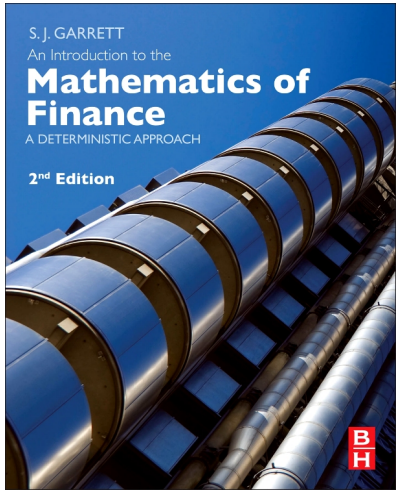
Essential Topic: Loan repayment schedules

Chapter 5



Annual repayments

- An important application of annuity notation and compound interest tables is loan calculations.
- In the simplest form, an investor lends an amount L in return for a series of n level payments of amount X, calculated on the basis of an effective interest rate of i.
- The present value of the money lend is simply L and the present value of the repayments is $Xa_{\overline{n}|}$.
- The *equation of the value* of the loan is therefore

$$L = Xa_{\overline{n}|}$$

Example

- If a 25-year loan of £200,000 is issued on the basis of an interest rate of 4% per annum, the annual repayment is calculated as

$$X = \frac{200,000}{a_{\overline{25}|}} = £12,802.38$$

- Implicit in the above is that repayments are in *arrears*. This represents that interest is paid for the prior use of capital.
- In each year, X covers the 4% interest on the capital outstanding and a contribution towards the capital repayment. These are such that the capital is entirely repaid after 25 years.
- The proportion of X which represents capital repayment grows with time. This reflects that the capital outstanding is being paid down year on year and so the interest component reduces.

Repayment schedules

- A *repayment schedule* illustrates the split of each repayment amount between interest and capital repayment.
- The above example has the following repayment schedule:

Year	Capital outstanding (at start)	Interest paid (at end)	Capital repaid (at end)
1st	200,000.00	8,000.00	4,802.38
2nd	195,197.62	7,807.90	4,994.47
...	...	...	...
12th	135,233.47	5,409.34	7,393.04
...	...	...	...
24th	24,147.13	965.89	11,836.49
25th	12,310.64	492.43	12,309.95

- Note that in the final year, the capital repaid at the end of the year matches the capital outstanding at the start of the year (subject to rounding errors).

pTHLY REPAYMENTS

- In practice repayments are made more frequently than annually, for example monthly.
- This can be incorporated easily into loan calculations using $p$thly paid annuities.
- For example, the *monthly* repayments on a 25-year loan of £200,000 issued on the basis of an interest rate of $i = 4\%$ per annum are x, such that

$$12x \times a_{\overline{25|}}^{(12)} = 200,000 \text{ i.e. } x = £1,047.79$$

- Note that $12x = £12,578.49 < X = £12,802.38$, and so the total annual repayment is lower when paid monthly. This reflects that the capital amount is being brought down sooner when paid more frequently than annually.
- Repayment schedules can be constructed as before but with monthly time steps.

Capital outstanding

- It is often useful to be able to determine the capital outstanding at any particular time without constructing a lengthy repayment schedule.
- The capital outstanding at any time can be calculated directly using either the *retrospective* or *prospective* methods.
- The retrospective method "looks backwards" and calculates the capital outstanding as the accumulation of the original loan minus the accumulation of all repayments to date.
- The prospective method "looks forwards" and calculates the capital outstanding as the present value of all future repayments under the original basis of the loan.

EXAMPLE

Calculate the capital outstanding at time $t = 11$ (start of the twelfth year) of a 25-year loan of £$200,000$, issued on the basis of an interest rate of 4% per annum.

Answer

The annual repayment was previously calculated to be $X = £12,802.38$.

- Retrospective method:

$$\begin{aligned}\text{Capital outstanding} =& 200,000 \times (1+i)^{11} - 12,802.38 \times s_{\overline{11}|} \\ =& £135,231.98\end{aligned}$$

- Prospective method:

$$\begin{aligned}\text{Capital outstanding} =& 12,802.38 a_{\overline{14}|} \\ =& £135,232.82\end{aligned}$$

Agreement is found with the previous repayment schedule (£135,233.47) subject to rounding errors.

Example

A 10-year loan of £100,000 is to be repaid by monthly payments calculated on the basis of $i = 3\%$ per annum. Immediately after the 24th repayment, the borrower requests that he repay the loan in full. Calculate the amount then due.

Answer

We demonstrate the retrospective and prospective methods, but usually only one method is required. Both approaches require the monthly payment amount, x

$$x = \frac{100,000}{12a_{\overline{10|}}^{(12)}} = £963.74$$

Example

- Retrospective method:

$$\begin{aligned}\text{Capital outstanding} =& 100,000 \times (1+i)^2 - 12 \times 963.74 \times s_{\overline{2|}}^{(12)} \\ =& £82,291.96\end{aligned}$$

- Prospective method:

$$\begin{aligned}\text{Capital outstanding} =& 12 \times 963.74 \times a_{\overline{8|}}^{(12)} \\ =& £82,292.31\end{aligned}$$

These two values are identical subject to rounding errors.

Summary

- Loan calculations are an important use of annuity notion and compound interest tables.
- The full repayment schedule can be calculated from a loan's basis using an iterative approach year on year.
- However, this is typically impractical and either of the *retrospective* or *prospective* approaches can be used to directly calculate the capital outstanding at any time.
- The retrospective approach involves accumulating all previous cash flows of the loan (including the initial capital lent).
- The prospective approach involves computing the present value of all future payments under the original basis of the loan.