Yaw stability

For ships (and submarines), we usually interpret 'yaw stability' (course keeping stability) as the ability to stay on a straight course without using controls (rudders). Yaw stability is desirable, but not mandatory. In fact, many modern ships with slender hulls (like containerships) are slightly yaw unstable.

The differential equations of motions (for ship without laying rudder):

$$\begin{aligned}(Y'_{\dot{v}} - m')\dot{v}' + (Y'_{\dot{r}} - m'x'_G)\dot{r}' + v'Y'_v + (Y'_r - m')r' &= 0 \\ (N'_{\dot{v}} - m'x'_G)\dot{v}' + (N'_{\dot{r}} - I'_{zz})\dot{r}' + v'N'_v + (N'_r - m'x'_G)r' &= 0\end{aligned}$$

have the following solution:

$$\begin{aligned}v'(t') &= v'_1 e^{\sigma'_1 t'} + v'_2 e^{\sigma'_2 t'} \\ r'(t') &= r'_1 e^{\sigma'_1 t'} + r'_2 e^{\sigma'_2 t'}\end{aligned}$$

The constants v'_1, v'_2, r'_1, and r'_2 can be determined from initial conditions for v, $\dot{v}$, r, and $\dot{r}$. Differentiation of the general solution and substitution into the differential equations yields the characteristic equation:

$$a\sigma'^2 + b\sigma' + c = 0$$

$$\begin{aligned}a &= (N'_{\dot{r}} - I'_{zz})(Y'_{\dot{v}} - m') - (Y'_{\dot{r}} - m'x'_G)(N'_{\dot{v}} - m'x'_G) \\ b &= (N'_{\dot{r}} - I'_{zz})Y'_v + (Y'_{\dot{v}} - m')(N'_r - m'x'_G) \\ &= -(Y'_r - m')(N'_{\dot{v}} - m'x'_G) - (Y'_{\dot{r}} - m'x'_G)N'_v \\ c &= (N'_r - m'x'_G)Y'_v - (Y'_r - m')N'_v\end{aligned}$$

The solutions of this quadratic equation are:

$$\sigma'_{1,2} = \frac{1}{2}\left(-\frac{b}{a} \pm \sqrt{\left(\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)}\right)$$

Yaw stability requires that both σ'_1 and σ'_2 have negative real parts. (Then both v' and r' will go to zero for increasing time, i.e. the ship will eventually assume again a straight path.) This condition is fulfilled if A, B, and C all have same sign. We generally have for ships:

Y'_v	negative and large	N'_v	uncertain sign and small
$Y'_{\dot{v}}$	negative and large	$N'_{\dot{v}}$	uncertain sign and small
Y'_r	uncertain sign and small	N'_r	negative and large
$Y'_{\dot{r}}$	uncertain sign and small	$N'_{\dot{r}}$	negative and large
m'	positive and large	I'_{zz}	positive and large
x'_G	uncertain sign and small		

Thus coefficients a and b are always positive for ships. Yaw stability requires than positive c, respectively:

$$\frac{N'_r - m'x'_G}{Y'_r - m'} - \frac{N'_v}{Y'_v} > 0$$