, $\sum_{m,n}^\prime$</td>
<td>Summation over all integral values of $n$ excluding $n = 0$, and summation over all integral values of $n$ and $m$ excluding $m = n = 0$, respectively.</td>
</tr>
<tr>
<td>$\sum$, $\prod$</td>
<td>An empty $\sum$ has value 0, and an empty $\prod$ has value 1</td>
</tr>
</tbody>
</table>

*continued on next page*
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{ij} = \begin{cases} 1 &amp; i = j \ 0 &amp; i \neq j \end{cases}$</td>
<td>Kronecker delta</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Theta function parameter (cf. 8.18)</td>
</tr>
<tr>
<td>$\times$ and $\wedge$</td>
<td>Vector product (cf. 10.11)</td>
</tr>
<tr>
<td>$.$</td>
<td>Scalar product (cf. 10.11)</td>
</tr>
<tr>
<td>$\nabla$ or “del”</td>
<td>Vector operator (cf. 10.21)</td>
</tr>
<tr>
<td>$\nabla^2$</td>
<td>Laplacian (cf. 10.31)</td>
</tr>
<tr>
<td>$\sim$</td>
<td>Asymptotically equal to</td>
</tr>
<tr>
<td>$\arg z$</td>
<td>The argument of the complex number $z = x + iy$</td>
</tr>
<tr>
<td>curl or rot</td>
<td>Vector operator (cf. 10.21)</td>
</tr>
<tr>
<td>div</td>
<td>Vector operator (divergence) (cf. 10.21)</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>Fourier transform (cf. 17.21)</td>
</tr>
<tr>
<td>$\mathcal{F}_c$</td>
<td>Fourier cosine transform (cf. 17.31)</td>
</tr>
<tr>
<td>$\mathcal{F}_s$</td>
<td>Fourier sine transform (cf. 17.31)</td>
</tr>
<tr>
<td>grad</td>
<td>Vector operator (gradient) (cf. 10.21)</td>
</tr>
<tr>
<td>$h_i$ and $g_{ij}$</td>
<td>Metric coefficients (cf. 10.51)</td>
</tr>
<tr>
<td>$^H$</td>
<td>Hermitian transpose of a vector or matrix (cf. 13.123)</td>
</tr>
<tr>
<td>$H(x) = \begin{cases} 0 &amp; x &lt; 0 \ 1 &amp; x \geq 0 \end{cases}$</td>
<td>Heaviside step function</td>
</tr>
<tr>
<td>$\Im z \equiv y$</td>
<td>The imaginary part of the complex number $z = x + iy$</td>
</tr>
<tr>
<td>$k$</td>
<td>The letter $k$ (when not used as an index of summation) denotes a number in the interval $[0, 1]$. This notation is used in integrals that lead to elliptic integrals. In such a connection, the number $\sqrt{1 - k^2}$ is denoted by $k'$.</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>Laplace transform (cf. 17.11)</td>
</tr>
<tr>
<td>$\mathcal{M}$</td>
<td>Mellin transform (cf. 17.41)</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>The natural numbers $(0, 1, 2, \ldots)$</td>
</tr>
<tr>
<td>$O(f(z))$</td>
<td>The order of the function $f(z)$. Suppose that the point $z$ approaches $z_0$. If there exists an $M &gt; 0$ such that $</td>
</tr>
<tr>
<td>Symbol</td>
<td>Meaning</td>
</tr>
<tr>
<td>--------</td>
<td>---------</td>
</tr>
<tr>
<td>$q$</td>
<td>The nome, a theta function parameter (cf. 8.18)</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>The real numbers</td>
</tr>
<tr>
<td>$R(x)$</td>
<td>A rational function</td>
</tr>
<tr>
<td>$\text{Re } z \equiv x$</td>
<td>The real part of the complex number $z = x + iy$</td>
</tr>
<tr>
<td>$S_n^m$</td>
<td>Stirling number of the first kind (cd. 9.74)</td>
</tr>
<tr>
<td>$\mathfrak{S}_n^m$</td>
<td>Stirling number of the second kind (cd. 9.74)</td>
</tr>
<tr>
<td>$\text{sign } x = \begin{cases} +1 &amp; x &gt; 0 \ 0 &amp; x = 0 \ -1 &amp; x &lt; 0 \end{cases}$</td>
<td>The sign (signum) of the real number $x$</td>
</tr>
<tr>
<td>$\mathbf{T}$</td>
<td>Transpose of a vector or matrix (cf. 13.115)</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>The integers ${0, \pm 1, \pm 2, \ldots}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_b$</td>
<td>Bilateral $z$ transform (cf. 18.1)</td>
</tr>
<tr>
<td>$\mathbb{Z}_u$</td>
<td>Unilateral $z$ transform (cf. 18.1)</td>
</tr>
</tbody>
</table>
Note on the Bibliographic References

The letters and numbers following equations refer to the sources used by Russian editors. The key to the letters will be found preceding each entry in the Bibliography beginning on page 1141. Roman numerals indicate the volume number of a multivolume work. Numbers without parentheses indicate page numbers, numbers in single parentheses refer to equation numbers in the original sources.

Some formulas were changed from their form in the source material. In such cases, the letter \( a \) appears at the end of the bibliographic references.

As an example, we may use the reference to equation 3.354–5:

\[
\text{ET I 118 (1) } a
\]

The key on page 1141 indicates that the book referred to is:

Erdélyi, A. et al., *Tables of Integral Transforms*.

The Roman numeral denotes volume one of the work; 118 is the page on which the formula will be found; (1) refers to the number of the formula in this source; and the \( a \) indicates that the expression appearing in the source differs in some respect from the formula in this book.

In several cases, the editors have used Russian editions of works published in other languages. Under such circumstances, because the pagination and numbering of equations may be altered, we have referred the reader only to the original sources and dispensed with page and equation numbers.