CHAPTER 1

Domain Perturbation for Linear and Semi-Linear Boundary Value Problems

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Abstract

This is a survey on elliptic boundary value problems on varying domains and tools needed for that. Such problems arise in numerical analysis, in shape optimisation problems and in the investigation of the solution structure of nonlinear elliptic equations. The methods are also useful to obtain certain results for equations on non-smooth domains by approximation by smooth domains.

Domain independent estimates and smoothing properties are an essential tool to deal with domain perturbation problems, especially for non-linear equations. Hence we discuss such estimates extensively, together with some abstract results on linear operators.

A second major part deals with specific domain perturbation results for linear equations with various boundary conditions. We completely characterise convergence for Dirichlet boundary conditions and also give simple sufficient conditions. We then prove boundary homogenisation results for Robin boundary conditions on domains with fast oscillating boundaries, where the boundary condition changes in the limit. We finally mention some simple results on problems with Neumann boundary conditions.

The final part is concerned about non-linear problems, using the Leray-Schauder degree to prove the existence of solutions on slightly perturbed domains. We also demonstrate how to use the approximation results to get solutions to nonlinear equations on unbounded domains.

Keywords: Elliptic boundary value problem, Domain perturbation, Semilinear equations, A priori estimates, Boundary homogenisation

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 6
Edited by M. Chipot

Contents

1. Introduction

The purpose of this survey is to look at elliptic boundary value problems

$$\begin{aligned} \mathcal{A}_n u &= f \quad \text{in } \Omega_n, \\ \mathcal{B}_n u &= 0 \quad \text{on } \partial\Omega_n \end{aligned}$$

with all major types of boundary conditions on a sequence of open sets Ω_n in $\mathbb{R}^N$ ($N \geq 2$). We then study conditions under which the solutions converge to a solution of a limit problem

$$\begin{aligned} \mathcal{A} u &= f \quad \text{in } \Omega, \\ \mathcal{B} u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

on some open set $\Omega \subset \mathbb{R}^N$. In the simplest case $\mathcal{A}, \mathcal{A}_n = -\Delta$ is the negative Laplace operator, and $\mathcal{B}_n, \mathcal{B}$ the Dirichlet, Robin or Neumann boundary operator, but we work with general non-selfadjoint elliptic operators in divergence form. We are interested in very singular perturbation, not necessarily of a type such that a change of variables can be applied to reduce the problem onto a fixed domain. For the theory of smooth perturbations rather complementary to ours we refer to [84] and references therein.

The main features of this exposition are the following:

- We present an L_p-theory of linear and semi-linear elliptic boundary value problems with domain perturbation in view.
- We establish domain perturbation results for linear elliptic problems with Dirichlet, Robin and Neumann boundary conditions, applicable to semi-linear problems.
- We show how to use the linear perturbation theory to deal with semi-linear problems on bounded and unbounded domains. In particular we show how to get multiple solutions for simple equations, discuss the issue of precise multiplicity and the occurrence of large solutions.
- We provide abstract perturbation theorems useful also for perturbations other than domain perturbations.
- We provide tools to prove results for linear and nonlinear equations on general domains by means of smoothing domains and operators (see Section 8).

Our aim is to build a domain perturbation theory suitable for applications to semi-linear problems, that is, problems where $f = f(x, u(x))$ is a function of x as well as the solution $u(x)$. For nonlinearities with growth, polynomial or arbitrary, we need a good theory for the linear problem in L_p for $1 < p < \infty$. Good in the context of domain perturbations means that in all estimates there is *control on domain dependence* of the constants involved. We establish such a theory in Section 2.1, where we also introduce precise assumptions on the operators. Starting from a definition of weak solutions we prove smoothing properties of the corresponding resolvent operators with control on domain dependence. The main results are Theorems 2.4.1 and 2.4.2. In particular we prove that the resolvent operators have smoothing properties independent of the domain for Dirichlet and Robin boundary conditions, but not for Neumann boundary conditions. To be able to work in a common space we consider the resolvent operator as a map acting on $L_p(\mathbb{R}^N)$, so that it becomes a pseudo-resolvent (see Section 2.5).

The smoothing properties of the resolvent operators enable us to reformulate a semi-linear boundary value problem as a fixed point problem in $L_p(\mathbb{R}^N)$. Which $p \in (1, \infty)$ we choose depends on the growth of the nonlinearity. The rule is, the faster the growth, the larger the choice of p. We also show that under suitable growth conditions, a solution in L_p is in fact in L_∞. Again, the focus is on getting control over the domain dependence of the constants involved. For a precise formulation of these results we refer to Section 3.

Let the resolvent operator corresponding to the linear problems be denoted by $R_n(\lambda)$ and $R(\lambda)$. The key to be able to pass from perturbations of the linear to perturbations of the nonlinear problem is the following property of the resolvents:

$$\text{If } f_n \rightharpoonup f \text{ weakly, then } R_n(\lambda) f_n \to R(\lambda) f \text{ strongly.}$$

Hence for all types of boundary conditions we prove such a statement. If $R(\lambda)$ is compact, it turns out that the above property is equivalent to convergence in the operator norm. An issue connected with that is also the convergence of the spectrum. We show that the above property implies the convergence of every finite part of the spectrum of the relevant differential operators. We also show here that it is sufficient to prove convergence of the resolvent in $L_p(\mathbb{R}^N)$ for some p and some λ to have them for all. These abstract results are collected in Section 4.

The most complete convergence results are known for the Dirichlet problem (Section 5). The limit problem is always a Dirichlet problem on some domain. We extensively discuss convergence in the operator norm. In particular, we look at necessary and sufficient conditions for pointwise and uniform convergence of the resolvent operators. As a corollary to the characterisation we see that convergence is independent of the operator chosen. We also give simple sufficient conditions for convergence in terms of properties of the set $\Omega_n \cap \bar{\Omega}^c$. The main source for these results is [58].

The situation is rather more complicated for Robin boundary conditions, where the type of boundary condition can change in the limit problem. In Section 6 we present three different cases. First, we look at problems with only a small perturbation of $\partial\Omega$. We can cut holes and add thin pieces outside Ω, connected to Ω only near a set of capacity zero. Second, we look at approximating domains with very fast oscillating boundary. In that case the limit problem has Dirichlet boundary conditions. Third, we deal with domains with oscillating boundary, such that the limit problem involves Robin boundary conditions with a different weight on the boundary. The second and third results are really *boundary homogenisation* results. These results are all taken from [51].

The Neumann problem is very badly behaved, and without quite severe restriction on the sequence of domains Ω_n we cannot expect the resolvent to converge in the operator norm. In particular the spectrum does not converge, as already noted in [40, page 420]. We only prove a simple convergence result fitting into the general framework established for the other boundary conditions.

After dealing with linear equations we consider semi-linear equations. A lot of this part is inspired by Dancer's paper [45] and related work. The approach is quite different since we treat linear equations first, and then use their properties to deal with nonlinear equations. The idea is to use degree theory to get solutions on a nearby domain, given a solution of the limit problem. The core of the argument is an abstract topological argument which may be useful also for other types of perturbations (see Section 9.2). We also discuss the issue of precise multiplicity of solutions and the phenomenon of large solutions. Finally, we show that the theory also applies to unbounded limit domains.

There are many other motivations to look at domain perturbation problems, so for instance variational inequalities (see [102]), numerical analysis (see [77,107,110,116–119]), potential and scattering theory (see [10,108,113,124]), control and optimisation (see [31,34,82,120]), Γ-convergence (see [24,42]) and solution structures of nonlinear elliptic equations (see [45,47,52,69]). We mention more references in the discussion on the specific boundary conditions. Some results go back a long time, see for instance [19] or [40]. The techniques are even older for the Dirichlet problem for harmonic functions with the pioneering work [93].

Finally, there are many results we do not even mention, so for instance for convergence in the L_∞-norm we refer to [8,9,14,23,26]. Furthermore, similar results can be proved for parabolic problems. The key for that are domain-independent heat kernel estimates. See for instance [7,17,47,52,59,78] and references therein. The above is only a small rather arbitrary selection of references.

2. Elliptic boundary value problems in divergence form

The purpose of this section is to give a summary of results on elliptic boundary value problems in divergence form with emphasis on estimates with control over the domain dependence.

2.1. *Weak solutions to elliptic boundary value problems*

We consider boundary value problems of the form

$$\begin{aligned} \mathcal{A}u &= f \quad \text{in } \Omega, \\ \mathcal{B}u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1.1}$$

on an open subset of $\mathbb{R}^N$, not necessarily bounded or connected. Here $\mathcal{A}$ is an elliptic operator in divergence form and $\mathcal{B}$ a boundary operator to be specified later in this section. The operator $\mathcal{A}$ is of the form

$$-\operatorname{div}(A_0\nabla u + au) + b\cdot\nabla u + c_0 u \tag{2.1.2}$$

with $A_0 \in L_\infty(\Omega, \mathbb{R}^{N\times N})$, $a, b \in L_\infty(\Omega, \mathbb{R}^N)$ and $c_0 \in L_\infty(\Omega)$. Moreover, we assume that $A_0(x)$ is positive definite, uniformly with respect to $x \in \Omega$. More precisely, there exists a constant $\alpha_0 > 0$ such that

$$\alpha_0|\xi|^2 \leq \xi^T A_0(x)\xi \tag{2.1.3}$$

for all $\xi \in \mathbb{R}^N$ and almost all $x \in \Omega$. We call α_0 the *ellipticity constant*.

REMARK 2.1.1. We only defined the operator $\mathcal{A}$ on Ω, but we can extend it to $\mathbb{R}^N$ by setting $a = b = 0$, $c_0 = 0$ and $A(x) := \alpha_0 I$ on Ω^c. Then the extended operator $\mathcal{A}$ also satisfies (2.1.3). In particular the ellipticity property (2.1.3) holds. Hence without loss of generality we can assume that $\mathcal{A}$ is defined on $\mathbb{R}^N$.

We further define the *co-normal derivative* associated with $\mathcal{A}$ on $\partial\Omega$ by

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} := (A_0(x)\nabla u + a(x)u) \cdot \nu,$$

where ν is the outward pointing unit normal to $\partial\Omega$. Assuming that $\partial\Omega$ is the disjoint union of Γ_1, Γ_2 and Γ_3 we define the boundary operator $\mathcal{B}$ by

$$\mathcal{B}u := \begin{cases} u|_{\Gamma_1} & \text{on } \Gamma_1 \text{ (Dirichlet b.c.)}, \\ \dfrac{\partial u}{\partial \nu_{\mathcal{A}}} & \text{on } \Gamma_2 \text{ (Neumann b.c.)}, \\ \dfrac{\partial u}{\partial \nu_{\mathcal{A}}} + b_0 u & \text{on } \Gamma_3 \text{ (Robin b.c.)} \end{cases} \tag{2.1.4}$$

with $b_0 \in L_\infty(\Gamma_3)$ nonnegative. If all functions involved are sufficiently smooth, then by the product rule

$$-v\operatorname{div}(A_0\nabla u + au) = (A_0\nabla u + au) \cdot \nabla v - \operatorname{div}(v(A_0\nabla u + au))$$

and therefore, if Ω admits the divergence theorem, then

$$\begin{aligned} &-\int_\Omega v\operatorname{div}(A_0\nabla u + au)\,dx \\ &\qquad = \int_\Omega (A_0\nabla u + au) \cdot \nabla v\,dx - \int_{\partial\Omega} (v(A_0\nabla u + au)) \cdot \nu\,d\sigma, \end{aligned}$$

where σ is the surface measure on $\partial\Omega$. Hence, if u is sufficiently smooth and $v \in C^1(\bar{\Omega})$ with $v = 0$ on Γ_1, then

$$\int_\Omega v\mathcal{A}u\,dx = \int_\Omega (A_0\nabla u + au) \cdot \nabla v + (a \cdot \nabla u + c_0 u)v\,dx + \int_{\Gamma_3} b_0 uv\,d\sigma.$$

The expression on the right-hand side defines a bilinear form. We denote by $H^1(\Omega)$ the usual Sobolev space of square integrable functions having square integrable weak partial derivatives. Moreover, $H_0^1(\Omega)$ is the closure of the set of test functions $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

DEFINITION 2.1.2. For $u, v \in H^1(\Omega)$ we set

$$a_0(u, v) := \int_\Omega (A_0\nabla u + au) \cdot \nabla v + (b \cdot \nabla u + c_0 u)v\,dx.$$

The expression

$$a(u, v) := a_0(u, v) + \int_{\Gamma_3} b_0 uv\,d\sigma$$

is called the *bilinear form associated with* $(\mathcal{A}, \mathcal{B})$.

If u is a sufficiently smooth solution of (2.1.1), then

$$a(u, v) = \langle f, v\rangle := \int_\Omega f v\,dx \tag{2.1.5}$$

for all $v \in C^1(\bar{\Omega})$ with $v = 0$ on Γ_1. Note that (2.1.5) does not just make sense for classical solutions of (2.1.1), but for $u \in H^1(\Omega)$ as long as the boundary integral is defined. We therefore generalise the notion of solution and just require that u is in a suitable subspace V of $H^1(\Omega)$ and (2.1.5) for all $v \in V$.

ASSUMPTION 2.1.3. We require that V be a Hilbert space such that V is dense in $L_2(\Omega)$, that

$$H_0^1(\Omega) \hookrightarrow V \hookrightarrow H^1(\Omega),$$

and that

$$\left\{u \in C^1(\bar{\Omega})\colon \operatorname{supp} u \subset \bar{\Omega} \setminus \Gamma_1, \int_{\Gamma_3} b_0|u|^2\,d\sigma < \infty\right\} \subset V.$$

If Γ_3 is nonsmooth we replace the surface measure σ by the $(N-1)$-dimensional Hausdorff measure so that the boundary integral makes sense. We also require that

$$\|u\|_V := (\|u\|^2_{H^1(\Omega)} + \alpha_0^{-1}\|u\sqrt{b_0}\|^2_{L_2(\Gamma_3)})^{1/2} \tag{2.1.6}$$

is an equivalent norm on V.

We next consider some specific special cases.

EXAMPLE 2.1.4. (a) For a homogeneous Dirichlet problem we assume that $\Gamma_1 = \partial\Omega$ and let $V := H_0^1(\Omega)$. For the norm we can choose the usual H^1-norm, but on bounded domains we could just use the equivalent norm $\|\nabla u\|_2$. More generally, on domains Ω lying between two hyperplanes of distance D, we can work with the equivalent norm $\|\nabla u\|_2$ because of Friedrich's inequality

$$\|u\|_2 \le D\|\nabla u\|_2 \tag{2.1.7}$$

valid for all $u \in H_0^1(\Omega)$ (see [111, Theorem II.2.D]).

(b) For a homogeneous Neumann problem we assume that $\Gamma_2 = \partial\Omega$ and let $V := H^1(\Omega)$ with the usual norm.

(c) For a homogeneous Robin problem we assume that $\Gamma_3 = \partial\Omega$. In this exposition we will always assume that Ω is a Lipschitz domain when working with Robin boundary conditions and choose $V := H^1(\Omega)$. On a bounded domain we can work with the equivalent norm

$$\|v\|_V := (\|\nabla u\|_2^2 + \|u\|^2_{L_2(\partial\Omega)})^{1/2} \tag{2.1.8}$$

(see [111, Theorem III.5.C] or [56]). It is possible to admit arbitrary domains as shown in [11,56].

We finally define what we mean by a weak solution of (2.1.1).

DEFINITION 2.1.5 (Weak solution). We say u is a weak solution of (2.1.1) if $u \in V$ and (2.1.5) holds for all $v \in V$. Moreover, we say that u is a weak solution of $\mathcal{A} = f$ in Ω if $u \in H^1_{\mathrm{loc}}(\Omega)$ such that (2.1.5) holds for all $v \in C_c^\infty(\Omega)$.

Note that a weak solution of $\mathcal{A} = f$ on Ω does not need to satisfy any boundary conditions. As we shall see, it is often easy to get a weak solution in Ω by domain approximation. The most difficult part is to verify that it satisfies boundary conditions. We next collect some properties of the form $a(\cdot\,,\cdot)$ on V. In what follows we use the norms

$$\|A\|_\infty := \Bigl(\sum_{i,j=1}^N \|a_{ij}\|_\infty^2\Bigr)^{1/2} \quad \text{and} \quad \|a\|_\infty := \Bigl(\sum_{i=1}^N \|a_i\|_\infty^2\Bigr)^{1/2}$$

for matrices $A = [a_{ij}]$ and vectors $a = (a_1, \dots, a_N)$.

PROPOSITION 2.1.6. *Suppose that* $(\mathcal{A}, \mathcal{B})$ *is defined as above. Then there exists* $M > 0$ *such that*

$$|a(u, v)| \le M\|u\|_V\|v\|_V \tag{2.1.9}$$

for all $u, v \in V$. *More precisely we can set*

$$M = \|A\|_\infty + \|a\|_\infty + \|b\|_\infty + \|c_0\|_\infty + \alpha_0.$$

Moreover, if we let

$$\lambda_{\mathcal{A}} := \|c_0^-\|_\infty + \frac{1}{2\alpha_0}\|a + b\|_\infty^2, \tag{2.1.10}$$

where α_0 *is the ellipticity constant from* (2.1.3)*, then*

$$\frac{\alpha_0}{2}\|\nabla u\|_2^2 \le a_0(u, u) + \lambda_{\mathcal{A}}\|u\|_2^2 \tag{2.1.11}$$

for all $u \in H^1(\Omega)$. *Finally, setting* $\lambda_0 := \lambda_{\mathcal{A}} + \alpha_0/2$ *we see that*

$$\frac{\alpha_0}{2}\|u\|_V^2 \le a(u, u) + \lambda_0\|u\|_2^2$$

for all $u \in V$.

PROOF. By the Cauchy–Schwarz inequality and the definition of $a_0(\cdot\,,\cdot)$

$$\begin{aligned}|a_0(u, v)| &\le \|A\nabla u\|_2\|\nabla v\|_2 + \|au\|_2\|\nabla v\|_2 + \|b\nabla u\|_2\|v\|_2 + \|c_0u\|_2\|v\|_2\\ &\le \|A\|_\infty\|\nabla u\|_2\|\nabla v\|_2 + \|a\|_\infty\|u\|_2\|\nabla v\|_2\\ &\qquad + \|b\|_\infty\|\nabla u\|_2\|v\|_2 + \|c_0\|_\infty\|u\|_2\|v\|_2\\ &\le (\|A\|_\infty + \|a\|_\infty + \|b\|_\infty + \|c_0\|_\infty)\|u\|_V\|v\|_V\end{aligned}$$

for all $u, v \in V$. Similarly for the boundary integral

$$\int_{\Gamma_3} b_0uv\,d\sigma \le \|u\sqrt{b_0}\|_{L_2(\Gamma_3)}\|v\sqrt{b_0}\|_{L_2(\Gamma_3)} \le \alpha_0\|u\|_V\|v\|_V$$

for all $u, v \in V$, where we used (2.1.6) for the definition of the norm in V. Combining the two inequalities, (2.1.9) follows. We next prove (2.1.11). Given $u \in H^1(\Omega)$, using (2.1.3) we get

$$\begin{aligned}\alpha_0\|\nabla u\|_2^2 &\leq \int_\Omega (A\nabla u)\cdot\nabla u\,dx\\ &\leq a_0(u,u) - \int_\Omega (a+b)u\cdot\nabla u + c_0^-|u|^2\,dx\\ &\leq a_0(u,u) + \|a+b\|_\infty\|u\|_2\|\nabla u\|_2 + \|c_0^-\|_\infty\|u\|_2^2\\ &\leq a_0(u,u) + \frac{1}{2\alpha_0}\|a+b\|_\infty^2\|u\|_2^2 + \frac{\alpha_0}{2}\|\nabla u\|_2^2 + \|c_0^-\|_\infty\|u\|_2^2\end{aligned}$$

if we use the elementary inequality $xy \leq x^2/2\varepsilon + \varepsilon y^2/2$ valid for $x, y \geq 0$ and $\varepsilon > 0$ in the last step. Rearranging the inequality we get (2.1.11). If we add $\alpha_0\|u\|_2^2/2$ and the boundary integral if necessary to (2.1.11), then the final assertion follows. □

2.2. *Abstract formulation of boundary value problems*

We saw in Section 2.1 that all boundary value problems under consideration have the following structure.

ASSUMPTION 2.2.1 (Abstract elliptic problem). There exist Hilbert spaces V and H such that $V \hookrightarrow H$ and V is dense in H. Suppose there exists a bilinear form

$$a(\cdot\,,\cdot)\colon V\times V\to\mathbb{R}$$

with the following properties. There exists a constant $M > 0$ such that

$$|a(u,v)| \leq M\|u\|_V\|v\|_V \tag{2.2.1}$$

for all $u, v \in V$. Also, there exist constants $\alpha > 0$ and $\lambda_0 \geq 0$ such that

$$\alpha\|u\|_V^2 \leq a(u,u) + \lambda_0\|u\|_H^2 \tag{2.2.2}$$

for all $u \in V$.

By assumption on V and H we have

$$V \hookrightarrow H \hookrightarrow V'$$

if we identify H with its dual H' by means of the Riesz representation theorem with both embeddings being dense. In particular

$$|\langle u, v\rangle| \leq \|u\|_{V'}\|u\|_V$$

for all $u, v \in H$ and duality coincides with the inner product in H. By (2.2.1) the map $v \to a(u, v)$ is bounded and linear for every fixed $u \in V$. If we denote that functional by $Au \in V'$, then

$$a(u,v) = \langle Au, v\rangle$$

for all $u, v \in V$. The operator $A\colon V \to V'$ is linear and by (2.2.1) we have $A \in \mathcal{L}(V, V')$ with

$$\|A\|_{\mathcal{L}(V,V')} \le M.$$

We say A is the operator induced by the form $a(\cdot\,,\cdot)$. We can also consider it as an operator on V' with domain V.

THEOREM 2.2.2. *Let $A \in \mathcal{L}(V, V')$ be defined as above. Then A is a densely defined closed operator on V' with domain V. Moreover,*

$$[\lambda_0, \infty) \subset \varrho(-A), \tag{2.2.3}$$

and

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(V',V)} \le \alpha^{-1} \tag{2.2.4}$$

for all $\lambda \ge \lambda_0$.

PROOF. From the Lax–Milgram theorem (see [62, Section VI.3.2.5, Theorem 7]) it follows that $(\lambda I + A)^{-1} \in \mathcal{L}(V', V)$ exists for every $\lambda \ge \lambda_0$. In particular $(\lambda_0 I + A)^{-1} \in \mathcal{L}(V', V')$ since $V \hookrightarrow V'$, so $(\lambda_0 I + A)^{-1}$ is closed on V'. As the inverse of a closed linear operator is closed we get that $\lambda_0 I + A$. Hence A is closed as an operator on V'. Since V is dense in H by assumption, V is dense in V' as well. Also from the above, (2.2.3) is true. Next let $f \in V'$ and $u \in V$ with $Au + \lambda u = f$. If $\lambda \ge \lambda_0$, then

$$\alpha\|u\|_V^2 \le a(u, u) + \lambda\|u\|_H^2 = \langle f, u\rangle \le \|f\|_{V'}\|u\|_V,$$

from which (2.2.4) follows by dividing by $\|u\|_V$. □

We now look at the abstract elliptic equation

$$Au + \lambda_0 u = f \quad \text{in } V', \tag{2.2.5}$$

which is equivalent to the "weak" formulation that u in V with

$$a(u, v) + \lambda_0\langle u, v\rangle = \langle f, v\rangle \tag{2.2.6}$$

for all $v \in V$. We admit $f \in V'$ in both cases, but note that for $f \in H$, the expression $\langle f, v\rangle$ is the inner product in H. The above theorem tells us that (2.2.5) has a unique solution for every $f \in V'$ whenever $\lambda \ge \lambda_0$. We now summarise the values for λ_0 for the various boundary conditions.

EXAMPLE 2.2.3. (a) If Ω is a bounded domain, or more generally if Ω is an open set lying between two parallel hyperplanes, then (2.1.7) shows that $\|\nabla u\|_2$ is an equivalent norm on $H_0^1(\Omega)$. Hence according to Proposition 2.1.6 we can choose $\lambda_0 := \lambda_{\mathcal{A}}$ in Theorem 2.2.2.

(b) In general, the Neumann problem has a zero eigenvalue. Hence by Proposition 2.1.6 we choose $\lambda_0 = \lambda_{\mathcal{A}} + \alpha_0/2$.

(c) In Example 2.1.4(c) we introduced the equivalent norm (2.1.8). If the boundary coefficient b_0 is bounded from below by a positive constant $\beta > 0$, then by Proposition 2.1.6

$$\begin{aligned}\|\nabla u\|_2^2 + \|u\|_{L_2(\partial\Omega)}^2 &\leq 2\max\left\{\frac{1}{\alpha_0}, \frac{1}{\beta}\right\}\Big(\frac{\alpha_0}{2}\|\nabla u\|_2^2 + \beta\|u\|_{L_2(\partial\Omega)}^2\Big)\\ &\leq 2\max\left\{\frac{1}{\alpha_0}, \frac{1}{\beta}\right\}\left(a_0(u,u) + \lambda_{\mathcal{A}}\|u\|_2^2 + \int_{\partial\Omega} b_0 u^2\,d\sigma\right).\end{aligned} \tag{2.2.7}$$

Hence in Theorem 2.2.2 we can choose $\lambda_0 := \lambda_{\mathcal{A}}$ if Ω is a bounded Lipschitz domain and $b_0 \geq \beta$ for some constant $\beta > 0$.

We frequently look at solutions in spaces other than V. To deal with such cases we look at the *maximal restriction* of the operator A to some other Banach E space with $E \hookrightarrow V'$. We let

$$D(A_E) := \{u \in V : Au \in E\}$$

with $A_E u := Au$ for all $u \in D(A_E)$ and call A_E the maximal restriction of A to E.

PROPOSITION 2.2.4. *Suppose that A_E is the maximal restriction of A to $E \hookrightarrow V'$ and that $(\lambda I + A)^{-1}(E) \subset E$ for some $\lambda \in \varrho(-A)$. Then A_E is closed and $\varrho(A) \subset \varrho(A_E)$.*

PROOF. We first prove A_E is closed. Suppose that $u_n \in D(A_E)$ with $u_n \to u$ and $Au_n \to v$ in E. As $E \hookrightarrow V'$ convergence is also in V'. Because A is closed in V' with domain E we conclude that $u \in V$ and $Au = v$. We know that $u, v \in E$, so $v \in D(A_E)$ and $A_E u = v$, proving that A_E is closed. By assumption $(\lambda I + A)^{-1}(E) \subset E$. Because $\lambda I + A_E$ is closed, also its inverse is a closed operator on E. Hence by the closed graph theorem $(\lambda I + A_E)^{-1} \in \mathcal{L}(E)$. In particular, the above argument shows that $\varrho(A) \subset \varrho(A_E)$. □

As a special case in the above proposition we can set $E := H$. Sometimes it is useful to prove properties of the operator A via the associated semigroup it generates on H. A proof of the following proposition can be found in [64, §XVII.6, Proposition 3].

PROPOSITION 2.2.5. *Under the above assumption, $-A_H$ generates a strongly continuous analytic semigroup e^{-tA_H} on H. Moreover,*

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(H)} \leq \frac{1}{\lambda - \lambda_0}$$

for all $\lambda > \lambda_0$.

PROOF. We only prove the resolvent estimate. If $u = (\lambda I + A)^{-1} f$ for some $\lambda > \lambda_0$ and $f \in H$, then

$$(\lambda - \lambda_0)\|u\|_H^2 \leq a(u,u) + \lambda\|u\|_H^2 = \langle f, u\rangle \leq \|f\|_H \|u\|_H.$$

If we rearrange the inequality, then the resolvent estimate follows. □

2.3. *Formally adjoint problems*

When working with non-selfadjoint problems it is often necessary to consider the adjoint problem. Suppose that $a(\cdot\,,\cdot)$ is a bilinear form on a Hilbert space $V \hookrightarrow H$ as in the previous section. We define a new bilinear form

$$a^\sharp\colon V \times V \to \mathbb{R}$$

by setting

$$a^\sharp(u, v) := a(v, u) \tag{2.3.1}$$

for all $u, v \in V$. If $a(\cdot\,,\cdot)$ satisfies (2.2.1) and (2.2.2), then clearly $a^\sharp(\cdot\,,\cdot)$ has the same properties with the same constants M, α and λ_0. We denote the operator induced on V by $A^\sharp$. It is given by

$$a^\sharp(u, v) = \langle A^\sharp u, v\rangle$$

for all $u, v \in V$. We now relate $A^\sharp$ to the dual A' of A.

PROPOSITION 2.3.1. *Suppose $a(\cdot\,,\cdot)$ is a bilinear form satisfying Assumption* 2.2.1 *and $a^\sharp(\cdot\,,\cdot)$ and $A^\sharp$ as defined above. Then $A' = A^\sharp \in \mathcal{L}(V, V')$. Moreover, $A'_H = A^\sharp_H \in \mathcal{L}(H, H)$ is the adjoint of the maximal restriction A_H. Finally, if $a(\cdot\,,\cdot)$ is a symmetric form, then A_H is self-adjoint.*

PROOF. Since every Hilbert space is reflexive $V'' = V$ and so $A' \in \mathcal{L}(V'', V') = \mathcal{L}(V, V')$. Now by definition of A and $A^\sharp$ we have

$$\langle Au, v\rangle = a(u, v) = a^\sharp(v, u) = \langle A^\sharp v, u\rangle$$

for all $u, v \in V$. Next look at the maximal restriction A_H. From the above

$$\langle A_H u, v\rangle = \langle u, A^\sharp_H v\rangle$$

for all $u \in D(A_H)$ and all $v \in D(A^\sharp_H)$, so $A'_H = A^\sharp$. Finally, since $A' = A^\sharp = A$ if A is a symmetric form, the maximal restrictions A_H and $A^\sharp_H$ are the same, so A_H is self-adjoint. □

Let us now look at boundary value problems $(\mathcal{A}, \mathcal{B})$ given by (2.1.2) and (2.1.4) satisfying the assumptions made in Section 2.1. Let $a(\cdot\,,\cdot)$ be the form associated with $(\mathcal{A}, \mathcal{B})$ as in Definition 2.1.2. Then $a^\sharp(\cdot\,,\cdot)$ is the form associated with the *formally adjoint boundary value problem* $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$ given by

$$\mathcal{A}^\sharp = -\operatorname{div}(A_0^T(x)\nabla u + b(x)u) + a(x)\cdot\nabla u + c_0 u \tag{2.3.2}$$

Table 2.1. Constants in (2.4.1) for Dirichlet problems

Condition on Ω	Value of d	Value of λ_0	Value of c_a
$N \geq 3$, any Ω	N	$\lambda_{\mathcal{A}}$	$c(N)/\alpha_0$
$N = 2$, any Ω	any $d \in (2, \infty)$	$\lambda_{\mathcal{A}} + \alpha_0/2$	$c(d)/\alpha_0$
$N = 2$, Ω between parallel hyperplanes of distance D	any $d \in (2, \infty)$	$\lambda_{\mathcal{A}}$	$c(d)(1 + D^2)/\alpha_0$

and

$$\mathcal{B}^\sharp u := \begin{cases} u|_{\Gamma_1} & \text{on } \Gamma_1 \text{ (Dirichlet b.c.)}, \\ \dfrac{\partial u}{\partial \nu_{\mathcal{A}^\sharp}} & \text{on } \Gamma_2 \text{ (Neumann b.c.)}, \\ \dfrac{\partial u}{\partial \nu_{\mathcal{A}^\sharp}} + b_0 u & \text{on } \Gamma_3 \text{ (Robin b.c.)}, \end{cases} \tag{2.3.3}$$

where

$$\frac{\partial u}{\partial \nu_{\mathcal{A}^\sharp}} := (A_0^T(x)\nabla u + b(x)u) \cdot \nu.$$

Note that $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$ has the same structure as $(\mathcal{A}, \mathcal{B})$ with A_0 replaced by its transposed A_0^T and the roles of a and b interchanged. If A and $A^\sharp$ are the corresponding operators induced on V and H, then all the assertions of Proposition 2.3.1 apply.

2.4. *Global a priori estimates for weak solutions*

In our treatment of domain perturbation problems, global L_p-L_q-estimates for weak solutions to (2.1.1) play an essential role, especially in the nonlinear case with f depending on u. We provide a simple test to obtain such estimates and apply them to the three boundary conditions. The estimates are only based on an embedding theorem for the space V of weak solutions introduced in Section 2.1.

As it turns out, the key to control the domain dependence of L_p-estimates for solutions of (2.1.1) is the constant in a Sobolev-type inequality. More precisely, let $a(\cdot\,,\cdot)$ be the form associated with the boundary value problem $(\mathcal{A}, \mathcal{B})$ as given in Assumption 2.1.2. We require that there are constants $d > 2$, $c_a > 0$ and $\lambda_0 \geq 0$ such that

$$\|u\|_{2d/(d-2)}^2 \leq c_a(a(u, u) + \lambda_0\|u\|_2^2) \tag{2.4.1}$$

for all $u \in V$, where V is the space of weak solutions associated with $(\mathcal{A}, \mathcal{B})$. We always require that V satisfies Assumption 2.1.3. Note that $d = N$ is the smallest possible d because of the optimality of the usual Sobolev inequality in $H_0^1(\Omega)$. To obtain control over domain dependence of the constants d, c_a and λ_0, we need to choose d larger in some cases. Explicit values for the three boundary conditions are listed in Tables 2.1–2.3. In these tables, the constant $c(\cdot)$ only depends on its argument, α_0 is the ellipticity constant from (2.1.3), and $\lambda_{\mathcal{A}}$ is given by (2.1.10). Proofs of (2.4.1) for the various cases are given

Table 2.2. Constants in (2.4.1) for Robin problems if $b_0 \geq \beta$ for some $\beta > 0$

Condition on Ω	Value of d	Value of λ_0	Value of c_a
$N \geq 2$, $\lvert\Omega\rvert < \infty$	$2N$	$\lambda_{\mathcal{A}}$	$c(N)(1 + \lvert\Omega\rvert^{1/N}) \max\{\frac{1}{\alpha_0}, \frac{1}{\beta}\}$
$N \geq 2$, $\lvert\Omega\rvert \leq \infty$	$2N$	$\lambda_{\mathcal{A}} + \alpha_0/2$	$c(N) \max\{\frac{1}{\alpha_0}, \frac{1}{\beta}\}$

Table 2.3. Constants in (2.4.1) for Neumann problems

Condition on Ω	Value of d	Value of λ_0	Value of c_a
$N \geq 3$, cone condition	N	$\lambda_{\mathcal{A}} + \alpha_0/2$	$c(d, \text{cone})/\alpha_0$
$N = 2$, cone condition	any $d \in (2, \infty)$	$\lambda_{\mathcal{A}} + \alpha_0/2$	$c(d, \text{cone})/\alpha_0$
$N \geq 2$, special class of Ω	$d > 2$ depending on Ω	$\lambda_{\mathcal{A}} + \alpha_0/2$	$c(d, \text{class})/\alpha_0$
No condition on Ω	"$d = \infty$" no smoothing	N/A	N/A

in Sections 2.4.2 and 2.4.3. It turns out that there are domain-independent estimates and smoothing properties for Dirichlet and Robin boundary conditions, but not for Neumann boundary conditions.

THEOREM 2.4.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open set and $u \in V$ is a weak solution of* (2.1.1). *If* (2.4.1) *holds, then there exists a constant $C > 0$ only depending on d and $p \geq 2$ such that*

$$\|u\|_{dp/(d-2p)} \leq c_a C(\|f\|_p + \lambda_0 \|u\|_p) \tag{2.4.2}$$

if $p \in [2, d/2)$, and

$$\|u\|_\infty \leq c_a C(\|f\|_p + \lambda_0 \|u\|_p) + \|u\|_p \tag{2.4.3}$$

or

$$\|u\|_\infty \leq c_a C(\|f\|_p + \lambda_0 \|u\|_p) + \|u\|_{2d/(d-2)} \tag{2.4.4}$$

if $p > d/2$. Moreover, if $\lambda_0 = 0$ or $u \in L_p(\Omega)$ (if $|\Omega| < \infty$ for instance), then the above estimates are valid for $p \in [2d/(d+2), 2)$ as well.

PROOF. Let A be the operator induced by the form $a(\cdot\,,\cdot)$ on V. Given $u \in V$ we set $u_q := |u|^{q-2}u$ for $q \geq 2$. Assuming that (2.4.1) holds, it follows from [57, Proposition 5.5] that

$$\|u\|_{dq/(d-2)}^q \leq c_a \frac{q}{2}(\langle Au, u_q\rangle + (q-1)\lambda_0\langle u, u_q\rangle)$$

for all $q \geq 2$ and $u \in V$ for which the expression on the right-hand side is finite. Then we apply [57, Theorem 4.5] to get the estimates. Compared to that reference, we have replaced the term $\|u\|_p$ by $\|u\|_{2d/(d-2)}$ in (2.4.4). We can do this by replacing q_0 in equation (4.11) in the proof of [57, Theorem 4.5] by $q_0 := 1 + \frac{2d}{p'(d-2)}$ and then complete the proof in a similar way. Finally, the limitation that $p \geq 2$ comes from proving that $u \in L_p(\Omega)$ first if $\lambda_0 \neq 0$. If $\lambda_0 = 0$, or if we know already that $u \in L_p(\Omega)$, then this is not necessary (see also [61, Theorem 2.5]), and we can admit $p \in [2d/(d+2), 2)$. □

The above theorem tells us that for $p \geq 2$ and $\lambda \geq \lambda_{\mathcal{A}}$

$$(\lambda I + A)^{-1} \colon L_p(\Omega) \cap L_2(\Omega) \to L_{m(p)}(\Omega)$$

if we set

$$m(p) := \begin{cases} \dfrac{dp}{d-2p} & \text{if } p \in (1, d/2), \\ \infty & \text{if } p > d/2 \end{cases} \tag{2.4.5}$$

and A is the operator associated with the problem (5.1.2) as constructed in Section 2.2.

We next want to derive domain-independent bounds for the norm of the resolvent operator by constructing an operator in $L_p(\Omega)$ for $p \in (1, \infty)$. Let A_2 denote the maximal restriction of A to $L_2(\Omega)$. By Proposition 2.2.5, the operator $-A_2$ is the generator of a strongly continuous analytic semigroup on $L_2(\Omega)$. Moreover, still assuming that (2.4.1) holds, e^{-tA_2} has an kernel satisfying pointwise Gaussian estimates and therefore interpolates to $L_p(\Omega)$ for all $p \in (1, \infty)$ (see [55]). Denote by $-A_p$ its infinitesimal generator. The dual semigroup is a strongly continuous analytic semigroup on $L_{p'}(\Omega)$ and its generator is A'_p. Let $A^\sharp_p$ be the corresponding operators associated with the formally adjoint problem. Since $(A^\sharp_2)' = A_2$ by Proposition 2.3.1 we get $(A^\sharp_p)' = A_{p'}$. We denote the exponent conjugate to p by p', that is,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

From [55, Theorem 5.1]

$$\|e^{-tA_p}\|_{\mathcal{L}(L_p)} \leq e^{\omega_p t}, \qquad \omega_p := \max\{p-1, p'-1\}\lambda_0 \tag{2.4.6}$$

for all $t > 0$ and $p \in (1, \infty)$.

Solutions of the abstract equation $(\lambda I + A_p)u = f$ with $f \in L_p(\Omega)$ are called generalised solutions of the corresponding elliptic problem in $L_p(\Omega)$. If $1 < p < 2N/(N+2)$ they are not weak solutions in general, but solutions in an even weaker sense. Note that we defined A_p by means of semigroup theory to be able to easily get a definition for domains with unbounded measure, because for such domains we cannot expect $L_p(\Omega) \subset V'$ for $p > 2$.

THEOREM 2.4.2. *Let A_p as defined above and $p \in (1, \infty)$. Then $(\omega_p, \infty) \subset \varrho(-A_p)$ with $\omega_p := \max\{p-1, p'-1\}\lambda_0$, and*

$$\|(\lambda I + A_p)^{-1}\|_{\mathcal{L}(L_p)} \leq \frac{1}{\lambda - \omega_p} \tag{2.4.7}$$

for all $\lambda > \omega_p$. Furthermore, for every $p > 1$, $p \neq N/2$, there exists a constant $C > 0$ only depending on d, p and c_a and λ, λ_0 such that

$$\|(\lambda I + A_p)^{-1}\|_{\mathcal{L}(L_p, L_{m(p)})} \leq C \tag{2.4.8}$$

for all $\lambda > \lambda_0$. If Ω is bounded, then $(\lambda I + A)^{-1} \colon L_p(\Omega) \to L_q(\Omega)$ is compact for all $q \in [p, m(p))$. Finally, if $A^\sharp_p$ is the operator associated with the formally adjoint problem, then $A'_p = A^\sharp_{p'}$.

PROOF. As $-A_p$ generates a strongly continuous semigroup, (2.4.6) implies that $(\omega_p, \infty) \subset \varrho(-A_p)$ and that

$$(\lambda I + A_p)^{-1} = \int_0^\infty e^{-tA_p} e^{-\lambda t}\, dt$$

for all $\lambda > \omega_p$ (see [125, Section IX.4]). Hence (2.4.7) follows if we take into account (2.4.6). Now for $f \in L_2(\Omega) \cap L_p(\Omega)$ we have $u := (\lambda I + A_p)^{-1} \in L_p(\Omega)$. Hence, if $2d/(d+2) \leq p < d/2$, then by (2.4.2)

$$\|u\|_{dp/(d-2p)} \leq c_a C(\|f + \lambda u\|_p + \lambda_0 \|u\|_p) \leq c_a C \left(1 + \frac{\lambda + \lambda_0}{\lambda - \omega_p}\right) \|f\|_p.$$

A similar estimate is obtained by using (2.4.3) if $p > d/2$. Now (2.4.8) follows since $L_2(\Omega) \cap L_p(\Omega)$ is dense in $L_p(\Omega)$ if we choose C appropriately. If $1 < p < 2d/(d+2)$, then we use a duality argument. In that case $q := m(p)' > 2d/(d-2)$ and a simple calculation reveals that $p = m(q)'$. Because

$$(\lambda I + A_q^\sharp)^{-1} \in \mathcal{L}(L_q(\mathbb{R}^N), L_{m(q)}(\mathbb{R}^N)),$$

by duality

$$((\lambda I + A_q^\sharp)^{-1})' = (\lambda I + (A_q^\sharp)')^{-1} = (\lambda I + A_p)^{-1} \in \mathcal{L}(L_p(\mathbb{R}^N), L_{m(p)}(\mathbb{R}^N))$$

with equal norm. Compactness of the resolvent on $L_p(\Omega)$ for $1 < p < \infty$ follows from [57, Section 7]. Now compactness as an operator from $L_p(\Omega)$ to $L_q(\Omega)$ for $q \in [p, m(p))$ follows from a compactness property of the Riesz–Thorin interpolation theorem (see [94]). □

REMARK 2.4.3. Note that the above theorem is not optimal, but it is sufficient for our purposes. In particular the condition $\lambda > \omega_p$ could be improved by various means. If A has compact resolvent, then the spectrum of A_p is independent of p because the above smoothing properties of the resolvent operator show that every eigenfunction is in $L^\infty(\Omega)$. Also if $p = N$, then the spectrum is independent of p by [95] because of Gaussian bounds for heat kernels (see [55]).

2.4.1. *Sobolev inequalities associated with Dirichlet problems*

If $N \geq 3$ there exists a constant $c(N)$ only depending on the dimension N such that

$$\|u\|_{2N/(N-2)} \leq c(N) \|\nabla u\|_2 \tag{2.4.9}$$

for all $u \in H^1(\mathbb{R}^N)$ (see [76, Theorem 7.10]). If $N = 2$, then for every $q \in [2, \infty)$ there exists a constant c_q only depending on q such that

$$\|u\|_q \leq c_q \|u\|_{H^1(\mathbb{R}^N)}$$

for all $u \in H^1(\mathbb{R}^2)$ (see [96, Theorem 8.5]). If $q \in (2, \infty)$ and $d := 2q/(q-2)$, then $q = 2d/(d-2)$. Hence for every $d > 2$ there exists $c_d > 0$ only depending on d such that

$$\|u\|_{2d/(d-2)} \leq c_d \|u\|_{H^1(\mathbb{R}^N)}. \tag{2.4.10}$$

If Ω is lying between two parallel hyperplanes of distance D, then using (2.1.7) we conclude that

$$\|u\|_{2d/(d-2)} \leq c_d \|u\|_{H^1(\mathbb{R}^N)} \leq c_d \sqrt{1+D^2} \|\nabla u\|_2 \tag{2.4.11}$$

for all $u \in H_0^1(\Omega)$. Combining the above with the basic inequalities in Proposition 2.1.6 we can summarise the constants appearing in (2.4.1) in Table 2.1.

The L_∞-estimates for Dirichlet problems are very well known, see for instance [76, Chapter 8]. The estimates for $p < N/2$ are not as widely known, and sometimes stated with additional assumptions on the structure of the operators, see [38, Appendix to Chapter 3], [112, Theorem 4.2] or without proof in [45, Lemma 1]. A complete proof is contained in [57].

2.4.2. *Maz'ya's inequality and Robin problems*

It may be surprising that solutions of the elliptic problem with Robin boundary conditions

$$\begin{aligned} \mathcal{A}u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}} + b_0 u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.4.12}$$

with $b_0 \geq \beta$ for some constant $\beta > 0$ satisfy domain-independent estimates similar to the ones for problems with Dirichlet boundary conditions. The estimates were first established in [56]. In the present discussion we will only work with bounded Lipschitz domains $\Omega \subset \mathbb{R}^N$, but note that the result could be generalised to arbitrary domains. We refer to [11,56] for details.

The weak solutions of the Robin problem on a Lipschitz domain are in $H^1(\Omega)$ as discussed in Example 2.1.4(c). In the usual Sobolev inequality $\|u\|_{2N/(N-2)} \leq c\|u\|_{H^1}$, the constant c depends on the shape of the domain as examples of domains with a cusp show (see [2, Theorem 5.35]). Hence only a weaker statement can be true. The key is an inequality due to Maz'ya from [100] (see [101, Section 3.6]) stating that

$$\|u\|_{N/(N-1)} \leq c(N)(\|\nabla u\|_1 + \|u\|_{L_1(\partial\Omega)})$$

for all $u \in W_1^1(\Omega) \cap C(\bar{\Omega})$, where $c(N)$ is the isoperimetric constant depending only on $N \geq 2$. Substituting u^2 into the above inequality we get

$$\begin{aligned} \|u\|_{2N/(N-1)}^2 &\leq c(N)(2\|u\nabla u\|_1 + \|u\|_{L_2(\partial\Omega)}^2) \\ &\leq c(N)(\|u\|_{H^1}^2 + \|u\|_{L_2(\partial\Omega)}^2). \end{aligned} \tag{2.4.13}$$

By the density of $H^1(\Omega) \cap C(\bar{\Omega})$ in $H^1(\Omega)$ the inequality is valid for all $u \in H^1(\Omega)$. If Ω has finite measure, then similarly

$$\|u\|_{2N/(N-1)}^2 \leq c(N)(1+|\Omega|)^{1/N}(\|\nabla u\|_2^2 + \|u\|_{L_2(\partial\Omega)}^2) \tag{2.4.14}$$

for all $u \in H^1(\Omega)$ with a constant $c(N)$ different from the original one, but only depending on N. Combining (2.2.7) with (2.4.13) or (2.4.14) we therefore get (2.4.1) with $d = 2N$ as displayed in Table 2.2.

2.4.3. *Sobolev inequalities associated with Neumann problems*
The smoothing properties for Dirichlet and Robin problems we established in the previous sections were based on the validity of a Sobolev-type inequality for functions in the space of weak solutions with a constant independent of the shape of the domain. The space of weak solutions for the Neumann problem

$$\begin{aligned} \mathcal{A}u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

is $H^1(\Omega)$. In the case of the Robin problem we could use an equivalent norm on $H^1(\Omega)$ involving a boundary integral. As pointed out in Section 2.4.2, the boundary integral is of higher order than the H^1-norm if the domain is bad. Hence for the Neumann problem the constant c in the Sobolev inequality

$$\|u\|_{2N/(N-2)} \leq c\|u\|_{H^1} \tag{2.4.15}$$

depends on the shape and not just the measure of Ω. There are no easy necessary and sufficient conditions for the inequality to be true. A sufficient condition is that Ω satisfies an (interior) cone condition, that is, there exists an open cone $C \subset \mathbb{R}^N$ with vertex at zero such that for every $x \in \partial\Omega$ there is an orthogonal transformation T such that $x+T(C) \subset \Omega$ (see [2, Definition 4.3]). The constant c in (2.4.15) depends on the length and the angle of the cone C (see [2, Lemma 5.12]). A cone condition is however not necessary for getting a Sobolev inequality uniformly with respect to a family of domains. Shrinking holes of fixed shape to a point is sufficient (see [53, Section 2]). Alternatively an extension property is also sufficient. This includes domains with fractal boundary (quasi-disks) as shown in [101, Section 1.5.1]. We could replace the cone C by a standard polynomial cusp and get an inequality of the form

$$\|u\|_{2d/(d-2)} \leq c\|u\|_{H^1} \tag{2.4.16}$$

with $d \geq N$ depending on the sharpness of the cusp (see [2, Theorem 5.35] or [101, Section 4.4]). For general domains there is no such $d > 2$, and there are no smoothing properties of the resolvent operator. This corresponds to the degenerate case "$d = \infty$" because $2d/(d-2) \to 2$ as $d \to \infty$. Combining Proposition (2.1.11) with the above we get (2.4.1) with constants as displayed in Table 2.3.

2.5. *The pseudo-resolvent associated with boundary value problems*

When dealing with varying domains we want to embed our problem into a fixed large space. In this section we want to explain how to do that. We define the inclusion $i_\Omega\colon L_p(\Omega) \to L_p(\mathbb{R}^N)$ to be the trivial extension

$$i_\Omega(u) := \begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \Omega^c. \end{cases} \tag{2.5.1}$$

We also sometimes write $\tilde{u} := i_\Omega(u)$ for the trivial extension. The above extension operator also acts as an operator

$$i_\Omega\colon H_0^1(\Omega) \to H^1(\mathbb{R}^N).$$

Indeed, by definition $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. We can identify $C_c^\infty(\Omega)$ with the set $\{u \in C_c^\infty(\mathbb{R}^N)\colon \operatorname{supp} u \subset \Omega\}$, and view $H_0^1(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $H^1(\mathbb{R}^N)$.

Furthermore, we let $r_\Omega\colon L_p(\mathbb{R}^N) \to L_p(\Omega)$ be the restriction

$$r_\Omega(u) := u|_\Omega. \tag{2.5.2}$$

If $f \in H^{-1}(\mathbb{R}^N)$, then we restrict the functional to the closed subspace $H_0^1(\Omega)$ of $H^1(\mathbb{R}^N)$ to get an element of $H^{-1}(\Omega)$. More formally we define the restriction operator $r_\Omega\colon H^{-1}(\mathbb{R}^N) \to H^{-1}(\Omega)$ by

$$r_\Omega(f) := f|_{H_0^1(\Omega)}. \tag{2.5.3}$$

On the subspace $L_2(\mathbb{R}^N)$ of $H^{-1}(\mathbb{R}^N)$, the two definitions coincide. We next prove that the operators i_Ω and r_Ω are dual to each other.

LEMMA 2.5.1. *Let $\Omega \subset \mathbb{R}^N$ be open. Let i_Ω and r_Ω as defined above and $1 < p < \infty$. Then*

$$i_\Omega \in \mathcal{L}(L_p(\Omega), L_p(\mathbb{R}^N)) \cap \mathcal{L}(H_0^1(\Omega), H^1(\mathbb{R}^N))$$

and $\|i_\Omega\| = \|r_\Omega\| = 1$. Moreover

$$i'_\Omega = r_\Omega \in \mathcal{L}(L_{p'}(\mathbb{R}^N), L_{p'}(\Omega)) \cap \mathcal{L}(H^{-1}(\mathbb{R}^N), H^{-1}(\Omega))$$

and $r'_\Omega = i_\Omega$, where p' is the conjugate exponent to p.

PROOF. The first assertion follows directly from the definition of the operators. If $f \in H^{-1}(\mathbb{R}^N)$ or $L_{p'}(\mathbb{R}^N)$, then by definition of i_Ω and r_Ω

$$\langle f, i_\Omega(u)\rangle = \langle r_\Omega(f), u\rangle$$

for all $u \in C_c^\infty(\Omega)$. By density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$ and $L_p(\Omega)$ we get $i'_\Omega = r_\Omega$. By the reflexivity of the spaces involved we therefore also have $r'_\Omega = i_\Omega$. □

Let $(\mathcal{A}_n, \mathcal{B}_n)$ and $(\mathcal{A}, \mathcal{B})$ be elliptic boundary value problems on open sets Ω_n and Ω, respectively. Suppose that A_n and A are the corresponding operators induced as discussed in Section 2.2. We can then embed the problems in $\mathbb{R}^N$ as follows.

DEFINITION 2.5.2. Let A_n, A defined as above. We set

$$R_n(\lambda) := i_{\Omega_n}(\lambda I + A_n)^{-1} r_{\Omega_n} \quad \text{and} \quad R(\lambda) := i_\Omega(\lambda I + A)^{-1} r_\Omega$$

whenever the inverse operators exist. Similarly we define $R_n^\sharp(\lambda)$ and $R^\sharp(\lambda)$ for the formally adjoint problem.

The family of operators $R(\lambda)$ and $R_n(\lambda)$ form a pseudo-resolvent as defined for instance in [125, Section VIII.4]. In particular they satisfy the resolvent equation

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$$

for all $\lambda, \mu \in \varrho(-A)$. Using Lemma 2.5.1 and the last assertion in Theorem 2.4.2 we get the following properties of the pseudo-resolvent.

LEMMA 2.5.3. *If $\lambda \in \varrho(A)$, then $R(\lambda)' = R^\sharp(\lambda)$.*

REMARK 2.5.4. From the above it also follows that we can replace $(\lambda + A)^{-1}$ by $R(\lambda)$ in (2.4.7) and (2.4.8) with all constants being the same.

3. Semi-linear elliptic problems

The purpose of this section is to formulate semi-linear boundary value problems as a fixed point equation in $L_p(\mathbb{R}^N)$. Then we derive an L_∞-estimate for solutions in terms of an L_p-norm, provided the nonlinearity satisfies a growth condition.

3.1. *Abstract formulation of semi-linear problems*

We now want to look at properties of weak solutions of the semi-linear boundary value problem

$$\begin{aligned} \mathcal{A}u &= f(x, u(x)) \quad && \text{in } \Omega, \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1.1}$$

where $(\mathcal{A}, \mathcal{B})$ are as discussed in Section 2.1. We use the smoothing properties from Section 2.4 to show that under suitable growth conditions on f, the boundary value problem (3.1.1) can be viewed as a fixed point equation in $L_p(\mathbb{R}^N)$ for a some range of $p \in (1, \infty)$.

We assume that $V \subset H^1(\Omega)$ is the space of weak solutions for the boundary conditions under consideration as introduced in Section 2.1 and discussed in detail for different boundary conditions in Section 2.4. We also assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function with properties to be specified. Given $u : \Omega \to \mathbb{R}$ we define the *superposition operator* $F(u)$ by

$$F(u)(x) := f(x, u(x)) \tag{3.1.2}$$

for all $x \in \Omega$, provided that $F(u) \in V'$. We call $u \in V$ a weak solution of (3.1.1) if

$$a(u, v) = \langle F(u), v\rangle$$

for all $v \in V$. Here $a(\cdot\,,\cdot)$ is the form associated with $(\mathcal{A}, \mathcal{B})$ as in Definition 2.1.2. If A is the operator induced by $(\mathcal{A}, \mathcal{B})$ we can rewrite (3.1.1) as

$$Au = F(u). \tag{3.1.3}$$

To be able to show that $F(u) \in V'$ we need to make some assumptions on f.

ASSUMPTION 3.1.1. Suppose that $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $f(\cdot, \xi): \Omega \to \mathbb{R}$ is measurable for all $\xi \in \mathbb{R}$, and $f(x, \cdot) \in C(\mathbb{R})$ for almost all $x \in \Omega$. Further suppose that there exist a function $g \in L_1(\Omega) \cap L_\infty(\Omega)$ and constants $1 \leq \gamma < \infty$ and $c \geq 0$ such that

$$|f(x, \xi)| \leq g(x) + c|\xi|^\gamma \tag{3.1.4}$$

for all $(x, \xi) \in \Omega \times \mathbb{R}$.

The above growth conditions on f lead to the following mapping properties of the superposition operator.

LEMMA 3.1.2. *Suppose that f satisfies Assumption* 3.1.1 *with* $1 \leq \gamma \leq p$. *Then the corresponding superposition operator F is in $C(L_p(\Omega), L_{p/\gamma}(\Omega))$. Moreover,*

$$\|F(u)\|_{p/\gamma} \leq \|g\|_1 + \|g\|_\infty + c\|u\|_p^\gamma$$

for all $u \in L_p(\Omega)$.

PROOF. By [6, Theorem 3.1] we have

$$\|F(u)\|_{p/\gamma} \leq \|g\|_{p/\gamma} + c\|u\|_p^\gamma \leq \|g\|_1^{\gamma/p} \|g\|_\infty^{1-\gamma/p} + c\|u\|_p.$$

Then use Young's inequality to get

$$\|g\|_1^{\gamma/p} \|g\|_\infty^{1-\gamma/p} \leq \frac{\gamma}{p}\|g\|_1 + \left(1 - \frac{\gamma}{p}\right)\|g\|_\infty \leq \|g\|_1 + \|g\|_\infty.$$

Continuity is proved in [6, Theorem 3.7]. □

Note that [6, Theorem 3.1] shows that the growth conditions are necessary and sufficient for F to map $L_p(\Omega)$ into $L_{p/\gamma}(\Omega)$. In particular, if F maps $L_p(\Omega)$ into itself and $p \in (1, \infty)$, then $\gamma = 1$, that is, f grows at most linearly.

From now on we assume that (2.4.1) is true for all $v \in V$. Then by Theorem 2.4.2

$$(\lambda I + A)^{-1} \in \mathcal{L}(L_p(\Omega)) \cap \mathcal{L}(L_p(\Omega), L_{m(p)}(\Omega))$$

for all $\lambda \in \varrho(-A)$ with $m(p)$ given by (2.4.5). If we fix $\lambda \in \varrho(-A)$, then we can rewrite (3.1.3) as $Au + \lambda u = F(u) + \lambda u$ and hence in form of the fixed point equation

$$u = (\lambda I + A)^{-1}(F(u) + \lambda u).$$

To be able to consider this as an equation in $L_p(\Omega)$ we need that the right-hand side is in $L_p(\Omega)$ if $u \in L_p(\Omega)$. Taking into account Lemma 3.1.2 and the smoothing property of the resolvent we need that $m(p/\gamma) \geq p$. When looking at boundedness of weak solutions and convergence properties with respect to the domain it is necessary to require $m(p/\gamma) > p$, or equivalently,

$$1 \leq \gamma < 1 + \frac{2p}{d}. \tag{3.1.5}$$

Because every weak solution of (3.1.1) lies in $L_{2d/(d-2)}(\Omega)$ we can assume that $p \geq 2d/(d-2)$. Then automatically $1 \leq \gamma \leq p$ as required in Lemma 3.1.2.

PROPOSITION 3.1.3. *Suppose that* $(\mathcal{A}, \mathcal{B})$ *is such that* (2.4.1) *holds for some* $d > 2$. *Moreover, let* $2d/(d-2) \leq p < \infty$ *such that Assumption* 3.1.1 *holds with* γ *satisfying* (3.1.5). *Fix* $\lambda \in \varrho(-A)$ *and set*

$$G(u) := (\lambda I + A)^{-1}(F(u) + \lambda u).$$

Then $G \in C(L_p(\Omega), L_p(\Omega))$ *and*

$$\begin{aligned}\|G(u)\|_p \leq \|R(\lambda)\|_{\mathcal{L}(L_{p/\gamma}, L_p)}(\|g\|_1 + \|g\|_\infty + c\|u\|_p^\gamma) \\ + |\lambda| \|R(\lambda)\|_{\mathcal{L}(L_p)} \|u\|_p \end{aligned} \tag{3.1.6}$$

for all $u \in L_p(\Omega)$. *Furthermore, if* Ω *is bounded, then* G *is compact, that is,* G *maps bounded sets of* $L_p(\Omega)$ *onto relatively compact sets of* $L_p(\Omega)$. *Finally,* $u \in L_p(\Omega) \cap V$ *is a weak solution of* (3.1.1) *if and only if* u *is a fixed point of*

$$u = G(u)$$

in $L_p(\Omega)$.

PROOF. By Lemma 3.1.2, $F \in C(L_p(\Omega), L_{p/\gamma}(\Omega))$ is bounded with

$$\|F(u)\|_{p/\gamma} \leq \|g\|_1 + \|g\|_\infty + c\|u\|_p^\gamma,$$

so F is bounded. Clearly (3.1.5) implies $m(p/\gamma) > p$, so Theorem 2.4.2 shows that $R(\lambda) \in \mathcal{L}(L_{p/\gamma}(\Omega), L_p(\Omega)) \cap \mathcal{L}(L_p(\Omega))$ with the operator being compact if Ω is bounded. Hence G is continuous as claimed and compact if Ω is bounded. From the definition of G

$$\|G(u)\|_p \leq \|R(\lambda)\|_{\mathcal{L}(L_{p/\gamma}, L_p)} \|F(u)\|_{p/\gamma} + |\lambda| \|R(\lambda)\|_{\mathcal{L}(L_p)} \|u\|_p.$$

Combining it with the estimate of $\|F(u)\|_{p/\gamma}$ from above we obtain (3.1.6). The last assertion is evident from the definition of G. □

REMARK 3.1.4. Because every weak solution lies in $L_{2d/(d-2)}(\Omega)$ the above condition is automatically satisfied if $\gamma < (d+2)/(d-2)$, that is, the growth is subcritical for the exponent d.

3.2. *Boundedness of weak solutions*

We apply results from Section 2.4 to show that weak solutions of (2.1.1) are in L_∞ if they are in $L_p(\mathbb{R}^N)$ and the nonlinearity satisfies a growth condition.

THEOREM 3.2.1. *Suppose that* $(\mathcal{A}, \mathcal{B})$ *is such that* (2.4.1) *holds for some* $d > 2$ *and* $\lambda_0 \geq 0$. *Moreover, let* $p \geq 2d/(d-2)$ *such that Assumption* 3.1.1 *holds with* γ *satisfying* (3.1.5). *Suppose that* $u \in V \cap L_p(\Omega)$ *is a weak solution of* (3.1.1). *If* $\lambda_0 = 0$, *then* $u \in L_\infty(\Omega)$ *and there exists an increasing function* $q\colon [0, \infty) \to [0, \infty)$ *such that*

$$\|u\|_\infty \leq q(\|u\|_p).$$

That function only depends on γ, p, *an upper bound for* $\|g\|_1 + \|g\|_\infty$ *and* c *from Assumption* 3.1.1 *and the constants* c_a, C *from Theorem* 2.4.1.

If $\lambda_0 > 0$ and in addition $u \in L_{p/\gamma}(\Omega)$, then $u \in L_\infty(\Omega)$ and

$$\|u\|_\infty \le q(\|u\|_p, \|u\|_{p/\gamma})$$

with the function q also depending on λ_0.

PROOF. Suppose that $u \in L_p(\Omega)$ is a solution of (3.1.1) with p and γ satisfying (3.1.5). We set

$$p_{k+1} := m(p_k/\gamma) \quad \text{and} \quad p_0 := p$$

and note that $p_0/\gamma = p/\gamma \ge 2d/(d+2)$. Using (2.4.2) with $\lambda_0 = 0$ and Lemma 3.1.2

$$\|u\|_{p_{k+1}} \le c_a C \|F(u)\|_{p_k/\gamma} \le c_a C(\|g\|_1 + \|g\|_\infty + \|u\|_{p_k}^\gamma) \tag{3.2.1}$$

as long as $p_k < \gamma d/2$. From (3.1.5) the sequence (p_k) is increasing. Hence, again using (3.1.5) we get

$$\begin{aligned} p_{k+1} - p_k &= \frac{dp_k}{d\gamma - 2p_k} - p_k = \left(\frac{d}{d\gamma - 2p_k} - 1\right) p_k \\ &\ge \left(\frac{d}{d\gamma - 2p} - 1\right) p > 0 \end{aligned}$$

as long as $p_k/\gamma < d/2$. Therefore we can choose $m \in \mathbb{N}$ such that $p_m < \gamma d/2 < p_{m+1}$. Then by (2.4.2) and Lemma 3.1.2

$$\begin{aligned} \|u\|_\infty &\le c_a C \|F(u)\|_{p_{m+1}/\gamma} + \|u\|_{p_{m+1}} \\ &\le c_a C(\|g\|_1 + \|g\|_\infty + \|u\|_{p_{m+1}}^\gamma) + \|u\|_{p_{m+1}} \end{aligned} \tag{3.2.2}$$

and we are done. We now obtain the L_∞-bound for u by applying (3.2.1) inductively to $k = 0, \dots, m$, and finally using (3.2.2). It is now obvious how to define the function q having the required properties. If $\lambda_0 > 0$, then (3.2.1) has to be replaced by

$$\begin{aligned} \|u\|_{p_{k+1}} &\le c_a C(\|F(u)\|_{p_k/\gamma} + \lambda_0 \|u\|_{p_k/\gamma}) \\ &\le c_a C(\|g\|_1 + \|g\|_\infty + \|u\|_{p_k}^\gamma + \lambda_0 \|u\|_{p_k/\gamma}). \end{aligned}$$

Now the assertion follows in a similar manner as in the case $\lambda_0 = 0$. □

REMARK 3.2.2. On domains with finite measure we often work with (3.1.4), where g is a constant. In that case dependence on $\|g\|_1 = g|\Omega|$ means dependence on the measure of the domain and the magnitude of g. Moreover, if Ω has finite measure, then the condition $u \in L_{p/\gamma}(\Omega)$ is automatically satisfied because $p/\gamma \le p$.

4. Abstract results on linear operators

Many convergence properties of the resolvents reduce to an abstract perturbation theorem. We collect these results here. The first is a characterisation of convergence in the operator norm if the limit is compact. Then we discuss a spectral mapping theorem and how to apply it to get continuity of the spectrum and the corresponding projections. Finally we use an interpolation argument to extend convergence in $L_p(\mathbb{R}^N)$ for some p to all p.

4.1. *Convergence in the operator norm*

The aim of this section is to prove a characterisation of convergence in the operator norm useful in the context of domain perturbations. Recall that a sequence of operators (T_n) on a Banach space E is called *strongly convergent* if $T_n f \to Tf$ for all $f \in E$.

PROPOSITION 4.1.1. *Suppose that E, F are Banach spaces, E is reflexive and that $T_n, T \in \mathcal{L}(E, F)$. Then the following assertions are equivalent.*
(1) *T is compact and $T_n \to T$ in $\mathcal{L}(E, F)$;*
(2) *$T_n f_n \to Tf$ in F whenever $f_n \rightharpoonup f$ weakly in E;*
(3) *$T_n \to T$ strongly and $T_n f_n \to 0$ in F whenever $f_n \rightharpoonup 0$ weakly in E.*

PROOF. We first prove that (1) implies (2). Assuming that $f_n \rightharpoonup f$ weakly in E we have

$$\|T_n f_n - Tf\|_F \leq \|T_n - T\|_{\mathcal{L}(E,F)} \|f_n\|_E + \|T(f_n - f)\|_F.$$

The first term on the right-hand side converges to zero because $T_n \to T$ in $\mathcal{L}(E, F)$ by assumption and weakly convergent sequences are bounded. By compactness of T and since $f_n - f \rightharpoonup 0$ weakly in E, also the second term converges to zero, proving (2).

Clearly (2) implies (3) so it remains to prove that (3) implies (1). We start by showing that T is compact. Because E is reflexive we only need to show that $Tf_n \to 0$ in F whenever $f_n \rightharpoonup 0$ weakly in E (see [39, Proposition VI.3.3]). Assume now that $f_n \rightharpoonup 0$ weakly in E. Clearly (2) in particular shows that $T_n \to T$ strongly, so $T_k f_n \to Tf_n$ as $k \to \infty$ for every fixed $n \in \mathbb{N}$. Hence for every $n \in \mathbb{N}$ there exists $k_n \geq n$ such that $\|T_{k_n} f_n - Tf_n\|_F \leq 1/n$, and thus

$$\begin{aligned}\|Tf_n\|_F &\leq \|Tf_n - T_{k_n} f_n\|_F + \|T_{k_n}(f_n - f_{k_n})\|_F + \|T_{k_n} f_{k_n}\|_F \\ &\leq \frac{1}{n} + \|T_{k_n}(f_n - f_{k_n})\|_F + \|T_{k_n} f_{k_n}\|_F.\end{aligned}$$

By assumption $\|T_{k_n} f_{k_n}\|_F \to 0$ as $n \to \infty$ since $f_n \rightharpoonup 0$ and likewise $\|T_{k_n}(f_n - f_{k_n})\|_F \to 0$ as $n \to \infty$ because $f_n - f_{k_n} \rightharpoonup 0$. Hence the right-hand side of the above inequality converges to zero as $n \to \infty$, so $Tf_n \to 0$ in F and thus T is compact.

To prove that T_n converges in $\mathcal{L}(E, F)$, we assume to the contrary that this is not the case and derive a contradiction. Then there exist $\varepsilon > 0$ and $f_n \in E$ with $\|f_n\| = 1$ such that $\varepsilon \leq \|T_n f_n - Tf_n\|_F$ for all $n \in \mathbb{N}$. As bounded sets in a reflexive space are weakly sequentially compact there exists a subsequence (f_{n_k}) such that $f_{n_k} \rightharpoonup f$ weakly in E. Therefore

$$\begin{aligned}0 < \varepsilon &\leq \|T_{n_k} f_{n_k} - Tf_{n_k}\|_F \\ &\leq \|T_{n_k}(f_{n_k} - f)\|_F + \|T_{n_k} f - Tf\|_F + \|T(f - f_{n_k})\|_F.\end{aligned} \tag{4.1.1}$$

The first term converges to zero by assumption as $f_{n_k} - f \rightharpoonup 0$ weakly in E. The second term converges to zero as $T_n \to T$ strongly, and the last term converges to zero as T is compact and $f - f_{n_k} \rightharpoonup 0$ weakly in E. However, this contradicts (4.1.1), showing that T_n must converge in $\mathcal{L}(E, F)$. Hence (1) holds, completing the proof of the proposition. □

Note that we do not require the T_n to be compact. Hence the sequence of operators is not necessarily collectively compact as in [5].

4.2. *A spectral mapping theorem*

When looking at domain perturbation problems we embedded the problems involved into one single space by making use of inclusions and restrictions as introduced in Section 2.5. The purpose of this section is to show that we can still apply the standard perturbation theory of linear operators to show continuity of the spectrum and the corresponding projections.

Suppose that E, F are Banach spaces, and that A is a closed densely defined operator on F with domain $D(A)$. Moreover, suppose that there exist $i \in \mathcal{L}(F, E)$ and $r \in \mathcal{L}(E, F)$ such that $ri = I_F$. For $\lambda \in \varrho(A)$ we consider the pseudo-resolvent

$$R(\lambda) := i(\lambda I - A)^{-1} r.$$

We then have the following spectral mapping theorem. A pseudo-resolvent is a family of linear operators satisfying the resolvent equation

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu)$$

for λ, μ in some open subset of $\mathbb{C}$.

PROPOSITION 4.2.1. *Suppose that $\lambda \in \varrho(A)$, and that $\mu \neq \lambda$. Then $\mu \in \varrho(A)$ if and only if $(\mu - \lambda)^{-1} \in \varrho(R(\lambda))$. If that is the case, then*

$$R(\mu) = \frac{1}{\mu - \lambda} R(\lambda) \left(\frac{1}{\mu - \lambda} I_E - R(\lambda) \right)^{-1}. \tag{4.2.1}$$

PROOF. Replacing A by $\lambda I_F - A$ we can assume without loss of generality that $\lambda = 0$ and thus $A^{-1} \in \mathcal{L}(F)$. Now $0 \neq \mu \in \varrho(A)$ if and only if $1/\mu \in \varrho(A^{-1})$ (see [92, Theorem III.6.15]), so we only need to prove that $1/\mu \in \varrho(R(0))$ if and only if $1/\mu \in \varrho(A^{-1})$. To do so we first split the equation

$$\frac{1}{\mu} u - R(0) u = f \tag{4.2.2}$$

into an equivalent system of equations. Observe that $P := ir$ is a projection. If we set $E_1 := P(E)$ and $E_2 := (I - P)(E)$, then $E = E_1 \oplus E_2$. By construction, the image of $R(0)$ is in E_1. As $r = rP$ we have $PR(0) = R(0)P$. Setting $u_1 := Pu$ and $u_2 := (I_E - P)u$, equation (4.2.2) is equivalent to the system

$$\left(\frac{1}{\mu} I_E - R(0) \right) u_1 = Pf, \tag{4.2.3}$$

$$\frac{1}{\mu} u_2 = (I - P) f. \tag{4.2.4}$$

Assume now that $\mu \in \varrho(A^{-1})$, and fix $f \in E$ arbitrary. It follows that $u_1 := i(\mu^{-1} I_F - A^{-1})^{-1} Pf$ is the unique solution of (4.2.3), and $u_2 := \mu(I - P)f$ is the unique solution of (4.2.4). Hence $u := u_1 + u_2$ is the unique solution of (4.2.2) and the map $f \to (u_1, u_2)$ is continuous, showing that $1/\mu \in \varrho(R(0))$.

Next assume that $1/\mu \in \varrho(R(0))$, and that $g \in F$ is arbitrary. Set $f := i(g)$ and note that $Pf = f$ in that case. By assumption (4.2.3) has a unique solution u_1. As $(I-P)f = 0$ the solution of (4.2.4) is zero. Hence $r(u_1)$ is the unique solution of $(\mu^{-1}I - A^{-1})u = g$, showing that $1/\mu \in \varrho(A^{-1})$. We finally prove identity (4.2.1), provided $\lambda, \mu \in \varrho(A)$. By the resolvent equation

$$(\lambda I - A)^{-1} = (\mu I - A)^{-1}(I_F - (\lambda - \mu)(\lambda I - A)^{-1}).$$

Using that $ri = I_F$ this yields

$$\begin{aligned} R(\lambda) &= i(\mu I_F - A)^{-1}(I_F r - (\lambda - \mu) r i(\lambda I - A)^{-1} r) \\ &= i(\mu I_F - A)^{-1} r(I_E - (\lambda - \mu) i(\lambda I - A)^{-1} r) \\ &= R(\mu)(I_E - (\lambda - \mu)R(\lambda)) \\ &= (\lambda - \mu)R(\mu)\left(\frac{1}{\lambda - \mu} I_E - R(\lambda)\right). \end{aligned}$$

As we know that $(\lambda - \mu)^{-1} \in \varrho(R(\lambda))$, identity (4.2.1) follows by rearranging the above equation. □

4.3. *Convergence properties of resolvent and spectrum*

We consider a situation similar to the one in Section 4.2, but with a sequence of closed operators A_n defined on Banach spaces F_n with domains $D(A_n)$. Moreover suppose that there exist a Banach space E and operators $i_n \in \mathcal{L}(F_n, E)$ and $r \in \mathcal{L}(E, F_n)$ such that $r_n i_n = I_{F_n}$. We also deal with a limit problem involving a closed densely defined operator A on a Banach space F. For $\lambda \in \varrho(A_n) \cap \varrho(A)$ we consider the *pseudo-resolvents*

$$R_n(\lambda) := i_n(\lambda I - A_n)^{-1} r_n \quad \text{and} \quad R(\lambda) := i(\lambda I - A)^{-1} r$$

similarly as in the concrete case of boundary value problems in Section 2.5. We then have the following theorem about convergence of the pseudo-resolvents.

THEOREM 4.3.1. *Suppose that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(E)$ for some $\lambda \in \mathbb{C}$. Then, for every $\mu \in \varrho(A)$ we have $\mu \in \varrho(A_n)$ for $n \in \mathbb{N}$ large enough, and $R_n(\mu) \to R(\mu)$ in $\mathcal{L}(E)$.*

PROOF. Suppose that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(E)$ for some $\lambda \in \mathbb{C}$, and that $\mu \in \varrho(A)$. By Proposition 4.2.1 we have $(\mu - \lambda)^{-1} \in \varrho(-R(\lambda))$ and so [92, Theorem IV.2.25] implies that $(\mu - \lambda)^{-1} \in \varrho(-R_n(\lambda))$ if only n is large enough. Applying Proposition 4.2.1 again we see that $\mu \in \varrho(A_n)$ if n is large enough. Using (4.2.1) we get

$$\begin{aligned} \lim_{n\to\infty} R_n(\mu) &= \lim_{n\to\infty} \frac{1}{\mu - \lambda} R_n(\lambda)\left(\frac{1}{\mu - \lambda} I_E - R_n(\lambda)\right)^{-1} \\ &= \frac{1}{\mu - \lambda} R(\lambda)\left(\frac{1}{\mu - \lambda} I_E - R(\lambda)\right)^{-1} = R(\mu) \end{aligned}$$

in $L(E)$. Here we use that the map $T \mapsto (\alpha I - T)^{-1}$ is continuous as a map from $\mathcal{L}(E)$ into itself if $\alpha \in \varrho(T)$ (see [121, Theorem IV.1.5]). This completes the proof of the theorem. □

From the above we get the upper semi-continuity of separated parts of the spectrum and in particular the continuity of every finite system of eigenvalues. Recall that a spectral set is a subset of the spectrum which is open and closed in the spectrum. To every spectral set we can consider the corresponding spectral projection (see [92, Section III.6.4]). The following properties of the spectral projections immediately follow from [92, Theorem IV.3.16] and Proposition 4.2.1.

COROLLARY 4.3.2. *Suppose that $R_n(\lambda) \to R(\lambda)$ in $L(E)$ for some $\lambda \in \mathbb{C}$, that $\Sigma \subset \sigma(-A_\Omega) \subset \mathbb{C}$ is a compact spectral set, and that Γ is a rectifiable closed simple curve enclosing Σ, separating it from the rest of the spectrum. Then, for n sufficiently large, $\sigma(A_n)$ is separated by Γ into a compact spectral set Σ_n and the rest of the spectrum. Denote by P and P_n the corresponding spectral projections. Then the dimension of the images of P and P_n are the same, and P_n converges to P in norm.*

REMARK 4.3.3. As a consequence of the above corollary we get the continuity of every finite system of eigenvalues (counting multiplicity) and of the corresponding spectral projection. In particular, we get the continuity of an isolated eigenvalue of simple algebraic multiplicity and its eigenvector when normalised suitably (see [92, Section IV.3.5] for these facts on perturbation theory).

In all cases of domain perturbation we look at, we have that the resolvents $R_n(\lambda)$ act on $L_p(\mathbb{R}^N)$ for all $p \in (1, \infty)$ with image in $L_{m(p)}(\mathbb{R}^N)$ with $m(p)$ given by (2.4.5).

THEOREM 4.3.4. *Suppose that for every $p \in (1, \infty)$ there exists $M, \lambda > 0$ such that*

$$\|R_n(\lambda)\|_{\mathcal{L}(L_p)} + \|R_n(\lambda)\|_{\mathcal{L}(L_p, L_{m(p)})} \leq M \tag{4.3.1}$$

for all $n \in \mathbb{N}$. If $R(\lambda)$ is compact on $L_p(\Omega)$ for some $p \in (1, \infty)$ and $\lambda \in \varrho(A)$, then it is compact for all $p \in (1, \infty)$ and all $\lambda \in \varrho(A)$. Moreover, the following assertions are equivalent:

(1) *There exist $p_0 \in (1, \infty)$ and $\lambda > 0$ such that $R_n(\lambda) f_n \to R(\lambda) f$ in $L_{p_0}(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$.*
(2) *There exist $p_0 \in (1, \infty)$ and $\lambda > 0$ such that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_{p_0}(\mathbb{R}^N))$.*
(3) *For every $\lambda \in \varrho(A)$ and $p \in (1, \infty)$ we have $R_n(\lambda) f_n \to R(\lambda) f$ in $L_q(\mathbb{R}^N)$ for all $q \in [p, m(p))$, whenever $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$.*
(4) *For every $\lambda \in \varrho(A)$ and $p \in (1, \infty)$*
$$R_n(\lambda) \to R(\lambda) \quad \text{in } \mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$$
for all $q \in [p, m(p))$.

Assertions (2) *and* (4) *are equivalent without the compactness of $R(\lambda)$.*

PROOF. The equivalence of (1) and (2) follow directly from Proposition 4.1.1. We show that (2) implies (4). Note that the argument does not make use of the compactness of $R(\lambda)$. Together with Theorem 4.3.1 it follows from (2) that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N))$ for all $\lambda \in \varrho(A)$. Fix $p \in (1, \infty)$ and then $p_1 \in (1, \infty)$ such that either $p_0 < p < p_1$ or $p_0 > p > p_0$. Choose $\lambda \in \varrho(A)$ such that (4.3.1) holds for $p = p_1$. Then by the Riesz–Thorin interpolation theorem

$$\begin{aligned}\|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_p)} &\le \|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_{p_0})}^{1-\theta} \|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_{p_1})}^{\theta} \\ &\le (2M)^\theta \|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_{p_0})}^{1-\theta}\end{aligned}$$

if we choose $\theta \in (0, 1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

(see [22, Theorem 1.1.1]). Hence $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N))$. If $p < q < m(p)$, then again by (4.3.1) and the Riesz–Thorin interpolation theorem

$$\begin{aligned}\|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_p, L_q)} &\le \|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_p)}^{1-\theta} \|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_p, L_{m(p)})}^{\theta} \\ &\le (2M)^\theta \|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_p)}^{1-\theta}\end{aligned}$$

with $\theta \in (0, 1)$ such that

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{m(p)}.$$

Hence $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [p, m(p))$. Recall that we had fixed λ for this argument. Let $\mu \in \varrho(A)$ be arbitrary with $\lambda \neq \mu$. Since the map $T \mapsto (\alpha I - T)^{-1}$ is continuous as a map from $\mathcal{L}(E)$ into itself if $\alpha \in \varrho(T)$ (see [121, Theorem IV.1.5])

$$S_n := \frac{1}{\mu - \lambda}\left(\frac{1}{\mu - \lambda} I_E - R_n(\lambda)\right)^{-1} \to S := \frac{1}{\mu - \lambda}\left(\frac{1}{\mu - \lambda} I_E - R(\lambda)\right)^{-1}$$

in $\mathcal{L}(L_p(\mathbb{R}^N))$. Hence, using the identity (4.2.1),

$$\begin{aligned}\|R_n(\mu) - R(\mu)\|_{\mathcal{L}(L_p, L_q)} &= \|R_n(\lambda)S_n - R(\lambda)S\|_{\mathcal{L}(L_p, L_q)} \\ &\le \|R_n(\lambda)\|_{\mathcal{L}(L_p, L_q)} \|S_n - S\|_{\mathcal{L}(L_p)} + \|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_p, L_q)} \|S\|_{\mathcal{L}(L_p)}\end{aligned}$$

for all $n \in \mathbb{N}$. Using what we have proved already, we get $R_n(\mu) \to R(\mu)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [p, m(p))$. Since all operators $R(\lambda)$ interpolate, a compactness property of the Riesz–Thorin interpolation theorem shows that $R(\lambda)$ is compact as an operator in $\mathcal{L}(L_p(\mathbb{R}^N))$ and $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $p \in (1, \infty)$ and $q \in [p, m(p))$ if $R(\lambda)$ is compact (see [94]). Now the equivalence of (3) and (4) follows from Proposition 4.1.1. □

5. Perturbations for linear Dirichlet problems

The most complete results on domain perturbation are for problems with Dirichlet boundary conditions. After stating the main assumptions we will give a complete characterisation of convergence of solutions for the Dirichlet problem on a domain Ω_n to a solution of the corresponding problem on Ω.

Theorem 5.2.4 is the main theorem on strong convergence and Theorem 5.2.6 the main result on convergence in the operator norm. Section 5.3 is then concerned with *necessary conditions* and Section 5.4 with *sufficient conditions* for convergence.

5.1. *Assumptions and preliminary results*

Given open sets $\Omega_n \subset \mathbb{R}^N$ $(N \geq 2)$ we ask under what conditions the solutions of

$$\begin{aligned} \mathcal{A}_n u + \lambda u &= f_n \quad \text{in } \Omega_n, \\ u &= 0 \quad \text{on } \partial\Omega_n \end{aligned} \tag{5.1.1}$$

converge to a solution of the corresponding problem

$$\begin{aligned} \mathcal{A} u + \lambda u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{5.1.2}$$

on a limit domain Ω as $n \to \infty$. Most of the results in this section are taken from [58], but here we allow perturbations of $\mathcal{A}$ as well. We make the following basic assumptions on the operators $\mathcal{A}_n$ below.

ASSUMPTION 5.1.1. We let $\mathcal{A}_n$ be operators of the form

$$-\operatorname{div}(A_{0n}(x)\nabla u + a_n(x)u) + b_n(x) \cdot \nabla u + c_{0n} u \tag{5.1.3}$$

with $A_{0n} \in L_\infty(\mathbb{R}^N, \mathbb{R}^{N\times N})$, $a_n, b_n \in L_\infty(\mathbb{R}^N, \mathbb{R}^N)$ and $c_{0n} \in L_\infty(\mathbb{R}^N)$. Moreover, assume that the ellipticity constant $\alpha_0 > 0$ can be chosen uniformly with respect to $n \in \mathbb{N}$, and that

$$\sup_{n\in\mathbb{N}}\{\|A_{0n}\|_\infty, \|a_n\|_\infty, \|b_n\|_\infty, \|c_{0n}\|_\infty\} < \infty. \tag{5.1.4}$$

We also assume that $\mathcal{A}$ is an operator of the form (2.1.2) and that

$$\lim_{n\to\infty} A_{0n} = A_0, \qquad \lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b, \qquad \lim_{n\to\infty} c_{0n} = c_0 \tag{5.1.5}$$

almost everywhere in $\mathbb{R}^N$.

The condition that $\mathcal{A}_n, \mathcal{A}$ be defined on $\mathbb{R}^N$ is no restriction since by Remark 2.1.1 we can always extend them to $\mathbb{R}^N$. The bilinear forms associated with the boundary value problem are given by

$$a_n(u, v) := \int_\Omega (A_{0n}\nabla u + a_n u) \cdot \nabla v + (b_n \cdot \nabla u + c_{0n} u)v\, dx \tag{5.1.6}$$

for all $u, v \in H_0^1(\Omega_n)$ and by

$$a(u, v) := \int_\Omega (A_0\nabla u + au) \cdot \nabla v + (b \cdot \nabla u + c_0 u)v\, dx.$$

for all $u, v \in H_0^1(\Omega)$. Applying Proposition 2.1.6 we get the following properties.

PROPOSITION 5.1.2. *Suppose that Assumption* 5.1.1 *is satisfied. Then there exists* $M > 0$ *such that*

$$|a_n(u, v)| \leq M\|u\|_{H_0^1}\|v\|_{H_0^1} \tag{5.1.7}$$

for all $u, v \in H_0^1(\mathbb{R}^N)$ *and all* $n \in \mathbb{N}$. *Moreover,*

$$\frac{\alpha_0}{2}\|\nabla u\|_2^2 \leq a_n(u, u) + \lambda\|u\|_2^2$$

for all $\lambda \in \mathbb{R}$ *with*

$$\lambda \geq \lambda_{\mathcal{A}} := \sup_{n\in\mathbb{N}} \left(\|c_{0n}^-\|_\infty + \frac{1}{2\alpha_0}\|a_n + b_n\|_\infty \right), \tag{5.1.8}$$

and

$$\frac{\alpha_0}{2}\|u\|_{H^1}^2 \leq a_n(u, u) + \lambda\|u\|_2^2$$

for all $\lambda \in \mathbb{R}$ *with*

$$\lambda \geq \lambda_0 := \lambda_{\mathcal{A}} + \frac{\alpha_0}{2} \tag{5.1.9}$$

for all $u \in H_0^1(\mathbb{R}^N)$ *and all* $n \in \mathbb{N}$. *Similar inequalities hold for* $a(\cdot\,,\cdot)$ *with the same constants. Finally,*

$$\lim_{n\to\infty} a_n(u_n, v_n) = a(u, v) \tag{5.1.10}$$

if $u_n \rightharpoonup u$ *weakly and* $v_n \to v$ *strongly in* $H^1(\mathbb{R}^N)$ *or vice versa.*

PROOF. The first properties follow from Proposition 2.1.6. For the last, note the following fact. If c_n is bounded in $L_\infty(\mathbb{R}^N)$ with $c_n \to c$ pointwise and $w_n \to w$ in $L_2(\mathbb{R}^N)$, then $c_n w_n \to cw$ in $L_2(\mathbb{R}^N)$ as well. Indeed,

$$\begin{aligned} \|c_n w_n - cw\|_2 &\leq \|c_n(w_n - w)\|_2 + \|(c_n - c)w\|_2 \\ &\leq \|c_n\|_\infty\|w_n - w\|_2 + \|(c_n - c)w\|_2, \end{aligned}$$

where the first term on the right-hand side converges to zero because $\|c_n\|_\infty$ is bounded and $w_n \to w$ in $L_2(\mathbb{R}^N)$. The second term converges to zero by the dominated convergence theorem. Hence under the given assumptions, every term in (5.1.6) is the L_2 inner product of a strongly and a weakly converging sequence and therefore (5.1.10) follows. □

Depending on the domains we can use $\|\nabla u\|_2$ as a norm on $H_0^1(\Omega)$, for instance if the measure of Ω_n is uniformly bounded, or if all Ω_n are contained between two parallel hyperplanes. Since we do not want to restrict ourselves to such a situation we will generally work with $\lambda \geq \lambda_0$ as given in (5.1.9). From the results in Section 2.2 we construct operators

$$A_n \in \mathcal{L}(H_0^1(\Omega_n), H^{-1}(\Omega_n)) \quad \text{and} \quad A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega)),$$

where by definition

$$H^{-1}(\Omega) = (H_0^1(\Omega))'.$$

Generally, the right-hand side f of (5.1.2) is in $H^{-1}(\Omega)$, so the linear functional f is defined on the closed subspace $H_0^1(\Omega)$ of $H^1(\mathbb{R}^N)$. By the Hahn–Banach theorem (see [125, Theorem 6.5.1]) there exists an extension $\tilde{f}$ of f with $\|\tilde{f}\|_{H^{-1}(\mathbb{R}^N)} = \|f\|_{H^{-1}(\Omega)}$. Hence we can assume without loss of generality that $f \in H^{-1}(\mathbb{R}^N)$.

Suppose that $R_n(\lambda)$, $R(\lambda)$ are given as in Definition 2.5.2. From Theorem 2.2.2 we conclude that

$$[\lambda_0, \infty) \subset \varrho(-A_n) \cap \varrho(-A),$$

and also the uniform estimate

$$\|R_n(\lambda)\|_{\mathcal{L}(H^{-1}, H_0^1)} \leq \frac{2}{\alpha_0} \tag{5.1.11}$$

for all $\lambda \geq \lambda_0$ and all $n \in \mathbb{N}$.

We summarise the results of this section in the following proposition. It is a uniform a priori estimate for weak solutions of (5.1.1) and (5.1.2).

PROPOSITION 5.1.3. *If $\lambda \geq \lambda_0$, then*

$$\|R_n(\lambda)\|_{\mathcal{L}(H^{-1}(\mathbb{R}^N), H^1(\mathbb{R}^N))} \leq \frac{2}{\alpha_0}$$

for all $n \in \mathbb{N}$. A similar estimate with the same constant hold for $R(\lambda)$.

PROOF. The claim follows by combining Lemma 2.5.3, Proposition 2.1.6, Theorem 2.2.2 and Proposition 5.1.2. □

5.2. *The main convergence result*

In this section we summarise the main convergence results for Dirichlet problems. The bulk of the proof will be given in Section 5.5.

When proving that the solutions of (5.1.1) converge to a solution of (5.1.2), the following two conditions appear very naturally.

ASSUMPTION 5.2.1. Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$. The weak limit points of every sequence $u_n \in H_0^1(\Omega_n)$ lie in $H_0^1(\Omega)$.

ASSUMPTION 5.2.2. Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and for every $u \in H_0^1(\Omega)$ there exists $u_n \in H_0^1(\Omega_n)$ such that $u_n \to u$ in $H^1(\mathbb{R}^N)$.

If the above conditions are satisfied it is often said that $\Omega_n \to \Omega$ in the sense of Mosco as this is equivalent to $H_0^1(\Omega_n) \to H_0^1(\Omega)$ as subspaces of $H^1(\mathbb{R}^N)$ in the sense of Mosco [102, Section 1]. The conditions also appear in a more disguised form in [116], and explicitly in [119]. A discussion in terms of capacity appears in [25].

DEFINITION 5.2.3 (Mosco convergence). We say $\Omega_n \to \Omega$ *in the sense of Mosco*, if the open sets $\Omega_n, \Omega \subset \mathbb{R}^N$ satisfy Assumption 5.2.1 and Assumption 5.2.2.

For the formulation of the main convergence result for Dirichlet problems we use the notation and framework introduced in Section 5.1. In particular, $R_n(\lambda)f$ and $R(\lambda)f$ are the weak solutions of (5.1.1) and (5.1.2) extended to $\mathbb{R}^N$ by zero. Also recall that we can choose $f_n, f \in H_0^1(\mathbb{R}^N)$ without loss of generality by extending the functionals by means of the Hahn–Banach Theorem if necessary.

THEOREM 5.2.4. *If $\lambda \geq \lambda_0$, then the following assertions are equivalent.*

(1) $\Omega_n \to \Omega$ *in the sense of Mosco;*
(2) $R_n(\lambda)f_n \rightharpoonup R(\lambda)f$ *weakly in* $H^1(\mathbb{R}^N)$ *whenever* $f_n \rightharpoonup f$ *weakly in* $H^{-1}(\mathbb{R}^N)$*;*
(3) $R_n(\lambda)f_n \to R(\lambda)f$ *in* $H^1(\mathbb{R}^N)$ *whenever* $f_n \to f$ *in* $H^{-1}(\mathbb{R}^N)$*;*
(4) $R_n(\lambda)f \rightharpoonup R(\lambda)f$ *weakly in* $H^1(\mathbb{R}^N)$ *for* f *in a dense subset of* $H^{-1}(\mathbb{R}^N)$.

The property that $R_n(\lambda)f \to R(\lambda)f$, at least in the case of the Laplace operator, is often called the γ-convergence of the solutions (see for instance [27]). Note that in particular, the above theorem implies that convergence is independent of the operator under consideration, a result that has been proved for a restricted class of operators in [15].

The above theorem does not say anything about convergence in the operator norm, it is only a theorem on the strong convergence of the resolvent operators. Strong convergence does not imply the convergence of the eigenvalues to the corresponding eigenvalues of the limit problem. However, according to Corollary 4.3.2 we get convergence of every finite part of the spectrum if the pseudo-resolvents converge in the operator norm.

If we assume that there is a bounded open set B such that $\Omega_n, \Omega \subset B$ for all $n \in \mathbb{N}$, then we get convergence in the operator norm.

COROLLARY 5.2.5. *Suppose that $\Omega_n \to \Omega$ in the sense of Mosco. Moreover, suppose that there exists a bounded open set B such that $\Omega_n, \Omega \subset B$ for all $n \in \mathbb{N}$. Finally let $\lambda \in \varrho(-A)$. Then $\lambda \in \varrho(-A_n)$ for n large enough, and $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(H^{-1}(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, 2d/(d-2))$.*

PROOF. If $\lambda \geq \lambda_0$, then from Theorem 5.2.4 we have that $u_n := R_n(\lambda)f_n \rightharpoonup u := R(\lambda)f$ weakly in $H^1(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $H^{-1}(\Omega)$. Since $u_n \in H_0^1(B)$ for all $n \in \mathbb{N}$ and B is bounded, Rellich's Theorem implies that $u_n \to u$ in $L_q(\mathbb{R}^N)$ for all $q \in [1, 2d/(d-2))$. Hence $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(H^{-1}(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, 2d/(d-2))$ by Proposition 4.1.1. The remaining assertions follow from Theorem 4.3.1. □

We now want to look at the situation where only Ω is bounded, but not necessarily Ω_n. We then get necessary and sufficient conditions for convergence in the operator norm. We denote by $\lambda_1(U)$ the *spectral bound* of $-\Delta$ on the open set U with Dirichlet boundary conditions. It is given by the variational formula

$$\lambda_1(U) = \inf_{\substack{u \in H_0^1(U) \\ u \neq 0}} \frac{\|\nabla u\|_{L_2(U)}^2}{\|u\|_{L_2(U)}^2}. \tag{5.2.1}$$

For convenience we set

$$\lambda_1(\emptyset) := \infty.$$

We then have the following characterisation of convergence in the operator norm.

THEOREM 5.2.6. *Suppose that Ω is bounded and that $\Omega_n \to \Omega$ in the sense of Mosco. Then the following assertions are equivalent.*

(1) *There exists $\lambda > 0$ such that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(H^{-1}(\mathbb{R}^N), L_2(\mathbb{R}^N))$.*

(2) *For every $\lambda \in \varrho(-A)$ we have $\lambda \in \varrho(-A_n)$ for n large enough, and $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [p, m(p))$ and all $p \in (1, \infty)$, where $m(p)$ is defined by* (2.4.5) *with $d = N$.*

(3) *There exists an open set B with $\bar{\Omega} \subset B$ such that $\lambda_1(\Omega_n \cap \bar{B}^c) \to \infty$ as $n \to \infty$.*

PROOF. We know that $R(\lambda) \in \mathcal{L}(H^{-1}(\mathbb{R}^N), H^1(\mathbb{R}^N))$. Since Ω is bounded, Rellich's Theorem implies that $R(\lambda) \in \mathcal{L}(H^{-1}(\mathbb{R}^N), L_2(\mathbb{R}^N))$ is compact. Hence by Proposition 4.1.1, assertion (1) is equivalent to the following statement:

(1') For some λ large enough $R_n(\lambda) f_n \to R(\lambda) f$ in $L_2(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$.

Note that the above implies that $R_n(\lambda) f_n \to R(\lambda) f$ in $L_2(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $L_2(\mathbb{R}^N)$. From Theorem 2.4.2 we have uniform a priori estimates for $\lambda > 0$ large enough, and therefore Theorem 4.3.4 shows that (1') is equivalent to (2). Hence it remains to show that (1') is equivalent to (3).

Suppose that (1') is true, but not (3). Then there exists a bounded open set B containing $\bar{\Omega}$ such that $\lambda_1(\Omega_n \setminus \bar{B}^c) \not\to \infty$. Hence for every $k \in \mathbb{N}$ there exist $n_k > k$ and $\varphi_{n_k} \in C_c^\infty(\Omega_{n_k} \setminus \bar{B}^c)$ and $c > 0$ such that $\|\varphi_{n_k}\|_2 = 1$ and

$$0 \leq \lambda_{n_k} \leq \|\nabla \varphi_{n_k}\|_2^2 \leq c$$

for all $k \in \mathbb{N}$. We define functionals $f_{n_k} \in H^{-1}(\mathbb{R}^N)$ by

$$\langle f_{n_k}, v \rangle := a_{n_k}(\varphi_{n_k}, v) + \lambda \langle \varphi_{n_k}, v \rangle$$

for all $v \in H^1(\mathbb{R}^N)$. By (5.1.7) and the choice of φ_{n_k} we have

$$\|f_{n_k}\|_{H^{-1}} \leq (M + \lambda) \|\varphi_{n_k}\|_{H^1} \leq (M + \lambda)\sqrt{1 + c^2}$$

for all $k \in \mathbb{N}$. This means that (f_{n_k}) is a bounded sequence in $H^{-1}(\mathbb{R}^N)$, and therefore has a subsequence converging weakly in $H^{-1}(\mathbb{R}^N)$ to some $f \in H^{-1}(\mathbb{R}^N)$. We denote that subsequence again by (f_{n_k}). By definition of f_{n_k} we have $\varphi_{n_k} = R_{n_k}(\lambda)f_{n_k}$ and by assumption (1')

$$\varphi_{n_k} = R_{n_k}(\lambda)f_{n_k} \to R(\lambda)f$$

in $L_2(\mathbb{R}^N)$. Since $\operatorname{supp}(\varphi_{n_k}) \cap \bar{\Omega} = \emptyset$, the definition of f_{n_k} implies that $f|_{H^{-1}(\Omega)} = 0$ and so $\varphi_{n_k} \to 0$ in $L_2(\mathbb{R}^N)$. However, this is impossible since we chose φ_{n_k} such that $\|\varphi_{n_k}\|_2 = 1$ for all $k \in \mathbb{N}$. Hence we have a contradiction, so (1') implies (3).

We finally prove that (3) implies (1'). Suppose that $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$. Then by Theorem 5.2.4

$$u_n := R_n(\lambda)f_n \rightharpoonup u := R(\lambda)f$$

weakly in $H^1(\mathbb{R}^N)$. Let B be an open set as in (3) and choose an open bounded set U with $\bar{B} \subset U$. Then by Rellich's theorem $u_n \to u$ in $L_2(U)$. Hence it remains to show that $u_n \to 0$ in $L_2(\mathbb{R}^N \setminus U)$. We choose a cutoff function $\psi \in C^\infty(\mathbb{R}^N)$ such that $0 \le \psi \le 1$, $\psi = 0$ on $\bar{B}$ and $\psi = 1$ on $\mathbb{R}^N \setminus U$. Then, $\psi u_n \in H^1(\Omega \cap \bar{B}^c)$, and setting $\lambda_n := \lambda_1(\Omega \setminus \bar{B})$ we get

$$\begin{aligned}\lambda_n \|u_n\|^2_{L_2(\mathbb{R}^N \setminus U)} &\le \lambda_n \|\psi u_n\|^2_{L_2(\mathbb{R}^N \setminus \bar{B})} \\ &\le \|\nabla(\psi u_n)\|^2_{L_2(\mathbb{R}^N \setminus \bar{B})} = \|\nabla(\psi u_n)\|_2^2 \qquad (5.2.2)\end{aligned}$$

for all $n \in \mathbb{N}$. Since (f_n) is bounded in $H^{-1}(\mathbb{R}^N)$, the sequence (u_n) is bounded in $H^1(\mathbb{R}^N)$. Hence

$$\|\nabla(\psi u_n)\|_2^2 \le (\|\psi\|_\infty^2 + \|\psi\|_\infty^2)\|u_n\|^2_{H^1}$$

is bounded. Because $\lambda_n \to \infty$ by assumption, (5.2.2) implies that $u_n \to 0$ in $L_2(\mathbb{R}^N \setminus U)$. Hence $u_n \to u$ in $L_2(\mathbb{R}^N)$ as claimed. This completes the proof of the theorem. □

REMARK 5.2.7. (a) Note that condition (3) in the above theorem is always satisfied if $\lambda_1(\Omega_n \setminus \bar{\Omega}^c) \to \infty$ as $n \to \infty$. Indeed, from the monotonicity of the first eigenvalue of the Dirichlet problem with respect to the domain

$$\lambda_1(\Omega_n \setminus \bar{\Omega}^c) \le \lambda_1(\Omega_n \setminus \bar{B}^c)$$

for every bounded set B with $\Omega \subset B$. The monotonicity is a consequence of the variational formula (5.2.1).

(b) Note that (3) is also satisfied if $|\Omega_n \cap \bar{\Omega}^c| \to 0$, whether or not Ω_n is bounded. To see this let B_n be a ball of the same volume as $\Omega_n \cap \bar{\Omega}^c$. As the measure goes to zero $\lambda_1(B_n) \to \infty$, and by the isoperimetric inequality for the first eigenvalue of the Dirichlet problem (see [20,83]) we get

$$\lambda_1(\Omega_n \cap \bar{\Omega}^c) \ge c\lambda_1(B_n) \to \infty.$$

EXAMPLE 5.2.8. We give a situation, where we get convergence in the operator norm, but Ω_n has unbounded measure for all $n \in \mathbb{N}$. We can take a disk and attach an infinite strip. We then let the width of the strip tend to zero. Then by Friedrich's inequality (2.1.7)

$$\lambda_1(\Omega_n \setminus \bar{\Omega}) \ge \frac{1}{D^2} \to \infty$$

if the width D of the strip goes to zero. The situation is depicted in Figure 5.1.

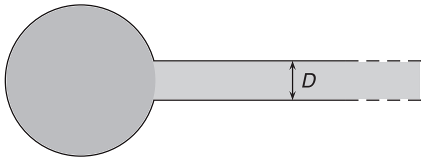

Fig. 5.1. Disc with an infinite strip attached.

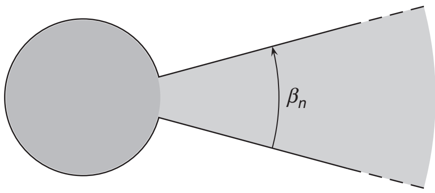

Fig. 5.2. Disc with an infinite cone attached.

In contrast, if we attach a cone of angle β_n rather than a strip, then convergence is not in the operator norm if $\beta_n \to 0$ as shown in Figure 5.2. The fact that these domains converge in the sense of Mosco follows from Theorem 5.4.5 below. Note that $\lambda_1(\Omega_n) = 0$ for all $n \in \mathbb{N}$, and therefore $\lambda_1(\Omega_n) \not\to \lambda_1(\Omega)$. This means that there is no convergence of the spectrum. More examples are given in [58, Section 8]. Examples where just part of the spectrum converges can be found in [107].

5.3. *Necessary conditions for convergence*

In this section we collect some necessary conditions for convergence in the sense of Mosco. For a convergence result such as the one in Theorem 5.2.4 we clearly need that the support of the limit function is in $\bar{\Omega}$. We give a simple characterisation of such a requirement in terms of the spectral bound of the Laplacian on bounded sets outside the limit set $\bar{\Omega}$.

THEOREM 5.3.1. *For open sets $\Omega_n, \Omega \subset \mathbb{R}^N$ the following assertions are equivalent.*

(1) *The weak limit points of every sequence $u_n \in H_0^1(\Omega_n)$, $n \in \mathbb{N}$, in $H^1(\mathbb{R}^N)$ have support in $\overline{\Omega}$;*

(2) *For every open bounded set B with $\bar{B} \subset\subset \mathbb{R}^N \setminus \overline{\Omega}$*
$$\lim_{n\to\infty} \lambda_1(\Omega_n \cap B) = \infty; \tag{5.3.1}$$

(3) *There exists an open covering $\mathcal{O}$ of $\mathbb{R}^N \setminus \overline{\Omega}$ such that* (5.3.1) *holds for all $B \in \mathcal{O}$.*

PROOF. Suppose that (1) holds and let B be a bounded open set with $\bar{B} \subset \mathbb{R}^N \setminus \overline{\Omega}$. Set $\lambda_n := \lambda_1(\Omega_n \cap B)$. Then, by the variational characterisation (5.2.1) of the spectral bound, for every $n \in \mathbb{N}$ there exists $v_n \in C_c^\infty(\Omega_n \cap B)$ with

$$(\lambda_n + 1)\|v_n\|_2^2 \geq \|\nabla v_n\|_2^2 = 1. \tag{5.3.2}$$

Since B is bounded (2.1.7) implies that (v_n) is bounded in $H_0^1(B)$. Hence there exists a subsequence (v_{n_k}) converging weakly to some v in $H_0^1(B)$. By assumption $\operatorname{supp}(v) \subset \bar{B} \subset\subset \mathbb{R}^N \setminus \overline{\Omega}$, and so (1) implies that $v = 0$. As B is bounded Rellich's Theorem shows that $\|v_{n_k}\|_2 \to 0$. Hence, (5.3.2) can only be true if $\lambda_{n_k} - 1 \to \infty$, implying that $\lambda_{n_k} \to \infty$ as $k \to \infty$. The above arguments apply to every weakly convergent subsequence of (v_n) and therefore (1) implies (2).

Clearly (2) implies (3) and so it remains to prove that (3) implies (1). Suppose that $u_n \in H_0^1(\Omega_n)$, and that $u_{n_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ as $k \to \infty$. Let $\mathcal{O}$ be an open covering of $\mathbb{R}^N \setminus \overline{\Omega}$ with the properties stated in (3). Fix $B \subset \mathcal{O}$ and let $\varphi \in C_c^\infty(B)$. Then $\varphi u_n \in H_0^1(\Omega_n \cap \overline{\Omega}^c \cap B)$. The map $u_n \to u_n\varphi$ is a bounded linear map from $H^1(\mathbb{R}^N)$ to $H_0^1(B)$ and therefore is weakly continuous. Hence, if $u_{n_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, then $\varphi u_{n_k} \rightharpoonup \varphi u$ weakly in $H_0^1(B)$. As B is bounded, Rellich's theorem implies that $\varphi u_{n_k} \to \varphi u$ in $L_2(\mathbb{R}^N)$. Now by (5.3.1) and the boundedness of the sequence $(\|\nabla(\varphi u_{n_k})\|_2)$

$$\|\varphi u\|_2^2 = \lim_{k\to\infty} \|\varphi u_{n_k}\|_2^2 \leq \lim_{k\to\infty} \frac{\|\nabla(\varphi u_{n_k})\|_2^2}{\lambda_1(\Omega_{n_k} \cap B)} = 0.$$

Hence $\varphi u = 0$ almost everywhere for all $\varphi \in C_c^\infty(B)$, so $u = 0$ almost everywhere in B. As $\mathcal{O}$ is a covering of $\mathbb{R}^N \setminus \overline{\Omega}$ it follows that $\operatorname{supp} u \subset \overline{\Omega}$ as claimed. □

REMARK 5.3.2. The above condition does not imply Assumption 5.2.1. The reason is that a function $u \in H^1(\mathbb{R}^N)$ with $\operatorname{supp}(u) \subset \bar{\Omega}$ does not need to be in $H_0^1(\Omega)$. We discuss conditions for that in the next subsection.

We next give a characterisation of Assumption 5.2.2 in terms of capacity. A related result appears in [108, Proposition 4.1] and a proof is given in [77, page 75] or [119, page 24]. Our exposition follows [58, Section 7]. Recall that the capacity (or more precisely (1, 2)-capacity) of a compact set $E \subset \mathbb{R}^N$ is given by

$$\operatorname{cap}(E) := \inf\{\|u\|_{H^1}^2 : u \in H_0^1(\mathbb{R}^N) \text{ and } u \geq 1 \text{ in a neighbourhood of } E\}$$

(see [80, Section 2.35]). We could also define capacity with respect to an open set U and define for $E \subset U$ compact $E \subset \mathbb{R}^N$ given by

$$\operatorname{cap}_U(E) := \inf\{\|u\|_{H^1}^2 : u \in H_0^1(U) \text{ and } u \geq 1 \text{ in a neighbourhood of } E\}.$$

It turns out that $\operatorname{cap}_U(E) = 0$ if and only if $\operatorname{cap}(E) = 0$. Moreover, we can work with $u \in C_c^\infty(\mathbb{R}^N)$ and $u \in C_c^\infty(U)$, respectively rather than the Sobolev spaces.

PROPOSITION 5.3.3. *Let $\Omega_n, \Omega \subset \mathbb{R}^N$ be open sets. Then the following conditions are equivalent.*

(1) *Assumption* 5.2.2*;*

(2) *For every open set $B \subset \mathbb{R}^N$ and every $\varphi \in C_c^\infty(\Omega \cap B)$ there exists $\varphi_n \in C_c^\infty(\Omega_n \cap B)$ such that $\varphi_n \to \varphi$ in $H_0^1(\Omega \cap B)$;*

(3) *For every compact set $K \subset \Omega$*

$$\lim_{n\to\infty} \operatorname{cap}(K \cap \Omega_n^c) = 0.$$

PROOF. We prove that (1) implies (3). Fix a compact set $K \subset \Omega$ and let $\varphi \in C_c^\infty(\Omega)$ with $0 \leq \varphi \leq 1$ and $\varphi = 1$ in a neighbourhood of K. By assumption there exists $u_n \in H_0^1(\Omega)$ such that $u_n \to \varphi$ in $H^1(\mathbb{R}^N)$. As $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ there exists $\varphi_n \in C_c^\infty(\Omega)$ such that $\|u_n - \varphi_n\|_{H^1} < 1/n$. Then

$$\|\varphi_n - \varphi\|_{H^1} \leq \|\varphi_n - u_n\|_{H^1} + \|u_n - \varphi\|_{H^1} \leq \frac{1}{n} + \|u_n - \varphi\|_{H^1} \to 0.$$

Now set $\psi_n := \varphi - \varphi_n$. Then by construction $\psi_n \in H^1(\mathbb{R}^N)$ and $\psi_n = 1$ in a neighbourhood of $K \cap \Omega_n^c$. Hence by definition of capacity

$$\operatorname{cap}(K \cap \Omega_n^c) \leq \|\psi_n\|_{H^1}^2 = \|\varphi_n - \varphi\|_{H^1}^2 \to 0$$

as claimed.

We next prove that (3) implies (2). We fix an open set $B \subset \mathbb{R}^N$. Clearly we only need to consider the case where $\Omega \cap B \neq \emptyset$. Let $\varphi \in C_c^\infty(\Omega \cap B)$. By definition of capacity there exists $\psi_n \in C_c(\Omega \cap B)$ such that $\psi_n = 1$ on $\operatorname{supp} \varphi \cap \Omega_n^c$ and such that

$$\|\psi_n\|_{H^1}^2 \leq \operatorname{cap}(\operatorname{supp} \varphi \cap \Omega_n^c) + \frac{1}{n}$$

for all $n \in \mathbb{N}$. Hence by assumption $\psi_n \to 0$ in $H^1(\mathbb{R}^N)$. We now set $\varphi_n := (1 - \psi_n)\varphi$. Then by construction $\varphi_n \in C_c^\infty(\Omega \cap B)$ and

$$\begin{aligned}
\|\varphi_n - \varphi\|_{H^1} = \|\varphi\psi_n\|_{H^1} &\leq \|\varphi\psi_n\|_2 + \|\psi_n \nabla\varphi + \varphi\nabla\psi_n\|_2 \\
&\leq (\|\varphi\|_\infty + \|\nabla\varphi\|_\infty)\|\psi_n\|_2 + \|\varphi\|_\infty \|\nabla\psi_n\|_2 \\
&\leq 2(\|\varphi\|_\infty + \|\nabla\varphi\|_\infty)\|\psi_n\|_{H^1} \to 0.
\end{aligned}$$

Hence we have found $\varphi_n \in C_c^\infty(\Omega \cap B)$ with $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$, proving (2).

We finally prove that (2) implies (1). For given $u \in H_0^1(\Omega)$ there exists $\varphi_k \in C_c^\infty(\Omega)$ such that $\varphi_n \to u$ in $H^1(\Omega)$. Now by (2) there exists $\varphi_{k,n} \in C_c^\infty(\Omega_n)$ such for every fixed $k \in \mathbb{N}$ we have $\varphi_{k,n} \to \varphi_k$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Hence for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\|\varphi_{k,n} - \varphi_k\|_{H^1} < 1/k$ for all $n > n_k$. We can also arrange that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Now we set $u_n := \varphi_{k,n}$ whenever $n_k < n \leq n_{k+1}$. Then $u_n \in H_0^1(\Omega_n)$ and our aim is to show that $u_n \to u$ in $H^1(\mathbb{R}^N)$. To do so fix $\varepsilon > 0$. As $\varphi_k \to u$ in $H^1(\mathbb{R}^N)$ there exists $k_0 \in \mathbb{N}$ such that $1/k + \|\varphi_k - u\|_{H^1} < \varepsilon$ for all $k > k_0$. Given $n > n_{k_0+1}$ there exists $k > k_0$ such that $n_k < n \leq n_{k+1}$ and so by construction $u_n = \varphi_{k,n}$ and

$$\|u_n - u\|_{H^1} \leq \|\varphi_{k,n} - \varphi_k\|_{H^1} + \|\varphi_k - u\|_{H^1} \leq 1/k + \|\varphi_k - u\|_{H^1} < \varepsilon.$$

This shows that $\|u_n - u\|_{H^1} < \varepsilon$ for all $n > n_{k_0+1}$. As $\varepsilon > 0$ was arbitrary we conclude that $u_n \to u$ in $H^1(\mathbb{R}^N)$ as claimed. □

5.4. *Sufficient conditions for convergence*

In this section we collect some simple sufficient conditions for $\Omega_n \to \Omega$ in the sense of Mosco. First we look at approximations of an open set Ω by open sets from the inside.

PROPOSITION 5.4.1. *Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets. If supposing that $\Omega_n \subset \Omega_{n+1} \subset \Omega$ for all $n \in \mathbb{N}$, and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$, then $\Omega_n \to \Omega$ in the sense of Mosco.*

PROOF. Since $H_0^1(\Omega_n) \subset H_0^1(\Omega)$ for all $n \in \mathbb{N}$, Assumption 5.2.1 is clearly satisfied. Suppose that $u \in H_0^1(\Omega)$. If $\varphi \in C_c^\infty(\Omega)$, then by assumption there exists $n_0 \in \mathbb{N}$ such that $\operatorname{supp}(\varphi) \subset \Omega_n$ for all $n \geq n_0$. We now choose $\phi_n \in C_c^\infty(\Omega_n)$ arbitrary for $1 \leq n \leq n_0$ and $\varphi_n := \varphi$ for all $n > n_0$. Then clearly $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ and Assumption 5.2.2 follows from Proposition 5.3.3. □

For approximations from the outside we need a weak regularity condition on the boundary of Ω. We define

$$H_0^1(\bar{\Omega}) := \{u \in H^1(\mathbb{R}^N) \colon u = 0 \text{ almost everywhere on } \bar{\Omega}^c\}$$

We make the following definition.

DEFINITION 5.4.2. We say the open set $\Omega \subset \mathbb{R}^N$ is *stable* if $H_0^1(\Omega) = H_0^1(\bar{\Omega})$.

The above notion of stability is the same as the stability of the Dirichlet problem for harmonic functions on Ω as introduced in Keldyš [93]. An excellent discussion of bounded stable sets is given in [79]. A discussion on the connections between stability of the Dirichlet problem for harmonic functions and the Poisson problem by more elementary means is presented in [9].

PROPOSITION 5.4.3. *An open set $\Omega \subset \mathbb{R}^N$ is stable if one of the following conditions is satisfied:*

(1) *Ω has the segment property except possibly on a set of capacity zero;*

(2) *for all $x \in \partial\Omega$ except possibly a set of capacity zero*

$$\liminf_{r\to 0} \frac{\operatorname{cap}(\bar{\Omega}^c \cap B(x,r))}{\operatorname{cap}(\Omega^c \cap B(x,r))} > 0,$$

where $B(x,r)$ is the ball of radius r centred at x.

The last condition is necessary and sufficient for the stability of Ω.

PROOF. For a proof of (1) we refer to [77, p. 77/78], [119, Section 3.2] or [124, Satz 4.8]), and for (2) to [1, Theorem 11.4.1]. □

More characterisations of stability are in [79, Theorem 11.9]. Note that, if Ω is Lipschitz (or even smoother), then Ω satisfies the segment condition and Ω is therefore stable. According to [67, Theorem V.4.4], the segment condition is equivalent to $\partial\Omega$ to be continuous.

PROPOSITION 5.4.4. *Suppose that $\Omega \subset \Omega_{n+1} \subset \Omega_n$ for all $n \in \mathbb{N}$, and that $\bigcap_{n\in\mathbb{N}} \Omega_n \subset \bar{\Omega}$. If Ω is stable, then $\Omega_n \to \Omega$ in the sense of Mosco.*

PROOF. Since $H_0^1(\Omega) \subset H_0^1(\Omega_n)$ for all $n \in \mathbb{N}$, Assumption 5.2.2 is clearly satisfied. Suppose now that (u_n) is a sequence in $H_0^1(\Omega_n)$. Since $\bigcap_{n\in\mathbb{N}} \Omega_n \subset \bar{\Omega}$ it follows that every weak limit point u of that sequence has support in $\overline{\Omega}$. Hence by the stability of Ω

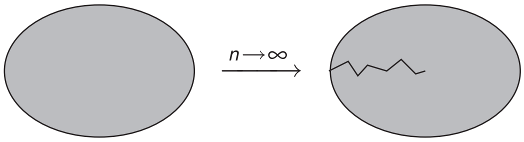

Fig. 5.3. Cracking domain.

we get that $u \in H_0^1(\Omega)$ as required in Assumption 5.2.1. Hence $\Omega_n \to \Omega$ in the sense of Mosco. □

THEOREM 5.4.5. *Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$ are open (not necessarily bounded) sets, and that Ω is stable. Then $\Omega_n \to \Omega$ in the sense of Mosco if and only if the following two conditions are satisfied.*

(1) $\operatorname{cap}(K \cap \Omega_n^c) \to 0$ *as $n \to \infty$ for all compact sets $K \subset \Omega$;*

(2) *There exists an open covering $\mathcal{O}$ of $\mathbb{R}^N \setminus \overline{\Omega}$ such that $\lambda_1(U \cap \Omega_n) \to \infty$ as $n \to \infty$ for all $U \in \mathcal{O}$;*

PROOF. The assertion follows from Proposition 5.3.3 and Theorem 5.3.1, together with the definition of stability. □

Note that the requirement that Ω be stable is too much in certain cases. An example is a cracking domain as in Figure 5.3. It is sufficient to require a condition on

$$\Gamma := \bigcap_{n \in \mathbb{N}} \left(\overline{\bigcup_{k \geq n} (\Omega_k \cap \partial\Omega)} \right) \subset \partial\Omega. \tag{5.4.1}$$

For instance for the cracking domain we consider, the set Γ consists of the end point of the crack. As that set is of capacity zero, we get convergence in the sense of Mosco. Also if $\Gamma \subset \partial\Omega$ satisfies a segment condition except at a set of capacity zero, we also get convergence of Ω_n. A discussion of this condition is given in [58, Section 7] or [119]. Examples of cracking domains also appear in [124].

5.5. *Proof of the main convergence result*

To prove Theorem 5.2.4 we will proceed as follows. First we prove that (1) implies (2) and that (2) implies (3). We then observe that Assumption 5.2.1 follows from (2), whereas Assumption 5.2.2 follows from (3). Hence we could try to prove that (3) implies (2) to get from (3) back to (1). However, this does not seem to be possible. Instead we prove that (3) implies a statement similar to (2), but for the formally adjoint problem. That still implies Assumption 5.2.2, so (3) implies (1). We then prove the equivalence to (4) separately by using the uniform estimate on the norm of $R_n(\lambda)$ and the density.

We start by proving that (1) implies (2).

PROPOSITION 5.5.1. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets with $\Omega_n \to \Omega$ in the sense of Mosco. If $\lambda \geq \lambda_0$, then $R_n(\lambda) f_n \rightharpoonup R(\lambda) f$ weakly in $H^1(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $H^{-1}(\Omega)$.*

PROOF. Let $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$. Set $u_n := R_n(\lambda) f_n$. By Proposition 5.1.3 we have

$$\|u_n\|_{H^1} \leq \frac{2}{\alpha_0} \|f_n\|_{H^{-1}}$$

for all $n \in \mathbb{N}$. Since (f_n) is weakly convergent and therefore bounded, the sequence (u_n) is bounded in $H^1(\mathbb{R}^N)$. Hence it has a weakly convergent subsequence (u_{n_k}) with limit v. By Assumption 5.2.1 we have $v \in H_0^1(\Omega)$. Given $\varphi \in C_c^\infty(\Omega)$ Assumption 5.2.2 implies that there exist $\varphi_n \in H_0^1(\Omega_n)$ with $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$. Because u_n is a weak solution of (5.1.1) we get

$$a_n(u_n, \varphi_n) = \langle f_n, \varphi_n \rangle$$

for all $n \in \mathbb{N}$. Since $u_n \rightharpoonup v$ weakly and $\varphi_n \to \varphi$ strongly we can use (5.1.10) to pass to the limit in the above identity. Hence

$$a_n(v, \varphi) = \langle f, \varphi \rangle$$

for all $\varphi \in C_c^\infty(\Omega)$, showing that v is a weak solution of (5.1.2). Since (5.1.2) has a unique solution we conclude that $v = R(\lambda) f$ and that the whole sequence converges. □

Next we prove that (2) implies (3).

PROPOSITION 5.5.2. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and that $\lambda \geq \lambda_0$. Moreover, suppose $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$ with $R_n(\lambda) f_n \rightharpoonup R(\lambda) f$ weakly in $H^1(\mathbb{R}^N)$. Then $R_n(\lambda) f_n \to R(\lambda) f$ in $H^1(\mathbb{R}^N)$.*

PROOF. Assume that $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$, so that $u_n := R_n(\lambda) f_n \rightharpoonup u := R(\lambda) f$ weakly in $H^1(\mathbb{R}^N)$. Hence because in every instance a strongly and a weakly convergent sequence is paired,

$$\lim_{n\to\infty} (a_n(u_n, u_n) + \lambda \|u_n\|_2^2) = \lim_{n\to\infty} \langle f_n, u_n \rangle = \langle f, u \rangle = a(u, u) + \lambda \|u\|_2^2,$$

and also

$$\lim_{n\to\infty} (a_n(u_n, u) + \lambda \langle u_n, u \rangle) = \lim_{n\to\infty} (a_n(u, u_n) + \lambda \langle u, u_n \rangle) = a(u, u) + \lambda \|u\|_2^2.$$

Therefore

$$\begin{aligned} a_n(u_n - u, u_n - u) + \lambda \|u_n - u\|_2^2 &= a_n(u_n, u_n) + \lambda \|u_n\|_2^2 \\ &- (a_n(u_n, u) + \lambda \langle u_n, u \rangle) - (a_n(u, u_n) + \lambda \langle u, u_n \rangle) + a(u, u) + \lambda \|u\|_2^2 \to 0. \end{aligned}$$

From Proposition 5.1.2 we get

$$\frac{2}{\alpha_0} \|u_n - u\|_{H^1}^2 \leq a_n(u_n - u, u_n - u) + \lambda \|u_n - u\|_2^2 \to 0,$$

showing that $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$. □

As remarked earlier we cannot prove directly the converse of the above proposition, but we can prove the corresponding weak convergence property for the formally adjoint problem we introduced in Section 2.3.

PROPOSITION 5.5.3. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and that $\lambda \geq \lambda_0$. Suppose that $R_n(\lambda)f_n \to R(\lambda)f$ in $H^1(\mathbb{R}^N)$ whenever $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$. Then $R_n^\sharp(\lambda)f_n \rightharpoonup R^\sharp(\lambda)$ weakly in $H^1(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$.*

PROOF. Assume that $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$ and fix $g \in H^{-1}(\mathbb{R}^N)$. Then by (3) $R_n(\lambda)g \to R(\lambda)g$ and so by Lemma 2.5.3

$$\langle g, R_n^\sharp(\lambda)f_n\rangle = \langle R_n(\lambda)g, f_n\rangle \to \langle R(\lambda)g, f\rangle = \langle g, R^\sharp(\lambda)f\rangle,$$

completing the proof of the proposition. □

We now prove that the weak convergence property (2) implies Assumption 5.2.2 and and the strong convergence property (3) implies Assumption 5.2.1.

LEMMA 5.5.4. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and that $\lambda \geq \lambda_0$. Suppose that $R_n(\lambda)f_n \rightharpoonup R(\lambda)f$ in $H^1(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ in $H^{-1}(\mathbb{R}^N)$. Then Assumption 5.2.1 holds.*

PROOF. Let $u_n \in H_0^1(\Omega_n)$ and define functionals $f_n \in H^{-1}(\mathbb{R}^N)$ by

$$\langle f_n, v\rangle := a_n(u_n, v) + \lambda\langle u_n, v\rangle$$

for all $v \in H^1(\mathbb{R}^N)$. By (5.1.7)

$$\|f_n\|_{H^{-1}} \leq (M + \lambda)\|u_n\|_{H^1}$$

for all $n \in \mathbb{N}$. Now assume that $u_{n_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$. We need to show that $u \in H_0^1(\Omega)$. Since (u_{n_k}) is bounded, the above shows that (f_{n_k}) is bounded in $H^{-1}(\mathbb{R}^N)$. Because every bounded sequence in a Hilbert space has a weakly convergent subsequence there exists a further subsequence, denoted again by (f_{n_k}), with $f_{n_k} \rightharpoonup f$. Now by assumption and the definition of f_n

$$u_{n_k} := R_{n_k}(\lambda)f_{n_k} \rightharpoonup R(\lambda)f = u.$$

Since $u \in H_0^1(\Omega)$ Assumption 5.2.1 follows. □

LEMMA 5.5.5. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and that $\lambda \geq \lambda_0$. Suppose that $R_n(\lambda)f_n \to R(\lambda)f$ in $H^1(\mathbb{R}^N)$ whenever $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$. Then Assumption 5.2.2 holds.*

PROOF. Fix $\varphi \in H_0^1(\Omega)$ and define $f \in H^{-1}(\mathbb{R}^N)$ by

$$\langle f, v\rangle := a(\varphi, v) + \lambda\langle \varphi, v\rangle$$

for all $v \in H^1(\mathbb{R}^N)$. Then by (3) and the definition of f

$$\varphi_n := R_n(\lambda)f \to R(\lambda)f = \varphi$$

in $H^1(\mathbb{R}^N)$. Since $\varphi_n \in H_0^1(\Omega_n)$, Assumption 5.2.2 follows. □

We finally need to prove that (4) is equivalent to the other assertions. Clearly (2) implies (4), so we only need to prove that (4) implies (3). The proof is an abstract argument just using a uniform bound on the norms of $R_n(\lambda)$.

LEMMA 5.5.6. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and that $\lambda \geq \lambda_0$. Suppose that $R_n(\lambda)f \rightharpoonup R(\lambda)f$ for f in a dense subset of $H^{-1}(\mathbb{R}^N)$, then $R_n(\lambda)f_n \to R(\lambda)f$ in $H^1(\mathbb{R}^N)$ whenever $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$.*

PROOF. Let V be the dense set of $H^{-1}(\mathbb{R}^N)$ for which $R_n(\lambda)g \rightharpoonup R(\lambda)g$ weakly in $H^1(\mathbb{R}^N)$ for all $g \in V$. Then by Proposition 5.5.2 convergence is actually in $H^1(\mathbb{R}^N)$, so $R_n(\lambda) \to R(\lambda)$ strongly on the dense subset V. Since the norms of $\|R_n(\lambda)\|$ is uniformly bounded by Proposition 5.1.3, we have strong convergence on $H^1(\mathbb{R}^N)$. Also because of the uniform bound and the strong convergence, $R_n(\lambda)f_n \to R(\lambda)f$ in $H^1(\mathbb{R}^N)$ whenever $f_n \to f$ in $H^{-1}(\Omega)$. □

6. Varying domains and Robin boundary conditions

An interesting feature of Robin problems is, that for a sequence of domains, the boundary condition can change in the limit. We consider three different cases.

6.1. *Summary of results*

Given open sets $\Omega_n \subset \mathbb{R}^N$ ($N \geq 2$) we consider convergence of solutions of the Robin problems

$$\begin{aligned} \mathcal{A}_n u + \lambda u &= f_n && \text{in } \Omega_n, \\ \frac{\partial}{\partial \nu_{\mathcal{A}}} u + b_{0n} u &= 0 && \text{on } \partial\Omega_n \end{aligned} \tag{6.1.1}$$

to a solution of a limit problem

$$\begin{aligned} \mathcal{A}u + \lambda u &= f && \text{in } \Omega, \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{6.1.2}$$

on a domain Ω as $n \to \infty$. On the operators $\mathcal{A}_n$ we make the same assumptions as in the case of the Dirichlet problem. We also need a positivity assumption on the boundary coefficient b_0.

ASSUMPTION 6.1.1. Suppose that $\mathcal{A}_n, \mathcal{A}$ satisfy Assumption 5.1.1. Moreover, let $b_{0n} \geq \beta$ for some constant $\beta > 0$.

The above conditions allow us to make use of the domain-independent a priori estimates proved in Section 2.4.2.

The situation is not as simple as in the case of Dirichlet boundary conditions, where the limit problem satisfies Dirichlet boundary conditions as well. Here, the boundary

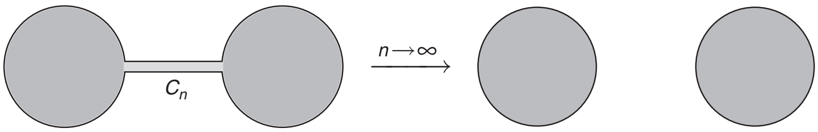

Fig. 6.1. Dumbbell like domain converging to two circles.

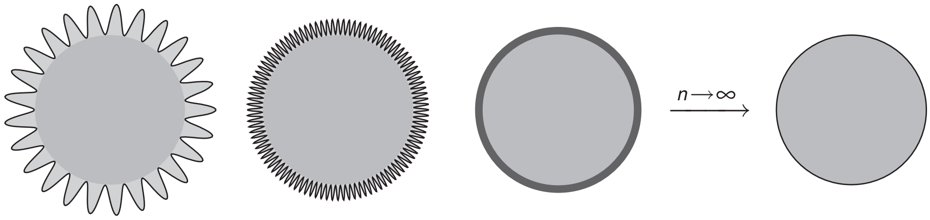

Fig. 6.2. Domains with fast oscillating boundaries approaching a disc.

conditions of the limit problem depends on how the domains Ω_n approach Ω. We consider the following three cases:

(1) The boundary $\partial\Omega$ is only modified in the neighbourhood of a very small set, namely a set of capacity zero. A prototype of such an approximation is a dumbbell with a handle C_n shrinking to a line. The limit set consists of two disconnected sets as shown in Figure 6.1. It is also possible to cut small holes and shrink them to a set of capacity zero. The limit problem is then a Robin problem with the same boundary conditions as the approximating problems. How much boundary we add is irrelevant. A precise statement is in Theorem 6.3.3.

(2) The boundaries of the approaching domains are wildly oscillating. If the oscillations, which do not necessarily need to be periodic, are very fast, then the limit problem turns out to be a Dirichlet problem. See Figure 6.2 for an example. A magnification of the boundary of the last domain shown in the sequence is displayed in Figure 6.3. The precise result is stated in Theorem 6.4.3.

(3) The boundaries of the approaching domains oscillate moderately, not necessarily in a periodic fashion. The limit domain can then be a Robin problem with a different coefficient b_0 in the boundary conditions. Again, an example is as shown in Figure 6.2, with boundary not oscillating quite as fast. The precise result is stated in Theorem 6.5.1.

To see that such phenomena are to be expected, look at the model of heat conduction. The boundary conditions describe a partially insulated boundary, where the loss of heat is proportional to the temperature at the boundary. If the boundary becomes longer the loss is bigger. If the additional boundary is only connected to the main body by a small set as for instance the handle in the case of the dumbbell, then its influence on the temperature inside the major parts of the body is negligible. This corresponds to case one. In the second case, the oscillating boundary will act like a radiator and cool the body better and better, the longer the boundary gets. As the length of the boundary goes to infinity, the

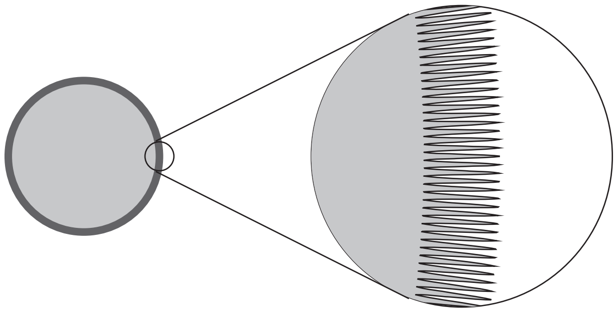

Fig. 6.3. Enlarged portion of a very fast oscillating boundary.

cooling becomes perfect and we get Dirichlet boundary conditions. In the third case the boundary oscillates, but its surface area does not go to infinity and therefore we just get a better cooling, meaning that we have Robin boundary conditions with a possibly larger different boundary coefficient. It is worth noting that the cooling can only get better, not worse.

The first case shows that in a way, the Robin problem behaves very similar to the Dirichlet problem, where we get convergence in the operator norm and therefore convergence of the spectrum. This is in sharp contrast to the Neumann problem. The last two phenomena are *boundary homogenisation results*, where we get the *effective boundary conditions* in the limit. Our exposition follows [51]. In parts we make stronger assumptions to avoid lengthy technical proofs. For periodic oscillations, using very different techniques, other boundary homogenisation results complementing ours are proved in [21,36,71,109] with very different methods. Problems with small holes (obstacles) were considered in [105,106,122,123]. There are applications to problems with nonlinear boundary conditions in [16].

According to Table 2.2 we can set $\lambda_0 = \lambda_{\mathcal{A}}$ given by (5.1.8). Therefore, by Theorem 2.2.2, the Robin problem (6.1.1) is uniquely solvable for all $\lambda \geq \lambda_{\mathcal{A}}$ and all $f_n \in L_p(\Omega_n)$ if $p \geq 2N/(N+1)$. Let $i_{\Omega_n}(f)$ be the extension of $f \in L_p(\Omega)$ by zero outside Ω_n and $r_{\Omega_n}(f)$ the restriction of $f \in L_p(\mathbb{R}^N)$ to Ω_n. Finally let A_n be the operator induced by the form $a_n(\cdot\,,\cdot)$ induced by (6.1.1). We let $R_n(\lambda)$ and $R(\lambda)$ be the pseudo-resolvents associated with the problems $(\mathcal{A}_n, \mathcal{B}_n)$ and the limit problem $(\mathcal{A}, \mathcal{B})$ as in Definition 2.5.2. In all three cases we show that, if $p \in (1, \infty)$, then

$$R_n(\lambda) \to R(\lambda)$$

in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, m(p))$, where $m(p)$ is given by

$$m(p) := \begin{cases} Np(N-p)^{-1} & \text{if } p \in (1, N), \\ \infty & \text{if } p > N. \end{cases} \tag{6.1.3}$$

Since $R(\lambda)$ is compact, Theorem 4.3.4 shows that the above is equivalent to

$$\lim_{n\to\infty} R_n(\lambda) f_n = R(\lambda) f$$

in $L_q(\mathbb{R}^N)$ for all $q \in [1, m(p))$, whenever $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$. The latter property is essential for dealing with semi-linear problems.

After proving some preliminary results we devote to each of the three cases a separate section, where we give a precise statement and a proof of the convergence theorems.

6.2. *Preliminary results*

To prove the three results mentioned in the previous section we make use of some preliminary results. They are about convergence to a solution of the limit problem on Ω without consideration of boundary conditions.

Given open sets Ω_n we work with functions in $u_n \in H^1(\Omega_n)$ and look at their convergence properties in an open set Ω. We do not assume that Ω_n has an extension property. Even if it has, then its norm does not need to be uniformly bounded with respect to $n \in \mathbb{N}$. Instead we extend u_n and ∇u_n by zero outside Ω_n and consider convergence in $L_2(\mathbb{R}^N)$. We denote these functions by $\tilde{u}_n$ and $\widetilde{\nabla u}_n$. The argument used is very similar, but slightly more complicated than the one given on the proof of Proposition 5.5.1. The complication arises because in the present case u_n cannot be considered as an element of $H^1(\mathbb{R}^N)$.

PROPOSITION 6.2.1. *Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$ satisfy Assumption* 5.2.2 *and $\mathcal{A}_n$ Assumption* 5.1.1. *Furthermore, suppose that $u_n \in H^1(\Omega_n)$ are weak solutions of $\mathcal{A}_n u_n = f_n$ with $f_n \in L_2(\mathbb{R}^N)$ and $f_n \rightharpoonup f$ weakly in $L_2(\mathbb{R}^N)$. If $\|u_n\|_{H^1}$ is uniformly bounded, then there exists a subsequence (u_{n_k}) and $u \in H^1(\Omega)$ such that $\tilde{u}_{n_k} \rightharpoonup u$ and $\widetilde{\nabla u}_{n_k} \rightharpoonup \nabla u$ weakly in $L_2(\Omega)$ and $L_2(\Omega, \mathbb{R}^N)$, respectively. Moreover, u is a weak solution of $\mathcal{A}u = f$ in Ω.*

PROOF. By assumption $\|u_n\|_{H^1}$ is uniformly bounded, and so the functions u_n and ∇u_n, extended by zero outside Ω_n are bounded sequences in $L_2(\mathbb{R}^N)$. Hence there exists a subsequence such that $u_{n_k} \rightharpoonup u$ weakly in $L_2(\Omega)$ and $\nabla u_{n_k} \rightharpoonup v$ weakly in $L_2(\Omega, \mathbb{R}^N)$. Renumbering the subsequence we assume that (u_n) and (∇u_n) converge. Fix now $\varphi \in C_c^\infty(\Omega)$. By assumption and Proposition 5.3.3 there exists $\varphi_n \in C_c^\infty(\Omega_n \cap \Omega)$ such that $\varphi_n \to \varphi$ in $H^1(\Omega)$. As $u_n, \varphi_n \in H^1(\Omega_n)$

$$\int_\Omega \nabla u_n \varphi_n \, dx = \int_{\Omega_n} \nabla u_n \varphi_n \, dx = -\int_{\Omega_n} u_n \nabla \varphi_n \, dx = -\int_\Omega u_n \nabla \varphi_n \, dx$$

for all $n \in \mathbb{N}$. Because $\tilde{u}_n, \widetilde{\nabla u}_n$ converge weakly and $\varphi_n, \nabla \varphi_n$ strongly we can pass to the limit and get

$$\int_\Omega v \varphi \, dx = -\int_\Omega u \nabla \varphi \, dx.$$

As $\varphi \in C_c^\infty(\Omega)$ was arbitrary, this means that $v = \nabla u$ is the weak gradient of u in Ω, so $u \in H^1(\Omega)$. Next we observe that, as in the proof of (5.1.10), we have $a_n(u_n, \varphi_n) \to a(u, \varphi)$. We know that $a_n(u_n, \varphi_n) = \langle f_n, \varphi_n \rangle$, and passing to the limit $a(u, \varphi) = \langle f, \varphi \rangle$. Hence $\mathcal{A}u = f$ as claimed. □

The above is a very weak result. In particular, there are no assumptions on Ω_n outside Ω. If we add some more assumptions we get strong convergence.

PROPOSITION 6.2.2. *In addition to the assumptions of Proposition* 6.2.1, *suppose that* (u_n) *is bounded in* $L_r(\mathbb{R}^N)$ *for some* $r > 2$. *Moreover, suppose that for every* $\varepsilon > 0$ *there exists a compact set* $K \subset \Omega$ *and* $n_0 \in \mathbb{N}$ *such that* $K \subset \Omega_n$ *and* $|(\Omega_n \cup \Omega) \setminus K| < \varepsilon$ *for all* $n > n_0$. *Then there exists a subsequence* $(\tilde{u}_{n_k})$ *such that* $\tilde{u}_{n_k} \to u$ *in* $L_q(\mathbb{R}^N)$ *for all* $q \in [2, r)$, *and* $u = 0$ *almost everywhere in* Ω^c.

PROOF. Proposition 6.2.1 guarantees that a subsequence of $(\tilde{u}_n)$ converges weakly in $L_2(\mathbb{R}^N)$ to a solution of $\mathcal{A}u = f$. Denote that subsequence again by $(\tilde{u}_n)$. We show that convergence takes place in the L_2-norm. Because $(\tilde{u}_n)$ is also bounded in $L_r(\mathbb{R}^N)$ we can also assume that the subsequence converges weakly in $L_r(\mathbb{R}^N)$, or weakly* if $r = \infty$. Hence $u \in L_r(\mathbb{R}^N)$.

Fix $\varepsilon > 0$ and a compact set $K \subset \Omega$ and $n_1 \in \mathbb{N}$ such that $K \subset \Omega_n$ and $|(\Omega_n \cup \Omega) \setminus K| < \varepsilon/2$ for all $n > n_1$. Then choose an open set $U \subset K$ with $|(\Omega_n \cup \Omega) \setminus U| < \varepsilon$ for all $n > n_1$. By assumption $(\tilde{u}_n)$ is bounded in $H^1(K)$ and so Rellich's theorem implies that $\tilde{u}_n \to u$ in $L_2(U)$. Hence there exists $n_2 \in \mathbb{N}$ such that $\|\tilde{u}_n - u\|_{L_2(U)} < \varepsilon$ for all $n > n_2$. Using that $u_n \in L_r(\mathbb{R}^N)$ we get by Hölder's inequality

$$\begin{aligned}\|\tilde{u}_n - u\|_2 &= \|\tilde{u}_n - u\|_{L_2(U)} + \|\tilde{u}_n - u\|_{L_2((\Omega_n \cup \Omega)\setminus U)} \\ &< \varepsilon + \left|(\Omega_n \cup \Omega) \setminus U\right|^{1/2-1/r} (\|\tilde{u}_n\|_r + \|u\|_r) < \varepsilon + \varepsilon^{1/2-1/r} (\|\tilde{u}_n\|_r + \|u\|_r)\end{aligned}$$

for all $n > n_0 := \max\{n_1, n_2\}$. By the uniform bound on $\|u_n\|_r$ we conclude that $\tilde{u}_n \to u$ in $L_2(\mathbb{R}^N)$. Since $\Omega^c \subset U^c$ the above argument also shows that

$$\|u_n\|_{L_2(\Omega^c)} \le \|u_n\|_{L_2((\Omega_n \cup \Omega)\setminus U)} < \varepsilon^{1/2-1/r} \|u_n\|_r$$

for all $n > n_0$. Hence $\tilde{u}_n \to 0$ in $L_2(\Omega^c)$ and thus $u = 0$ on Ω^c almost everywhere. By the uniform bound on $\|u_n\|_r$ and interpolation

$$\|\tilde{u}_n - u\|_q \le \|\tilde{u}_n - u\|_2^{\theta} \|\tilde{u}_n - u\|_r^{1-\theta} \to 0$$

for $q \in [2, r)$, where $\theta = \frac{2(r-q)}{q(r-2)}$. Hence $\tilde{u}_n \to u$ in $L_q(\mathbb{R}^N)$ for all $q \in [2, \infty)$. □

We now use the a priori estimates for the solutions of the Robin problem to verify the boundedness assumptions made in the previous propositions.

COROLLARY 6.2.3. *Suppose* Ω_n, Ω *are bounded Lipschitz domains* $p > N$ *and* $f_n \rightharpoonup f$ *in* $L_p(\mathbb{R}^N)$. *Let* u_n *be the weak solution of* (6.1.1). *If* $\lambda \ge \lambda_0$, *then there exists a constant* M *independent of* $n \in \mathbb{N}$ *such that*

$$\|u_n\|_{V_n} + \|u_n\|_\infty \le M. \tag{6.2.1}$$

Moreover, if $\Omega \subset \Omega_n$ *for all* $n \in \mathbb{N}$ *and* $|\Omega_n \setminus \Omega| \to 0$, *then there exists a subsequence* (u_{n_k}) *of* (u_n) *converging to a weak solution of*

$$\mathcal{A}u + \lambda u = f$$

in $L_q(\mathbb{R}^N)$ *for all* $q \in [2, \infty)$. *Finally,* $u = 0$ *on* $\bar{\Omega}^c$ *almost everywhere.*

PROOF. By Theorem 2.4.1 with constants from Table 2.2 there exists a constant C only depending on N, p and an upper bound for $|\Omega_n|$ such that

$$\|u_n\|_\infty \le C \max\left\{\frac{1}{\alpha_0}, \frac{1}{\beta}\right\} \|f_n\|_p. \tag{6.2.2}$$

Similarly, using the norm

$$\|u_n\|_{V_n} := \left(\|\nabla u_n\|_2 + \int_{\partial\Omega_n} u_n^2 \, d\sigma\right)^{1/2}$$

we get from Theorem 2.2.2 that

$$\|u_n\|_{V_n} \le C_1 \|f_n\|_{V_n'} \le C_2 \|f_n\|_p$$

for all $n \in \mathbb{N}$ with constants C_1, C_2 independent of $n \in \mathbb{N}$. Using that weakly convergent sequences in $L_p(\mathbb{R}^N)$ are bounded we get the existence of a constant M independent of $n \in \mathbb{N}$ such that (6.2.1) holds for all $n \in \mathbb{N}$. The second part follows from Proposition 6.2.2 because Assumption 5.2.2 and all other assumptions are clearly satisfied. □

6.3. *Small modifications of the original boundary*

Without further mentioning we use the notation and setup from Section 6.1 and suppose $(\mathcal{A}_n, \mathcal{B}_n)$ satisfy Assumption 6.1.1. We look at a situation where the original boundary remains largely unperturbed. How much boundary we add outside the domain or inside as holes is almost irrelevant. What we mean by small modifications of $\partial\Omega$ we specify as follows.

ASSUMPTION 6.3.1. Suppose that $\Omega_n \subset \mathbb{R}^N$ are bounded open sets satisfying a Lipschitz condition. Let Ω be an open set and $K \subset \bar{\Omega}$ be a compact set of capacity zero such that for every neighbourhood U of K there exists $n_0 \in \mathbb{N}$ such that

$$\bar{\Omega} \cap \left(\overline{\Omega_n \cap (\Omega \cup U)^c}\right) = \emptyset \quad \text{and} \quad \Omega \subset \Omega_n \cup U \tag{6.3.1}$$

for all $n > n_0$. Moreover, assume that

$$\lim_{n\to\infty} |\Omega_n \cap \Omega^c| = 0.$$

Note that the first condition in (6.3.1) means that U allows us to separate $\bar{\Omega}$ from $\overline{\Omega_n \cap (\Omega \cup U)^c}$ which is the part of $\bar{\Omega}_n$ outside $\bar{\Omega}$ as shown in Figure 6.4. Note that the above assumption also allows us to cut holes in Ω shrinking to a set of capacity zero as $n \to \infty$. If (6.3.1) holds, then also

$$\partial\Omega \cap U^c \subset \partial\Omega_n. \tag{6.3.2}$$

This means that $\partial\Omega$ is contained in $\partial\Omega_n$ except for a very small set. To see this let $x \in \partial\Omega \cap U^c$. If W is a small enough neighbourhood of x, then in particular $W \cap \Omega \subset W \cap \Omega_n$ by (6.3.1) and so $W \cap \Omega_n \ne \emptyset$ for every neighbourhood of x. Moreover,

$$\partial\Omega \cap \Omega_n \cap U^c \subset \Omega^c \cap \Omega_n \cap U^c \subset \overline{\Omega_n \cap (\Omega \cup U)^c}$$

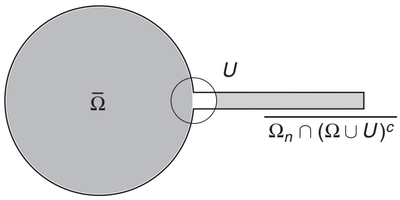

Fig. 6.4. Separation of the part of Ω_n from $\bar{\Omega}$ by U.

and because $\partial\Omega \subset \bar{\Omega}$ (6.3.1) implies that $\partial\Omega \cap \Omega_n \cap U^c = \emptyset$. Hence (6.3.2) follows. We next use the above to construct a sequence of cutoff functions.

LEMMA 6.3.2. *Suppose Ω_n, Ω satisfy Assumption* 6.3.1. *Then there exists a sequence $\psi_n \in C^\infty(\bar{\Omega}) \cap C^\infty(\bar{\Omega}_n)$ such that $0 \le \psi_n \le 1$, $\psi_n \to 0$ on $\Omega_n \cap \bar{\Omega}^c$ and $\psi_n \to 1$ in $H^1(\Omega)$.*

PROOF. Fix a sequence U_k of bounded open sets such that $K \subset \bar{U}_{k+1} \subset U_k$ for all $k \in \mathbb{N}$, that $\bigcap U_k = K$ and that $\operatorname{cap}(\bar{U}_k) \to 0$ converges to zero. We can make such a choice because K is a compact set with $\operatorname{cap}(K) = 0$. By (6.3.1), for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$\bar{\Omega} \cap \big(\overline{\Omega_n \cap (\Omega \cup U_k)^c}\big) = \emptyset \quad \text{and} \quad \Omega \subset \Omega_n \cup U_k$$

for all $n > n_k$. Let now $\theta_k \in C_c^\infty(U_{k+1})$ be such that $0 \le \theta_k \le 1$ and $\theta_k = 1$ on $\bar{U}_k$. Since $\operatorname{cap}(\bar{U}_k)$ converges to zero we can make that choice such that $\theta_k \to 0$ in $H^1(\mathbb{R}^N)$. For $n_k < n \le n_{k+1}$ we now set

$$\psi_n(x) := \begin{cases} 1 - \theta_k(x) & \text{if } x \in \bar{\Omega} \cup \bar{U}_k, \\ 0 & \text{if } x \in (\bar{\Omega} \cup \bar{U}_k)^c. \end{cases}$$

By choice of n_k and θ_k the function ψ_n is well defined with the required properties. □

We now prove the main theorem of this section.

THEOREM 6.3.3. *Suppose Ω_n are bounded Lipschitz domains satisfying Assumption* 6.3.1 *and that $b_{0n} \rightharpoonup b_0$ weakly in $L_r(\partial\Omega)$ for some $r \in (1, \infty)$ or weakly* in $L_\infty(\partial\Omega)$. If $1 < p < \infty$ and $\lambda \in \varrho(-A)$, then for n large enough $\lambda \in \varrho(-A_n)$ and $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, m(p))$. Here $m(p)$ is given by* (6.1.3) *and $R(\lambda)$ is the resolvent operator associated with the problem*

$$\begin{aligned} \mathcal{A}u + \lambda u &= f && \text{in } \Omega, \\ \frac{\partial}{\partial \nu_{\mathcal{A}}} u + b_0 u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{6.3.3}$$

PROOF. We only need to consider $p > N$ and $\lambda > \lambda_{\mathcal{A}}$ with $\lambda_{\mathcal{A}}$ defined by (5.1.8). Given that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$, we show that $R_n(\lambda) f_n \to R(\lambda) f$ in $L_p(\mathbb{R}^N)$. Since $R(\lambda)$ is compact Theorem 4.3.4 and the uniform a priori estimates from Theorem 2.4.1 in conjunction with Table 2.2 complete the proof.

Let $p > N$, $\lambda > \lambda_{\mathcal{A}}$ and suppose that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$. Set $u_n := R(\lambda) f_n$ and $u := R(\lambda) f$. Then Corollary 6.2.3 implies the uniform bound (6.2.1). Also note that Assumption 6.3.1 implies Assumption 5.2.2 and hence Proposition 6.2.2 guarantees that a subsequence of $u_n := R_n(\lambda) f_n$ converges to some $u \in H^1(\Omega)$ in $L_q(\mathbb{R}^N)$ for all $q \in (1, \infty)$. Moreover, $u = 0$ in $\mathbb{R}^N \setminus \Omega$ almost everywhere and u satisfies $\mathcal{A}u + \lambda u = f$ in Ω. Since (6.3.3) has a unique solution, u is the only possible accumulation point for the sequence (u_n), and therefore the whole sequence converges. To simplify notation we therefore assume that u_n converges.

We assumed that Ω_n are Lipschitz domains and that for every open set U containing K the set boundary $\partial\Omega$ is contained in $\partial\Omega_n \cup U$ for n large enough. Hence $\partial\Omega$ satisfies a Lipschitz condition except possibly at $K \cap \partial\Omega$. Because the latter set has capacity zero the restrictions of $C_c^\infty(\mathbb{R}^n)$ to Ω are dense in $H^1(\Omega)$. Fix $\varphi \in C_c^\infty(\mathbb{R}^N)$ and set $\varphi_n := \psi_n \varphi$ with ψ_n the cutoff functions from Lemma 6.3.2. Then by construction $\varphi_n \in H^1(\Omega_n) \cap H^1(\Omega)$ with $\varphi_n \to \varphi$ in $H^1(\Omega)$. In particular

$$a_n(u_n, \varphi_n) + \lambda\langle u_n, \varphi_n\rangle = \langle f_n, \varphi_n\rangle$$

for all $n \in \mathbb{N}$. Clearly $\langle u_n, \varphi_n\rangle \to \langle u, \varphi\rangle$ and $\langle f_n, \varphi_n\rangle \to \langle f, \varphi\rangle$, so we only have to deal with $a_n(u_n, \varphi_n)$. An argument similar to that used in the proof of Proposition 5.1.2 shows that

$$\lim_{n\to\infty} a_{0n}(u_n, \varphi_n), = a_0(u, \varphi_n)$$

where $a_{0n}(\cdot\,,\cdot)$ is the form corresponding to $(\mathcal{A}_n, \mathcal{B}_n)$ excluding the boundary integral as in Definition 2.1.2. We only need to show that

$$\lim_{n\to\infty} \int_{\partial\Omega_n} b_{0n} u_n \varphi_n \, d\sigma = \lim_{n\to\infty} \int_{\partial\Omega} b_{0n} u_n \varphi_n \, d\sigma = \int_{\partial\Omega} b_0 u \varphi \, d\sigma \tag{6.3.4}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$. The first equality is because of the properties of the cutoff functions ψ_n, so we prove the second one. Fix an open set U such that $K \cap \partial\Omega \subset U$. Because $\partial\Omega \cap U^c$ is Lipschitz, the trace operator from $H^1(\Omega)$ into $L_2(\partial\Omega \cap U^c)$ is compact (see [104, Théorème 2.6.2]). As $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ the corresponding traces converge in $L_2(\partial\Omega \cap U^c)$ and by the uniform bound (6.2.1) and interpolation in $L_s(\partial\Omega \cap U^c)$ for all $s \in [1, \infty)$. By construction, for big enough n, we have $\varphi_n = \varphi$ on $\partial\Omega \cap U^c$. Hence by the assumptions on b_{0n}

$$\lim_{n\to\infty} \int_{\partial\Omega\cap U^c} b_{0n} u_n \varphi_n \, d\sigma = \lim_{n\to\infty} \int_{\partial\Omega\cap U^c} b_{0n} u_n \varphi \, d\sigma = \int_{\partial\Omega\cap U^c} b_0 u \varphi \, d\sigma.$$

The assumptions on b_{0n} also imply that (b_{0n}) is bounded in $L_r(\partial\Omega)$, and therefore

$$\Big|\int_{\partial\Omega\cap U} b_{0n} u_n \varphi \, d\sigma\Big| \leq \|u_n \varphi_n\|_\infty \|b_{0n}\|_{L_r(\partial\Omega)} \sigma(\partial\Omega \cap U).$$

Since we can choose the measure $\sigma(\partial\Omega \cap U)$ to be arbitrarily small and $\|u_n \varphi_n\|_\infty$ is uniformly bounded (6.3.4) follows, completing the proof of the theorem. □

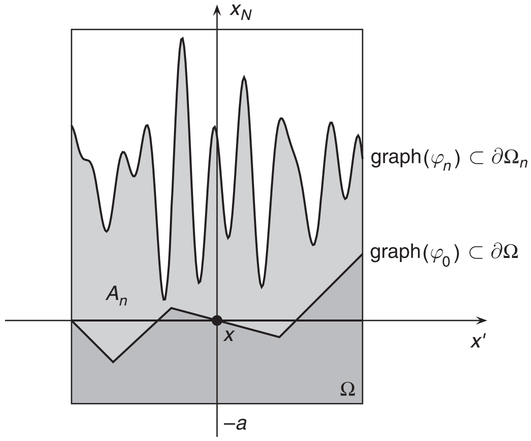

Fig. 6.5. $\partial\Omega$, $\partial\Omega_n$ are locally graphs with respect to the same coordinate system.

6.4. *Boundary homogenisation: Limit is a Dirichlet problem*

In this section we look at a sequence of problems of the form (5.1.1), where $\partial\Omega_n$ is different to $\partial\Omega$ on large parts of the domain. We assume that $\partial\Omega_n$ and $\partial\Omega$ are, at least after a change of coordinates, the graph of a function with respect to the same coordinate system for all $n \in \mathbb{N}$ as illustrated in Figure 6.5. We make this more precise as follows.

ASSUMPTION 6.4.1. Let Ω, Ω_n be domains in $\mathbb{R}^N$ such that $\Omega \subset \Omega_n$ and

$$\lim_{n\to\infty} |\Omega_n \setminus \Omega| = 0.$$

We further assume that for every $x \in \partial\Omega$ there exists a coordinate system with x at the centre and a cylinder $Z = B \times (-a, \infty) \subset \mathbb{R}^{N-1} \times \mathbb{R}$ for some $a > 0$ such that

$$\Omega_n \cap Z = \{(x', x_N) \in B \times (-a, \infty) \colon x_N < \varphi_n(x')\}$$

and

$$\Omega \cap Z = \{(x', x_N) \in B \times (-a, \infty) \colon x_N < \varphi_0(x')\}$$

with $\varphi_n \colon B \to \mathbb{R}$ Lipschitz continuous for all $n \in \mathbb{N}$.

REMARK 6.4.2. Note that $\varphi_n \to \varphi_0$ in $L_1(B)$. To see this note that because $\Omega \subset \Omega_n$ we have $\varphi_n \geq \varphi_0$ and hence

$$\|\varphi_n - \varphi_0\|_{L_1(B)} = \int_B \varphi_n(x') - \varphi_0(x')\, dx' = |Z \cap (\Omega_n \setminus \Omega)| \leq |\Omega_n \setminus \Omega| \to 0$$

as $n \to \infty$.

The assumption that $\Omega \subset \Omega_n$ is only for simplicity to avoid overly technical proofs. It is sufficient to assume that for every compact set $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n > n_0$. The argument is given in [51, Remark 5.10(a)].

We define $R_n(\lambda)$ and $R(\lambda)$ as in Definition 2.5.2, where $R(\lambda)$ is associated with the Dirichlet problem (5.1.2).

THEOREM 6.4.3. *Suppose* Ω_n, Ω *are bounded Lipschitz domains satisfying Assumption* 6.4.1. *Moreover, assume that for every* $x \in \partial\Omega$ *the corresponding functions* $\varphi_n : B \to \mathbb{R}$ *satisfy*

$$\lim_{n\to\infty} \big|\{y \in B : |\nabla\varphi_n(y)| < t\}\big| = 0$$

for all $t > 0$. *Finally suppose that* $b_{0n} \geq \beta$ *for some constant* $\beta > 0$. *If* $1 < p < \infty$ *and* $\lambda \in \varrho(-A)$, *then for n large enough* $\lambda \in \varrho(-A_n)$ *and* $R_n(\lambda) \to R(\lambda)$ *in* $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ *for all* $q \in [1, m(p))$. *Here* $R(\lambda)$ *is the resolvent associated with the Dirichlet problem* (5.1.2) *and* $m(p)$ *is given by* (6.1.3).

PROOF. As in the proof of Theorem 6.3.3 we only need to consider $p > N$ and $\lambda > \lambda_{\mathcal{A}}$, where $\lambda_{\mathcal{A}}$ is given by (5.1.8). The other cases follow from Theorem 4.3.4 and the uniform a priori estimates from Theorem 2.4.1 in conjunction with Table 2.2. Hence assume that $p > N$, that $\lambda > \lambda_{\mathcal{A}}$ and that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$. We set $u_n := R_n(\lambda) f_n$. Convergence of a subsequence to a solution u of $\mathcal{A}u + \lambda u = f$ in $L_p(\mathbb{R}^N)$ follows from Corollary 6.2.3. Hence we only need to show that u satisfies Dirichlet boundary conditions, that is, $u \in H_0^1(\Omega)$. Since the Dirichlet problem has a unique solution, u is the only possible accumulation point for the sequence (u_n), and therefore the whole sequence converges.

The boundary conditions are local, so we only need to look at a neighbourhood of every boundary point. Fix a cylinder Z and functions φ_n as in Assumption 6.4.1. Because the domains are Lipschitz domains, it is sufficient to show that u has zero trace on $\partial\Omega \cap Z$ (see [104, Théorème 2.4.2]). We know that $u_n \rightharpoonup u$ weakly in $H^1(\Omega \cap Z)$. By the compactness of the trace operator $\gamma \in \mathcal{L}(H^1(\Omega \cap Z), L_2(\partial\Omega \cap Z))$ (see [104, Théorème 2.6.2]) we know that $\gamma(u_n) \to \gamma(u)$ in $L_2(\partial\Omega \cap Z)$. From now on we simply write u_n, u for the traces. We need to show that

$$\lim_{n\to\infty} \|u_n\|_{L_2(\partial\Omega\cap Z)} = 0. \tag{6.4.1}$$

We do that in two steps. To express the boundary integrals in the coordinates chosen we need the Jacobians

$$g_n := \sqrt{1 + |\nabla\varphi_n|^2}.$$

By Rademacher's theorem (see [68, Section 3.1.2]) the gradient exists almost everywhere. We can write

$$\|u_n\|_{L_2(\partial\Omega\cap Z)} = \left(\int_B \big|u_n(x', \varphi_0(x'))\big|^2 g_0(x')\, dx'\right)^{1/2} \tag{6.4.2}$$

$$\leq \|g_0\|_\infty^{1/2} \left(\int_B \big|u_n(x', \varphi_n(x')) - u_n(x', \varphi_0(x'))\big|^2 dx'\right)^{1/2} \tag{6.4.3}$$

$$+ \|g_0\|_\infty^{1/2} \left(\int_B \big|u_n(x', \varphi_n(x'))\big|^2 dx'\right)^{1/2}. \tag{6.4.4}$$

We show separately that each term on the right-hand side of the above inequality converges to zero as $n \to \infty$. For the first term we use Fubini's theorem and the fundamental theorem of calculus to write

$$\int_B \left|u_n(x', \varphi_n(x')) - u_n(x', \varphi_0(x'))\right| dx' = \int_B \left|\int_{\varphi_0(x')}^{\varphi_n(x')} \frac{\partial}{\partial x_N} u_n(x', x_N)\, dx_N\right| dx'$$
$$\leq \int_B \int_{\varphi_0(x')}^{\varphi_n(x')} \left|\nabla u_n(x', x_N)\right| dx_N\, dx' \leq |A_n|^{1/2} \|\nabla u_n\|_{L_2(\Omega\cap Z)}, \tag{6.4.5}$$

where A_n is the region between graph(φ_0) and graph(φ_n) as shown in Figure 6.5. By assumption $|\Omega_n \setminus \Omega| \to 0$ and so $|A_n| \to 0$. Furthermore, by (6.2.1) and the definition of u_n the sequence $\|\nabla u_n\|_{L_2(\partial\Omega\cap Z)}$ is bounded. Since u_n is not necessarily continuously differentiable, (6.4.5) needs to be justified. First note that u_n is continuous on $Z \cap \Omega_n$ (see [76, Theorem 8.24]). Because Ω_n is Lipschitz, u_n can be extended to a function in $H^1(Z)$. Since such functions can be represented by a function such that $u_n(x', \cdot)$ is absolutely continuous in the coordinate directions, we can indeed apply the fundamental theorem of calculus as done above (see [101, Section 1.1.3] or [68, Section 4.9.2]). Because

$$\left(\int_B \left|u_n(x', \varphi_n(x')) - u_n(x', \varphi_0(x'))\right|^2 dx'\right)^{1/2}$$
$$\leq 2\|u_n\|_\infty^{1/2} \left(\int_B \left|u_n(x', \varphi_n(x')) - u_n(x', \varphi_0(x'))\right| dx'\right)^{1/2}$$

the bound (6.2.1) implies that (6.4.3) converges to zero.

We next prove that (6.4.4) converges to zero as well. For $t > 0$ we set

$$[g_n \geq t] := \{y \in B : g_n(y) \geq t\} \quad \text{and} \quad [g_n < t] := \{y \in B : g_n(y) < t\}$$

and write

$$\int_B |u_n(x', \varphi_n(x'))|^2\, dx'$$
$$= \int_{[g_n \leq t]} |u_n(x', \varphi_n(x'))|^2\, dx' + \int_{[g_n > t]} |u_n(x', \varphi_n(x'))|^2\, dx'.$$

By (6.2.1) there exists a constant $M_1 > 0$ such that

$$\int_{[g_n \leq t]} \left|u_n(x', \varphi_n(x'))\right|^2 dx' \leq \|u_n\|_\infty^2 |[g_n \leq t]| \leq M_1 |[g_n \leq t]|$$

and that

$$\int_{[g_n > t]} \left|u_n(x', \varphi_n(x'))\right|^2 dx'$$
$$\leq \frac{1}{t} \int_B \left|u_n(x', \varphi_n(x'))\right|^2 g_n(x')\, dx' \leq \frac{1}{t} \|u_n\|^2_{L_2(\partial\Omega\cap Z)} \leq \frac{M_1}{t}$$

for all $t > 0$. Therefore

$$\int_B \left|u_n(x', \varphi_n(x'))\right|^2 dx' \leq M_1 \left(|[g_n \leq t]| + \frac{1}{t}\right).$$

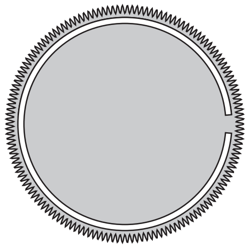

Fig. 6.6. Perturbation of a disc with limit problem a Robin problem.

Fix now $\varepsilon > 0$ and choose $t > 0$ such that $M_1/t < \varepsilon/2$. By assumption there exists $n_0 \in \mathbb{N}$ such that $|[g_n < t]| < \varepsilon/2M_1$ for all $n > n_0$. Hence from the above

$$\int_B \left|u_n(x', \varphi_n(x'))\right|^2 dx' < \varepsilon$$

for all $n > n_0$. Therefore (6.4.4) converges to zero as claimed and (6.4.1) follows. □

REMARK 6.4.4. (a) The assumption that $\partial\Omega_n$ is a graph over $\partial\Omega$ is essential in the above theorem. The domain shown in Figure 6.6 has a very fast oscillating outside boundary. If we shrink the connection from the outside ring with the disc to a point and the ring itself to a circle, then the assumptions of Theorem 6.3.3 are satisfied and the limit problem is the Robin problem (6.3.3).

(b) It is possible to work with diffeomorphisms flattening the boundary locally as shown in Figure 6.7, but the proof is more complicated. We refer to [51] for details.

6.5. *Boundary homogenisation: Limit is a Robin problem*

As in Section 6.4 we look at oscillating boundaries. The oscillations however are slower, and it turns out that the limit problem of (5.1.1) is a Robin problem of the form

$$\begin{aligned} \mathcal{A}u + \lambda u &= f \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu_{\mathcal{A}}} u + g b_0 u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{6.5.1}$$

with first order coefficient gb_0 rather than just b_0. Now the resolvent $R(\lambda)$ is the pseudo-resolvent associated with (6.5.1). Suppose that Assumption 6.4.1 is satisfied and that φ_n are the functions associated with the parametrisation of $\partial\Omega$ and $\partial\Omega_n$. The function g is associated with the limit of the Jacobians

$$g_n := \sqrt{1 + |\nabla\varphi_n|^2}.$$

The assumption that $\Omega \subset \Omega_n$ is again only for simplicity, see [51, Remark 5.10(a)] on how to overcome it. The assumption that $\partial\Omega_n$ is a graph over $\partial\Omega$ is again essential with a similar example as in Remark 6.4.4(a). This can, in a generalised sense, be as shown in Figure 6.7 (see [51] for details).

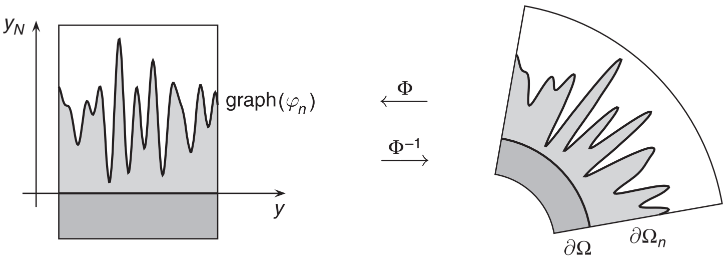

Fig. 6.7. Simultaneously parametrised domains.

THEOREM 6.5.1. *Suppose Ω_n, Ω are bounded Lipschitz domains satisfying Assumption* 6.4.1 *and that $b_0 \in C(\mathbb{R}^N)$. Set $b_{0n} := b_0|_{\partial\Omega_n}$ with $b_0 \geq \beta_0$ for some constant $\beta > 0$. Moreover, assume that $g \in L_\infty(\partial\Omega)$ such that for every $x \in \partial\Omega$ the corresponding functions $\varphi_n \colon B \to \mathbb{R}$ satisfy*

$$\lim_{n\to\infty} \frac{\sqrt{1+|\nabla\varphi_n|^2}}{\sqrt{1+|\nabla\varphi_0|^2}} = \lim_{n\to\infty} \frac{g_n}{g_0} = g$$

weakly in $L_r(B)$ for some $r \in (1,\infty)$, weakly in $L_\infty(B)$ or strongly in $L_1(B)$. If $1 < p < \infty$ and $\lambda \in \varrho(-A)$, then for n large enough $\lambda \in \varrho(-A_n)$ and $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, m(p))$. Here $R(\lambda)$ is the resolvent associated with the Robin problem* (6.5.1) *and $m(p)$ is given by* (6.1.3).

PROOF. As in the previous cases considered we only need to look at $p > N$ and $\lambda > \lambda_{\mathcal{A}}$ with $\lambda_{\mathcal{A}}$ defined by (5.1.8). Given that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$, we show that $R_n(\lambda) f_n \to R(\lambda) f$ in $L_p(\mathbb{R}^N)$. Since $R(\lambda)$ is compact, Theorem 4.3.4 and the uniform a priori estimates from Theorem 2.4.1 in conjunction with Table 2.2 complete the proof. Assume that $p > N$, that $\lambda > \lambda_{\mathcal{A}}$ and that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$. We set $u_n := R_n(\lambda) f_n$. By Corollary 6.2.3 the sequence u_n is bounded in $H^1(\Omega) \cap L_\infty(\mathbb{R}^N)$. It has a subsequence converging to some function $u \in H^1(\Omega)$ in $L_p(\mathbb{R}^N)$. Since (6.5.1) has a unique solution it follows that the whole sequence converges if we can show that u solves (6.5.1). To simplify notation we assume that u_n converges.

We assumed that Ω is a Lipschitz domain and therefore the restrictions of functions in $C_c^\infty(\mathbb{R}^n)$ to Ω are dense in $H^1(\Omega)$. An argument similar to that used in the proof of Proposition 5.1.2 shows that

$$\lim_{n\to\infty} a_{0n}(u_n, \psi) = a_0(u, \psi)$$

where $a_{0n}(\cdot\,,\cdot)$ is the form corresponding to $(\mathcal{A}_n, \mathcal{B}_n)$ excluding the boundary integral as in Definition 2.1.2. We only need to show that

$$\lim_{n\to\infty} \int_{\partial\Omega_n} b_{0n} u_n \psi \, d\sigma = \int_{\partial\Omega} b_0 g u \psi \, d\sigma.$$

By a partition of unity it is sufficient to consider $\psi \in C_c^\infty(\mathbb{R}^N)$ in a cylinder $Z = B \times (-a, \infty)$ as in Assumption 6.4.1. In these coordinates the above becomes

$$\lim_{n\to\infty} \int_B (b_0 u_n \psi)(x', \varphi_n(x')) g_n(x')\, dx' = \int_B (b_0 g u \psi)(x', \varphi_0(x')) g_0(x')\, dx', \tag{6.5.2}$$

where $(b_0 u_n \psi)(x', \varphi_n(x'))$ is the product of the functions $b_0 u_n \psi$ evaluated at the point $(x', \varphi_n(x'))$. First note that

$$u_n(x', \varphi_n(x')) \to u(x', \varphi_0(x'))$$

in $L_1(B)$ by a similar argument as the one used to prove that (6.4.3) converges to zero. The product $b_0\psi$ is continuous and bounded on B and therefore the corresponding superposition operator on $L_1(B)$ is continuous (see [6, Theorem 3.1 and 3.7]). Since $\varphi_n \to \varphi$ in $L_1(B)$ by Remark 6.4.2 we therefore conclude that

$$(b_0\psi)(x', \varphi_n(x')) \to (b_0\psi)(x', \varphi_0(x'))$$

in $L_1(B)$. Since b_0, ψ and u_n are bounded in $L_\infty(B)$ it follows that

$$(b_0 u_n \psi)(x', \varphi_n(x')) \to (b_0 u \psi)(x', \varphi_0(x'))$$

in $L_s(B)$ for all $s \in [1, \infty)$. By assumption

$$g_n(x') = \frac{g_n(x')}{g_0(x')} g_0(x') \rightharpoonup g(x, \varphi_0(x')) g_0(x')$$

weakly in $L_r(B)$ for some $r \in (1, \infty)$, weakly* in $L_\infty(B)$ or strongly in $L_1(B)$. If we combine everything, then (6.5.2) follows. □

REMARK 6.5.2. (a) From the above the function new weight g is larger than or equal to one, so $b_0 g \geq b_0$ always. This reflects the physical description mentioned in Section 6.1, where we argued that a larger surface area of the oscillating boundary will lead to better cooling. We cannot approach a Neumann problem for that reason. The best we can do is to have $g = 1$. If the oscillations are too fast, then heuristically we have "$g = \infty$" and the limit problem is a Dirichlet problem as in Theorem 6.4.3.

(b) Given a Lipschitz domain it is possible to construct a sequence of C^∞ domains satisfying the assumptions of the above theorem in such a way that $g = 1$, that is, the boundary conditions of the limit domain are unchanged. For details see Section 8.3.

7. Neumann problems on varying domains

7.1. *Remarks on Neumann problems*

We saw in Section 2.4.3 that there are no smoothing properties for the Neumann problem uniformly with respect to the domains. This makes dealing with Neumann boundary conditions rather more difficult. In particular, we saw that for Dirichlet and Robin problems the resolvent operators converge in the operator norm. This is not in general the case for Neumann problems. In particular, we cannot expect the spectrum to converge as in the case of the other boundary conditions, which means the resolvent operator only converges strongly (that is, pointwise) in $\mathcal{L}(L_2(\mathbb{R}^N))$. In this exposition we only prove a result similar to those for Dirichlet and Robin problems. It is beyond the scope of these notes to give a comprehensive treatment of the other phenomena. We refer to the literature, in particular to the work of Arrieta [13,17], the group of Bucur, Varchon and Zolésio [29,30] with necessary and sufficient conditions for domains in the plane in [28]. There is other work by Jimbo [87–90] and references therein. Other references include [32,33,47,49,81].

7.2. *Convergence results for Neumann problems*

Given open sets $\Omega_n \subset \mathbb{R}^N$ $(N \geq 2)$ we consider convergence of solutions of the Neumann problems

$$\begin{aligned} \mathcal{A}_n u + \lambda u &= f_n \quad \text{in } \Omega_n, \\ \frac{\partial}{\partial \nu_{\mathcal{A}}} u &= 0 \quad \text{on } \partial\Omega_n \end{aligned} \tag{7.2.1}$$

to a solution the Neumann problem

$$\begin{aligned} \mathcal{A} u + \lambda u &= f \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu_{\mathcal{A}}} u &= 0 \quad \text{on } \partial\Omega_n \end{aligned} \tag{7.2.2}$$

on a domain Ω as $n \to \infty$. On the operators $\mathcal{A}_n, \mathcal{A}$ we make the same assumptions as in the case of the Dirichlet and Robin problems, namely those stated in Assumption 5.1.1. To get a result in the spirit of the others proved so far we make the following assumptions on the domains. The conditions are far from optimal, but given the difficulties mentioned in Section 7.1, we refer to the literature cited there for more general conditions.

ASSUMPTION 7.2.1. Suppose that Ω_n, Ω are bounded open sets with the following properties.

(1) There exists a compact set $K \subset \bar{\Omega}$ of capacity zero such that for every neighbourhood U of K there exists $n_0 \in \mathbb{N}$ with $\Omega \subset \Omega \cup U$ for all $n > n_0$.
(2) $|\Omega_n \cap \Omega^c| \to 0$ as $n \to \infty$.
(3) $\{u|_\Omega : u \in C_c^\infty(\mathbb{R}^N)\}$ is dense in $H^1(\Omega)$.
(4) There exists $d > 2$, $c_a > 0$ and $\lambda_0 \geq 0$ such that
$$\|u\|_{2d/(d-2)}^2 \leq c_a(a_n(u,u) + \lambda_0 \|u\|_2)$$
for all $u \in H^1(\Omega_n)$ and all $n \in \mathbb{N}$. Here $a_n(\cdot\,,\cdot)$ is given as in Definition 2.1.2 without the boundary integral.

REMARK 7.2.2. (a) Condition (1) allows to cut holes into Ω shrinking to a set K of capacity zero. The holes cannot be arbitrary since otherwise (4) is violated. However

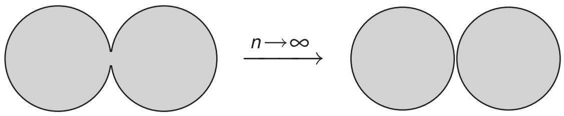

Fig. 7.1. Approximation of touching spheres, preserving a uniform cone condition.

if the holes have a fixed shape and are just contracted by a scalar factor, then (4) is satisfied (see [53, Section 2]).

(b) Condition (4) is satisfied if for instance all domains Ω_n, Ω satisfy a cone condition uniformly with respect to $n \in \mathbb{N}$, that is, the angle and length of the cone defining the cone condition is the same for all $n \in \mathbb{N}$ (see [2, Lemma 5.12]). But as the example with the holes in (a) shows this is only a sufficient condition for (4). Such an approach was used in [37] for instance.

(c) We might think that under the above condition we can have examples like the dumbbell shaped domains in Figure 6.1. As it turns out, condition (4) cannot be satisfied for such a case, because (4) implies convergence in the operator norm as we prove below. Convergence in the operator norm, by Corollary 4.3.2, every finite system of eigenvalues converges, but for dumbbell shaped domains or other exterior perturbations this is not the case (see [13,87]). However, if we replace the dumbbell by two touching balls opened up slightly near the touching balls, then we can ensure that a uniform cone condition is satisfied. Moreover, (3) holds because the union of two balls has a smooth boundary except at a set of capacity zero, where the balls touch (See Figure 7.1). The other conditions are obviously also satisfied.

(d) To get convergence of solutions (but not necessarily of the spectrum) we could work with conditions similar to the Mosco conditions stated in Assumption 5.2.1 and 5.2.2 in the case of the Dirichlet problem. The conditions are explicitly used and stated in [28, Section 2].

We define the resolvent operators $R_n(\lambda)$ and $R(\lambda)$ as in Definition 2.5.2 with $(\mathcal{A}_n, \mathcal{B}_n)$ and $(\mathcal{A}, \mathcal{B})$ being the operators associated with (7.2.1) and (7.2.2), respectively. By Theorem 2.2.2, the Neumann problem (6.1.1) is uniquely solvable for all $\lambda \geq \lambda_0 := \lambda_{\mathcal{A}} + \alpha_0/2$ and all $f_n \in L_2(\Omega_n)$. This means $R_n(\lambda)$ and $R(\lambda)$ is well defined for $n \geq n_0$.

THEOREM 7.2.3. *Suppose that Assumption* 7.2.1 *holds. If* $\lambda \in \varrho(-A)$, *then* $\lambda \in \varrho(A_n)$ *for all n large enough. Moreover, for every* $p \in (1, \infty)$ *we have* $R_n(\lambda) \to R(\lambda)$ *in* $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ *for all* $q \in [p, m(p))$, *where* $m(p)$ *is defined by* (2.4.5).

PROOF. Fix $\lambda \geq \lambda_{\mathcal{A}} + \alpha_0/2$ and let $f_n \rightharpoonup f$ weakly in $L_2(\mathbb{R}^N)$. Set $u_n := R_n(\lambda) f_n$ and $u := R(\lambda) f$. Note that by assumption and (2.4.7) the operator $R(\lambda)$ is compact. By Theorem 4.3.4 and (2.4.2) it is therefore sufficient to prove that $u_n \to u$ in $L_2(\mathbb{R}^N)$. To do so first note that by Theorem 2.2.2 we have

$$\|u_n\|_{H^1(\Omega_n)} \leq \frac{2}{\alpha_0} \|f_n\|_{(H^1(\Omega_n))'} \leq \frac{2}{\alpha_0} \|f_n\|_{L_2(\Omega_n)}$$

for all $n \in \mathbb{N}$. Hence $\|u_n\|_{H^1(\Omega)}$ is uniformly bounded. Hence there is a subsequence (u_{n_k}) such that $(\tilde{u}_{n_k})$ and also $(\widetilde{\nabla u}_{n_k})$ converge weakly in $L_2(\mathbb{R}^N)$. Here $\tilde{u}_{n_k}$ and $\widetilde{\nabla u}_{n_k}$ are the extensions of u_{n_k} and ∇u_{n_k} by zero outside Ω_n. If we can show that $u \in H^1(\Omega)$ and that u solves (7.2.2), then the whole sequence converges since the limit problem admits a unique solution. For simplicity we denote the subsequence chosen again by (u_n).

First note that our assumptions make it possible to apply Proposition 6.2.2 and thus $u \in H^1(\Omega)$ and $u_n \to u$ in $L_2(\mathbb{R}^N)$. Note also that this function has support in $\bar{\Omega}$. We show that u solves (7.2.2). By assumption $\operatorname{cap}(\bar{\Omega} \cap \Omega_n^c) \to 0$ and so there exists $\varphi_n \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \varphi_n \le 1$, $\varphi_n = 1$ in a neighbourhood of $\bar{\Omega} \cap \Omega_n^c$ and $\varphi_n \to 0$ in $H^1(\mathbb{R}^N)$. Let now $\psi \in C_c^\infty(\Omega)$ and set $\psi_n := \psi(1 - \psi_n)$. Then $\psi_n \to \psi$ in $H^1(\mathbb{R}^N)$. Now

$$a_n(u_n, \psi_n) + \langle u_n, \psi_n \rangle = \langle f_n, \psi_n \rangle$$

for all $n \in \mathbb{N}$. Every term in the above identity involves a pair of a strongly and a weakly convergent sequence and therefore an argument similar to that in the proof of Proposition 5.1.2 shows that we can pass to the limit to get

$$a(u, \psi) + \langle u, \psi \rangle = \langle f, \psi \rangle.$$

Because the restrictions of functions in $C_c^\infty(\Omega)$ to Ω are dense in $H^1(\Omega)$ we conclude that u is the weak solution of (7.2.2) as claimed. □

8. Approximation by smooth data and domains

The above can be used to approximate problems on nonsmooth domains by a sequence of problems on smooth domains. This is a useful tool to get results for nonsmooth domains, using results on smooth domains. Such techniques were for instance central in [12,54,86, 93,98]. The technique can be used to prove isoperimetric inequalities for eigenvalues, given they are known for smooth domains and involve constants independent of the geometry of the domain. A recent collection of such inequality for which the technique could be applied appears in [18]. Such an approach was also used in [60] for Robin boundary conditions.

8.1. *Approximation by operators having smooth coefficients*

Consider an operator $\mathcal{A}$ as in Section 2.1 with diffusion matrix $A_0 = [a_{ij}]$, drift terms $a = (a_1, \dots, a_N)$ and $b = (b_1, \dots, b_N)$, and potential c_0 in $L^\infty(\mathbb{R}^N)$. Also assume that $\mathcal{A}$ satisfies the ellipticity condition (2.1.3). If $\mathcal{A}$ is only given on an open set Ω, then extend it to an operator on $\mathbb{R}^N$ as in Remark 2.1.1.

Define the nonnegative function $\varphi \in C_c^\infty(\mathbb{R}^N)$ by

$$\varphi(x) := \begin{cases} c \exp\left(\dfrac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1 \end{cases}$$

with $c > 0$ chosen such that

$$\int_{\mathbb{R}^N} \varphi(x)\,dx = 1.$$

For all $n \in \mathbb{N}$ define φ_n by $\varphi_n(x) := n^N \varphi(nx)$. Then $(\varphi_n)_{n\in\mathbb{N}}$ is an approximate identity in $\mathbb{R}^N$. For all $n \in \mathbb{N}$ and $i, j = 1, \dots, N$ we set

$$a_{ij}^{(n)} := \varphi_n * a_{ij}, \qquad a_i^{(n)} := \varphi_n * a_i, \qquad b_i^{(n)} := \varphi_n * b_i \quad \text{and} \quad c_{0n} := \varphi_n * c_0.$$

We then define $\mathcal{A}_n$ by

$$\mathcal{A}_n u := -\operatorname{div}(A_n \nabla u + a_n u) + b_n \cdot \nabla u + c_{0n} u \tag{8.1.1}$$

with $A_n = [a_{ij}^{(n)}]$, $a_n = (a_1^{(n)}, \dots, a_N^{(n)})$, $b_n = (b_1^{(n)}, \dots, b_N^{(n)})$. The following proposition shows that the family of operators $\mathcal{A}_n$ in particular satisfies Assumption 5.1.1.

PROPOSITION 8.1.1. *The family of operators $\mathcal{A}_n$ as defined above has coefficients of class C^∞ and satisfies Assumption* 5.1.1. *Moreover, $\mathcal{A}_n$ has the same ellipticity constant α_0 as $\mathcal{A}$ and*

$$\lambda_{\mathcal{A}_n} := \|c_{0n}^-\|_\infty + \frac{1}{2\alpha_0}\|a_n + b_n\|_\infty^2 \le \lambda_{\mathcal{A}} \tag{8.1.2}$$

for all $n \in \mathbb{N}$, where $\lambda_{\mathcal{A}}$ is defined by (2.1.10).

PROOF. Let $g \in L_\infty(\mathbb{R}^N)$. By the properties of convolution

$$\begin{aligned} -\|g^-\|_\infty &= -\int_{\mathbb{R}^N} \varphi_n(y)\|g^-\|_\infty\,dy \le \int_{\mathbb{R}^N} \varphi_n(y) g(y-x)\,dy \\ &= \varphi_n * g \le \int_{\mathbb{R}^N} \varphi_n(y)\|g^+\|_\infty\,dy = \|g^+\|_\infty \end{aligned}$$

for all $n \in \mathbb{N}$ if we use that $\|\varphi_n\|_1 = 1$. Here $g^+ := \max\{u, 0\}$ and $g^- := \max\{-u, 0\}$ are the positive and negative parts of g. In particular, from the above

$$\|\varphi_n * g\|_\infty \le \|g\|_\infty \quad \text{and} \quad \|(\varphi_n * g)^-\|_\infty \le \|g^-\|_\infty.$$

Assuming that the ellipticity condition (2.1.3) holds we get

$$\begin{aligned} \xi^T A_n(x)\xi &= \sum_{i,j=1}^N a_{ij}^{(n)}(x)\xi_j\xi_i = \sum_{i,j=1}^N \int_{\mathbb{R}^N} \varphi_n(y) a_{ij}(x-y)\,dy\,\xi_j\xi_i \\ &= \int_{\mathbb{R}^N} \varphi_n(y) \sum_{i,j=1}^N a_{ij}(x-y)\xi_j\xi_i\,dy \ge \int_{\mathbb{R}^N} \varphi_n(y)\alpha_0|\xi|^2\,dy = \alpha_0|\xi|^2 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\mathcal{A}_n$ is elliptic with the same ellipticity constant as $\mathcal{A}$. Using the smoothing properties of convolution, the coefficients of $\mathcal{A}_n$ are of class C^∞. Moreover, they converge to the corresponding coefficients of $\mathcal{A}$ almost everywhere (see [68, Section 4.2.1, Theorem 1]). □

8.2. *Approximation by smooth domains from the interior*

The purpose of this section is to prove that every open set in $\mathbb{R}^N$ can be exhausted by a sequence of smoothly bounded open sets. This fact is frequently used, but proofs are often not given, only roughly sketched or very technical (see [63, Section II.4.2, Lemma 1] and [67, Theorem V.4.20]). We give a simple proof based on the existence of suitable cutoff functions and Sard's lemma. The idea is similar to the proof of [67, Theorem V.4.20], where the existence of a sequence of approximating domains with analytic boundary is shown. For open sets U, V we write

$$U \subset\subset V$$

if U is bounded and $\bar{U} \subset V$.

PROPOSITION 8.2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then there exists a sequence of bounded open sets Ω_n with boundary of class C^∞ such that $\Omega_n \subset\subset \Omega_{n+1} \subset\subset \Omega$ for all $n \in \mathbb{N}$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$. If Ω is connected we can choose Ω_n to be connected as well.*

PROOF. Given an open set $\Omega \subset \mathbb{R}^N$, define

$$V_n := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > 1/n,\ |x| < n\}$$

for all $n \in \mathbb{N}$. Then $U_n := V_n$ is open and $U_n \subset\subset U_{n+1} \subset\subset \Omega$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} U_n = \Omega$. Next choose cutoff functions $\psi_n \in C_c^\infty(U_{n+1})$ such that $0 \le \psi \le 1$ with $\psi = 1$ on $\bar{U}_n$. By Sard's lemma (see [85, Theorem 3.1.3]) we can choose regular values $t_n \in (0, 1)$ of ψ_n for every $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ we set

$$E_n := \{x \in \Omega \colon \psi_n(x) > t_n\}.$$

Let Ω_n consist of the connected components of E_n containing U_n. With this choice $U_n \subset\subset \Omega_n \subset\subset U_{n+1}$, and since t_n is a regular value of ψ_n, by the implicit function theorem, $\partial\Omega_n$ is of class C^∞. By the properties of U_n also $\Omega_n \subset\subset \Omega_{n+1} \subset\subset \Omega$ for all $n \in \mathbb{N}$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$.

If Ω is connected we can choose a sequence U_n of connected subsets of V_n with $U_n \subset\subset U_{n+1} \subset\subset \Omega$ for all $n \in \mathbb{N}$. Then the Ω_n constructed above are connected as required. We finish this proof by showing that we can indeed choose U_n connected if Ω is connected. There exists $n_0 \in \mathbb{N}$ such that $V_n \neq \emptyset$ for all $n \ge n_0$. Fix $x_0 \in V_{n_0}$ and denote by U_{n_0} the connected component of V_{n_0} containing x_0. For $n > n_0$ we inductively define U_n to be the connected component of V_n containing U_{n-1}. Then $U := \bigcup_{n\in\mathbb{N}} U_n$ is a nonempty open set. If we show that $\Omega \setminus U$ is open, then the connectedness of Ω implies that $U = \Omega$. Let $x \in \Omega \setminus U$. Since Ω is open there exists $m \in \mathbb{N}$ such that $B(x, 2/m) \subset \Omega$. Hence $B(x, 1/m) \subset V_m$ and so $B(x, 1/m) \cap U_n = \emptyset$ for all $n \in \mathbb{N}$, since otherwise $B(x, 1/m) \subset U_n$ for some n. Therefore $B(x, 1/m) \subset \Omega \setminus U$ and thus $\Omega \setminus U$ is open. □

8.3. *Approximation from the exterior for Lipschitz domains*

Approximation from the inside by smooth domains is a useful technique for the Dirichlet problem. The situation is more difficult for Robin problems, where we saw in Section 6 that the limit problem very much depends on the boundary of the domains Ω_n. We want to state an existence theorem on a sequence of smooth domains where the boundary measure converges to the correct measure of the limit domain. The result originally goes back to Nečas [103]. We state it as proved in [66, Theorem 5.1].

THEOREM 8.3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then there exists a sequence of domains Ω_n of class C^∞ satisfying Assumption* 6.4.1. *Moreover, the functions $\varphi_n \colon B \to \mathbb{R}$ from Assumption* 6.4.1 *can be chosen such that*

(1) $\varphi_n \in C^\infty(B)$ *for* $n \geq 1$;

(2) $\varphi_n \to \varphi_0$ *uniformly on* B;

(3) $\|\varphi_n\|_\infty$ *is uniformly bounded with respect to* $n \in \mathbb{N}$;

(4) $\nabla\varphi_n \to \nabla\varphi_0$ *in* $L_p(B)$ *for all* $p \in [1, \infty)$.

The situation is shown in Figure 6.5. Note that Condition (4) is the most difficult to achieve.

REMARK 8.3.2. We can apply the above theorem to get a sequence of smooth domains approaching a given Lipschitz domain by smooth domains in such a way that Theorem 6.5.1 applies with $g = 1$, that is, the boundary conditions on the limit domain are the same as on the domains Ω_n.

Let φ_n be as in the above theorem and set

$$g_n := \sqrt{1 + |\nabla\varphi_n|^2}.$$

Since for $0 \leq b < a$ the function $t \mapsto \sqrt{t + a^2} - \sqrt{t + b^2}$ is decreasing as a function of $t \geq 0$ we get

$$\sqrt{1 + a^2} - \sqrt{1 + b^2} \leq a - b.$$

Therefore

$$\left|\sqrt{1 + |\nabla\varphi_n|^2} - \sqrt{1 + |\nabla\varphi_0|^2}\right| \leq \big||\nabla\varphi_n| - |\nabla\varphi_0|\big| \leq |\nabla\varphi_n - \nabla\varphi_0|,$$

and so from (4) in the above theorem

$$\left\|\frac{g_n}{g_0} - 1\right\|_p \leq \|g_n - g_0\|_p \leq \|\nabla\varphi_n - \nabla\varphi_0\|_p \to 0$$

as $n \to \infty$ for all $p \in [1, \infty)$. Hence

$$\lim_{n\to\infty} \frac{g_n}{g_0} = 1$$

in $L_p(\Omega)$ for all $p \in [1, \infty)$ as required in Theorem 6.5.1.

9. Perturbation of semi-linear problems

9.1. *Basic convergence results for semi-linear problems*

Consider the boundary value problem

$$\begin{aligned} \mathcal{A}u &= f(x, u(x)) && \text{in } \Omega, \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{9.1.1}$$

and perturbations

$$\begin{aligned} \mathcal{A}_n u &= f(x, u(x)) && \text{in } \Omega_n, \\ \mathcal{B}_n u &= 0 && \text{on } \partial\Omega_n. \end{aligned} \tag{9.1.2}$$

We assume throughout that $\mathcal{A}_n, \mathcal{A}$ satisfy Assumption 5.1.1. We considered linear problems with various boundary operators $\mathcal{B}_n$. Let $R_n(\lambda)$ and $R(\lambda)$ be the resolvent operators corresponding to the boundary value problems $(\mathcal{A}_n, \mathcal{B}_n)$ and the relevant limit problem $(\mathcal{A}, \mathcal{B})$ as given in Definition 2.5.2. Finally, let A_n, A be the operators induced by $(\mathcal{A}_n, \mathcal{B}_n)$ and $(\mathcal{A}, \mathcal{B})$ as defined in Section 2.2. In all cases we proved that the following assumptions hold.

ASSUMPTION 9.1.1. Let Ω be bounded. For every $\lambda \in \varrho(-A)$ there exists $n_0 \in \mathbb{N}$ such that $\lambda \in \varrho(-A_n)$ for all $n > n_0$. Moreover, for every $p \in (1, \infty)$

$$\lim_{n\to\infty} R_n(\lambda) = R(\lambda)$$

in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N)) \cap \mathcal{L}(L_p(\mathbb{R}^N))$ for all $q \in [p, m(p))$. Here $m(p)$ is given by (2.4.5) with d depending on to the type of problem under consideration (see Tables 2.1–2.3).

We summarise the various cases below.

(1) Dirichlet boundary conditions on Ω_n and the limit domain Ω (see Section 5.2, in particular Theorem 5.2.6).
(2) Robin boundary conditions on Ω_n and Ω, perturbing the original boundary $\partial\Omega$ only very slightly (see Section 6.3). The dumbbell problem shown in Figure 6.1 is an example.
(3) Robin boundary conditions on Ω_n with very fast oscillating boundary. On the limit domain Ω we have Dirichlet boundary conditions (see Section 6.4).
(4) Robin boundary conditions on Ω_n with moderately fast oscillating boundary. On the limit domain Ω we have Robin boundary conditions with a different boundary coefficient (see Section 6.5).
(5) Neumann boundary conditions and an additional assumption on the domains Ω_n. In the limit we also have Neumann boundary conditions (see Section 7).

As seen in Section 3.1 we can rewrite (9.1.1) and (9.1.2) as fixed point equations in $L_p(\Omega)$ and $L_p(\Omega_n)$, respectively. We want write the problems as equations on $L_p(\mathbb{R}^N)$.

Suppose that $f: \mathbb{R}^N \times \mathbb{R}$ satisfies Assumption 3.1.1 and let F be the corresponding superposition operator. In the spirit of Proposition 3.1.3 we let

$$G(u) := R(\lambda)(F(u) + \lambda u) \quad \text{and} \quad G_n(u) := R_n(\lambda)(F(u) + \lambda u)$$

for $\lambda \geq 0$ big enough so that $R_n(\lambda)$, $R(\lambda)$ are well defined. The only difference to the definition given in Proposition 3.1.3 is, that G acts on $L_p(\mathbb{R}^N)$ rather than $L_p(\Omega)$, and similarly G_n. This allows us to work in a big space independent of Ω. By Proposition 3.1.3, finding a solution of (9.1.1) in $L_p(\Omega)$ is equivalent to finding a solution to the fixed point equation

$$u = G(u) \tag{9.1.3}$$

in $L_p(\mathbb{R}^N)$ for $p \geq 2d/(d-2)$. Similarly, (9.1.2) is equivalent to

$$u = G_n(u) \tag{9.1.4}$$

in $L_p(\mathbb{R}^N)$. We now prove some basic properties of G_n, G.

PROPOSITION 9.1.2. *Suppose that* $f: \mathbb{R}^N \times \mathbb{R}$ *satisfies Assumption* 3.1.1, *and that* γ, p *are such that* (3.1.5) *holds. Let* Ω *be bounded and suppose that Assumption* 9.1.1 *is satisfied. Then*

(1) $G, G_n \in C(L_p(\mathbb{R}^N), L_p(\mathbb{R}^N))$ *for all* $n \in \mathbb{N}$;
(2) *If* $u_n \to u$ *in* $L_p(\mathbb{R}^N)$, *then* $G_n(u_n) \to G(u)$ *in* $L_p(\mathbb{R}^N)$;
(3) *For every bounded sequence* (u_n) *in* $L_p(\mathbb{R}^N)$, *the sequence* $(G_n(u_n))_{n\in\mathbb{N}}$ *has a convergent subsequence in* $L_p(\mathbb{R}^N)$.

Moreover, the map G_n *is compact if* Ω_n *is bounded.*

PROOF. By the growth condition (3.1.5) and Assumption 9.1.1 it follows that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_{p/\gamma}(\mathbb{R}^N)) \cap \mathcal{L}(L_p(\mathbb{R}^N))$. The first assertion (1) then directly follows from Proposition 3.1.3. Also the compactness of G and G_n if Ω and Ω_n are bounded follows from Proposition 3.1.3. Statement (2) follows from the continuity of F proved in Lemma 3.1.2 and the assumptions on $R_n(\lambda)$. To prove (3) note that by Proposition 3.1.3 the sequence $(F(u_n))_{n\in\mathbb{N}}$ is bounded for every bounded sequence (u_n) in $L_p(\Omega)$. Hence there exists a subsequence such that $F(u_{n_k}) + \lambda u_{n_k} \rightharpoonup g$ weakly in $L_{p/\gamma}(\mathbb{R}^N) + L_p(\mathbb{R}^N)$. By Theorem 4.3.4

$$G_{n_k}(u_{n_k}) = R_{n_k}(\lambda)(F(u_{n_k}) + \lambda u_{n_k}) \to R(\lambda)g$$

in $L_p(\mathbb{R}^N)$, completing the proof of the proposition. □

REMARK 9.1.3. Sometimes, when working with positive solutions and comparison principles it is useful to make sure $f(x, \xi) + \lambda\xi$ is increasing as a function of ξ, and that $R(\lambda)$ is a positive operator. Both can be achieved by choosing $\lambda > 0$ large enough if $f \in C^1(\mathbb{R}^N \times \mathbb{R})$.

REMARK 9.1.4. For the assertions of the above propositions to be true Assumption 3.1.1 is not necessary. We discuss some examples. For instance consider a linear f of the form

$$f(x, u) = w_n u$$

with $w_n \rightharpoonup w$ weakly in $L_q(\mathbb{R}^N)$ for some $q > d/2$. Such cases arise in dealing with semi-linear problems (see for instance [50, Lemma 2]). More generally, suppose that γ, p satisfy (3.1.5) and that $q > p/(\gamma - \delta)$. If

$$f(x, u) = w_n(x)|u|^{\delta-1}u$$

with $w_n \rightharpoonup w$ weakly in $L_q(\mathbb{R}^N)$, then the assertions of the above propositions hold in $L_p(\mathbb{R}^N)$.

THEOREM 9.1.5. *Suppose that* $f : \mathbb{R}^N \times \mathbb{R}$ *satisfies Assumption* 3.1.1*, and that* γ*,* p *are such that* (3.1.5) *holds. Further suppose that* u_n *are solutions of* (9.1.2) *such that the sequence* (u_n) *is bounded in* $L_p(\mathbb{R}^N)$*. Finally suppose that Assumption* 9.1.1 *is satisfied. Then there exists a subsequence* (u_{n_k}) *converging to a solution* u *of* (9.1.1) *in* $L_q(\mathbb{R}^N)$ *for all* $q \in [p, \infty)$*.*

PROOF. We know from the above discussion that $u_n = G_n(u_n)$ for all $n \in \mathbb{R}^N$. Since (u_n) is bounded, Proposition 9.1.2 property (3) implies that there exists a subsequence (u_{n_k}) such that

$$u_{n_k} = G_{n_k}(u_{n_k}) \to u$$

in $L_p(\mathbb{R}^N)$. Now by Proposition 9.1.2 property (2) we get that $G_{n_k}(u_{n_k}) \to G(u)$ in $L_p(\mathbb{R}^N)$. Theorem 3.2.1 implies that (u_n) is bounded in $L_\infty(\mathbb{R}^N)$. Hence by interpolation, convergence is in $L_q(\mathbb{R}^N)$ for all $q \in [p, \infty)$. □

REMARK 9.1.6. (a) If we assume that $1 \le \gamma < (d+2)/(d-2)$, then a bound in $L_2(\mathbb{R}^N)$ is sufficient in the above theorem.

(b) Solutions to semi-linear problems are in general not unique, so we do not expect the whole sequence to converge. If we happen to know by some means that (u_n) has at most one limit, then the whole sequence converges.

We next look at problems *without growth conditions* on the nonlinearity, but instead assume that the solutions on Ω_n are bounded uniformly with respect to $n \in \mathbb{N}$. We still assume that $f : \mathbb{R}^N \times \mathbb{R}$ is Carathéodory (see Assumption 3.1.1 for what this means).

THEOREM 9.1.7. *Suppose that* $f : \mathbb{R}^N \times \mathbb{R}$ *is a Carathéodory function, and that* $\sup_{n\in\mathbb{N}} |\Omega_n| < \infty$*. Further suppose that* u_n *are solutions of* (9.1.2) *such that the sequence* (u_n) *is bounded in* $L_\infty(\mathbb{R}^N)$*. Finally suppose that Assumption* 9.1.1 *is satisfied. Then there exists a subsequence* (u_{n_k}) *converging to a solution* u *of* (9.1.1) *in* $L_p(\mathbb{R}^N)$ *for all* $p \in [1, \infty)$*.*

PROOF. Let $M := \sup_{n\in\mathbb{N}} \|u_n\|_\infty$. Let $\psi \in C^\infty(\mathbb{R})$ be a monotone function with $\psi(\xi) = \psi$ if $\xi \le M$ and $\psi(\xi) = M + 1$ if $|\xi| \ge M + 1$. Then define $\tilde{f}(x, \xi) := f(x, \psi(\xi))$ and by $\tilde{F}$ the corresponding superposition operator. Then $\tilde{f}$ is bounded and therefore satisfies

Assumption 3.1.1 with $\gamma = 1$. Since $\|u_n\|_\infty \le M$ we clearly have $\tilde{F}(u_n) = F(u_n)$ and so we can replace F by $\tilde{F}$ in the definition of G_n, G without changing the solutions. Hence we can apply Theorem 9.1.5 to conclude that $u_n \to u$ in $L_2(\mathbb{R}^N)$. By the uniform L_∞-bound convergence is in $L_p(\mathbb{R}^N)$ for all $p \in [2, \infty)$, and by the uniform boundedness of $|\Omega_n|$ also for $p \in [1, 2)$. □

REMARK 9.1.8. An L_∞-bound as required above follows from an L_p-bound under suitable growth conditions on the nonlinearity f as shown in Section 3.2.

9.2. *Existence of nearby solutions for semi-linear problems*

Suppose that the limit problem (9.1.1) has a solution. In this section we want to prove that under certain circumstances the perturbed problem (9.1.2) has a solution nearby. In the abstract framework this translates into the question whether the fixed point equation (9.1.4) has, at least for n large enough, a solution near a given solution of (9.1.3). Of course, we do not expect this for arbitrary solutions. Note that the results here do not just apply to domain perturbation problems, but to other types of perturbations having similar properties as well.

One common technique to prove existence of such solutions is by means of the *Leray–Schauder degree* (see [65, Chapter 2.8] or [99, Chapter 4]). We assume that G is a compact operator, that is, if it maps bounded sets onto relatively compact sets. Then, if $U \subset E$ is an open bounded set such that $u \neq G_\Omega(u)$ for all $u \in \partial U$, then the Leray–Schauder degree, $\deg(I - G_\Omega, U, 0) \in \mathbb{Z}$, is well defined. If we deal with positive solutions we can use the degree in cones as in [3,44]. In order to do that we need some more assumptions. These are satisfied for the concrete case of semi-linear boundary value problems as shown in Proposition 9.1.2 with $E = L_p(\mathbb{R}^N)$ for some $p \in (1, \infty)$.

ASSUMPTION 9.2.1. Suppose E is a Banach space and suppose
(1) $G, G_n \in C(E, E)$ are compact for all $n \in \mathbb{N}$;
(2) If $u_n \to u$ in E, then $G_n(u_n) \to G(u)$ in E.
(3) For every bounded sequence (u_n) in E the sequence $(G_n(u_n))_{n\in\mathbb{N}}$ has a convergent subsequence in E.

The following is the main result of this section. The basic idea of the proof for specific domain perturbation problems goes back to [45]. The proof given here is a more abstract version of the ones found in [52,8] for some specific domain perturbation problems.

THEOREM 9.2.2. *Suppose that G_n, G satisfy Assumption* 9.2.1. *Moreover, let $U \subset E$ be an open bounded set such that $G(u) \neq u$ for all $u \in \partial U$. Then there exists $n_0 \in \mathbb{N}$ such that $G_n(u) \neq u$ for all $u \in \partial U$ and*

$$\deg(I - G, U, 0) = \deg(I - G_n, U, 0) \tag{9.2.1}$$

for all $n \ge n_0$.

PROOF. We use the homotopy invariance of the degree (see [65, Section 2.8.3] or [99, Theorem 4.3.4]) to prove (9.2.1). We define the homotopies $H_n(t, u) := tG_n(u) + (1-t)G(u)$

for $t \in [0,1]$, $u \in E$ and $n \in \mathbb{N}$. To prove (9.2.1) it is sufficient to show that there exists $n_0 \in \mathbb{N}$ such that

$$u \neq H_n(t,u) \tag{9.2.2}$$

for all $n \geq n_0$, $t \in [0,1]$ and $u \in \partial U$. Assume to the contrary that there exist $n_k \to \infty$, $t_{n_k} \in [0,1]$ and $u_{n_k} \in \partial U$ such that $u_{n_k} = H_{n_k}(t_{n_k}, u_{n_k})$ for all $k \in \mathbb{N}$. As U is bounded in E, Assumption 9.2.1 (1) and (3) guarantee that $t_{n_k} \to t_0$ in $[0,1]$, $G_{n_k}(u_{n_k}) \to v$ and $G(u_{n_k}) \to w$ in E for some $v, w \in E$ if we pass to a further subsequence. Hence

$$\begin{aligned} u_{n_k} &= H_{n_k}(t_{n_k}, u_{n_k}) = t_{n_k} G_{n_k}(u_{n_k}) + (1 - t_{n_k}) G(u_{n_k}) \\ &\xrightarrow{k \to \infty} u := t_0 v + (1 - t_0) w \end{aligned}$$

in E and so $u_{n_k} \to u$ in E and $u \in \partial U$ since ∂U is closed. Now the continuity of G and Assumption 9.2.1 part (2) imply that $v = w = G(u)$, so that $u_{n_k} \to u = t_0 G(u) + (1 - t_0) G(u) = G(u)$. Hence $u = G(u)$ for some $u \in \partial U$, contradicting our assumptions. Thus there exists $n_0 \in \mathbb{N}$ such that (9.2.2) is true for all $n \geq n_0$, completing the proof of the theorem. □

Of course, we are most interested in the case $\deg(I - G_\Omega, U, 0) \neq 0$. Then, by the solution property of the degree (see [99, Theorem 4.3.2]), (9.1.3) has a solution in U. As a corollary to Theorem 9.2.2 we get the existence of a solution of (9.1.4) in U.

COROLLARY 9.2.3. *Suppose that G_n, G satisfy Assumption* 9.2.1 *and that $U \subset E$ is open and bounded with $u \neq G(u)$ for all $u \in \partial U$. If* $\deg(I - G, U, 0) \neq 0$*, then there exists $n_0 \in \mathbb{N}$ such that* (9.1.4) *has a solution in U for all $n \geq n_0$.*

Now we consider an isolated solution u_0 of (9.1.3) and recall the definition of its *index*. Denote by $B_\varepsilon(u_0)$ the open ball of radius $\varepsilon > 0$ and centre u_0 in E. Then $\deg(I - G, B_\varepsilon(u_0), 0)$ is defined for small enough $\varepsilon > 0$. Moreover, by the excision property of the degree $\deg(I - G, B_\varepsilon(u_0), 0)$ stays constant for small enough $\varepsilon > 0$. Hence the fixed point index of u_0,

$$\mathrm{i}(G, u_0) := \lim_{\varepsilon \to 0} \deg(I - G, B_\varepsilon(u_0), 0)$$

is well defined.

THEOREM 9.2.4. *Suppose that G_n, G satisfy Assumption* 9.2.1. *If u_0 is an isolated solution of* (9.1.3) *with* $\mathrm{i}(G, u_0) \neq 0$*, then for n large enough there exist solutions u_n of* (9.1.4) *such that $u_n \to u_0$ in E as $n \to \infty$.*

PROOF. By assumption there exists $\varepsilon_0 > 0$ such that

$$\mathrm{i}(G, u_0) = \deg(I - G, B_\varepsilon(u_0), 0) \neq 0$$

for all $\varepsilon \in (0, \varepsilon_0)$. Hence by Corollary 9.2.3 problem (9.1.4) has a solution in $B_\varepsilon(u_0)$ for all $\varepsilon \in (0, \varepsilon_0)$ if only n large enough. Hence a sequence as required exists. □

Without additional assumptions it is possible that there are several different sequences of solutions of (9.1.4) converging to u_0. However, if $G \in C^1(E, E)$ and u_0 is *nondegenerate*, that is, the linearised problem

$$v = DG(u_0)v \tag{9.2.3}$$

has only the trivial solution, then u_n is unique for large $n \in \mathbb{N}$.

THEOREM 9.2.5. *Suppose that* $G_n, G \in C^1(E, E)$ *satisfy Assumption* 9.2.1. *Further assume that* $DG_n(u_n) \to DG(u)$ *in* $\mathcal{L}(E)$. *If* u_0 *is a nondegenerate solution of* (9.1.3), *then there exists* $\varepsilon > 0$ *such that* (9.1.4) *has a unique solution in* $B_\varepsilon(u_0)$ *for all* n *large enough and this solution is nondegenerate.*

PROOF. As u_0 is nondegenerate $I - DG(u_0)$ is invertible with bounded inverse. Moreover, since G is compact [65, Proposition 8.2] implies that $DG(u_0)$ is compact as well. By [99, Theorem 5.2.3 and Theorem 4.3.14] $\mathrm{i}(G, u_0) = \pm 1$, so by Theorem 9.2.4 there exists a sequence of solutions u_n of (9.1.3) with $u_n \to u$ as $n \to \infty$. As the set of invertible linear operators is open in $\mathcal{L}(E)$ we conclude that $I - DG_n(u_n)$ is invertible for n sufficiently large. Hence u_n is nondegenerate for n sufficiently large.

We now show uniqueness. Suppose to the contrary that there exist solutions u_n and v_n of (9.1.4) converging to u_0 with $u_n \neq v_n$ for all $n \in \mathbb{N}$ large enough. As $G_n \in C^1(E, E)$ we get

$$G(u_n) - G(v_n) = \int_0^1 DG_n(tu_n + (1-t)v_n)\,dt\,(u_n - v_n).$$

Hence if we set $w_n := \frac{u_n - v_n}{\|u_n - v_n\|}$, then $\|w_n\| = 1$ and

$$w_n = \frac{G_n(u_n) - G_n(v_n)}{\|u_n - v_n\|} = \int_0^1 DG_n(tu_n + (1-t)v_n)\,dt\; w_n \tag{9.2.4}$$

for all $n \in \mathbb{N}$. As $u_n, v_n \to u_0$ we get $tu_n + (1-t)v_n \to u_0$ in E for all $t \in [0, 1]$ and hence by assumption $DG_n(tu_n + (1-t)v_n) \to DG(u_0)$ in $\mathcal{L}(E)$ for all $t \in [0, 1]$. By the continuity of DG_n and the compactness of $[0, 1]$ there exists $t_n \in [0, 1]$ such that

$$\sup_{t \in [0,t]} \left\| DG_n(tu_n + (1-t)v_n) \right\| = \left\| DG_n(t_n u_n + (1-t_n)v_n) \right\|.$$

Since $t_n u_n + (1-t_n)v_n \to u_0$ by assumption $DG_n(tu_n + (1-t)v_n) \to DG(u_0)$ in $\mathcal{L}(E)$ and hence

$$\sup_{\substack{t \in [0,t] \\ n \in \mathbb{N}}} \left\| DG_n(tu_n + (1-t)v_n) \right\| < \infty.$$

By the dominated convergence theorem

$$\int_0^1 DG_n(tu_n + (1-t)v_n)\,dt \to DG(u_0)$$

in $\mathcal{L}(E)$. As the set of invertible operators is open we see that

$$I - \int_0^1 DG_n(tu_n + (1-t)v_n)\,dt$$

has a bounded inverse for n sufficiently large. But this contradicts (9.2.4) and therefore uniqueness follows. □

9.3. *Applications to boundary value problems*

We now discuss how to apply the abstract results in the previous section to boundary value problems. In addition to the assumptions made on f in Section 9.1 we assume that the corresponding superposition operator is differentiable. Conditions for that can be found in [6, Theorem 3.12]. We also assume that there exists a bounded set B such that

$$\Omega_n, \Omega \subset B$$

for all $n \in \mathbb{N}$.

THEOREM 9.3.1. *Suppose that $f \in C(\mathbb{R}^N \times \mathbb{R})$ satisfies Assumption 3.1.1, and that γ, p are such that (3.1.5) holds. Further suppose that Assumption 9.1.1 is satisfied, and that $u \in L_p(\mathbb{R}^N)$ is a nondegenerate isolated solution of (9.1.1). Then, there exists $\varepsilon > 0$ such that, for n large enough, equation (9.1.2) has a unique solution $u_n \in L_p(\Omega)$ with $\|u_n - u\|_p < \varepsilon$. Moreover, $u_n \to u \in L_q(\mathbb{R}^N)$ for all $q \in (1, \infty)$.*

PROOF. By Proposition 9.1.2 all assumptions of Theorem 9.2.5 are satisfied if we choose $E = L_p(\mathbb{R}^N)$. Hence there exist $\varepsilon > 0$ and $u_n \in L_p(\mathbb{R}^N)$ as claimed. By Theorem 3.2.1 the sequence (u_n) is bounded in $L_\infty(\mathbb{R}^N)$ and therefore the convergence is in $L_q(\mathbb{R}^N)$ for all $q \in [p, \infty)$. Since the measure of Ω_n is uniformly bounded, convergence takes place in $L_q(\mathbb{R}^N)$ for $q \in [1, p)$ as well. □

REMARK 9.3.2. (a) We could for instance choose $f(u) := |u|^{\gamma-1}u$, or a nonlinearity with that growth behaviour. Then we can choose p big enough such that (3.1.5) is satisfied. If $\gamma < (d+2)/(d-2)$, then the theorem applies to all nondegenerate weak solutions.

(b) The above theorem does not necessarily imply that (9.1.2) has only one solution near the solution u of (9.1.1) in $L_2(\mathbb{R}^N)$, because a solution in $L_2(\mathbb{R}^N)$ does not need to be close to u in $L_p(\mathbb{R}^N)$. However, if $\gamma < (d+2)/(d-2)$, then u_n is the unique weak solution of (9.1.2) in the ε-neighbourhood of u in $L_2(\mathbb{R}^N)$.

We next want to look at a problem without any growth conditions on f. In such a case we have to deal with solutions in $L_\infty(\mathbb{R}^N)$ only. The idea is to truncate the nonlinearity and apply Theorem 9.3.1.

THEOREM 9.3.3. *Suppose that $f \in C(\mathbb{R}^N \times \mathbb{R})$ satisfies Assumption 3.1.1, and that γ, p are such that (3.1.5) holds. Further suppose that Assumption 9.1.1 is satisfied, and that $u \in L_\infty(\mathbb{R}^N)$ is a nondegenerate isolated solution of (9.1.1). Then, there exists $\varepsilon > 0$ such that, for n large enough, equation (9.1.2) has a unique solution $u_n \in L_\infty(\Omega)$ with $\sup_{n\in\mathbb{N}} \|u_n\|_\infty < \infty$ and $u_n \to u \in L_q(\mathbb{R}^N)$ for all $q \in [1, \infty)$.*

PROOF. Fix $p > d/2$ with d from Assumption 9.1.1. Because $u \in L_\infty(\mathbb{R}^N)$ and $f \in C(\bar{B} \times \mathbb{R})$ we have $F(u) \in L_p(\mathbb{R}^N)$, where F is the superposition operator induced by f. By Theorem 2.4.1 and the assumptions on $(\mathcal{A}_n, \mathcal{B}_n)$ there exists a constant C independent of $n \in \mathbb{N}$ such that

$$\|u\|_\infty \leq M := C(\|F(u)\|_p + \lambda_0\|u\|_p) + \|u\|_p.$$

Let $\psi \in C^\infty(\mathbb{R})$ be a monotone function with

$$\psi(\xi) = \begin{cases} \xi & \text{if } \xi \leq M+1, \\ \operatorname{sgn}\xi(M+2) & \text{if } |\xi| \geq M+2. \end{cases}$$

Then define $\tilde{f}(x,\xi) := f(x,\psi(\xi))$. Then $\tilde{f}$ is a bounded function on $B \times \mathbb{R}$ and $\tilde{f}(x,\xi) = f(x,\xi)$ whenever $|\xi| \leq M+1$. As $\tilde{f}$ is bounded we can apply Theorem 9.3.1. Hence there exists $\varepsilon > 0$ such that, for n large enough, equation (9.1.2) has a unique solution $u_n \in L_p(\Omega)$ with $\|u_n - u\|_p < \varepsilon$. Moreover, $u_n \to u \in L_p(\mathbb{R}^N)$. By what we have seen above

$$\|u_n\|_\infty \leq C(\|\tilde{F}(u_n)\|_p + \lambda_0\|u_n\|_p) + \|u_n\|_p$$

for all $n \in \mathbb{N}$. Because $\tilde{f}$ is bounded, Lemma 3.1.2 shows that $\tilde{F}(u_n) \to \tilde{F}(u)$ in $L_p(\mathbb{R}^N)$. Hence, for n large enough

$$\|u_n\|_\infty < M + 1.$$

By definition of $\tilde{F}$ it follows that $\tilde{F}(u_n) = F(u_n)$ for n large enough, so u_n is a solution of (9.1.2). Convergence in $L_q(\mathbb{R}^N)$ for $q \in [p,\infty)$ follows by interpolation and for $q \in [1,p)$ since B has bounded measure. □

The main difficulty in applying the above results is to show that the solution of (9.1.1) is nondegenerate. A typical example is to look at a problem on two or more disjoint balls and get multiple solutions by taking different combinations of solutions. For instance if we have a trivial and a nontrivial solution, then on two disjoint balls we get four solutions: the trivial solution on both, the trivial on one and the nontrivial on the other and vice versa, and the nontrivial solution on both. We can then connect the domains either as a dumbbell domain with a small strip (see Figure 6.1) or touching balls (see Figure 7.1). The dumbbell example works for Dirichlet and Robin boundary conditions, but not for Neumann boundary conditions. For Neumann boundary conditions, the example of the touching spheres applies.

To illustrate the above we give an example to the Gelfand equation from combustion theory (see [72, §15]) due to [45]. The example shows that a simple equation can have multiple solutions on simply connected domains.

EXAMPLE 9.3.4. Consider the Gelfand equation

$$\begin{aligned} -\Delta u &= \mu e^u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{9.3.1}$$

on a bounded domain of class C^2. If $\mu > 0$, then $\mu e^u > 0$ for all u and thus by the maximum principle every solution of (9.3.1) is positive. By [73, Theorem 1] positivity implies that all solutions are radially symmetric if Ω is a ball.

It is well known that there exists $\mu_0 > 0$ such that (9.3.1) has a minimal positive solution for $\mu \in [0, \mu_0]$ and no solution for $\mu > \mu_0$ (see [4,41]). Moreover, for $\mu \in (0, \mu_0)$ this minimal solution is nondegenerate (see [41, Lemma 3]). Let now $\Omega = B_0 \cup B_1$ be the union of two balls B_0 and B_1 of the same radius and Ω_n the dumbbell-like domains as shown in Figure 6.1. If $N = 2$ and $\mu \in (0, \mu_0)$, then there exists a second solution for the problems on B_0 and B_1. In fact, the two solutions are the only solutions on a ball if $N = 2$ (see [91, p. 242] or [72, §15, p. 359]). Note that this is not true for $N \geq 3$ as shown in [91]. Equation (8) on page 415 together with the results in Section 2 in [43] imply that there is bifurcation from every degenerate solution. Since we know that there is no bifurcation in the interval $(0, \mu_0)$, it follows that the second solution is also nondegenerate.

We now show that there are possibly more than two solutions on (simply connected) domains other than balls. Looking at $\Omega = B_0 \cup B_1$ we have four nontrivial nondegenerate solutions of (9.3.1). Hence by Theorem 9.3.3 there exist at least four nondegenerate solutions of (9.3.1) on Ω_n for n large. Note that similar arguments apply to the nonlinearity $|u|^{p-1}u$ for p subcritical as discussed in [45].

9.4. *Remarks on large solutions*

If all solutions of (9.1.1) are nondegenerate it is tempting to believe that the number of solutions of (9.1.2) is the same for n sufficiently large. However, this is not always true, and Theorem 9.3.1 only gives a lower bound for the number of solutions of the perturbed problem. If there are more solutions on Ω_n, then Theorem 9.1.5 implies that their L_∞-norm goes to infinity as otherwise they converge to one of the solutions on Ω and hence are unique. To prove precise multiplicity of solutions of the perturbed problem, the task is to find a universal bound on the L_∞ norm valid for all solutions to the nonlinear problem. This is quite different from the result in Theorem 3.2.1 which just shows that under suitable growth conditions on the nonlinearity, a bound in $L_p(\mathbb{R}^N)$ implies a bound in $L_\infty(\mathbb{R}^N)$. However, it is still unclear whether in general there is a uniform bound on the L_p-norm for some p. Such uniform bounds are very difficult to get in general. We prove their existence for solutions to

$$\begin{aligned} -\Delta u &= f(u(x)) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{9.4.1}$$

provided f is sublinear in the sense that

$$\lim_{|\xi| \to \infty} \frac{|f(\xi)|}{|\xi|} = 0. \tag{9.4.2}$$

The above class clearly includes all bounded nonlinearities.

PROPOSITION 9.4.1. *Suppose that $f \in C(\mathbb{R})$ satisfies* (9.4.2). *Then there exists a constant M depending only on the function f and the diameter of Ω such that $\|u\|_\infty < M$ for every weak solution of* (9.4.1).

PROOF. By (9.4.2), for every $\varepsilon > 0$ there exists $\alpha > 0$ such that

$$\frac{|f(\xi)|}{|\xi|} < \varepsilon$$

whenever $|\xi| > \alpha$. Setting $m_\varepsilon := \sup_{|\xi| \le \alpha} |f(\xi)|$ we get

$$|f(\xi)| \le m_\varepsilon + \varepsilon|\xi|$$

for all $\xi \in \mathbb{R}$. In particular f satisfies (3.1.4) with $\gamma = 1$, and so by Theorem 3.2.1 it is sufficient to prove a uniform L_2-bound for all weak solutions of (9.4.1). If u is a weak solution of (9.4.1), then from the above

$$\begin{aligned}\|\nabla u\|_2^2 &= \int_\Omega f(u)u\,dx \le \int_\Omega |f(u)||u|\,dx \le \int_\Omega (m_\varepsilon + \varepsilon|u|)|u|\,dx \\ &\le \int_\Omega m_\varepsilon|u|\,dx + \varepsilon\|u\|_2^2 \le (m_\varepsilon|\Omega|)^{1/2}\|u\|_2 + \varepsilon\|u\|_2^2.\end{aligned}$$

Using an elementary inequality we conclude that

$$\|\nabla u\|_2^2 \le \varepsilon^{-1} m_\varepsilon|\Omega| + 2\varepsilon\|u\|_2^2.$$

If D denotes the diameter of Ω, then from Friedrich's inequality (2.1.7)

$$\|u\|_2^2 \le D^2\|\nabla u\|_2^2 \le \varepsilon^{-1} D^2 m_\varepsilon|\Omega| + 2D^2\varepsilon\|u\|_2^2.$$

We next choose $\varepsilon := 1/4D^2$ and so we get

$$\|u\|_2^2 \le 8D^4 m_\varepsilon|\Omega|$$

for every weak solution of (9.4.1). The right-hand side of the above inequality only depends on the quantities listed in the proposition, and hence the proof is complete. □

A priori estimates similar to the above can be obtained also for superlinear problems, but they involve the shape of the domain. An extensive discussion of the phenomena can be found in [45, Section 5] as well as [46,48,35]. The techniques to prove uniform a priori bounds are derived from Gidas and Spruck [75,74].

We complete this section by showing a simple example where large solutions occur. More sophisticated examples are in [46,48,35], in particular the example of the dumbbell as in Figure 6.1, where large solutions occur for the equation

$$\begin{aligned} -\Delta u &= |u|^{p-1}u \quad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned} \tag{9.4.3}$$

for $1 < p < (N+2)/(N-2)$. We give a simple example for the above equation, where large solutions occur, similar to examples given in [45, Section 5].

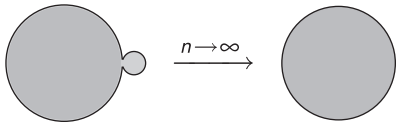

Fig. 9.1. Attaching a small shrinking ball.

EXAMPLE 9.4.2. By [114, Section I.2] the above equation has a positive solution u_r of $\Omega = B_r$ which is a ball of radius r. By a simple rescaling, it turns out that $\|u_r\|_\infty \to \infty$ as $r \to 0$. That solution is unique and nondegenerate by [45, Theorem 5], if $N = 2$. Then consider a domain Ω_n constructed from two touching balls $B_1 \cup B_{1/n}$ with a small connection as shown in Figure 9.1. When we let $n \to \infty$, then from Theorem 5.4.5 we get that $\Omega_n \to \Omega := B_1$ in the sense of Mosco. Since on every ball there are two solutions, the trivial solution and a nontrivial solution, there are four solutions on the union $B_1 \cup B_{1/n}$. If we make a small enough connection between the balls, then by Theorem 9.3.1 there are still at least four solutions. Hence we can construct a sequence of domains Ω_n with the required property. However, we know that there are precisely two solutions on Ω, the trivial and the nontrivial solution. The solutions on Ω_n not converging to one of the two solutions on Ω are such that $\|u_n\|_\infty \to \infty$.

9.5. *Solutions by domain approximation*

Consider now the nonlinear Dirichlet problem

$$\begin{aligned} \mathcal{A}u &= f(u) \quad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{9.5.1}$$

on some domain Ω, bounded or unbounded. We assume that $f \in C^1(\mathbb{R})$. We then let Ω_n be open sets such that $\Omega_n \to \Omega$ and consider the sequence of problems

$$\begin{aligned} \mathcal{A}_n u &= f(u) \quad \text{in } \Omega_n, \\ u &= 0 \qquad \text{on } \partial\Omega_n. \end{aligned} \tag{9.5.2}$$

The difference to the results in Section 9.3 is that we allow unbounded domains Ω. One possibility is to use the technique to prove results on nonsmooth or unbounded domains by approximation by smooth bounded domains from inside applying the results from Section 8.

THEOREM 9.5.1. *Suppose that $f \in C^1(\mathbb{R})$ and that $\Omega_n \to \Omega$ in the sense of Mosco. Suppose that u_n are weak solutions of* (9.5.2) *and that the sequence (u_n) is bounded in $H^1(\mathbb{R}^N) \cap L_\infty(\mathbb{R}^N)$. Then there exists a subsequence (u_{n_k}) converging to a solution u of* (9.5.1) *weakly in $H^1(\mathbb{R}^N)$ and strongly in $L_{p,\mathrm{loc}}(\mathbb{R}^N)$ for all $p \in [2, \infty)$.*

PROOF. By the boundedness of (u_n) in $H^1(\mathbb{R}^N)$ there exists a subsequence (u_{n_k}) converging to $u \in H^1(\mathbb{R}^N)$ weakly. Since $H^1(\mathbb{R}^N)$ is compactly embedded into

$L_{2,\mathrm{loc}}(\mathbb{R}^N)$ we have $u_{n_k} \to u$ in $L_{2,\mathrm{loc}}(\mathbb{R}^N)$ as $k \to \infty$. Because (u_n) is bounded in $L_\infty(\mathbb{R}^N)$, convergence takes place in $L_{p,\mathrm{loc}}(\mathbb{R}^N)$ for all $p \in [2,\infty)$. Since we have a uniform bound on (u_n) which is bounded in $L_\infty(\mathbb{R}^N)$ we can also truncate the nonlinearity f as in the proof of Theorem 9.3.3 and assume it is bounded. Then from Lemma 3.1.2 we get that $f(u_{n_k}) \to f(u)$ in $L_{2,\mathrm{loc}}(\mathbb{R}^N)$.

Now fix $\varphi \in C_c^\infty(\Omega)$. Then there exists a ball B with $\operatorname{supp}\varphi \subset B$. As $\Omega_n \to \Omega$, using Proposition 5.3.3, there exists $\varphi_n \in C_c^\infty(\Omega_n \cap B)$ such that $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$. Hence by using Proposition 5.1.2 and the fact that the support of φ_n is in B for all $n \in \mathbb{N}$ we get

$$\lim_{k\to\infty} \langle f(u_{n_k}), \varphi_n\rangle = \langle f(u), \varphi\rangle.$$

Since $u_{n_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ and $\varphi_n \to \varphi$ in $H^1(B)$ we have

$$\lim_{k\to\infty} a_{n_k}(u_{n_k}, \varphi_n) = a(u, \varphi).$$

Finally, by assumption $a(u_{n_k}, \varphi_{n_k}) = \langle f(u_{n_k}), \varphi_{n_k}\rangle$ for all $k \in \mathbb{N}$ we get $a(u,\varphi) = \langle f(u), \varphi\rangle$ for all $\varphi \in C_c^\infty(\Omega)$, that is, u is a weak solution of (9.5.1). □

Note that in general we cannot expect the solution u whose existence the above theorem proves to be nonzero if Ω_n or Ω are unbounded, or more precisely if the resolvents of the linear problems do not converge in the operator norm.

REMARK 9.5.2. Note that under suitable growth conditions, a uniform L_p-bound or even an L_2-bound on the solutions of (9.5.2) implies a uniform L_∞-bound by Theorem 3.2.1.

9.6. *Problems on unbounded domains*

Suppose that $\Omega \subset \mathbb{R}^N$ is an unbounded domain. By approximation by a sequence of bounded domains Ω_n we want to construct a nonnegative weak solution of

$$\begin{aligned} -\Delta u &= |u|^{p-2}u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{9.6.1}$$

with subcritical growth $p \in (2, 2N/(N-2))$. We also assume that the spectral bound of the Dirichlet Laplacian is positive. It is given as the infimum of the Rayleigh coefficient

$$s = \inf_{u\in H_0^1(\Omega)} \frac{\|\nabla u\|_2^2}{\|u\|_2^2} > 0. \tag{9.6.2}$$

Examples of such a domain are infinite strips of the form $\mathbb{R} \times U$ with U a domain in $\mathbb{R}^{N-1}$.

To construct a solution we let Ω_n be bounded open sets such that $\Omega_n \subset \Omega_{n+1} \subset \Omega$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. We then show that a subsequence of the positive solutions of

$$\begin{aligned} -\Delta u &= |u|^{p-2}u \quad \text{in } \Omega_n \\ u &= 0 \quad\quad\quad\; \text{on } \partial\Omega_n \end{aligned} \tag{9.6.3}$$

converges to a solution of (9.6.1) as $n \to \infty$. The solution of (9.6.3) can be obtained by means of a constrained variational problem. We let

$$M_n := \{u \in H_0^1(\Omega_n) \colon u > 0 \text{ and } \|u\|_p^p = p\}$$

and the functional

$$J(u) := \frac{1}{2}\int_{\Omega_n} |\nabla u|^2 \, dx.$$

Then J has a minimiser $v_n \in M_n$ for every $n \in \mathbb{N}$, and that minimiser is a positive weak solution of

$$\begin{aligned} -\Delta v &= \mu_n |v|^{p-2}v \quad \text{in } \Omega_n \\ v &= 0 \quad\quad\quad\quad\; \text{on } \partial\Omega_n \end{aligned}$$

with

$$\mu_n = 2J(v_n) = \|\nabla v_n\|_2^2.$$

The function

$$u_n := \mu_n^{1/(p-1)} v_n$$

then solves (9.6.3). For a proof of these facts we refer to [114, Section I.2]. We also get a bound on the solutions u_n, namely

$$\begin{aligned} \|\nabla u_n\|_2^2 &= \mu_n^{2/(p-1)} \|\nabla v_n\|_2^2 = \|\nabla v_n\|_2^{4/(p-1)} \|\nabla v_n\|_2^2 \\ &= \|\nabla v_n\|^{2(p+1)/(p-1)} = (2J(v_n))^{(p+1)/(p-1)} \end{aligned} \tag{9.6.4}$$

since $M_n \subset M_{n+1}$ for all $n \in \mathbb{N}$ we also have $J(v_{n+1}) \leq J(v_n)$. The sequences (∇v_n) and therefore (∇u_n) are bounded in $L_2(\Omega, \mathbb{R}^N)$. By (9.6.2) it follows that (u_n) and (v_n) are bounded sequences in $H_0^1(\Omega)$.

Note that the nonlinearity $|u|^{p-2}u$ satisfies Assumption 3.1.1 with $g = 0$ and $\gamma = p - 1 < (N+2)/(N-2)$. Since we can choose λ_0 (see Table 2.1), Proposition 3.2.1 shows that (u_n) is a bounded sequence in $L_\infty(\Omega)$. Hence there exists $S > 0$ such that $\|u\|_\infty^{p-2} \leq S$ for all $n \in \mathbb{N}$. By (9.6.2) and since u_n is a weak solution of (9.6.3)

$$s\|\nabla u_n\|_2^2 = s\|u_n\|_p^p \leq s\|u_n\|_\infty^{p-2}\|u_n\|_2^2 \leq \|u_n\|_\infty^{p-2}\|\nabla u_n\|_2^2.$$

Hence $0 < s \leq \|u_n\|_\infty^{p-2}$ for all $n \in \mathbb{N}$. We have thus proved the following proposition.

PROPOSITION 9.6.1. *Let s, S be as above. Then for every $n \in \mathbb{N}$ the problem* (9.6.3) *has a positive solution u_n such that (u_n) is bounded in $H_0^1(\Omega) \cap L_\infty(\Omega)$. Moreover, $\|\nabla u_n\|_2$ is decreasing and $0 < s \leq \|u_n\|_\infty^{p-2} \leq S$ for all $n \in \mathbb{N}$.*

The next task is to show that u_n converges to a solution of (9.6.1). We have seen that (u_n) is bounded in $H_0^1(\Omega) \cap L_\infty(\Omega)$. It therefore follows from Theorem 9.5.1 that a subsequence of (u_n) converges to a solution of (9.6.1).

PROPOSITION 9.6.2. *Let u_n be the solution of* (9.6.3) *as constructed above. Then there exists a subsequence of (u_{n_k}) converging weakly in $H^1(\Omega)$ to a solution u of* (9.6.1).

The main question is whether the solution u is nonzero. If the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact this is easy to see. If the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ is not compact, then we cannot expect u to be nonzero in general. However, if we assume that Ω is symmetric like for instance an infinite strip with cross-section of finite measure, we will prove the existence of a nonzero solution of (9.6.1) by exploiting the symmetry to generate compactness of a minimising sequence. We look at domains of the form

$$\Omega = \mathbb{R} \times U,$$

where $U \subset \mathbb{R}^{N-1}$ is a bounded open set. If we let $\Omega_n = (-n, n) \times U$, then by a result of Gidas–Ni–Nirenberg (see [70, Theorem 3.3]), the solutions u_n of (9.6.3) is symmetric with respect to the plane $\{0\} \times \mathbb{R}^N$ and decreasing as $|x_1|$ increases. By [97, Théorème III.2] the sequence (u_n) is compact in $L_q(\Omega)$ for $2 < q < 2N/(N-2)$. In the case of $N = 2$ a similar result appears in [115, Section 4]. It therefore follows that there exists a subsequence (u_{n_k}) converging strongly in $L_p(\Omega)$. Using u_{n_k} as a test function we get

$$\lim_{k\to\infty} \|\nabla u_{n_k}\|_2^2 = \lim_{k\to\infty} \|u_{n_k}\|_p^p = \|u\|_p^p = \|\nabla u\|_2^2.$$

As $u_{n_k} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ it follows that $u_{n_k} \to u$ strongly in $H_0^1(\Omega)$. By Proposition 9.6.1 the sequence (u_n) is bounded in $L_\infty(\Omega)$ and so (u_n) is bounded in $C^\mu(\Omega)$ (see [76, Theorem 8.22]) as well. Hence, by Arzela–Ascoli's theorem the subsequence also converges locally uniformly on Ω. The symmetry guarantees that the maximum of (u_n) is in $U \times \{0\}$ and therefore the lower bound on $\|u_n\|_\infty^{p-2}$ from Proposition 9.6.1 implies that $u \neq 0$.

However, without the symmetry, the solution may converge to zero. As an example look at the semi-strip $\Omega = (0, \infty) \times U$ which we exhaust by the domains $\Omega_n = (0, 2n) \times U$. Then the above proposition applies. The solutions on Ω_n are just translated solutions on $(-n, n) \times U$. However, the maximum of the function u_n is in $\{n\} \times U$, and moves to infinity as $n \to \infty$. Because the solutions decrease away from the maximum, they converge to zero in $L_{2,\mathrm{loc}}((0, \infty) \times U)$. This shows that the symmetry was essential for concluding that the limit solution is nonzero. The lower bound on $\|u_n\|_\infty^{p-2}$ from Proposition 9.6.1 and local uniform convergence do not help to get a nonzero solution.

Similarly we could look at (9.6.1) on the whole space $\Omega = \mathbb{R}^N$. We can then write Ω as a union of concentric balls Ω_n, and try to use the spherical symmetry to get some compactness from [97] similarly as above. However, since by [75] the equation (9.6.1) has no positive solution on $\mathbb{R}^N$, such an attempt must fail.

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