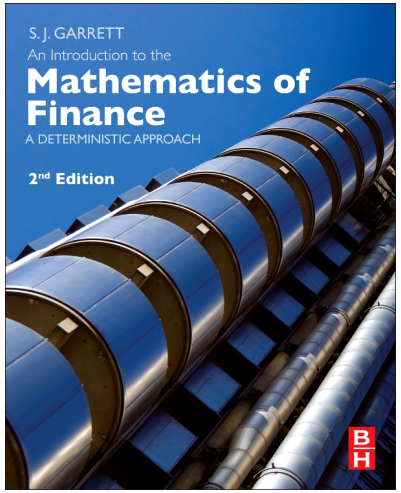
Essential Topic: Stochastic interest rates

Chapter 12



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Motivation

- We have so far been concerned with fixed, piecewise constant, or term-structured interest rates.
- In each case, the rates have been known in advance and all cash-flow analyses have been *deterministic*.
- However, in reality interest rates are not known in advance. This motivates the use of *stochastic* models.
- A simplification is to assume the *varying interest rate model* with *i.i.d.* rates across the years.

Varying interest rate model

- Under the varying interest rate model, the single rate that applies in each year is unknown in advance. However, once it has been determined (at the start of the year) it is fixed throughout that year.
- Furthermore, we assume that the interest rates that apply in each year are *independent and identically distributed* (i.i.d.) across all years.
- Essentially we are assuming that the interest rate in each year takes an *unknown* realization of a *known* distribution.
- Properties of the distribution of the random rate I are known in advance and

$$E[I] = j \qquad\qquad \mathrm{var}(I) = s^2.$$

Accumulation of a single investment

- Consider a unit investment made at time $t = 0$ which will accumulate under the i.i.d. varying interest rate model for n years.
- The *accumulation after n years* is denoted S_n.
- As the interest rate that applies in each year is random $\{i_t\}$, the accumulation S_n will also be random.
- We seek to determine the mean and variance of S_n in terms of properties of the interest rate distribution.

Accumulation of a single investment

- The expected value of S_n is calculated as

$$
\begin{aligned}
E[S_n] =& E\left[(1+I_1)\times(1+I_2)\times\cdots\times(1+I_n)\right] \\
=& E\left[1+I_1\right]\times E\left[1+I_2\right]\times\cdots\times E\left[1+I_n\right] \text{ (ind.)} \\
=& \prod_{t=1}^{n}(1+E[I_t]) \\
\implies E[S_n] =& (1+j)^n \text{ (if identically distributed)}
\end{aligned}
$$

- This gives the expected value of the accumulation under the i.i.d. varying interest rate model.

Accumulation of a single investment

- The variance of S_n is calculated from the standard expression

$$\mathrm{var}(S_n) = E\left[S_n^2\right] - E\left[S_n\right]^2$$

- We know that $E[S_n]^2 = (1+j)^{2n}$
- Further

$$\begin{aligned}
E[S_n^2] =& E\left[(1+I_1)^2 \times (1+I_2)^2 \times \cdots \times (1+I_n)^2\right] \\
=& E\left[(1+I_1)^2\right] \times E\left[(1+I_2)^2\right] \times \cdots \times E\left[(1+I_n)^2\right] \text{ (ind.)} \\
=& \prod_{t=1}^{n} \left(1 + 2E[I_t] + E[I_t^2]\right) \\
=& (1+2j+s^2+j^2)^n \text{ (if identically distributed)}
\end{aligned}$$

- The expression for the variance of the accumulation under the model is then

$$\mathrm{var}(S_n) = \left((1+j)^2 + s^2\right)^n - (1+j)^{2n}.$$

Example

The annual interest earned on an investment each year is a collection of i.i.d. random variables with mean 0.05 and standard deviation 0.15. Calculate the expected value and standard deviation of the accumulation of £10 after 5 years.

Answer

We have that $j = 0.05$ and $s^2 = 0.15^2$. The expected value and standard deviation of the accumulation are then

$$\begin{aligned} E\left[10S_5\right] =& 10 \times 1.05^5 = £12.76 \\ \operatorname{var}\left(10S_5\right) =& 10^2 \times \left((1.05^2 + 0.15^2)^5 - 1.05^{10}\right) = (4.16)^2 \\ \implies \text{s.d.} =& £4.16 \end{aligned}$$

Accumulation of annual investments

- We now consider the n-year accumulation of a unit investment made at the start of each year for n years under the i.i.d. varying interest rate model.
- This accumulation is a random variable, A_n, such that

$$\begin{aligned} A_n &= (1+I_1)(1+I_2)(1+I_3)\dots(1+I_n) \\ &\quad +(1+I_2)(1+I_3)\dots(1+I_n) \\ &\quad +(1+I_3)\dots(1+I_n) \\ &\quad \vdots \\ &\quad +(1+I_n). \end{aligned}$$

- To make progress towards expressing properties of A_n in terms of properties of I, we note the recursive relationship

$$A_n = (1+I_n)(1+A_{n-1}).$$

Accumulation of annual investments

- We take the expectation of the recursive relationship to find a recursive relationship for the *expected value of the n-year accumulation*, μ_n

$$\mu_n = E[A_n] = E[(1+I_n)(1+A_{n-1})] \text{ (ind.)}$$

- For identically distributed rates, this reduces to

$$\begin{aligned}\mu_n =& (1+j)(1+\mu_{n-1}) \\ =& \ddot{s}_{\overline{n|}} \quad \text{at rate } j\end{aligned}$$

- Similarly the *second moment*, $m_n = E[A_n^2]$, can be formed in terms of a recursive relationship

$$\begin{aligned}m_n &= \left[\text{var}(1+I_n) + E[1+I_n]^2\right] E\left[1+2A_{n-1}+A_{n-1}^2\right] \text{ (ind.)} \\ &= \left[s^2 + (1+j)^2\right]\left[1+2\mu_{n-1}+m_{n-1}\right] \text{ (iden. dist.)}\end{aligned}$$

- The *variance of the n-year accumulation* is then

$$\text{var}(A_n) = m_n - \mu_n^2.$$

EXAMPLE

The annual rate of interest earned on a particular fund is a random variable with i.i.d. annual rates. In all years the expected value of interest is 5% and the standard deviation 8%. Calculate the mean and standard deviation of the total accumulated amount at the end of 2 years if £1 is invested at the start of the first and second years.

Answer

We have that $j = 0.05$ and $s^2 = 0.08^2$ and require $E[A_2]$ and $\sqrt{\text{var}(A_2)}$.

$$\begin{aligned}
E[A_2] &= \mu_2 = \ddot{s}_{\overline{2|}} = £2.15 \\
\text{var}(A_2) &= m_2 - \mu_2^2 = \left(s^2 + (1+j)^2\right)(1 + 2\mu_1 + m_1) - 2.15^2 \\
\text{with } m_1 &= (1+j)^2 + s^2 = 1.1089 \text{ and } \mu_1 = 1.05 \\
\implies \text{var}(A_2) &= \left(0.08^2 + 1.05^2\right)(1 + 2 \times 1.05 + 1.1089) - 2.15^2 \\
&= (£0.21)^2.
\end{aligned}$$

LOG-NORMAL DISTRIBUTION

- An important distribution for the i.i.d. varying interest rate model is the *log-normal distribution*

$$1 + I \sim \ln N(\mu, \sigma^2)$$

  where μ and σ are parameters of the distribution.
- The properties of the distribution are such that

$$\begin{aligned}\ln S_n &= \sum_{t=1}^{n} \ln(1 + I_t) \\ &\sim N(n\mu, n\sigma^2)\end{aligned}$$

- The distribution parameters, μ and σ^2, are linked to j and s^2 via

$$1 + j = e^{\mu + \frac{\sigma^2}{2}} \qquad \text{and} \qquad s^2 = e^{2\mu + \sigma^2} \times \left(e^{\sigma^2} - 1\right).$$

- These form two simultaneous equations for the distribution parameters in terms of j and s.

Example

A single investment of £2000 is made into a fund that pays interest under the i.i.d. varying interest rate model with a log-normal distribution. The annual rates have mean 5% and standard deviation 10%.

a.) Calculate the expected value of the accumulation after 10 years.

b.) Calculate the probability that the accumulated value after 10 years will be less than 90% of the expected value from a.).

Example

Answer

a.) We have $j = 5\%$ and so

$$E[2000S_{10}] = 2000 \times (1.05)^{10} = £3257.79$$

b.) We can determine that $\mu = 0.0443$ and $\sigma^2 = 0.095^2$. Therefore

$$\begin{aligned}
P\left(S_{10} < 0.90 \times 1.05^{10}\right) &= P\left(S_{10} < 1.4660\right) \\
&= P\left(\ln S_{10} < \ln 1.4660\right) \\
&= P\left(\frac{\ln S_{10} - 10\mu}{\sqrt{10\sigma^2}} < \frac{\ln 1.4660 - 10\mu}{\sqrt{10\sigma^2}}\right) \\
&= P\left(Z < \frac{\ln 1.4660 - 10 \times 0.0433}{\sqrt{10 \times 0.095^2}}\right) \\
&= P\left(Z < -0.201\right) \approx 42\%
\end{aligned}$$

SUMMARY

- Under the *i.i.d. varying interest rate model* the interest rate in any year is unknown in advance but is a realization of the random variable I with known distribution. We have $E[I] = j$ and $\mathrm{var}(I) = s^2$.
- S_n denotes the *n-year accumulation of a unit investment* under the i.i.d. varying interest rate model. We have

$$E[S_n] = (1+j)^n \quad \text{and} \quad \mathrm{var}(S_n) = \left((1+j)^2 + s^2\right)^2 - (1+j)^{2n}.$$

- A_n denotes the *n-year accumulation of an annual unit investment made in advance*. We have

$$E[A_n] = \mu_n = \ddot{s}_{\overline{n}|} \quad \text{and} \quad \mathrm{var}(A_n) = m_n - \mu_n^2$$

where $m_n = \left[s^2 + (1+j)^2\right]\left[1 + 2\mu_{n-1} + m_{n-1}\right]$.

- The log-normal distribution is an important example of a distribution for I

$$1 + I \sim \ln N(\mu, \sigma^2) \implies \ln S_n \sim N(n\mu, n\sigma^2).$$