Nonlinear manoeuvring model

Linear theory has severe limitations:

– It fails to predict accurately the characteristics of many manoeuvres.

– It cannot predict the manoeuvres of yaw unstable ships.

– It decouples the surge equation from the sway and yaw equations. It does not reflect the speed reduction of ships in turns (in reality small for slow tankers, but up to 40% for fast ships like combatants).

To overcome these limitations, manoeuvring models must be extended to include non-linear terms. Often velocity terms up to third order in the Taylor expansions are considered in non-linear models, while higher-order terms and cross coupling terms of velocity and acceleration terms are regarded as insignificant. For the sway equation, port/starboard symmetry yields $Y'_u = Y'_{uu} = Y'_{uuu} = 0$. Several other terms are zero. (E.g. the sway force must respect the sign of the sway velocity, thus $Y'_{vv} = 0$.) The remaining terms yield for example:

$$\begin{aligned}
&(Y'_{\dot v} - m')\dot v' + (Y'_{\dot r} - m'x'_G)\dot r' + Y'_v v' + (Y'_r - m')r' + Y'_\delta \delta \\
&+Y'_0 + Y'_{0u}\Delta u' + Y'_{0uu}(\Delta u')^2 + \frac{1}{6}Y'_{vvv}v'^3 + \frac{1}{2}Y'_{vrr}v'r'^2 + \frac{1}{2}Y'_{v\delta\delta}v'\delta'^2 + Y'_{vu}v'\Delta u' + \frac{1}{2}Y'_{vuu}v'(\Delta u')^2 \\
&+\frac{1}{6}Y'_{rrr}r'^3 + \frac{1}{2}Y'_{rvv}r'v'^2 + \frac{1}{2}Y'_{r\delta\delta}r'\delta'^2 + Y'_{ru}r'\Delta u' + \frac{1}{2}Y'_{ruu}r'(\Delta u')^2 + \frac{1}{6}Y'_{\delta\delta\delta}\delta'^3 \\
&+\frac{1}{2}Y'_{\delta vv}\delta' v'^2 + \frac{1}{2}Y'_{\delta rr}\delta' r'^2 + Y'_{\delta u}\delta'\Delta u' + \frac{1}{2}Y'_{\delta uu}\delta'(\Delta u')^2 + Y'_{vr\delta}v'r'\delta' = 0
\end{aligned}$$

$$\begin{aligned}
&(N'_{\dot v} - m'x'_G)\dot v' + (N'_{\dot r} - I'_{xx})\dot r' + N'_v v' + (N'_r - m'x'_G)r' + N'_\delta \delta \\
&+N'_0 + N'_{0u}\Delta u' + N'_{0uu}(\Delta u')^2 + \frac{1}{6}N'_{vvv}v'^3 + \frac{1}{2}N'_{vrr}v'r'^2 + \frac{1}{2}N'_{v\delta\delta}v'\delta'^2 + N'_{vu}v'\Delta u' + \frac{1}{2}N'_{vuu}v'(\Delta u')^2 \\
&+\frac{1}{6}N'_{rrr}r'^3 + \frac{1}{2}N'_{rvv}r'v'^2 + \frac{1}{2}N'_{r\delta\delta}r'\delta'^2 + N'_{ru}r'\Delta u' + \frac{1}{2}N'_{ruu}r'(\Delta u')^2 + \frac{1}{6}N'_{\delta\delta\delta}\delta'^3 \\
&+\frac{1}{2}N'_{\delta vv}\delta v'^2 + \frac{1}{2}N'_{\delta rr}\delta r'^2 + N'_{\delta u}\delta\Delta u' + \frac{1}{2}N'_{\delta uu}\delta(\Delta u')^2 + N'_{vr\delta}v'r'\delta = 0
\end{aligned}$$

The terms with subscript 0 represent the influence of the propeller.

Following similar arguments, the surge equation is written as:

$$\begin{aligned}
&(X'_{\dot u} - m')\dot u' + X'_u\Delta u' + X'_n\Delta n' \\
&+X'_{uu}(\Delta u')^2 + X'_{nn}(\Delta n')^2 + \frac{1}{2}X'_{vv}v'^2 + \left(\frac{1}{2}X'_{rr}r'^2 + m'x'_G\right)r'^2 + \frac{1}{2}X'_{\delta\delta}\delta^2 + \frac{1}{2}X'_{vvu}v'^2\Delta u' \\
&+\frac{1}{2}X'_{rru}r'^2\Delta u' + \frac{1}{2}X'_{\delta\delta u}\delta^2\Delta u' + (X'_{vr} + m')v'r' + X'_{v\delta}v'\delta + X'_{r\delta}r'\delta + X'_{vru}v'r'\Delta u' \\
&+X'_{v\delta u}v'\delta\Delta u' + X'_{r\delta u}r'\delta\Delta u' = 0
\end{aligned}$$

Because of symmetry, only terms with even powers of v', r', and δ appear, because the same longitudinal force (resistance) should occur for either port or starboard motion. For single-screw ships, the longitudinal forces for rudder angles to port and starboard are not exactly symmetric, but the differences are neglected here. The added resistance terms due to ship motions and rudder action will slow the ship down.

The three equations for surge, sway, and yaw can be solved for the accelerations $\dot u'$, $\dot v'$, and $\dot r'$. The resulting equations can be integrated numerically for any given time history of $\delta(t')$ to predict ship manoeuvres, e.g. using Euler integration of the form $u'(t' + \Delta t') = u'(t') + \dot u'(t')\Delta t'$.

This simple integration scheme is usually sufficient due to the large inertia of the ship and the subsequent small accelerations. Integrating in turn the velocities yields then similarly updates for position and yaw angle.