Derivation of manoeuvring motion equation

Newton's law (Force = mass times acceleration) holds only in an inertial (non-accelerating) reference system. Let us denote such a global, earth-fixed coordinate system by x_0, y_0 with origin in the center of gravity of the ship. Then Newton's law gives:

$$\begin{aligned} m \cdot \ddot{x}_0 &= X_0 \\ m \cdot \ddot{x}_0 &= Y_0 \\ I_{zz} \cdot \ddot{\psi} &= N_0 \end{aligned}$$

X_0, Y_0, N_0 denote the forces in x_0 and y_0 direction and the yaw moment about the z_0 axis, respectively. Now a coordinate transform is necessary to move to a ship-fixed reference system x, y. The transformation of forces between two coordinate systems turned by an angle ψ follows from:

$$\begin{aligned} X_0 &= X \cos\psi - Y \sin\psi \\ Y_0 &= X \sin\psi + Y \cos\psi \end{aligned}$$

Similarly, we write for the velocity vector:

$$\begin{aligned} \dot{x}_0 &= u \cos\psi - v \sin\psi \\ \dot{y}_0 &= u \sin\psi + v \cos\psi \end{aligned}$$

u is the surge velocity, v the sway velocity, and $r = \dot{\psi}$ the yaw velocity.

Then the acceleration vector can be written using $r = \dot{\psi}$:

$$\begin{aligned} \ddot{x}_0 &= (\dot{u} - vr) \cos\psi - (\dot{v} + ur) \sin\psi \\ \ddot{y}_0 &= (\dot{u} - vr) \sin\psi + (\dot{v} + ur) \cos\psi \end{aligned}$$

Using vector and matrix notation, we can then write the initial equations for forces, now expressed by quantities in the ship-fixed system:

$$\begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} m\dot{u} - mvr \\ m\dot{v} + mur \end{Bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix}$$

As the matrices on both sides are identical, the solution is obvious:

$$\begin{aligned} m\dot{u} - mvr &= X \\ m\dot{v} + mur &= Y \end{aligned}$$

The equation for the yaw moment is simply:

$$I_{zz} \cdot \ddot{\psi} = N$$

Often, we work with a ship-fixed system with origin amidships. Then terms reflecting centrifugal forces need to be added (assuming $y_G = 0$, i.e. symmetric mass distribution with respect to the centreplane):

$$\begin{aligned} m\dot{u} - mvr - mx_G r^2 &= X \\ m\dot{v} + mur + mx_G \dot{r} &= Y \\ I_{zz} \cdot \dot{r} + mx_G(\dot{v} + ur) &= N \end{aligned}$$

The forces and moment on the r.h.s. of above equation can be due to various causes:

1. Hull forces due to the flow around the manoeuvring ship

2. Forces due to manoeuvring equipment (rudders, propellers, thrusters)
3. External forces due to environment (wind, waves, current)

In special cases, other forces like tow forces via tow lines can be added.

Let us for the moment consider only the first group of hydrodynamic hull forces. These forces are assumed to be proportional to velocities (u, v, r) and accelerations $(\dot{u}, \dot{v}, \dot{r})$. (There are no hydrostatic restoring forces for surge, sway and yaw as discussed in ship seakeeping. Therefore, the forces will not depend on deflections or yaw angle as such.)

In many cases, we can consider small deviation from the initial condition, i.e. the behaviour of the ship at the beginning of a manoeuvre. This is for example important in evaluation the course keeping ability of the ship (yaw stability). The initial condition of the ship is given by speed U in x-direction, all accelerations and other velocity components are zero. Let us denote a partial derivative with respect to a variable by an index of that variable, i.e.

$$Y_{\dot{v}} = \frac{\partial Y}{\partial \dot{v}}$$

denotes the partial derivative of the side force with respect to the side acceleration.

Expanding the side force in a Taylor series around the initial condition (index 0) and retaining only linear terms, yields then:

$$Y = Y_0 + (u - U)Y_u + vY_v + rY_r + \dot{u}Y_{\dot{u}} + \dot{v}Y_{\dot{v}} + \dot{r}Y_{\dot{r}}$$

Initially, the ship is in equilibrium (straight-ahead course at constant speed), i.e. $Y_0 = 0$. For usual symmetric ships, the motion in x-direction cannot cause any forces in y-direction, so $Y_u = 0$ and $Y_{\dot{u}} = 0$. Thus:

$$Y = vY_v + rY_r + \dot{v}Y_{\dot{v}} + \dot{r}Y_{\dot{r}}$$

Y_r and $Y_{\dot{r}}$ are usually small, non-zero numbers, because ships are generally not symmetric with respect to amidships. Similarly, we get for the longitudinal force and the yaw moment:

$$X = (u - U)X_u + \dot{u}X_{\dot{u}}$$

$$N = vN_v + \dot{v}N_{\dot{v}} + rN_r + \dot{r}N_{\dot{r}}$$

If we substitute these linearised equations for X, Y, and N in the fundamental motion equations, again expanded in a Taylor series and neglecting quadratically small terms like vr and r^2, we obtain:

$$\begin{aligned}
(X_{\dot{u}} - m)\dot{u} + (u - U)X_u &= 0 \\
(Y_{\dot{v}} - m)\dot{v} + (Y_{\dot{r}} - mx_G)\dot{r} + vY_v + (Y_r - mU)r &= 0 \\
(N_{\dot{v}} - mx_G)\dot{v} + (N_{\dot{r}} - I_{zz})\dot{r} + vN_v + (N_r - mx_GU)r &= 0
\end{aligned}$$

So far, we have have not considered the propeller, the rudder or any other manoeuvring device like a side thruster. Changing the propeller revolutions by Δn will not induce a side force or a yaw moment, but will change the longitudinal force. Laying rudder to one side will induce a resistance. However, the resistance will be the (approximately) the same for positive and negative rudder angle δ. A linear force coefficient will give same absolute value, but opposite sign for $\pm\delta$. Thus the linear coefficient X_δ must be zero. However, there will be additional terms for side force and yaw moment:

$$\begin{aligned}
(X_{\dot{u}} - m)\dot{u} + (u - U)X_u + X_n\Delta n &= 0 \\
(Y_{\dot{v}} - m)\dot{v} + (Y_{\dot{r}} - mx_G)\dot{r} + vY_v + (Y_r - mU)r + Y_\delta\delta &= 0 \\
(N_{\dot{v}} - mx_G)\dot{v} + (N_{\dot{r}} - I_{zz})\dot{r} + vN_v + (N_r - mx_GU)r + N_\delta\delta &= 0
\end{aligned}$$

Now we non-dimensionalise:

$$m' = \frac{m}{\frac{1}{2}\rho L^3}, \quad I'_{zz} = \frac{I_{zz}}{\frac{1}{2}\rho L^5}, \quad x'_G = \frac{x_G}{L},$$

$$\Delta u' = \frac{u-U}{U}, \quad v' = \frac{v}{U}, \quad r' = \frac{rL}{U}, \quad \dot{u}' = \frac{\dot{u}L}{U^2}, \quad \dot{v}' = \frac{\dot{v}L}{U^2}, \quad \dot{r}' = \frac{\dot{r}L^2}{U^2}$$

$$X'_u = \frac{X_u}{\frac{1}{2}\rho L^2 U}, \quad X'_{\dot{u}} = \frac{X_{\dot{u}}}{\frac{1}{2}\rho L^3}, \quad X'_n = \frac{X_n}{\frac{1}{2}\rho L^3 U}$$

$$Y'_v = \frac{Y_v}{\frac{1}{2}\rho L^2 U}, \quad Y'_{\dot{v}} = \frac{Y_{\dot{v}}}{\frac{1}{2}\rho L^3}, \quad Y'_r = \frac{Y_r}{\frac{1}{2}\rho L^3 U}, \quad Y'_{\dot{r}} = \frac{Y_{\dot{r}}}{\frac{1}{2}\rho L^4}, \quad Y'_\delta = \frac{Y_\delta}{\frac{1}{2}\rho L^2 U^2},$$

$$N'_v = \frac{N_v}{\frac{1}{2}\rho L^3 U}, \quad N'_{\dot{v}} = \frac{N_{\dot{v}}}{\frac{1}{2}\rho L^4}, \quad N'_r = \frac{N_r}{\frac{1}{2}\rho L^4 U}, \quad N'_{\dot{r}} = \frac{N_{\dot{r}}}{\frac{1}{2}\rho L^5}, \quad N'_\delta = \frac{N_\delta}{\frac{1}{2}\rho L^3 U^2}$$

This yields then the non-dimensionalised equations:

$$\begin{aligned}
(X'_{\dot{u}} - m')\dot{u}' + \Delta u' X'_u + X'_n \Delta n &= 0 \\
(Y'_{\dot{v}} - m')\dot{v}' + (Y'_{\dot{r}} - m' x'_G)\dot{r}' + v' Y'_v + (Y'_r - m')r' + Y'_\delta \delta &= 0 \\
(N'_{\dot{v}} - m' x'_G)\dot{v}' + (N'_{\dot{r}} - I'_{zz})\dot{r}' + v' N'_v + (N'_r - m' x'_G)r' + N'_\delta \delta &= 0
\end{aligned}$$