1 Elementary electrical circuit analysis

1.1 Introduction
Analogue electronic circuits deal with signal processing techniques such as amplification and filtering of electrical and electronic signals. Such signals are voltages or currents. In order to understand how these signals can be processed we need to appreciate the basic relationships associated with electrical currents and voltages in each electrical component as well as in any combination that make the complete electrical circuit. We start by defining the basic electrical quantities – voltage and current – and by presenting the main passive electrical devices; resistors, capacitors and inductors.

The fundamental tools for electrical circuit analysis – Kirchhoff’s laws – are discussed in section 1.4. Then, three very important electrical network theorems; Thévenin’s theorem, Norton’s theorem and the superposition theorem are presented.

The unit system used in this book is the International System of Units (SI) [1]. The relevant units of this system will be mentioned as the physical quantities are introduced. In this book detailed definition of the different units is not provided as this can be found in other sources, for example [2, 3], which address the physical and electromagnetic nature of circuit elements. At this stage it is relevant to mention that the SI system incorporates the decimal prefix to relate larger and smaller units to the basic units using these prefixes to indicate the various powers of ten. Table 1.1 shows the powers of ten most frequently encountered in circuit analysis.

<table>
<thead>
<tr>
<th>femto- (f)</th>
<th>pico- (p)</th>
<th>nano- (n)</th>
<th>micro (μ)</th>
<th>milli- (m)</th>
<th>kilo- (k)</th>
<th>mega- (M)</th>
<th>giga (G)</th>
<th>tera (T)</th>
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<tr>
<td>$10^{-15}$</td>
<td>$10^{-12}$</td>
<td>$10^{-9}$</td>
<td>$10^{-6}$</td>
<td>$10^{-3}$</td>
<td>$10^3$</td>
<td>$10^6$</td>
<td>$10^9$</td>
<td>$10^{12}$</td>
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Table 1.1: Powers of ten.

1.2 Voltage and current
By definition electrical current is the rate of flow (with time) of electrical charges passing a given point of an electrical circuit. This definition can be expressed as follows:

$$i(t) = \frac{dq(t)}{dt} \quad (1.1)$$

1The term ‘electronic signals’ is sometimes used to describe low-power signals. In this book, the terms ‘electrical’ and ‘electronic’ signals are used interchangeably to describe signals processed by a circuit.
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where \( i(t) \) represents the electrical current as a function of time represented by \( t \). The unit for the current is the ampere (A). \( q(t) \) represents the quantity of flowing electrical charge as a function of time and its unit is the coulomb (C). The elementary electrical charge is the charge of the electron which is equal to \( 1.6 \times 10^{-19} \text{ C} \).

At this stage it is relevant to mention that in this chapter we represent constant quantities by uppercase letters while quantities that vary with time are represented by the lower case. Hence, a constant electrical current is represented by \( I \) while an electrical current varying with time is represented by \( i(t) \).

Electrical current has a a very intuitive hydraulic analogue; water flow. Figure 1.1 a) shows a voltage source which is connected to a resistance, \( R \), creating a current flow, \( I \), in this circuit. Figure 1.1 b) shows an hydraulic equivalent system. The water pump together with the water reservoirs maintain a constant water pressure across the ends of the pipe. This pressure is equivalent to the voltage potential difference at the resistance terminals generated by the voltage source. The water flowing through the pipe is a consequence of the pressure difference. It is common sense that the narrower the pipe the greater the water resistance and the lower the water flow through it. Similarly, the larger the electrical resistance the smaller the electrical current flowing through the resistance. Hence, it is clear that the equivalent to the pipe water resistance is the electrical resistance \( R \).

The electrical current, \( I \), is related to the potential difference, or voltage \( V \), and to the resistance \( R \) by Ohm’s law:

\[
I = \frac{V}{R} \tag{1.2}
\]

The unit for resistance is the ohm (also represented by the Greek symbol \( \Omega \)). The unit for the potential difference is the volt (or simply V)\(^2\). Ohm’s law states that the current that flows through a resistor is inversely proportional to the value of that resistance and directly proportional to the voltage across the resistance. This law is of fundamental importance for electrical and electronic circuit analysis.

Now we discuss voltage and current sources. The main purpose of each of these sources is to provide power and energy to the circuit to which the source is connected.

1.2.1 Voltage sources

Figure 1.2 a) shows the symbols used to represent voltage sources. The plus sign, the anode terminal, indicates the higher potential and the minus sign, the cathode terminal, indicates the lower potential. The positive flow of current supplied by a voltage source is from the anode, through the exterior circuit, such as the resistance in figure 1.1 a), to the cathode. Note that the positive current flow is conventionally taken to be in the opposite direction to the flow of electrons. An ideal constant voltage source has a voltage–current, \( V–I \),

\(^2\)It is common practice to use the letter \( V \) to represent the voltage, as well as its unit. This practice is followed in this book.
characteristic like that illustrated in figure 1.2 b). From this figure we observe that an ideal voltage source is able to maintain a constant voltage \( V \) across its terminals regardless of the value of the current supplied to (positive current) or the current absorbed from (negative current) an electrical circuit.

When a voltage source, such as that shown in figure 1.1 a), provides a constant voltage at its terminals it is called a direct current (or DC) voltage source. No practical DC voltage source is able to maintain the same voltage across its terminals when the current increases. A typical \( V-I \) characteristic of a practical voltage source is as shown in figure 1.3 a). From this figure we observe that as the current \( I \) increases up to a value \( I_x \) the voltage drops from \( V_s \) to \( V_x \) in a linear manner. A practical voltage source can be modelled according to the circuit of figure 1.3 b) which consists of an ideal voltage source and a resistance \( R_s \) whose value is given by:

\[
R_s = \frac{V_s - V_x}{I_x} \tag{1.3}
\]

This resistance is called the ‘source output resistance’. Examples of DC voltage sources are the batteries used in radios, in cellular phones and automobiles.

An alternating (AC) voltage source provides a time varying voltage at its terminals which is usually described by a sine function as follows:

\[
v_s(t) = V_s \sin(\omega t) \tag{1.4}
\]

where \( V_s \) is the amplitude and \( \omega \) is the angular frequency in radians per second. An ideal AC voltage source has a \( V-I \) characteristic similar to that of the ideal DC voltage source in the sense that it is able to maintain the AC voltage regardless of the amount of current supplied or absorbed from a circuit. In practice AC voltage sources have a non-zero output resistance. An example of an AC voltage source is the domestic mains supply.

**Example 1.2.1** Determine the output resistance of a voltage source with \( V_s = 12 \) V, \( V_x = 11.2 \) V and \( I_x = 34 \) A.

**Solution**: The output resistance is calculated according to:

\[
R = \frac{V_s - V_x}{I_x} = 0.024 \, \Omega = 24 \, \text{m}\Omega
\]

---

### 1.2.2 Current sources

Figure 1.4 a) shows the symbol for the ideal current source\(^3\). The arrow indicates the positive flow of the current. Figure 1.4 b) shows the current–voltage,

\(^3\)Although the symbol of a current source is shown with its terminals in an open-circuit situation, the practical operation of a current source requires an electrical path between its terminals or the output voltage will become infinite.
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$I-V$, characteristic of an ideal current source. From this figure it is clear that an ideal current source is able to provide a given current regardless of the voltage at its terminals. Practical current sources have an $I-V$ characteristic like that represented in figure 1.5 a). As the voltage across the current source increases up to a value $V_x$ the current tends to decrease in a linear fashion. Figure 1.5 b) shows the equivalent circuit for a practical current source including a resistance $R_s$ which is once again called the ‘source output resistance’. The value of this resistance is:

$$R_s = \frac{V_x}{I_s - I_x} \quad (1.5)$$

Examples of simple current sources are difficult to provide at this stage. Most current sources are implemented using active devices such as transistors. Active devices are studied in detail in Chapter 6 where it is shown that, for example, the field-effect transistor, in specific configurations, displays current source behaviour.

**Example 1.2.2** Determine the output resistance of a current source whose output current falls from 2 A to 1.99 A when its output voltage increases from 0 to 100 V.

**Solution**: The output resistance is calculated according to:

$$R_s = \frac{V_x}{I_s - I_x} = \frac{100}{2 - 1.99} = 10 \text{ kΩ}$$

---

### Power supplied by a source

As mentioned previously, the main purpose of a voltage or current source is to provide power to a circuit. The instantaneous power delivered by either source is given by the product of the current supplied with the voltage at its terminals, that is,

$$p_s(t) = v(t) i(t) \quad (1.6)$$

The unit for power is the watt (W) when the voltage is expressed in volts (V) and the current is expressed in amperes (A). If the voltage and current are constant then eqn 1.6 can be written as:

$$P_s = V I \quad (1.7)$$

Often it is of interest to calculate the average power, $P_{AV_s}$, supplied by a source during a period of time $T$. This average power can be calculated by the successive addition of all values of the instantaneous power, $p_s(t)$, during the time
interval $T$ and then dividing the outcome by the time interval $T$. That is, $P_{AV_s}$ can be calculated as follows$^1$:

$$
P_{AV_s} = \frac{1}{T} \int_{t_o}^{t_o+T} p_s(t) \, dt
= \frac{1}{T} \int_{t_o}^{t_o+T} v(t) i(t) \, dt
$$  \hspace{1cm} (1.8)

where $t_o$ is a chosen instant of time. For a periodic signal (voltage or current) $T$ is usually chosen as the period of the signal.

**Example 1.2.3** A 12 volt DC source supplies a transistor circuit with periodic current of the form:

$$
i(t) = 3 + 2 \cos(2 \pi 100 t) \text{ mA}.
$$

Plot the instantaneous power and the average power supplied by this source in the time period $0 < t < 0.01 \text{ s}$.

**Solution:** The instantaneous power is calculated using eqn 1.6:

$$
p_s(t) = 12 \times [3 + 2 \cos(2 \pi 100 t)] 10^{-3}
= 36 + 24 \cos(2 \pi 100 t) \text{ (mW)}
$$

This is plotted in figure 1.6. The average power is calculated according to eqn 1.8:

$$
P_{AV_s} = \frac{1}{0.01} \int_{0}^{0.01} 12 \times [3 + 2 \cos(2 \pi 100 t)] 10^{-3} \, dt
= 100 \times 12 \times 10^{-3} \left[ 3t + \frac{2}{2 \pi 100} \sin(2 \pi 100 t) \right]_{0}^{0.01}
= 36 \text{ mW}
$$

Note that the same average power will be obtained if eqn 1.8 is applied over any time interval $T$ as long as $T$ is a multiple of the period of the waveform.

### 1.3 Electrical passive elements

The main passive electrical elements are the resistor, the capacitor and the inductor. For each of these elements we study the voltage–current relationship and we also present hydraulic analogies as suggested by Wilmshurst [4].

#### 1.3.1 Resistance and conductance

The resistance$^5$ has been presented in the previous section. Ohm’s law relates the voltage at the terminals of a resistor with the current which flows through it according to eqn 1.2. The hydraulic analogue for a resistance has also been presented above in figure 1.1. It is worth mentioning that if the voltage varies

$^1$Recall that the integral operation is basically an addition operation.

$^5$Strictly speaking, the suffix -or designates the name of the element (like resistor) while the suffix -ance designates the element property (like resistance). Often these two are used interchangeably.
with time then the current varies with time in exactly the same manner, as illustrated in figure 1.7. Therefore, the resistance appears as a scaling factor which relates the amplitude of the two electrical quantities; current and voltage. So, we can generalise eqn 1.2 as follows:

\[ i(t) = \frac{v(t)}{R} \]  

(1.9)

and

\[ v(t) = R i(t) \]  

(1.10)

Figure 1.7 illustrates this concept.

**Example 1.3.1** Consider a current \( i(t) = 0.5 \sin(\omega t) \) A flowing through a resistor of 10 Ω. Determine an expression for the voltage across the resistor.

**Solution:** Using eqn 1.10 we obtain the voltage \( v(t) \) as

\[ v(t) = R i(t) \]

\[ v(t) = 5 \sin(\omega t) \] V

Often it is useful to express Ohm’s law as follows:

\[ I = GV \]  

(1.11)

where \( G = R^{-1} \) is known as the ‘conductance’. The unit of the conductance is the siemen (S) and is equal to \((1 \text{ ohm})^{-1}\).

A resistance dissipates power and generates heat. When a resistance is driven by a DC source this power dissipation, \( P_R \), is given by:

\[ P_R = V I \]  

(1.12)

where \( V \) represents the voltage across the resistance terminals and \( I \) is the current that flows through it. Using Ohm’s law we can express eqn 1.12 as follows:

\[ P_R = RI^2 \]  

(1.13)

\[ = \frac{V^2}{R} \]  

(1.14)

These two eqns (1.13 and 1.14) appear to be contradictory in terms of the role the resistance plays in determining the level of power dissipation. Does the dissipated power increase with increasing the resistance (eqn 1.13) or does it decrease (eqn 1.14)? The answer to this question relates to the way we view the circuit and to what quantity we measure across the resistor. Let us consider the case where a resistor is connected across the terminals of an ideal voltage source. Here, the stimulus is the voltage that results in a current through the resistor. In this situation the larger the resistance the smaller the current is and,
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According to eqn 1.12, there is less power dissipation in the resistance. Note that, in the extreme situation of \( R \to \infty \) the resistance behaves as an open circuit (the current is zero) and there is no power dissipation. On the other hand, if the resistance is driven by an ideal current source, then the voltage across the resistor is the resulting effect. Hence, the larger the resistance, the larger the voltage developed across its terminals and, according to eqn 1.12, the power dissipation increases.

For time-varying sources the instantaneous power dissipated in a resistor is given by:

\[
p_R(t) = v(t) i(t) \quad (1.15) \\
= Ri^2(t) \quad (1.16) \\
= \frac{v^2(t)}{R} \quad (1.17)
\]

where \( v(t) \) represents the voltage across the resistance and \( i(t) \) is the current that flows through it. The average power dissipated in a period of time \( T \) can be expressed as follows:

\[
P_{AVn} = \frac{1}{T} \int_{t_a}^{t_a+T} p_R(t) \, dt \quad (1.18) \\
= \frac{R}{T} \int_{t_a}^{t_a+T} i^2(t) \, dt \quad (1.19) \\
= \frac{1}{RT} \int_{t_a}^{t_a+T} v^2(t) \, dt \quad (1.20)
\]

where \( t_a \) is a chosen instant of time. When the resistor is driven by a periodic signal, \( T \) is normally chosen to be its period.

**Example 1.3.2** An AC voltage \( v(t) = A \sin(\omega t) \), with \( A = 10 \) V, is applied to a resistance \( R = 50 \Omega \). Determine the average power dissipated.

**Solution:** According to eqn 1.20 we can write:

\[
P_{AVn} = \frac{1}{TR} \int_0^T V_A^2 \sin^2(\omega t) \, dt \\
= \frac{1}{T} \frac{V_A^2}{R} \int_0^T 1 - \cos(2\omega t) \, dt \\
= \frac{V_A^2}{2TR} \left[ t - \frac{1}{2\omega} \sin(2\omega t) \right]_0^T
\]

Since the period of the AC waveform is \( T = 2\pi/\omega \), the last eqn can be written as:

\[
P_{AVn} = \frac{1}{R} \frac{V_A^2}{2} \quad (1.21)
\]

= 1 W
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1.3.2 Capacitance

Figure 1.8 a) shows a capacitor connected to a voltage source. A capacitor is usually implemented using two metal plates separated by an insulator. This means that the capacitor does not allow the passage of direct current. However, when a voltage is applied across the terminals of an uncharged capacitor electronic charge can be added to one of the metal plates and removed from the other. This charge is proportional to the applied voltage and defines the capacitance according to the following eqn:

\[ q(t) = C \, v(t) \]  

(1.22)

where \( C \) is the capacitance and \( v(t) \) is the voltage applied across its terminals. The capacitance can be seen to have the ability to accumulate charge. The unit for the capacitance is the farad (F) when, in eqn 1.22, \( v(t) \) is expressed in volts and \( q(t) \) is expressed in coulombs. In other words, a capacitor of one farad will store one coulomb of charge if a potential difference of one volt exists across its plates. From eqns 1.1 and 1.22 we can relate the current through a capacitor with the voltage across its terminals according to the following eqn:

\[ i(t) = C \frac{d v(t)}{dt} \]  

(1.23)

Figure 1.8 b) shows a hydraulic analogue for the capacitor which is an elastic membrane covering the section of a water pipe. In this analogy the voltage becomes the water pressure, the capacitor becomes the membrane elasticity, and the charge becomes the water volume displaced by the membrane. Let us consider that there is water on both sides of the elastic membrane. This membrane does not allow the direct crossing of water between the two sides of the pipe. However, if there is a pulsed increase of the water pressure in one side of the membrane, as illustrated in fig 1.8 b), the stretching of this membrane causes an effective travelling of the pulse from one side of the membrane to the other (with no water passing through the membrane!). This process is conceptually similar to the flow of charges (current) in the metal plates of a capacitor. Note that if there is too much pressure on the membrane it will eventually breakdown and the same can happen to a capacitor if too much voltage is applied to it.

Integrating eqn 1.23 we obtain an expression for the voltage in terms of the current, that is:

\[ v(t) = \frac{1}{C} \int_0^t i(t) \, dt + V_{co} \]  

(1.24)

where \( V_{co} \) represents the initial voltage across the terminals of the capacitor at \( t = 0 \).

Unlike the resistor the capacitor does not dissipate any power. In fact because this circuit element is able to accumulate charge on its metal plates it stores energy. To illustrate this we return to the hydraulic analogue; if two shutters are inserted across each side of the pipe in order to stop the stretched...
membrane relaxing to its original state this is equivalent to the disconnection of a charged capacitor from the circuit. The energy stored in a capacitor can be written as

$$E_C = \frac{1}{2} CV^2 \text{ (joule)} \quad (1.25)$$

where $V$ is the voltage across the capacitor terminals.

### 1.3.3 Inductance

Figure 1.9 a) shows an inductor connected to a current source. The inductor is usually formed by a coil of a metal wire. The passage of current, $i(t)$, in a metal wire induces a magnetic flux which in turn (according to Faraday’s law) results in a voltage, $v(t)$, developing across the terminals, such that:

$$v(t) = L \frac{di(t)}{dt} \quad (1.26)$$

where $L$ is the inductance of the wire. The unit for the inductance is the henry (H) when, in eqn 1.26, $v(t)$ is expressed in volts and the current rate of change, $di(t)/dt$, is expressed in amperes per second. Thus, an inductance of one henry will have a potential difference of one volt across its terminals when the current passing through it is changing at a rate of one ampere per second. Figure 1.9 b) shows a hydraulic analogue for the inductor which consists of paddles connected to a flywheel. In this analogy the voltage becomes the water pressure and the current becomes the rate of water flow. The inductance becomes the flywheel moment of inertia. The flywheel requires water pressure to change the speed of the paddles which, in turn, change the rate of the water flow. In figure 1.9 b) we illustrate the situation where the application of a pulse of water pressure causes an increase in the speed of the water flow which will be maintained constant by the flywheel inertia until a different level of water pressure is applied. In electrical terms this means that the voltage difference at the terminals of an inductor is proportional to the rate of variation of the current that flows through it.

Integrating eqn 1.26 we obtain an expression for the current that flows in the inductor in terms of the voltage at its terminals, that is:

$$i(t) = \frac{1}{L} \int_0^t v(t) \, dt + I_{i_0} \quad (1.27)$$

where $I_{i_0}$ represents the initial current in the inductor at $t = 0$.

Like the capacitor, the inductor does not dissipate energy and is capable of storing energy. However, the energy is now stored in terms of the magnetic flux created by the current. This energy can be written as

$$E_L = \frac{1}{2} LI^2 \text{ (joule)} \quad (1.28)$$

where $I$ is the current through the inductor.
1.4 Kirchhoff’s laws

Kirchhoff’s laws provide the basis of all circuit analysis techniques as long as such circuits can be described by lumped elements such as resistors, capacitors, etc. There are two Kirchhoff’s laws: the current law and the voltage law. These two laws are quite simple in terms of concept. However, the application of these laws requires careful attention to the algebraic sign conventions of the current and voltage.

The current law

The current law states that the sum of currents entering a node is equal to the sum of the currents leaving the node. A node is a point at which two or more electrical elements have a common connection. Figure 1.10 a) shows an example of a node where the currents $I_1$ and $I_2$ are entering the node while the current $I_3$ is leaving the node. According to the current law, we can write:

$$I_1 + I_2 = I_3$$

which can also be expressed as:

$$I_1 + I_2 + (-I_3) = 0$$

This is equivalent to reversing the direction of the current $I_3$, as shown in figure 1.10 b). Hence, the current law can also be stated as follows: the sum of all currents flowing into a node, taking into account their algebraic signs, is zero.

Example 1.4.1 Consider the circuit of figure 1.11. Determine the currents $I_1$, $I_2$ and $I_3$.

Solution: From figure 1.11 we can write the following eqns for nodes $X$, $Y$ and $Z$, respectively:

$$6 + 7 + I_1 = 3 + 4 + 5$$
$$3 + 4 = I_2$$
$$5 + I_2 = I_3$$

Solving to obtain $I_1$, $I_2$ and $I_3$ we obtain:

$$I_1 = -1 \text{ A}$$
$$I_2 = 7 \text{ A}$$
$$I_3 = 12 \text{ A}$$

Note that the current $I_1$ is negative which means that the direction of the current is the opposite of that shown in the figure.

The voltage law

The voltage law states that the sum of all voltages around any closed electrical loop, taking into account polarities\(^6\), is zero. Figure 1.12 shows a circuit with

\(^6\)In this book we use curved arrows to indicate the potential difference between two points in a circuit, with the arrow head pointing to the lower potential.
two closed loops. By applying the voltage law we can write the following equations for loop 1 and loop 2, respectively:

\[
\begin{align*}
    v_R(t) + v_L(t) - v_s(t) &= 0 \\
    v_C(t) - v_R(t) &= 0
\end{align*}
\]

It should be noted that since the connections of the elements are assumed as ideal (zero resistance), voltage differences are observed only across the various elements. Hence, for example, node \( z \) is, from an electrical point-of-view, the same at the low end of the voltage source and at the low-end of the inductor.

Example 1.4.2 Consider the circuit of figure 1.13. Determine the voltages \( V_1 \) and \( V_2 \).

Solution: From figure 1.13 and starting from point \( x \) we can apply the voltage law to the upper and lower loop, respectively, as indicated below:

\[
\begin{align*}
    V_2 + 5 - 2 &= 0 \\
    2 + V_1 - 6 &= 0
\end{align*}
\]

Solving we get:

\[
\begin{align*}
    V_2 &= -3 \text{ V} \\
    V_1 &= 4 \text{ V}
\end{align*}
\]

We observe that \( V_2 \) is negative meaning that the polarity of the voltage drop is opposite to that chosen originally.

1.4.1 Series and parallel combinations of passive elements

Two connected elements are said to be in series if the same current flows through each and in parallel if they share the same voltage across their terminals. We study now the series and parallel combinations of resistors, capacitors and inductors.

Resistance

Figure 1.14 a) shows two resistors in a series connection. These two resistors can be replaced by a resistor with an equivalent resistance, \( R_{eq} \), as shown in figure 1.14 b). Hence, \( R_{eq} \) must draw the same amount of current \( I \) as the series combination. Applying Kirchhoff’s voltage law to the circuit of figure 1.14 a) we obtain:

\[
V_s + V_1 + V_2 = 0 \\

\text{that is}
\]

\[
V_s = V_1 + V_2
\]
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Since the current that flows through both resistors is the same we can write the last eqn as follows:

\[ V_s = R_1 I + R_2 I = (R_1 + R_2) I \] (1.31)

Applying Ohm’s law to the circuit of figure 1.14 b) we obtain:

\[ V_s = R_{eq} I \] (1.32)

therefore

\[ R_{eq} = R_1 + R_2 \] (1.33)

It can be shown (see problem 1.4) that the above can be generalised for the series combination of \( N \) resistors as follows:

\[ R_{eq} = \sum_{k=1}^{N} R_k \] (1.34)

that is, the equivalent resistance is obtained by the addition of all resistances that make the series connection.

Figure 1.14: a) Series combination of two resistors. b) Equivalent resistance.

Figure 1.15 a) shows two resistors in a parallel connection. Each resistor \( R_k \) can be expressed as a conductance \( G_k = R_k^{-1} \). Applying Kirchhoff’s current law to the circuit of figure 1.15 a) we obtain:

\[ I_s = I_1 + I_2 \] (1.35)

Since the voltage across each conductance is the same we can write the last eqn as follows:

\[ I_s = G_1 V + G_2 V = (G_1 + G_2) V \] (1.36)

Applying Ohm’s law to the circuit of figure 1.15 b) we obtain:

\[ I_s = G_{eq} V \] (1.37)

so that

\[ G_{eq} = G_1 + G_2 \] (1.38)

It can be shown (see problem 1.5) that this result can be generalised for the parallel combination of \( N \) resistors as follows:

\[ G_{eq} = \sum_{k=1}^{N} G_k \] (1.39)

that is, the equivalent conductance is the addition of each conductance that composes the parallel connection.
The equivalent resistance, \( R_{eq} \), for the parallel combination of two resistances can be obtained by re-writing eqn 1.38 as follows:

\[
\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}
\]  

or,

\[
R_{eq} = \frac{R_1 R_2}{R_1 + R_2}
\]

Often, we use the notation \( || \) to indicate the parallel connection of resistances, that is, \( R_1 || R_2 \) means \( R_1 \) in parallel with \( R_2 \).

**Example 1.4.3** Consider two resistances \( R_1 = 3 \, k\Omega \) and \( R_2 = 200 \, \Omega \).

1. Determine the equivalent resistance, \( R_{eq} \), for the series connection of \( R_1 \) and \( R_2 \).

2. Determine the equivalent resistance, \( R_{eq} \), for the parallel connection of \( R_1 \) and \( R_2 \).

**Solution:**

1. According to eqn 1.33 the equivalent resistance is \( R_{eq} = 3.2 \, k\Omega \).

2. According to eqn 1.41 the equivalent resistance is \( R_{eq} = 188 \, \Omega \).

It should be remembered that for the series connection of resistances the equivalent resistance is larger than the largest resistance in the series chain whilst for the parallel connection the equivalent resistance is smaller than the smallest resistance in the connection.

**Capacitance**

Figure 1.16 a) shows two capacitors in a *series* connection. These can be replaced by a capacitor with an equivalent capacitance, \( C_{eq} \), as shown in figure 1.16 b). Applying Kirchhoff’s voltage law to the circuit of figure 1.16 a) we obtain:

\[
v_s(t) = v_1(t) + v_2(t)
\]

Since the current that flows through both capacitors is the same then, according to eqn 1.24, we can rewrite the above eqn as follows:

\[
v_s(t) = \frac{1}{C_1} \int_0^t i(t) \, dt + \frac{1}{C_2} \int_0^t i(t) \, dt
\]

\[
= \left( \frac{1}{C_1} + \frac{1}{C_2} \right) \int_0^t i(t) \, dt
\]
Here we assume that the initial voltage across each capacitor is zero. Applying eqn 1.24 to the capacitor of the circuit of figure 1.16 b) we obtain:

\[ v_s(t) = \frac{1}{C_{eq}} \int_0^t i(t) \, dt \]  

(1.44)

From eqns 1.44 and 1.43 we conclude that

\[ \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \]  

(1.45)

that is:

\[ C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \]  

(1.46)

It can be shown (see problem 1.7) that eqn 1.45 can be generalised for the series combination of \( N \) capacitors as follows:

\[ \frac{1}{C_{eq}} = \sum_{k=1}^{N} \frac{1}{C_k} \]  

(1.47)

Figure 1.17 a) shows two capacitors in a parallel connection. Applying Kirchhoff’s current law to the circuit of figure 1.17 a) we obtain:

\[ i_s(t) = i_1(t) + i_2(t) \]  

(1.48)

Since the voltage across each capacitor is the same then, using eqn 1.23, we can write the above eqn as follows:

\[ i_s(t) = C_1 \frac{d v_1(t)}{dt} + C_2 \frac{d v_2(t)}{dt} = (C_1 + C_2) \frac{d v(t)}{dt} \]  

(1.49)

Applying eqn 1.23 to the capacitor of the circuit of figure 1.15 b) we obtain:

\[ i_s(t) = C_{eq} \frac{d v(t)}{dt} \]  

(1.50)

so that

\[ C_{eq} = C_1 + C_2 \]  

(1.51)

It can be shown (see problem 1.8) that the above eqn can be generalised for the parallel combination of \( N \) capacitors as follows:

\[ C_{eq} = \sum_{k=1}^{N} C_k \]  

(1.52)
Example 1.4.4 Consider two capacitances $C_1 = 0.3 \ \mu F$ and $C_2 = 1 \ \mu F$.

1. Determine the equivalent capacitance, $C_{eq}$, for the series connection of $C_1$ and $C_2$.

2. Determine the equivalent capacitance, $C_{eq}$, for the parallel connection of $C_1$ and $C_2$.

Solution:

1. According to eqn 1.46 the equivalent capacitance is $C_{eq} = 0.2 \ \mu F$.

2. According to eqn 1.51 the equivalent capacitance is $C_{eq} = 1.3 \ \mu F$.

It should be noted that, for capacitors, the series connection results in a decreased value for the equivalent capacitance while the parallel connection results in an increased value for the equivalent capacitance.

Inductance

Figure 1.18 a) shows two inductors\(^7\) in a series connection. These two inductors can be replaced by a single inductor with an equivalent inductance, $L_{eq}$, as shown in figure 1.18 b). Applying Kirchhoff’s voltage law to the circuit of figure 1.18 a) we obtain:

$$v_s(t) = v_1(t) + v_2(t) \quad (1.53)$$

Since the current that flows through both inductors is the same then, according to eqn 1.26, we can rewrite the last eqn as follows:

$$v_s(t) = L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt}$$

$$= (L_1 + L_2) \frac{di(t)}{dt} \quad (1.54)$$

Applying eqn 1.26 to the equivalent inductor in figure 1.14 b) we obtain:

$$v_s(t) = L_{eq} \frac{di(t)}{dt} \quad (1.55)$$

Comparing this eqn with eqn 1.54 we conclude that

$$L_{eq} = L_1 + L_2 \quad (1.56)$$

It can be shown (see problem 1.10) that the last eqn can be generalised for the series combination of $N$ inductors as follows:

$$L_{eq} = \sum_{k=1}^{N} L_k \quad (1.57)$$

\(^7\)We assume the inductors to be uncoupled, that is they are sufficiently far apart or mounted in such a way so that their magnetic fluxes do not interact.
Figure 1.19 a) shows two inductors in a parallel connection. Applying Kirchhoff’s current law to the circuit of figure 1.19 a) we obtain:

\[ i_s(t) = i_1(t) + i_2(t) \] (1.58)

Since the voltage across each inductor is the same, we can apply eqn 1.27 to eqn 1.58 as follows:

\[ i_s(t) = \frac{1}{L_1} \int_0^t v(t) \, dt + \frac{1}{L_2} \int_0^t v(t) \, dt \]
\[ = \left( \frac{1}{L_1} + \frac{1}{L_2} \right) \int_0^t v(t) \, dt \] (1.59)

where we assume that the initial current in each inductor is zero. Applying eqn 1.27 to the equivalent inductor of figure 1.15 b) we obtain:

\[ i_s(t) = \frac{1}{L_{eq}} \int_0^t v(t) \, dt \] (1.60)

Comparing the last eqn with eqn 1.59 we conclude that

\[ \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \] (1.61)

The equivalent inductance, \( L_{eq} \), for the parallel combination of two inductances can be obtained from eqn 1.61 as follows:

\[ L_{eq} = \frac{L_1 L_2}{L_1 + L_2} \] (1.62)

It can be shown (see problem 1.11) that the above eqn can be generalised for the parallel combination of \( N \) inductors as follows:

\[ \frac{1}{L_{eq}} = \sum_{k=1}^{N} \frac{1}{L_k} \] (1.63)

**Example 1.4.5** Consider two inductances \( L_1 = 1.5 \text{ mH} \) and \( L_2 = 1 \text{ mH} \).

1. Determine the equivalent inductance, \( L_{eq} \), for the series connection of \( L_1 \) and \( L_2 \).

2. Determine the equivalent capacitance, \( L_{eq} \), for the parallel connection of \( L_1 \) and \( L_2 \).

**Solution**:

1. According to eqn 1.56 the equivalent inductance is \( C_{eq} = 2.5 \text{ mH} \).

2. According to eqn 1.62 the equivalent inductance is \( C_{eq} = 0.6 \text{ mH} \).

It is interesting to note that like resistors (but unlike capacitors) the series connection of inductors increases the value of the equivalent inductance while the parallel connection decreases it.
### 1.4.2 Other types of circuit element connections

Circuit elements can be connected in combinations which are neither parallel nor series. For example, take the circuit of figure 1.20 a) for which we want to determine the equivalent resistance, $R_{eq}$, between terminals $A$ and $B$. In this circuit there is not a single combination of two resistances which share the same current through or the same voltage across their terminals and, therefore, there is not a single parallel or series connection. This means that we cannot directly apply the rules discussed previously, for parallel and series connections of resistances, to determine $R_{eq}$. However, the calculation of the equivalent resistance can be done by applying a test voltage source, $V_t$, to the terminals of the circuit as shown in figure 1.20 b). Then, we determine the current $I_t$ supplied by this source. Finally, by calculating the ratio $V_t/I_t$ we can find the equivalent resistance, $R_{eq}$, effectively applying Ohm’s law. Alternatively, we can apply a test current source, $I_t$, to the circuit as illustrated in figure 1.20 c). Again, by calculating the ratio $V_t/I_t$ we obtain $R_{eq}$. In general, this procedure can be applied to any circuit.

---

**Example 1.4.6** Determine the equivalent resistance, $R_{eq}$, of the circuit of figure 1.20 between terminals $A$ and $B$.

**Solution**: We apply a test voltage source to the circuit as shown in figure 1.21. This figure also shows the definition of the voltages across and currents through each resistance. Applying Kirchhoff’s current law we can write:

\[
\begin{align*}
I_t &= I_1 + I_2 \\
I_t &= I_4 + I_3 \\
I_1 &= I_4 + I_5 \\
I_3 &= I_2 + I_5
\end{align*}
\]  

(1.64)
and applying Kirchhoff’s voltage law we can write
\[
\begin{align*}
V_t &= V_1 + V_4 \\
V_t &= V_2 + V_3
\end{align*}
\] (1.65)

These two sets of eqns can be rewritten as:
\[
\begin{align*}
I_t &= \frac{V_1}{R_1} + \frac{V_2}{R_2} \\
I_t &= \frac{V_4}{R_4} + \frac{V_3}{R_3} \\
\frac{V_1}{R_1} &= \frac{V_4}{R_4} + \frac{V_5}{R_5} \\
\frac{V_4}{R_4} &= \frac{V_2}{R_2} + \frac{V_3}{R_3} \\
V_2 &= V_1 + V_4 \\
V_t &= V_2 + V_3
\end{align*}
\] (1.66)

Solving, to obtain \(V_t/I_t = R_{eq}\), we get:
\[
\frac{V_t}{I_t} = \frac{R_2R_3(R_1 + R_4) + R_1R_2(R_3 + R_4)}{R_2(R_5 + R_3 + R_4) + R_3(R_4 + R_1 + R_3) + R_1(R_4 + R_3)} + \frac{R_2(R_5 + R_3 + R_4) + R_3(R_4 + R_1 + R_3) + R_1(R_4 + R_3)}{R_3R_4(R_2 + R_5) + R_1R_3(R_4 + R_5)}
\]
\[
= 49.5 \, \Omega
\]

### 1.4.3 Electrical network analysis – Nodal analysis

There are various electrical analysis methods, all derived from Kirchhoff’s laws, to analyse electrical circuits. One of most effective and computationally efficient is the Nodal analysis method. Therefore, we now illustrate the application of this method to resistive electrical networks.

Figure 1.22 a) shows a circuit for which we want to determine the current \(I_A\). Since the resistance \(R_2\) is short-circuited the voltage across this resistance is zero and, according to Ohm’s law, the current that flows through \(R_2\) is zero. Hence, the circuit of figure 1.22 a) can be replaced by its equivalent represented in figure 1.22 b). First, we indicate the voltages at each node. These voltages indicate the potential difference between the node being considered and a reference node which can be chosen arbitrarily. This node is traditionally called ‘node zero’ (0) or the ‘ground terminal’ and is often chosen as the node with the highest number of attached electrical elements. For this circuit there are three nodes \((X, Y, Z)\) plus the reference node zero, as shown in figure 1.22 b). Then, we consider, in an arbitrary manner, the current direction in each branch,
as indicated in figure 1.22 b). The current that flows through each resistance can be expressed, according to Ohm’s law, as the ratio of the voltage across that resistance and the resistance value. Figure 1.23 illustrates this procedure for the current $I_1$ that flows through $R_1$; since the voltages $V_X$ and $V_Y$ have been defined, referenced to ground, then by applying Kirchhoff’s voltage law we can express the voltage across $R_1$, $V_1$, as the difference between $V_X$ and $V_Y$. Hence we can write:

$$I_1 = \frac{V_X - V_Y}{R_1}$$  (1.67)

Applying this technique to the remaining currents, as defined in the circuit of figure 1.22 b), we can write:

$$I_3 = \frac{V_Z - V_Y}{R_3}$$  (1.68)

$$I_4 = \frac{V_Y}{R_4}$$  (1.69)

$$I_5 = \frac{V_Z}{R_5}$$  (1.70)

Applying Kirchhoff’s current law to nodes $Y$ and $Z$ we can write:

$$I_1 + I_Y + I_3 = I_4$$  (1.71)

$$I_Y + I_3 + I_5 = 0$$  (1.72)

These two eqns can be expressed, using eqns 1.67–1.70, as follows:

$$\frac{V_X - V_Y}{R_1} + I_Y + \frac{V_Z - V_Y}{R_3} = \frac{V_Y}{R_4}$$  (1.73)

$$I_Y + \frac{V_Z - V_Y}{R_3} + \frac{V_Z}{R_5} = 0$$  (1.74)
Note that the only two unknown quantities are $V_Y$ and $V_Z$. Solving the last two eqns in order to obtain $V_Y$ and $V_Z$ we get:

$$V_Y = \frac{R_4 R_X + I_Y R_3 R_4 + V_X R_5}{R_3 R_4 + R_4 R_5 + R_4 R_4 + R_4 R_5 + R_4 R_5}$$

$$V_Z = \frac{R_5 R_X - R_3 R_4 I_Y - I_Y R_3 R_4}{R_3 R_4 + R_4 R_5 + R_4 R_4 + R_4 R_5 + R_4 R_5}$$

Since $I_A = I_3$, eqn 1.68 gives us $I_A = -6 \text{ mA}$. 

1.4.4 Resistive voltage and current dividers

Resistive voltage and current dividers are simple yet very important circuits which allow us to obtain fractions of a source voltage or current, respectively. In addition, these circuits play a major role in the calculation of voltage and current gains in electronic amplifier analysis.

Resistive voltage divider

Figure 1.24 a) shows the resistive voltage divider formed by resistances $R_1$ and $R_2$. For this circuit we observe that the current flowing through $R_1$ and $R_2$ is the same. Hence, we can write:

$$\frac{V_s - V_o}{R_1} = \frac{V_o}{R_2}$$

Solving this eqn in order to obtain $V_o$ we get:

$$V_o = V_s \frac{R_2}{R_1 + R_2}$$

We observe that if $R_1 = R_2$ then $V_o = 0.5 V_s$. Also, if $R_1 >> R_2$ then $V_o$ tends to zero. On the other hand, if $R_1 << R_2$ then $V_o$ tends to $V_s$.

Resistive current divider

Figure 1.24 b) shows the resistive current divider formed by resistances $R_1$ and $R_2$. For this circuit the voltage across each resistor is the same. Hence, we can write the following set of eqns:

$$I_s = \frac{V}{R_1} + \frac{V}{R_2}$$

$$I_o = \frac{V}{R_2}$$

Rearranging, we obtain $I_o$:

$$I_o = I_s \frac{R_1}{R_1 + R_2}$$

We can conclude that if $R_1 = R_2$ then $I_o = 0.5 I_s$. Also, if $R_1 >> R_2$ then $I_o$ tends to $I_s$. On the other hand, if $R_1 << R_2$ then $I_o$ tends to zero.
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1.4.5 Controlled sources

The voltage and the current sources presented in section 1.2 are called independent sources. Other types can be controlled by either a voltage or current existing elsewhere in the circuit. These controlled (or dependent) sources are often used to model the gain of transistors operating in their linear region and also to model the gain of linear electronic amplifiers, as will be discussed in Chapter 6. There are four types of controlled sources, drawn as diamond shapes, as illustrated in figure 1.25.

- **Voltage-controlled voltage sources:** the output of the source is a voltage, $V_o$, and the quantity that controls it is also a voltage, $V_i$. $A_v$ is the ratio $V_o/V_i$ and is called the ‘voltage gain’ of the source. $A_v$ is dimensionless;

- **Voltage-controlled current sources:** the output of the source is a current, $I_o$, but the quantity that controls it is a voltage, $V_i$. $G_m$ is the ratio $I_o/V_i$ and is called the ‘transconductance gain’ of the source. The dimension of $G_m$ is the siemen;

- **Current-controlled voltage sources:** the output of the source is a voltage, $V_o$, but the controlling quantity is a current $I_i$. $R_m$ is the ratio $V_o/I_i$ and is called the ‘transresistance gain’ of the source. The dimension of $R_m$ is the ohm;

- **Current-controlled current sources:** the output of the source is a current, $I_o$, and the controlling quantity is also a current, $I_i$. $A_i$ is the ratio $I_o/I_i$ and is called the ‘current gain’ of the source. $A_i$ is dimensionless.

It is important to note that the output characteristics of each of the dependent sources is exactly the same as those of its corresponding independent source (see figures 1.2 b) and 1.4 b)) but with $I$ and $V$ values being controlled by a quantity occurring somewhere else in the circuit.

**Example 1.4.7** Consider the circuit of figure 1.26 containing a voltage-controlled current source. Determine the voltage across the resistance $R_2$.

**Solution:** According to Kirchhoff’s current law the current that flows through $R_2$ is the sum of $I_s$ with the current supplied by the voltage-controlled current source, $G_m V_1$, that is:

$$I_2 = I_s + G_m V_1$$

Since $V_1 = R_1 I_s$ we can write the last eqn as follows:

$$I_2 = I_s (1 + G_m R_1)$$

Finally:

$$V_2 = R_2 I_2$$

$$= R_2 I_s (1 + G_m R_1)$$

$$= 81 \text{ V}$$
1.5 Thévenin’s theorem

Thévenin’s theorem states that any two-terminal electrical network, such as that depicted in figure 1.27 a) consisting of resistances and independent sources (voltage or current or both) can be replaced by an equivalent ideal voltage source in series with an equivalent resistance, as shown in figure 1.27 b). The value of the equivalent voltage source, \( V_{Th} \), known as the Thévenin voltage, is the open-circuit voltage at the output terminals. \( R_{Th} \) is the Thévenin resistance ‘looking into’ the terminals of the network when all independent voltage sources are replaced by short-circuits and all independent current sources are replaced by open-circuits.

**Example 1.5.1** Determine the Thévenin equivalent circuit for the circuit of figure 1.28 a) where the two terminals to be considered are \( X \) and \( Y \).

**Solution:** The calculation of the Thévenin voltage can be performed by analysis of the circuit of figure 1.28 b). Note that \( V_{Th} \) is equal to \( V_A \). We can write the following set of eqns:

\[
\begin{align*}
I_r &= I_2 + I_1 \\
V_s &= V_B - V_A \\
I_1 &= I_3
\end{align*}
\]

These can be rewritten as follows:

\[
\begin{align*}
I_r &= \frac{V_C + V_C - V_B}{R_2} \\
V_s &= \frac{V_B - V_A}{R_1} \\
V_A &= \frac{V_B - V_D}{R_1}
\end{align*}
\]

Solving, in order to obtain the voltages \( V_A \), \( V_B \) and \( V_C \) we get:

\[
V_A = \frac{R_3}{R_3 + R_1 + R_2} (1.82)
\]

\( = 1.1 \text{ V} \)
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\[ V_B = \frac{I_r R_2 R_3 + R_1 V_s + R_2 V_s}{R_3 + R_1 + R_2} \]  \hspace{1cm} (1.83)

\[ V_C = \frac{R_2 I_r R_3 + V_s + I_r R_1}{R_3 + R_1 + R_2} \]  \hspace{1cm} (1.84)

\[ = 8.1 \text{ V} \]

\[ = 8.6 \text{ V} \]

Since the Thévenin voltage is the voltage across points \( X \) and \( Y \) it is the same as \( V_A \), that is, \( V_{Th} = 1.1 \text{ V} \). The calculation of the Thévenin equivalent resistance can be carried out by analysing the circuit of figure 1.28 c) where it can be seen that the voltage source, \( V_s \), has been replaced by a short-circuit and the current source, \( I_r \), has been replaced by an open-circuit. The calculation of the Thévenin resistance can be calculated by applying a test voltage, \( V_t \), between terminals \( X \) and \( Y \), the terminals where the resistance is to be determined. The ratio between \( V_t \) and the current supplied by this test voltage source, \( I_t \), is the required resistance. Applying Kirchhoff’s current law to the circuit of figure 1.28 c) we can write:

\[ I_t = I_1 + I_3 \]  \hspace{1cm} (1.85)

From this figure we can also observe that the voltage \( V_t \) is applied to \( R_3 \) and also to the series combination of \( R_1 \) and \( R_2 \). Hence we can write:

\[ I_t = \frac{V_t}{R_1 + R_2} + \frac{V_t}{R_3} \]  \hspace{1cm} (1.86)

Solving for \( V_t/I_t \) we obtain:

\[ R_{Th} = \frac{R_3 (R_1 + R_2)}{R_1 + R_2 + R_3} \]  \hspace{1cm} (1.87)

\[ = 49.5 \text{ } \Omega \]

It should be noted that this resistance could also be calculated by close inspection of figure 1.28 c) after recognising that the resistance \( R_{Th} \) is the parallel combination of \( R_3 \) with the series combination of \( R_1 \) and \( R_2 \).

1.6 Norton’s theorem

Norton’s theorem states that any two-terminal electrical network, such as that depicted in figure 1.29 a) consisting of resistances and independent sources (voltage or current or both) can be replaced by an equivalent independent current source in parallel with an equivalent resistance, as shown in figure 1.29 b). The value of the current source is the current flowing from \( X \) to \( Y \) when \( X \) to \( Y \) are short-circuited. The equivalent resistance (Norton resistance), \( R_{Nt} \), is the resistance ‘looking into’ the terminals of the network when the independent voltage sources are replaced by short-circuits and the independent current sources are replaced by open-circuits. Norton’s theorem is the dual of Thévenin’s since the equivalent voltage source, \( V_{Th} \), is replaced by an equivalent current source, \( I_{Nt} \), and the series resistance, \( R_{Th} \), is replaced by a parallel resistance, \( R_{Nt} \).
Example 1.6.1 Determine the Norton equivalent circuit for the circuit of figure 1.28 a) where the two terminals to be considered are $X$ and $Y$.

**Solution:** The calculation of the Norton short-circuit current, $I_{Nt}$, can be performed by analysing the circuit of figure 1.30 a) for which we can write the following set of eqns:

\[
I_{Nt} = I_1 
\]
\[
I_r = I_1 + I_2 
\]
\[
V_B = V_s
\]

Note that since $R_3$ is short-circuited, the voltage across its terminals is zero and there is no current flowing through this resistance. Eqn 1.89 can be written as follows:

\[
I_r = \frac{V_C - V_B}{R_1} + \frac{V_C}{R_2}
\]

Since $V_B = V_s$ we have that:

\[
I_r = \frac{V_C - V_s}{R_1} + \frac{V_C}{R_2}
\]

Solving in order to obtain $V_C$ we can write:

\[
V_C = \frac{R_1 R_2}{R_1 + R_2} I_r + \frac{R_2}{R_1 + R_2} V_s
\]

\[
= 7.7 \text{ V}
\]

The Norton equivalent short-circuit current is,

\[
I_{Nt} = \frac{V_C - V_B}{R_1} = 0.023 \text{ A}
\]

The Norton equivalent resistance is determined in a fashion similar to that used to calculate the Thévenin resistance in example 1.5.1. Consequently, the Norton equivalent circuit for the circuit of 1.28 a) is as shown in figure 1.30 b).

It is useful to note the straightforward equivalence between the Thévenin and Norton theorems. If, for example, the Thévenin equivalent circuit is known then the equivalent Norton circuit can be obtained as shown in figures 1.31 a) and 1.31 b), that is:

\[
I_{Nt} = \frac{V_{Th}}{R_{Th}}
\]
\[
R_{Nt} = R_{Th}
\]
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Figure 1.31: a) Thévenin circuit. b) Equivalent Norton circuit. c) Norton circuit. d) Equivalent Thévenin circuit.

On the other hand, if the Norton equivalent circuit is known then the Thévenin equivalent circuit can be obtained as shown in figures 1.31 c) and 1.31 d), that is:

\[ V_{Th} = I_{Nt} R_{Nt} \]  
\[ R_{Th} = R_{Nt} \]

(1.97)  
(1.98)

It is left to the reader to prove these equivalences.

We emphasise that the discussion of Thévenin and Norton theorems presented above applies to circuit networks containing only independent sources (voltage and current). However, there are algebraic techniques which allow us to obtain Thévenin and Norton equivalent circuits when the networks include dependent sources [5].

1.7 Superposition theorem

The superposition theorem is of considerable importance since it can provide useful insight into the relative contribution of a given independent source to the current flowing through or the voltage across a given circuit element. The superposition theorem also plays a major role in the frequency domain circuit analysis, discussed in Chapter 3, and in the noise analysis of linear electronic circuits presented in Chapter 8.

The superposition theorem applies to linear circuits and it can be stated as follows: “In a network containing several current and/or voltage sources, the voltage across (or the current flowing through) any circuit element can be obtained from the algebraic sum of the voltages (currents) caused by each independent source considered individually with all other independent voltage sources considered as short-circuits and all other independent current sources considered as open-circuits”. As stated above, the superposition theorem does not allow for the substitution of controlled voltage sources and controlled current sources by short-circuits and open-circuits, respectively. In this respect the superposition theorem applies only to independent sources.[8]

Example 1.7.1 Apply the superposition theorem to determine the equivalent Thévenin voltage of the circuit of figure 1.28 a) where the two terminals to be

[8]Using mathematical manipulation techniques it possible to apply the superposition theorem to dependent sources [5].
considered are the $X$ and $Y$ terminals.

Solution: Figure 1.32 a) shows the equivalent circuit for the calculation of the contribution of $V_s$ to the Thévenin voltage across the terminals $X$ and $Y$. Note that the positive current supplied by the voltage source flows as indicated in this figure. Since all resistances are in series the current can be determined, according to Ohm’s law, as follows:

$$I = \frac{V_s}{R_1 + R_2 + R_3} = 33.3 \text{ mA}$$

The contribution to the Thévenin voltage is the voltage across $R_3$ and can be expressed as follows:

$$V_{Th_a} = -IR_3 = -\frac{V_s R_3}{R_1 + R_2 + R_3} = -2.7 \text{ V}$$

Figure 1.32 b) shows the equivalent circuit derived to calculate the contribution of $I_r$ to the Thévenin voltage. The current that flows through $R_3$ can be calculated using the current divider concept. Hence, since $R_2$ is in a series connection with $R_3$ we can write:

$$I_1 = I_r \frac{R_2}{R_1 + R_2 + R_3} = 47.6 \text{ mA}$$

and this contribution to the Thévenin voltage, the voltage across $R_3$, can be expressed as follows:

$$V_{Th_b} = I_1 R_3 = I_r \frac{R_2 R_3}{R_1 + R_2 + R_3} = 3.8 \text{ V}$$

Adding $V_{Th_a}$ and $V_{Th_b}$ we obtain the Thévenin voltage as:

$$V_{Th} = 1.1 \text{ V}$$

Note that the superposition theorem allows us to identify the contribution of each independent source in a clear manner. While the contribution of $V_s$ to the Thévenin voltage is negative ($-2.7$ V), the contribution of $I_r$ to $V_{Th}$ is positive (3.8 V). This results in a net voltage $V_{Th}$ of 1.1 V.

1.8 Bibliography


1.9 Problems

1.1 A voltage $v(t) = 10 \sin(2\pi \cdot 100 \cdot t + \pi/4)$ volts is applied across the terminals of a 1 $\mu$F capacitor. Sketch the current through the capacitor as a function of time from $t = 0$ to $t = 20$ ms.

1.2 A current $i(t) = 20 \cos(2\pi \cdot 5000 \cdot t)$ mA flows through a 3 mH inductor. Sketch the voltage across the inductor as a function of time from $t = 0$ to $t = 500$ $\mu$s.

1.3 Find the current through and the voltage across each resistance for the circuits of figure 1.33. Take $V_1 = 2$ V, $V_2 = 3$ V, $I_1 = 0.2$ A and $I_2 = 0.5$ A.

Figure 1.33: Circuits of problem 1.3.

1.4 Show that the equivalent resistance for the series combination of $N$ resistances is given by eqn 1.34.
1.5 Show that the equivalent conductance for the parallel combination of \( N \) resistances satisfies eqn 1.39.

1.6 For each circuit of figure 1.34 determine the equivalent resistance and conductance between points \( A \) and \( B \).

![Circuits of problem 1.6.](image)

1.7 Show that the equivalent capacitance for the series combination of \( N \) capacitances satisfies eqn 1.47.

1.8 Show that the equivalent capacitance for the parallel combination of \( N \) capacitances satisfies eqn 1.52.

1.9 For each circuit of figure 1.35 determine the equivalent capacitance between points \( A \) and \( B \).

1.10 Show that the equivalent inductance for the series combination of \( N \) inductors is given by eqn 1.57.

1.11 Show that the equivalent inductance for the parallel combination of \( N \) inductors satisfies eqn 1.63.
1. Elementary electrical circuit analysis

1.12 For each circuit of figure 1.36 determine the equivalent inductance between points $A$ and $B$.

1.13 For each circuit of figure 1.37 determine the voltage across and the current through $R_o$.

1.14 For each circuit of figure 1.38 determine the voltage across and the current through $R_1$.

1.15 For the circuits b) and c) of figure 1.38 determine the Thévenin equivalent circuits at points $A$ and $B$. 
1.16 For the circuits b) and c) of figure 1.38 determine the Norton equivalent circuits at points A and B.

1.17 For the circuits of figure 1.39 determine the voltage across and the current through $R_3$. Use values of $A_i = 12$, $G_m = 0.5$ S, $A_v = 10$ and $R_m = 40$ $\Omega$.

1.18 Apply the superposition theorem to the circuits of figure 1.40 to determine the voltage across and the current through $R_2$. $G_m = 0.9$ S, $A_v = 10$. 

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**Figure 1.38: Circuits of problems 1.14, 1.15 and 1.16.**

**Figure 1.39: Circuits of problem 1.17.**
Figure 1.40: Circuits of problem 1.18.
2 Complex numbers: An introduction

2.1 Introduction

Complex numbers play a major role in alternating current (AC) circuit analysis through the use of the phasor concept and associated analysis. This simplifies the analysis of circuits by representing voltage and current quantities in terms of magnitude and phase. Phasor analysis is also the foundation of frequency domain signal analysis and is used extensively in the remaining chapters of this book.

Phasors are basically a convenient representation of complex numbers. In this chapter we introduce complex numbers and the different ways of representing them. Following this introduction, we define complex numbers. In section 2.3 we describe the elementary algebraic operations for these types of numbers. Then, in section 2.4 we discuss the polar representation of complex numbers and in section 2.5 we introduce the exponential representation which is basically the phasor representation. Finally, in section 2.6, we present the calculation of powers and roots of complex numbers.

2.2 Definition

Real numbers can be integers (e.g. \(-1, 0, +2\)), fractional numbers (e.g. \(-1/2, 1/3, 5/6\)) and irrational numbers (e.g. \(\sqrt{3}, \pi\)). We can represent all real numbers on a single axis, the so-called real axis as illustrated in figure 2.1 a).

Complex numbers are quantities which are represented in a plane as shown in figure 2.1 b). This plane is called the ‘complex plane’ and it is defined by two orthogonal axes, \(X\) and \(Y\); the real axis and the imaginary axis, respectively. The representation of the complex plane using two orthogonal axes is also called the Argand diagram. Every complex number, \(z\), can be defined by a pair of real numbers (or pair of coordinates), \(x\) and \(y\), which identify the position of \(z\) in the complex plane:

\[
z = (x, y)
\]

In eqn 2.1 \(x\) is called the ‘real part’ of the complex number while \(y\) is called the ‘imaginary part’ of the complex number. In figure 2.1 b) we illustrate the representation of the complex numbers \(z_1 = (1, 2)\) and \(z_2 = (-\sqrt{3}, 4/3)\). It should be noted that all real numbers can be represented as complex numbers where the \(y\) coordinate is zero and they have the general form \((x, 0)\).

\footnote{The imaginary axis is also represented here by \(j\ Y\).}