Nomoto Equation

The yaw stability index is not always easy to evaluate particularly in early design, as it involves differences of numbers of similar magnitude. Among the simpler models, Nomoto's model is most notable. Nomoto's model employ's two constants K and T which are easily estimated from sea trials or model tests.

The differential equations of motions for side force and yaw moment are:

$$\begin{aligned}(Y'_{\dot{v}} - m')\dot{v}' + (Y'_{\dot{r}} - m'x'_G)\dot{r}' + v'Y'_v + (Y'_r - m')r' + Y'_\delta\delta &= 0 \\ (N'_{\dot{v}} - m'x'_G)\dot{v}' + (N'_{\dot{r}} - I'_{zz})\dot{r}' + v'N'_v + (N'_r - m'x'_G)r' + N'_\delta\delta &= 0\end{aligned}$$

These equations can be re-written (here in dimensional form) as:

$$\begin{aligned}T_1T_2\ddot{r} + (T_1 + T_2)\dot{r} + r &= K\delta + KT_3\dot{\delta} \\ T_1T_2\ddot{v} + (T_1 + T_2)\dot{v} + v &= K_v\delta + K_vT_4\dot{\delta}\end{aligned}$$

The first equation expresses the relationship between turning rate r and rudder angle δ. The equation is commonly called Nomoto's second-order equation. The coefficients follow from:

$$\begin{aligned}T_1T_2 &= \frac{(Y_{\dot{v}} - m)(N_{\dot{r}} - I_{zz}) - (Y_{\dot{r}} - mx_G)(N_{\dot{v}} - mx_G)}{Y_v(N_r - mx_GU) - N_v(Y_r - mU)} \\ T_1 + T_2 &= \frac{(Y_{\dot{v}} - m)(N_r - mx_GU) + (N_{\dot{r}} - I_{zz})Y_v}{Y_v(N_r - mx_GU) - N_v(Y_r - mU)} \\ T_3 &= \frac{(N_{\dot{v}} - mx_G)Y_\delta - (Y_{\dot{v}} - m)N_\delta}{N_vY_\delta - Y_vN_\delta} \\ T_4 &= \frac{(N_{\dot{r}} - I_{zz})Y_\delta - (Y_{\dot{r}} - mx_G)N_\delta}{(N_r - mx_GU)Y_\delta - (Y_r - mU)N_\delta} \\ K &= \frac{N_vY_\delta - Y_vN_\delta}{(N_r - mx_GU)Y_v - (Y_r - mU)N_v} \\ K_v &= \frac{(N_r - mx_GU)Y_\delta - (Y_r - mU)N_\delta}{(N_r - mx_GU)Y_v - (Y_r - mU)N_v}\end{aligned}$$

Since $|T_2| \ll |T_1|$, the Nomoto's second-order equation can be simplified to Nomoto's first-order equation:

$$T\dot{r} + r = K\delta$$

K is a rudder effectiveness constant and T is a time constant (responsiveness)

A simple case will illustrate the meaning of K and T: The rudder is put suddenly to a rudder angle δ, where it is kept fixed. Then the solution of Nomoto's equation gives:

$$r = K\delta(1 - e^{-t/T})$$

Thus the yaw rate r increases with time, but at an exponentially decaying rate dependent on T, approaching a steady value $K\delta$. A larger K produces thus a larger turning rate for given rudder angle. A larger T does not influence the final turning rate, but the time required to achieve a turning rate. Non-dimensional K and T are good parameters to compare for ships in design and easy to determine in model or full-scale trials.