

Chapter 32

MATHIEU FUNCTIONS

When PDEs such as Laplace's, Poisson's, and the wave equation are solved with cylindrical or spherical boundary conditions by separating variables in a coordinate system appropriate to the problem, we find radial solutions, which are usually the Bessel functions of Chapter 14, and angular solutions, which are $\sin m\varphi$, $\cos m\varphi$ in cylindrical cases and spherical harmonics in spherical cases. Examples are electromagnetic waves in resonant cavities, vibrating circular drumheads, and cylindrical wave guides.

When in cylindrical problems the circular boundary condition is changed to elliptical we are led to what are now known as Mathieu functions, which, might alternatively have been called elliptic cylinder functions. In fact, in 1868 Mathieu developed the leading terms of series solutions of the vibrating elliptical drumhead, and Whittaker and others in the early 1900s derived higher-order terms as well.

The goal of this chapter is to give an introduction to the rich and complex properties of Mathieu functions.

32.1 MATHIEU DIFFERENTIAL EQUATIONS

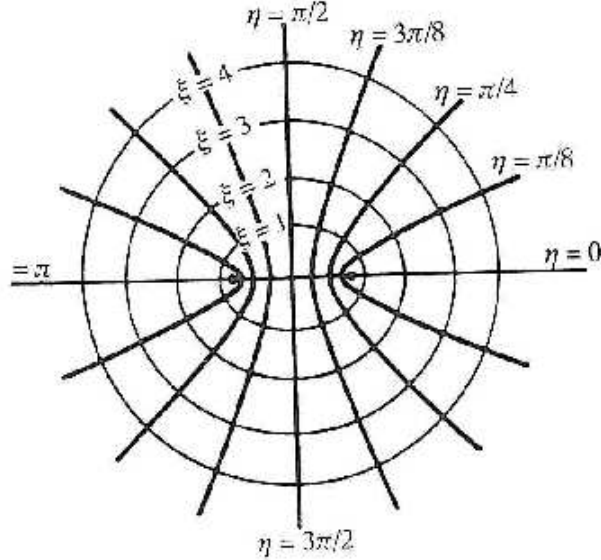
To see how Mathieu functions arise in problems with elliptical boundary conditions, we need to formulate the relevant PDEs in elliptical coordinates. The essential steps in this process are to identify the properties of the coordinate system, obtain the Laplacian operator in the elliptical coordinates, and to carry out a separation of variables in the elliptical coordinates.

ELLIPTICAL COORDINATE SYSTEM

Elliptical cylinder coordinates ξ , η , z , which are appropriate for elliptical boundary conditions, are expressed in rectangular coordinates as

$$\begin{aligned}x &= c \cosh \xi \cos \eta, & y &= c \sinh \xi \sin \eta, & z &= z, \\0 &\leq \xi < \infty, & 0 &\leq \eta < 2\pi,\end{aligned}\tag{32.1}$$

with $c > 0$ a parameter that we will later identify as locating the foci of the elliptical coordinates. The essential characteristics of this coordinate system are illustrated by the lines of constant ξ and constant η on the plane corresponding to an arbitrary value of z , as shown in Fig. 32.1. The figure suggests that the lines of constant ξ

Figure 32.1: Elliptical coordinates ξ, η .

are **confocal** ellipses (those with common foci), and that the lines of constant η are quarter-hyperbolas with the same foci. A quarter-hyperbola is half of one hyperbolic branch; the half-branch corresponding to a given value of η is the part of the branch in the quadrant corresponding to η .

To confirm that the lines of constant ξ (for fixed z) are confocal ellipses, we can combine the x and y equations of Eq. (32.1), taking for convenience $z = 0$ (the xy -plane), obtaining

$$\frac{x^2}{c^2 \cosh^2 \xi} + \frac{y^2}{c^2 \sinh^2 \xi} = 1. \quad (32.2)$$

For any fixed value of ξ , Eq. (32.2) describes an ellipse with foci at $(\pm c, 0)$ and with major and minor half-axes

$$a = c \cosh \xi, \quad b = c \sinh \xi, \quad (32.3)$$

respectively. Thus, we have identified the parameter c as half the distance between the foci of the confocal set of ellipses.

We next look at the ratio b/a of the axes of our ellipses:

$$\frac{b}{a} = \tanh \xi = \sqrt{1 - \frac{1}{\cosh^2 \xi}} \equiv \sqrt{1 - e^2}. \quad (32.4)$$

where $e = 1/\cosh \xi$, which must lie in the range $0 \leq e \leq 1$, is known as the **eccentricity** of the ellipse. As $\xi \rightarrow \infty$, $e \rightarrow 0$ and the ellipses become circles, as can be seen in Fig. 32.1. As $\xi \rightarrow 0$, the ellipse becomes more elongated until, at $\xi = 0$, it has shrunk to the line segment between the foci.

The limiting behavior of the elliptic coordinates as $c \rightarrow 0$ is that essentially all nonzero (x, y) correspond to very large ξ values; in this limit $e = 0$ and the lines of

constant ξ become circles. To make the limiting process explicit, we could replace $c \cosh \xi \approx c \sinh \xi$ by ρ , thereby recovering the usual circular polar coordinates.

Moving next to the lines of constant η , we return to Eq. (32.1) and eliminate ξ from the x and y equations, obtaining

$$\frac{x^2}{c^2 \cos^2 \eta} - \frac{y^2}{c^2 \sin^2 \eta} = 1, \quad (32.5)$$

For any fixed η Eq. (32.5) describes a hyperbola with foci at $(\pm c, 0)$. The different partial branches are reached by choosing the signs of x and y .

Differentiating the equation for the constant- ξ ellipse, we obtain

$$\frac{x dx}{\cosh^2 \xi} + \frac{y dy}{\sinh^2 \xi} = 0,$$

equivalent to

$$\frac{dy}{dx} = -\frac{x \sinh^2 \xi}{y \cosh^2 \xi} \quad \text{or} \quad (dx, dy) = (-y \cosh^2 \xi, x \sinh^2 \xi), \quad (32.6)$$

the formula for a vector tangent to the ellipse at the point (x, y) .

From the equation for the constant- η hyperbola, we obtain in a similar fashion

$$\frac{x dx}{\cos^2 \eta} - \frac{y dy}{\sin^2 \eta} = 0, \quad \text{leading to} \quad (dx, dy) = (y \cos^2 \eta, x \sin^2 \eta). \quad (32.7)$$

If we now take the scalar product of these vectors at any point x, y , we get

$$-y^2 \cosh^2 \xi \cos^2 \eta + x^2 \sinh^2 \xi \sin^2 \eta = -\frac{y^2 x^2}{c^2} + \frac{x^2 y^2}{c^2} = 0. \quad (32.8)$$

This result means that these confocal ellipses and hyperbolas have tangents that are orthogonal at their intersection points, and we therefore have an orthogonal coordinate system, in the sense of Section 3.10.

To extract the scale factors h_ξ, h_η from the differentials of the elliptical coordinates

$$\begin{aligned} dx &= c \sinh \xi \cos \eta d\xi - c \cosh \xi \sin \eta d\eta, \\ dy &= c \cosh \xi \sin \eta d\xi + c \sinh \xi \cos \eta d\eta, \end{aligned} \quad (32.9)$$

we sum their squares, finding

$$\begin{aligned} dx^2 + dy^2 &= c^2(\sinh^2 \xi \cos^2 \eta + \cosh^2 \xi \sin^2 \eta)(d\xi^2 + d\eta^2) \\ &= c^2(\cosh^2 \xi - \cos^2 \eta)(d\xi^2 + d\eta^2) \equiv h_\xi^2 d\xi^2 + h_\eta^2 d\eta^2 \end{aligned} \quad (32.10)$$

and yielding

$$h_\xi = h_\eta = c(\cosh^2 \xi - \cos^2 \eta)^{1/2}. \quad (32.11)$$

Note that there is no cross term involving $d\xi d\eta$, showing again that we are dealing with orthogonal coordinates.

Since it is our objective to solve PDE's in the elliptical coordinates, we proceed now to develop the Laplacian. Letting q_1, q_2, q_3 stand for ξ, η, z , we have $h_1 = h_2 = c(\cosh^2 \xi - \cos^2 \eta)^{1/2}$ and $h_3 = 1$. Then, noting that the ratio h_1/h_2 is unity and that neither h_1 nor h_2 depend upon q_3 , we find that the general expression for the Laplacian in curvilinear coordinates, Eq. (3.142), reduces to

$$\nabla^2 = \frac{1}{c^2(\cosh^2 \xi - \cos^2 \eta)} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) + \frac{\partial^2}{\partial z^2}. \quad (32.12)$$

SEPARATION OF VARIABLES IN PDEs

We begin our study of Mathieu equations by considering an example in which these ODEs naturally occur.

Example 32.1.1. Elliptical Drum

An elliptical drumhead with vertical displacement $z = z(x, y, t)$ has oscillations governed by the wave equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}, \quad (32.13)$$

with the wave velocity squared given by $v^2 = T/d$, where the drumhead is under tension T (force per unit length at the boundary) and d is its mass per unit area, with both T and d assumed constant. We first separate the harmonic time dependence, writing

$$z(x, y, t) = u(x, y)w(t),$$

where $w(t) = \cos(\omega t + \delta)$, with ω the frequency and δ a constant phase. Substituting this functional form for $z(x, y, t)$ into Eq. (32.13) yields

$$\frac{1}{u} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{1}{v^2 w} \frac{\partial^2 w}{\partial t^2} = -\frac{\omega^2}{v^2} = -k^2 = \text{constant}, \quad (32.14)$$

which is the two-dimensional Helmholtz equation for the displacement u . Since the drumhead is elliptical, it is desirable to use elliptical coordinates such that $\xi = \xi_0$ describes the boundary of the drumhead, on which the vertical displacement will have the value zero. We therefore use Eq. (32.12) to convert the Laplacian ∇^2 to the elliptical coordinates, then dropping the z -coordinate. This gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = \frac{1}{c^2(\cosh^2 \xi - \cos^2 \eta)} \left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} \right) + k^2 u = 0,$$

which can be rearranged to the usual expression for the Helmholtz equation in elliptical ξ, η coordinates:

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + c^2 k^2 (\cosh^2 \xi - \cos^2 \eta) u = 0. \quad (32.15)$$

Lastly, we separate the variables ξ and η in our Helmholtz equation, writing $u(\xi, \eta) = R(\xi)\Phi(\eta)$, reaching

$$\frac{1}{R} \frac{d^2 R}{d\xi^2} + c^2 k^2 \cosh^2 \xi = \lambda + \frac{1}{2} c^2 k^2, \quad (32.16)$$

$$c^2 k^2 \cos^2 \eta - \frac{1}{\Phi} \frac{d^2 \Phi}{d\eta^2} = \lambda + \frac{1}{2} c^2 k^2.$$

We have chosen $\lambda + c^2 k^2/2$ as the separation constant because we can move its second term to the left-hand side of each of these equations, obtaining $c^2 k^2 (\cosh^2 \xi - \frac{1}{2})$ and $c^2 k^2 (\cos^2 \eta - \frac{1}{2})$, which can then be written in the respective forms $\frac{1}{2} \cosh 2\xi$ and $\frac{1}{2} \cos 2\eta$. To bring our analysis to the usual notation, we now define a quantity

$$q = \frac{1}{4} c^2 k^2 = \frac{c^2 \omega^2}{4v^2}, \quad (32.17)$$

in terms of which the Φ equation assumes the form

$$\frac{d^2\Phi}{d\eta^2} + (\lambda - 2q \cos 2\eta) \Phi(\eta) = 0. \quad (32.18)$$

This ODE is known as the **Mathieu equation**, and in problems with elliptical symmetry the solutions which are nonsingular (with period π or 2π in the variable η) will normally be required. These periodic solutions are referred to as **Mathieu functions**. They are elliptic analogs of the sine and cosine functions that are the angular solutions in systems with circular symmetry. The Mathieu equation also has solutions that are nonperiodic; they are of less interest in physics and for the most part will not be discussed here.

The R equation of Eq. (32.16) has a form quite similar to Eq. (32.18):

$$\frac{d^2R}{d\xi^2} - (\lambda - 2q \cosh 2\xi) R(\xi) = 0. \quad (32.19)$$

In fact, it becomes a Mathieu equation with η replaced by $i\xi$. Because a similar change of variables relates the modified Bessel functions $I_n(x)$ and $K_n(x)$ to the ordinary Bessel functions $J_n(x)$ and $Y_n(x)$, the R equation is called the **modified Mathieu equation**. Thus, if we identify a specific solution to the Mathieu equation as $\Phi(\lambda, q; \eta)$, a corresponding solution of the modified Mathieu equation will be

$$R(\lambda, q; \xi) = \Phi(\lambda, q; i\xi). \quad (32.20)$$

The solutions $R(\lambda, q; \eta)$ that correspond (by $\eta \rightarrow i\xi$) with periodic functions Φ are referred to as **modified Mathieu functions**. It will be found that they are oscillatory but (for real ξ) not periodic.

Although we defer a detailed solution of this drum problem until after some general properties of Mathieu functions have been discussed, we can now sketch our solution strategy. Because the R and Φ equations have been obtained from the process of separation of variables, they must both contain the same value of λ and also a common value of q . Neither q nor λ is directly given by the input to our problem; in fact, different values of q will correspond to different oscillation frequencies, as can be seen from Eq. (32.17). As already mentioned, the angular functions $\Phi(\lambda, q; \eta)$ will need to be periodic, and for any given value of q this condition will only be possible for certain discrete λ values. In other words, Φ will be a solution to a q -dependent boundary value problem with eigenvalue λ .

We will also need to solve the R equation subject to the boundary condition $R(\lambda, q; \xi_0) = 0$ and subject to a requirement of continuity as the line segment $\xi = 0$ is crossed; note that the (ξ, η) values $(0, \eta)$ and $(0, -\eta)$ describe the same point. These solutions to the R equation are elliptical analogs to the Bessel functions J_n that arise in problems with circular symmetry. The situation is more complicated here, however, because the locations of the zeros of R depend upon both λ and q . Combining our discussions of the radial and angular equations, we see that it will be necessary to find values of λ and q that simultaneously solve both the angular and the radial boundary value problems. The practical difficulties of carrying out this process have in the past made it rather difficult to work with the Mathieu equation, but these difficulties have been overcome when the problem under study was sufficiently important. Fortunately, computers (and computer software) have now evolved to where Mathieu functions can be used with only moderate intellectual strain.

This example indicates how the interrelation of the eigenvalues λ and the continuous parameter q cause Mathieu functions to be among the most difficult special

functions used in physics. ■

Although the presence of the parameter q and its associated trigonometric term make it complicated to deal with Mathieu functions, they do have the redeeming feature that they are solutions of Sturm-Liouville systems, because the Mathieu equation is self-adjoint and we normally have appropriate boundary conditions.

Sometimes Mathieu functions arise in problems which do not have both a radial and an angular equation, in contrast to the previous example. One such situation is illustrated by the following example.

Example 32.1.2. The Quantum Pendulum

A plane pendulum of length l and mass m with gravitational potential $V(\theta) = -mgl \cos \theta$ is called a **quantum pendulum** if its wave function Ψ obeys the Schrödinger equation

$$-\frac{\hbar^2}{2ml^2} \frac{d^2\Psi}{d\theta^2} + [V(\theta) - E]\Psi = 0,$$

where the variable θ is the angular displacement from the vertical direction. For further details and illustrations we refer to Gutiérrez-Vega *et al.* in the Additional Readings. A boundary condition requires Ψ to be single-valued; that is, $\Psi(\theta + 2\pi) = \Psi(\theta)$. Substitution of

$$\theta = 2\eta, \quad \lambda = \frac{8Eml^2}{\hbar^2}, \quad q = -\frac{4m^2gl^3}{\hbar^2}$$

into the Schrödinger equation yields the Mathieu ODE with the boundary condition $\Psi(2(\eta + \pi)) = \Psi(2\eta)$. In other words, the boundary condition on the wave function expressed as a function of η is that it be periodic, with period π .

In this problem q does not appear as a parameter that must be chosen to make the radial and angular equations consistent, but is instead given by the problem input. And λ is obtained from a single Mathieu boundary-value problem and is proportional to the energy of the quantum state being described. ■

For many other applications involving Mathieu functions we refer to Ruby in the Additional Readings.

ALGEBRAIC FORM

It is sometimes useful to bring the Mathieu ODE to a form in which the coefficients are algebraic. This can be accomplished by making the so-called Lindemann-Stieltjes substitution $z = \cos^2 \eta$, with $dz/d\eta = -\sin 2\eta$. This substitution converts the Mathieu ODE to an equation in z . Writing

$$\frac{d}{d\eta} = \frac{dz}{d\eta} \frac{d}{dz} = -\sin 2\eta \frac{d}{dz} \quad \text{and} \quad \frac{d^2}{d\eta^2} = -2 \cos 2\eta \frac{d}{dz} + \sin^2 2\eta \frac{d^2}{dz^2},$$

we obtain

$$4z(1-z) \frac{d^2\Phi}{dz^2} + 2(1-2z) \frac{d\Phi}{dz} + [\lambda + 2q(1-2z)] \Phi = 0. \quad (32.21)$$

Before transformation, the Mathieu equation had no singularities in the finite complex plane; only an irregular singularity at infinity. The transformation leading to Eq. (32.21) has produced an equation with regular singularities at $z = 0$ and $z = 1$; an irregular singularity remains at infinity. By comparison, the hypergeometric ODE has three regular singularities. We note in closing that not all ODEs with two regular singularities and one essential singularity can be transformed into an ODE of the Mathieu type.

Exercises

- 32.1.1.** Find equations giving the elliptic coordinates ξ and η as explicit functions of x and y .

32.2 GENERAL PROPERTIES

Now that we have introduced the Mathieu ODE and identified some problems for which it is useful, we proceed to a survey of the general properties of solutions to the Mathieu ODE, with particular emphasis on the periodic solutions that are needed for both the problems we have already introduced as examples.

EXPANSION ABOUT $q = 0$

Although one often requires solutions to Mathieu's ODE for large values of the parameter q , valuable insight can be obtained by studying the behavior of this ODE for small q , writing all the quantities appearing in the ODE as expansions about $q = 0$. If we set $q = 0$ and let η be an angular coordinate with range $0 \leq \eta < 2\pi$, our ODE becomes

$$\frac{d^2\psi}{d\eta^2} + \lambda\psi = 0. \quad (32.22)$$

After applying the periodic boundary conditions, this equation has solutions

$$\begin{aligned} c_m(\eta) &= \cos m\eta \quad \text{with} \quad \lambda = m^2, \quad m = 0, 1, 2, \dots, \\ s_m(\eta) &= \sin m\eta \quad \text{with} \quad \lambda = m^2, \quad m = 1, 2, \dots, \end{aligned} \quad (32.23)$$

where the solutions of index m will exhibit m complete oscillations in the interval $0 \leq \eta < 2\pi$. Because of the circular symmetry, the sine and cosine functions of the same m have identical values of the eigenvalue λ and are similar in form except for a displacement in phase. Moreover, linear combinations of the sine and cosine solutions for the same λ are also solutions of our boundary value problem, a consequence of the complete circular symmetry.

Continuing to the more general case of the Mathieu ODE with $q \neq 0$, the term $-2q \cos 2\eta$ reduces the symmetry of the ODE to that of an interval of length π (but with reflection symmetry about $\eta = 0, \pi/2, \pi$, and $3\pi/2$). This reduced symmetry causes the ODE to have some periodic solutions that are even functions of η , while others are odd. It is no longer possible to have solutions that are identical except for phase. In fact, the even solutions (elliptic analogs of the cosine solution) correspond to different λ values than the odd solutions (analogous of the sine), with differences in λ that increase with $|q|$. For $q > 0$, the directions $\eta = 0$ and $\eta = \pi$ correspond to maxima in $2q \cos 2\eta$. Since these are directions of maximum importance for the

cosine-like solutions of the ODE, they will have larger λ values than the corresponding sine-like solutions. A qualitative analysis of this point is the topic of Exercise 32.2.3. See also the power series expansions below.

Proceeding now to a formal treatment of the power-series expansion, we write, for the cosine-like solution which for $q = 0$ has $\lambda = m^2$ with $m=1$ and which we choose to scale such that its $\cos \eta$ term has for all q the coefficient unity,

$$\Phi(q; \eta) = \cos \eta + c_1(\eta)q + c_2(\eta)q^2 + c_3(\eta)q^3 + \cdots, \quad (32.24)$$

$$\lambda(q) = 1 + \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3 + \cdots, \quad (32.25)$$

where the α_i and $c_i(\eta)$ are to be determined under the condition that Φ be an even function of η and periodic in η (with period 2π , like $\cos \eta$).

Substituting these expansions into the individual terms of the Mathieu ODE, we get

$$\frac{d^2 \Phi}{d\eta^2} = -\cos \eta + c_1''(\eta)q + c_2''(\eta)q^2 + c_3''(\eta)q^3 + \cdots. \quad (32.26)$$

$$\begin{aligned} \lambda \Phi &= \cos \eta + (c_1(\eta) + \alpha_1 \cos \eta)q + (c_2(\eta) + \alpha_1 c_1(\eta) + \alpha_2 \cos \eta)q^2 \\ &\quad + (c_3(\eta) + \alpha_1 c_2 + \alpha_2 c_1 + \alpha_3 \cos \eta)q^3 + \cdots \end{aligned} \quad (32.27)$$

$$\begin{aligned} -(2q \cos 2\eta)\Phi &= -2 \cos 2\eta \cos \eta q - 2 \cos 2\eta c_1(\eta)q^2 - 2 \cos 2\eta c_2(\eta)q^3 - \cdots \\ &= -(\cos \eta + \cos 3\eta)q - 2 \cos 2\eta c_1(\eta)q^2 - 2 \cos 2\eta c_2(\eta)q^3 - \cdots. \end{aligned} \quad (32.28)$$

In the second line of the expression for $-(2q \cos 2\eta)\Phi$ we have used the trigonometric formula $\cos n\eta \cos \eta = (\cos(n+1)\eta + \cos(n-1)\eta)/2$ to obtain terms of types arising in a Fourier series.

The Mathieu ODE is satisfied only if the overall coefficient of each power of q vanishes. The q^0 term is automatically zero since we chose the expansion form to be exact at $q = 0$, but the q^1 and higher terms provide meaningful conditions on the coefficients. The first two such terms yield

$$\text{Coefficient of } q^1: \quad c_1''(\eta) + c_1(\eta) - \cos 3\eta + (\alpha_1 - 1) \cos \eta = 0, \quad (32.29)$$

$$\text{Coefficient of } q^2: \quad c_2''(\eta) + c_2(\eta) + \alpha_1 c_1(\eta) - 2c_1(\eta) \cos 2\eta + \alpha_2 \cos \eta = 0. \quad (32.30)$$

These ODEs for the coefficients are inhomogeneous. Taking first the coefficient of q^1 , it will have a particular integral (specific solution to the inhomogeneous equation) which is the sum of that corresponding to $-\cos 3\eta$ and that corresponding to $(\alpha_1 - 1) \cos \eta$. The particular integral for $-\cos 3\eta$ is $c_1 = -(\cos 3\eta)/8$; that for $(\alpha_1 - 1) \cos \eta$ is $(1 - \alpha_1)\eta \sin \eta$. Since it is our intention to obtain a periodic solution, we must set $\alpha_1 = 1$ to eliminate the nonperiodic quantity $\eta \sin \eta$. The general solution to the homogeneous equation for c_1 is $A \cos \eta + B \sin \eta$; $\sin \eta$ must be rejected because it is odd and we are constructing an even solution; a $\cos \eta$ contribution would change the scaling from our agreed-upon setting and must also be rejected. Thus, we must have

$$c_1(\eta) = -\frac{\cos 3\eta}{8}. \quad (32.31)$$

Continuing to the coefficient of q^2 , we insert the already established values of c_1 and α_1 , and after rewriting $\cos 2\eta \cos 3\eta$ as $(\cos 5\eta + \cos \eta)/2$, we reach

$$c_2''(\eta) + c_2(\eta) - \frac{\cos 3\eta}{8} + \frac{\cos 5\eta}{8} + \left(\alpha_2 + \frac{1}{8}\right) \cos \eta = 0. \quad (32.32)$$

Once again, the particular integral corresponding to $\cos \eta$ is nonperiodic, causing us to set $\alpha_2 = -1/8$, and we discard the general solution to the homogeneous equation for the same reason we did so for c_1 . The particular integral for $(-\cos 3\eta + \cos 5\eta)/8$ leads to

$$c_2(\eta) = -\frac{\cos 3\eta}{64} + \frac{\cos 5\eta}{192}. \quad (32.33)$$

The above process may be continued, leading to power series expansions for the Φ and λ describing the solution to Mathieu's ODE that for $q = 0$ corresponds to $\cos \eta$. This solution is denoted ce_1 , where ce stands for cosine-elliptic, and the first few terms of its expansion and that of the corresponding λ value (denoted a_1 , and sometimes referred to as the **characteristic number** corresponding to ce_1) are

$$\begin{aligned} ce_1(q; \eta) = & \cos \eta - \frac{1}{8} q \cos 3\eta + \frac{1}{64} q^2 \left(-\cos 3\eta + \frac{1}{3} \cos 5\eta \right) \\ & - \frac{1}{512} q^3 \left(\frac{1}{3} \cos 3\eta - \frac{4}{9} \cos 5\eta + \frac{1}{18} \cos 7\eta \right) + \dots, \end{aligned} \quad (32.34)$$

$$a_1(q) = 1 + q - \frac{1}{8} q^2 - \frac{1}{64} q^3 - \frac{1}{1536} q^4 + \frac{11}{36864} q^5 + \dots$$

In a similar fashion we can generate the functions $ce_m(q; \eta)$ that correspond at $q = 0$ with $\cos m\eta$, with $m = 0, 2, 3, \dots$, with respective λ values $a_m(q)$. Those of even m have symmetry such that they have period π ; those of odd m have period 2π . Graphs of the ce_m reveal that for arbitrary q they exhibit m complete oscillations in the range $0 \leq \eta < 2\pi$. See Fig. 32.2. However, note that the individual oscillations are not periodic; the only periodicity is that on the interval π or 2π .

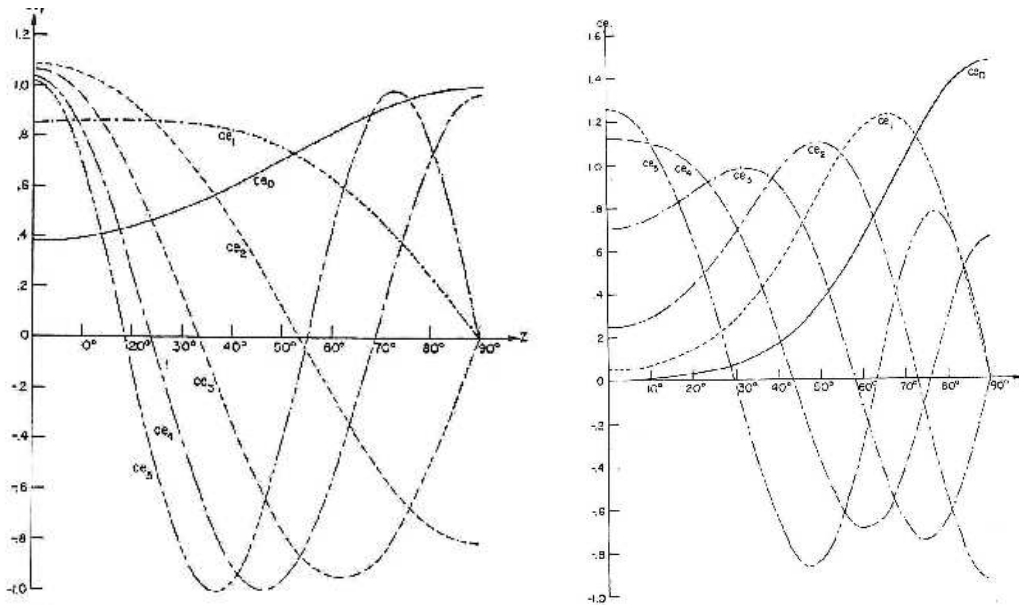
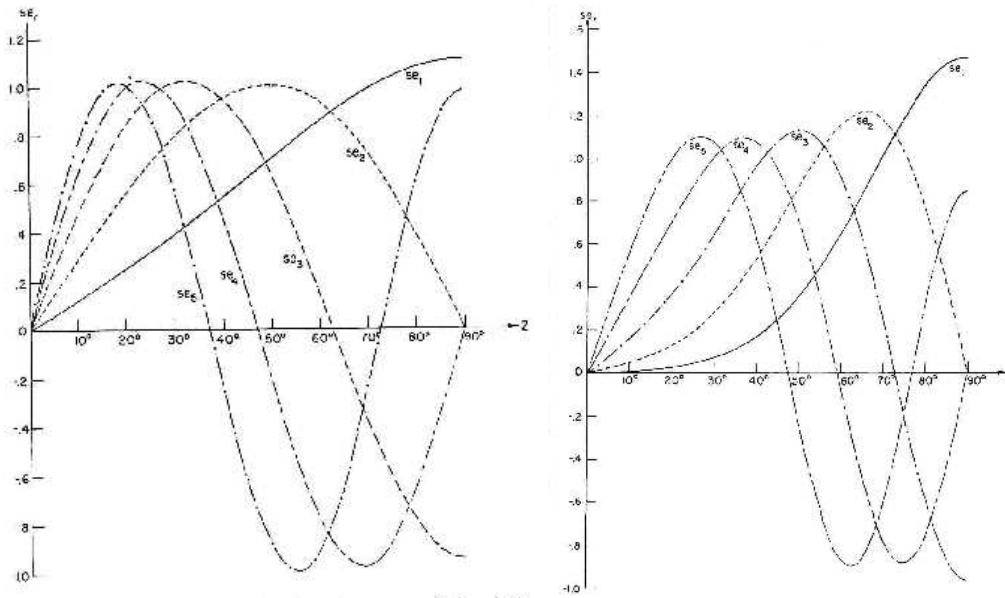
There are also Mathieu functions that correspond at $q = 0$ to $\sin m\eta$. These functions are designated $se_m(q; \eta)$ (se for sine-elliptic), with corresponding λ values (characteristic numbers) denoted $b_m(q)$. The expansion for se_1 takes the form

$$\begin{aligned} se_1(q; \eta) = & \sin \eta - \frac{1}{8} q \sin 3\eta + \frac{1}{64} q^2 \left(\sin 3\eta + \frac{1}{3} \sin 5\eta \right) \\ & - \frac{1}{512} q^3 \left(\frac{1}{3} \sin 3\eta + \frac{4}{9} \sin 5\eta + \frac{1}{18} \sin 7\eta \right) + \dots, \end{aligned} \quad (32.35)$$

$$b_1(q) = 1 - q - \frac{1}{8} q^2 + \frac{1}{64} q^3 - \frac{1}{1536} q^4 - \frac{11}{36864} q^5 + \dots$$

The se_m have period π if m is even and period 2π if m is odd. The first few se_m are plotted in Fig. 32.3. For the power series expansions of additional ce_m and se_m the reader is referred to AMS-55 (Abramowitz and Stegun, Additional Readings).

Notice that the characteristic numbers a_1 and b_1 are not equal, so ce_1 and se_1 are not two independent solutions of the same Mathieu equation. In fact, the second solution corresponding to ce_1 is a nonperiodic function, as is also the second solution corresponding to se_1 . It will be shown in the next section that for $q \neq 0$ and any m the eigenvalues a_m and b_m are unequal, and that the second solutions corresponding to ce_m and se_m are nonperiodic.

Figure 32.2: ce_m , $m = 0 \cdots 5$. Left, $q = 1$; right, $q = 10$.Figure 32.3: se_m , $m = 1 \cdots 5$. Left, $q = 1$; right, $q = 10$.

FOURIER EXPANSIONS

The power series expansions show that the Mathieu functions ce_m and se_m can be viewed as Fourier series with q -dependent coefficients. These expansions take the

generic forms (for $m \geq 0$)

$$ce_{2m}(q; \eta) = \sum_{n=0}^{\infty} A_{2n}^{(2m)}(q) \cos 2n\eta, \quad (32.36)$$

$$ce_{2m+1}(q; \eta) = \sum_{n=0}^{\infty} A_{2n+1}^{(2m+1)}(q) \cos(2n+1)\eta, \quad (32.37)$$

$$se_{2m+1}(q; \eta) = \sum_{n=0}^{\infty} B_{2n+1}^{(2m+1)}(q) \sin(2n+1)\eta, \quad (32.38)$$

$$se_{2m+2}(q; \eta) = \sum_{n=0}^{\infty} B_{2n+2}^{(2m+2)}(q) \sin(2n+2)\eta. \quad (32.39)$$

By insertion of these expansions into Mathieu's ODE we can obtain recurrence formulas connecting the coefficients A or B in the above expansions. Illustrating for ce_{2m} :

$$\begin{aligned} \left(\frac{d^2}{d\eta^2} + \lambda \right) ce_{2m}(q, \eta) &= \sum_{n=0}^{\infty} (-4n^2 + a_{2m}(q)) A_{2n}^{(2m)}(q) \cos 2n\eta, \quad (32.40) \\ -2q \cos(2\eta) ce_{2m}(q, \eta) &= \sum_{n=0}^{\infty} A_{2n}^{(2m)}(q) \cos 2n\eta \\ &= -2q A_0^{(2m)}(q) \cos 2\eta - 2q \sum_{n=1}^{\infty} A_{2n}^{(2m)}(q) \cos 2\eta \cos 2n\eta \\ &= -2q A_0^{(2m)}(q) \cos 2\eta - q \sum_{n=1}^{\infty} A_{2n}^{(2m)}(q) (\cos(2n+2)\eta + \cos(2n-2)\eta). \quad (32.41) \end{aligned}$$

Equating coefficients of $\cos 2n\eta$ for $n = 0, 1, \dots$, we find, abbreviating $a_{2m}(q)$ to a and $A_{2n}^{(2m)}(q)$ to A_{2n} ,

$$\begin{aligned} \text{For } ce_{2m}(q, \eta) : \quad & aA_0 - qA_2 = 0, \\ & (a-4)A_2 - q(A_4 + 2A_0) = 0, \quad (32.42) \\ & (a-4n^2)A_{2n} - q(A_{2n+2} + A_{2n-2}) = 0, \quad n \geq 2. \end{aligned}$$

Similar procedures for ce_{2m+1} , se_{2m+1} , and se_{2m+2} , with b an abbreviation for $b_m(q)$ and B_n short for $B_n^{(2m+1)}(q)$ or $B_n^{(2m+2)}(q)$ yield

$$\begin{aligned} \text{For } ce_{2m+1}(q, \eta) : \quad & (a-1-q)A_1 - qA_3 = 0, \quad (32.43) \\ & [a - (2n+1)^2] A_{2n+1} - q(A_{2n+3} + A_{2n-1}) = 0, \quad n \geq 1, \end{aligned}$$

$$\begin{aligned} \text{For } se_{2m+1}(q, \eta) : \quad & (b-1+q)B_1 - qB_3 = 0, \quad (32.44) \\ & [b - (2n+1)^2] B_{2n+1} - q(B_{2n+3} + B_{2n-1}) = 0, \quad n \geq 1, \end{aligned}$$

$$\begin{aligned} \text{For } se_{2m+2}(q, \eta) : \quad & (b-4)B_2 - qB_4 = 0, \quad (32.45) \\ & (b-4n^2)B_{2n} - q(B_{2n+2} + B_{2n-2}) = 0, \quad n \geq 2. \end{aligned}$$

EVALUATION OF CHARACTERISTIC NUMBERS

The recurrence relations for the Fourier coefficients permit the effective generation of expressions for ce_m , se_m , but before they can be used for this purpose we need to have values for the characteristic numbers a_m and b_m that enter the recurrence formulas. One way to proceed is to work with ratios of the coefficients. Illustrating for ce_{2m} , we rewrite the second of Eqs. (32.42) as

$$\begin{aligned} (a_{2m}(q) - 4)r_0 - qr_2r_0 - 2q &= 0, \quad \text{equivalent to} \\ r_0 &= -\frac{q}{2} \frac{1}{1 - \frac{1}{4}(a_{2m}(q) - qr_2)}, \end{aligned} \quad (32.46)$$

where $r_n = A_{n+2}^{(2m)}/A_n^{(2m)}$. Notice that we wrote $A_4^{(2m)}/A_0^{(2m)}$ as r_2r_0 .

We next note that the instance of r_2 in the denominator of Eq. (32.46) can be written in terms of r_4 by using the third of Eqs. (32.42). We first convert that equation to the form

$$r_{2n-2} = -\frac{q}{4n^2} \frac{1}{1 - \frac{1}{4n^2}(a_{2m}(q) - qr_{2n})}, \quad n \geq 2,$$

after which we use it for $n = 2$. Equation (32.46) becomes

$$r_0 = -\frac{q}{2} \frac{1}{1 - \frac{1}{4}a_{2m}(q) - \frac{q^2}{64} \frac{1}{1 - \frac{1}{16}(a_{2m}(q) - qr_4)}}.$$

Repeating this process indefinitely, we obtain r_0 as a **continued fraction**:

$$r_0 = -\frac{q}{2} \frac{1}{1 - \frac{1}{4}a_{2m}(q) - \frac{q^2}{64} \frac{1}{1 - \frac{1}{16}a_{2m}(q) - \frac{q^2}{576} \frac{1}{1 - \frac{1}{36}a_{2m}(q) - \dots}}}$$

This is an awkward notation, and it has become customary to write continued fractions in a somewhat idiosyncratic more condensed form. Rewriting the above, and shortening $a_{2m}(q)$ to a , the usual notation is

$$r_0 = -\frac{\frac{1}{2}q}{1 - \frac{1}{4}a} - \frac{\frac{1}{64}q^2}{1 - \frac{1}{16}a} - \frac{\frac{1}{576}q^2}{1 - \frac{1}{36}a} - \dots - \frac{q^2/16n^2(n-1)^2}{1 - a/4n^2} - \dots, \quad (32.47)$$

where n refers to the n -th term on the right-hand side of Eq. (32.47).

Now that we have an expression for r_0 , we can return to the first of Eqs. (32.42) and write

$$a_{2m}(q) = qr_0 = -\frac{\frac{1}{2}q^2}{1 - \frac{1}{4}a} - \frac{\frac{1}{64}q^2}{1 - \frac{1}{16}a} - \frac{\frac{1}{576}q^2}{1 - \frac{1}{36}a} - \dots. \quad (32.48)$$

Equation (32.48) defines $a_{2m}(q)$ but is not in solved form. However, it is easily solved by a successive approximation procedure, and can lead to accurate values of a_{2m} even

when q is large. Once $a_{2m}(q)$ has been computed to sufficient accuracy, we can then return to Eqs. (32.42) to obtain values of the coefficients $A_n^{(2m)}(q)$. Because Mathieu's ODE is homogeneous, its solutions are arbitrary as to scale and we can ordinarily choose any constant or even q -dependent value we like for $A_0^{(2m)}(q)$.¹

The analysis we have given for $ce_{2m}(q)$ can also be carried out (with suitable modifications) for $ce_{2m+1}(q)$, $se_{2m+1}(q)$, and $se_{2m+2}(q)$. The resulting continued fractions for $a_{2m+1}(q)$, $b_{2m+1}(q)$ and $b_{2m+2}(q)$ can be found in AMS-55 (Additional Readings). This and other sources also provide tabulations of the characteristic numbers for a range of m and q . Figure 32.4 shows graphically the behavior of a_m and b_m as a function of q . The figure illustrates the key issues of the present discussion. We see that at $q = 0$, a_m and b_m are equal, but as q increases, a_m becomes larger than b_m . Because we will show that the Mathieu equation can have (for given λ and q) at most one solution of period π or 2π these curves cannot cross, so for any $q > 0$ they will satisfy

$$a_0 < b_1 < a_1 < b_2 < a_2 \cdots . \quad (32.49)$$

We do not prove it, but we also see that at large q , the curves for a_m and b_{m+1} approach each other asymptotically.

Example 32.2.1. Numerical Evaluation of $a_2(q)$ for $q = 10$

This example shows how the value of a_2 can be extracted from its implicit representation as a continued fraction. Our starting point will be a guess for $a_2(10)$, which we can read from Fig. 32.4. Since the same continued fraction applies to all a_{2m} , an inappropriate starting point can lead us to a characteristic number a_{2m} other than a_2 . If we had absolutely no knowledge of the approximate value of a_{2m} , we could start from its value at $q = 0$ (namely $4m^2$), and increase q in small steps to ensure that we are converging upon the desired result.

A reasonable starting point for $a_2(10)$ is 8. Using this value and keeping the first three terms of Eq. (32.48) (thus approximating r_6 as zero), we find, working backward

$$\begin{aligned} -\frac{qr_4}{16} &\approx \frac{q^2/576}{1 - a/36} = \frac{100/576}{1 - 8/36} = 0.2232, \\ -\frac{qr_2}{4} &\approx \frac{q^2/64}{1 - a/16 - 0.2232} = \frac{100/64}{1 - 8/16 - 0.2232} = 5.6449, \\ a = qr_0 &\approx -\frac{q^2/2}{1 - a/4 - 5.6449} = -\frac{100/2}{1 - 8/4 - 5.6449} = 7.5246. \end{aligned}$$

Taking the average of this result and our initial estimate, $(7.5246 + 8)/2 = 7.7623$

¹This statement is an oversimplification. For all nonzero m there are specific q values for which $A_0(q)$ vanishes and an arbitrary value of $A_0(q)$ cannot be selected. For further discussion of this point, see McLachlan, Additional Readings.

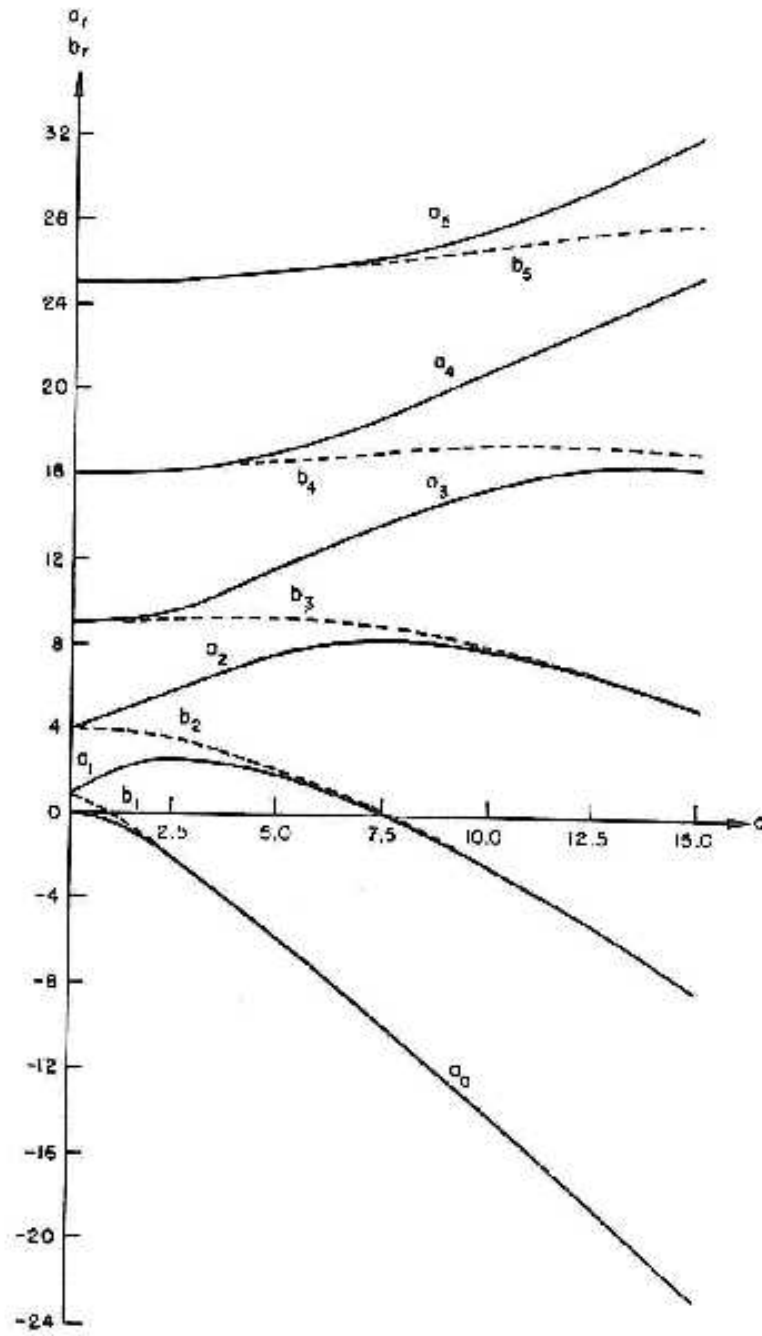


Figure 32.4: a_m (characteristic number for ce_m) and b_m (characteristic number of se_m).

and repeating the process, this time keeping six terms:

$$\begin{aligned} -\frac{qr_{10}}{100} &\approx \frac{q^2/14400}{1 - a/144} = \frac{100/14400}{1 - 7.7623/144} = 0.0073401, \\ -\frac{qr_8}{64} &\approx \frac{q^2/6400}{1 - a/100 - 0.0073401} = \frac{100/6400}{1 - 7.7623/100 - 0.0073401} = 0.0170758 \\ -\frac{qr_6}{36} &\approx \frac{q^2/2304}{1 - a/64 - 0.0170758} = \frac{100/2304}{1 - 7.7623/64 - 0.0170758} = 0.0503724, \\ -\frac{qr_4}{16} &\approx \frac{q^2/576}{1 - a/36 - 0.0503724} = \frac{100/576}{1 - 7.7623/36 - 0.0503724} = 0.2365248, \\ -\frac{qr_2}{4} &\approx \frac{q^2/64}{1 - a/16 - 0.2365248} = \frac{100/64}{1 - 7.7623/16 - 0.2365248} = 5.6138104, \\ a = qr_0 &\approx -\frac{q^2/2}{1 - a/4 - 5.6138104} = -\frac{100/2}{1 - 7.7623/4 - 5.6138104} = 7.62848. \end{aligned}$$

Averaging 7.7623 and 7.6285, we get 7.6954, and two more iterations bring us to 7.7124 as the average of the last input and output. These numbers are converging toward the accurate value of $a_2(10)$, which is 7.7174.

■

Example 32.2.2. Computation of Coefficients

Here we illustrate the use of the recurrence formulas, Eqs. (32.42)–(32.45) for computation of the coefficients in the Fourier series for the Mathieu functions. For this purpose we compute the first few coefficients for $ce_2(q, \eta)$ for $q = 10$, the same parameter values that were studied in Example 32.2.1. From that example (or elsewhere) we need as input the value of the characteristic number $a_2(10)$; we use the accurate value $a = 7.7174$.

In principle one could choose an arbitrary value (such as unity) for the initial coefficient $A_0^{(2)}$ (hereinafter abbreviated to A_0), after which Eqs. (32.42) could be used to obtain successively A_2 , A_4 , etc. This procedure works fine for a few steps, but on continuing further extremely severe differencing errors are encountered. That is to be expected, as the A_{2n} should approach zero as n increases because the Fourier series for ce_2 is convergent. In the present example, we take $A_0 = 1$, then (doing the arithmetic to five significant figures) we find

$$\begin{aligned} A_2 &= \frac{aA_0}{q} = \frac{(7.7174)(1)}{10} = 0.77174, \\ A_4 &= \frac{(a-4)A_2}{q} - 2A_0 = \frac{(3.7174)(0.77174)}{10} - 2(1) = -1.7131, \\ A_6 &= \frac{(a-16)A_4}{q} - A_2 = \frac{(-8.2826)(-1.7131)}{10} - 0.77174 = 0.64716. \end{aligned}$$

Continuing, the next few coefficients are $A_8 = -0.1173$, $A_{10} = 0.013$, $A_{12} = -0.003$, $A_{14} = 0.03$, and differencing errors make all further coefficients completely meaningless.

An alternate approach is to use the ratios r_{2n} which were found useful in computing the characteristic numbers. From the procedure outlined in Example 32.2.1 (but using data more accurate than were given in that example) we can get values of the r_n , from which, starting with $A_0 = 1$, we can form $A_2 = r_0 A_0$, $A_4 = r_2 A_2$, etc. Doing the arithmetic to five significant figures, we get the numeric values

$$\begin{aligned} A_2 &= 0.77174, & A_4 &= -1.7131, & A_6 &= 0.64717, & A_8 &= -0.11726, \\ A_{10} &= 0.01281, & A_{12} &= -0.00094, & A_{14} &= 0.00005, & \dots & \end{aligned}$$

We are now basically running the recursion sequence in descending order, starting from a coefficient A_{2N} which is assumed negligible. In the present illustrative case, it would be difficult to get all the coefficients needed to give ce_2 to five-digit accuracy by a straightforward upward recursion. ■

MODIFIED MATHIEU FUNCTIONS

Upon replacing the angular elliptic variable η by $i\xi$, the Mathieu ODE, Eq. (32.18), becomes the modified Mathieu equation, Eq. (32.19). and is the ODE for the radial variable ξ in problems like the elliptical drum. For this reason the modified Mathieu ODE is sometimes referred to as the **radial Mathieu equation**.

The replacement $\eta \rightarrow i\xi$ motivates the definitions of modified Mathieu functions as

$$\begin{aligned} Ce_n(q; \xi) &= ce_n(q; i\xi), & n &= 0, 1, \dots, \\ Se_n(q; \xi) &= -ise_n(q; i\xi), & n &= 1, 2, \dots \end{aligned}$$

The factor $-i$ in the second of these equations causes Se to be real. Because these functions have Maclaurin expansions, they correspond to the regular solutions of the modified Mathieu ODE. It can be shown that they are no longer periodic but remain oscillatory. Notice that Ce_n is computed using the value $\lambda = a_n(q)$ that was needed to make ce_n periodic. In other words, λ is found by considering the η equation, **not** the ξ equation. Corresponding remarks apply to Se_n .

In physical problems involving elliptical coordinates, the radial Mathieu ODE, Eq. (32.19), plays a role corresponding to Bessel's ODE in circular cylindrical geometry. Because there are four families of independent Bessel functions, namely the regular solutions J_n and irregular Neumann functions Y_n , along with the modified Bessel functions I_n and K_n , we expect four kinds of modified Mathieu functions to be relevant for describing ξ dependence. For $q > 0$ (the usual equation), many authors label the regular (or first-kind) solutions

$$Je_n(q; \xi) = Ce_n(q; \xi), \quad Jo_n(q; \xi) = Se_n(q; \xi).$$

The irregular or second-kind solutions have been denoted²

$$Ne_n(q; \xi), \quad No_n(q; \xi).$$

²Here "N" stands for Neumann; this naming convention corresponds to an older notation for Bessel functions which uses N where most authors now use Y , and no consensus has yet emerged as to the preferred notation for these Mathieu functions.

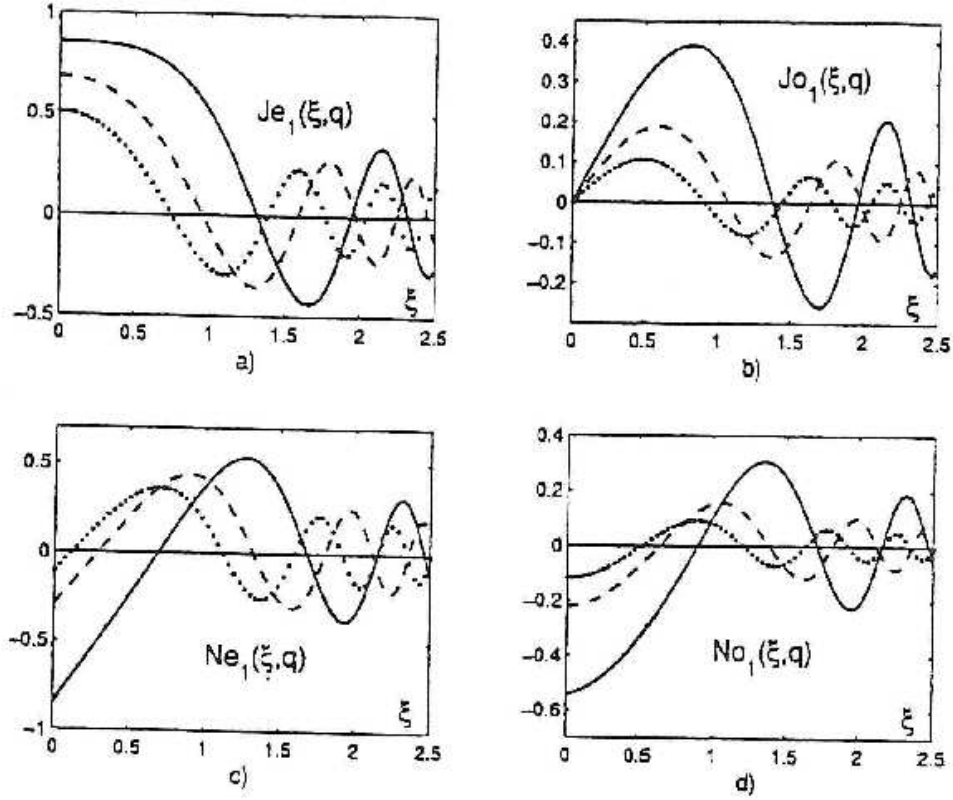


Figure 32.5: Modified Mathieu functions, $q = 1$ (solid line), $q = 2$ (dashed line), $q = 3$ (dotted line). [From Gutiérrez-Vega et al., *Am. J. Phys.* **71**: 233 (2003).]

All the above functions are oscillatory; some of them are plotted in Fig. 32.5.

The replacement of η by $i\xi$ makes the expansions given for ce_n and se_n completely unsatisfactory, and alternate formulations are essential. We give, without proof, the following expansions of Ce_m and Se_m in terms of Bessel functions (see AMS-55, Additional Readings). These formulas are at arbitrary scale. That reference also gives additional expansions that may for some parameter values be more rapidly convergent.

$$Ce_{2m}(q; \xi) = \sum_{k=0}^{\infty} A_{2k} J_{2k}(2\sqrt{q} \sinh \xi), \quad (32.50)$$

$$Ce_{2m+1}(q; \xi) = \sum_{k=0}^{\infty} (-1)^{k+1} A_{2k+1} J_{2k+1}(2\sqrt{q} \cosh \xi), \quad (32.51)$$

$$Se_{2m+1}(q; \xi) = \sum_{k=0}^{\infty} B_{2k+1} J_{2k+1}(2\sqrt{q} \sinh \xi), \quad (32.52)$$

$$Se_{2m+2}(q; \xi) = \tanh \xi \sum_{k=1}^{\infty} (-1)^k 2k B_{2k} J_{2k}(2\sqrt{q} \cosh \xi). \quad (32.53)$$

In the above equations the coefficients A_p are abbreviations for $A_p^r(q)$, where r is the subscript appearing on the equation's left-hand side; the notations B_p are similar abbreviations.

FUNCTIONS WITH $q < 0$

For problems involving oscillations, the parameter q in the Mathieu ODE will be positive, since it is proportional to the square of an oscillation frequency. Compare Eq. (32.17). However, there are some problems in which negative values of q are encountered. For the (unmodified) Mathieu equation, a change in the sign of q corresponds to the replacement of η by $\frac{1}{2}\pi - \eta$. Applying this replacement to the Fourier series expansion corresponding to ce_{2m} , and defining the result as shown below, we find

$$ce_{2m}(-q; \eta) = (-1)^m ce_{2m}(q; \frac{1}{2}\pi - \eta) = (-1)^m \sum_{n=0}^{\infty} (-1)^n A_{2n}^{2m}(q) \cos 2n\eta. \quad (32.54)$$

The factor $(-1)^n$ appears because

$$\cos 2n(\frac{1}{2}\pi - \eta) = \cos(n\pi - 2n\eta) = \cos n\pi \cos 2n\eta = (-1)^n \cos 2n\eta.$$

The factor $(-1)^m$ is included in the definition to cause $ce_{2m}(-q)$ to have the same sign as $ce_{2m}(+q)$ in the limit $q \rightarrow 0$. Noticing that the recurrence relations for A_{2n} remain unchanged if we replace q by $-q$ and A_{2n} by $(-1)^n A_{2n}$, we see that the value of a_{2m} obtained from the equations will remain unchanged, meaning that

$$a_{2m}(q) = a_{2m}(-q). \quad (32.55)$$

Thus the expansion of a_{2m} in powers of q will contain only even powers.

An analysis similar to that leading to Eq. (32.54) shows that $se_{2m+2}(-q; \eta)$ can be defined consistently as $(-1)^m se_{2m+2}(q; \frac{1}{2}\pi - \eta)$, and

$$se_{2m+2}(-q; \eta) = (-1)^m \sum_{n=0}^{\infty} (-1)^n B_{2n+2}^{2m+2}(q) \sin(2n+2)\eta. \quad (32.56)$$

Moreover,

$$b_{2m+2}(-q) = b_{2m+2}(q). \quad (32.57)$$

For ce_{2m+1} the situation is different. The substitution $\eta \rightarrow \frac{1}{2}\pi - \eta$ now changes the cosine into the sine, and a consistent definition of ce_{2m+1} is obtained from $(-1)^m se_{2m+1}(q, \frac{1}{2}\pi - \eta)$:

$$ce_{2m+1}(-q; \eta) = (-1)^m \sum_{n=0}^{\infty} (-1)^n B_{2n+1}^{2m+1}(q) \cos(2n+1)\eta. \quad (32.58)$$

The recurrence formula for se_{2m+1} is invariant under the substitutions $q \rightarrow -q$ and $B_{2n+1} \rightarrow (-1)^n B_{2n+1}$, so the characteristic number associated with $ce_{2m+1}(-q; \eta)$ is $b_{2m+1}(q)$, or

$$a_{2m+1}(-q) = b_{2m+1}(q). \quad (32.59)$$

A corresponding definition for se_{2m+1} yields

$$se_{2m+1}(-q; \eta) = (-1)^m \sum_{n=0}^{\infty} (-1)^n A_{2n+1}^{2m+1}(q) \sin(2n+1)\eta. \quad (32.60)$$

The characteristic number $b_{2m+1}(-q)$ is given by Eq. (32.59) with $q \rightarrow -q$.

For $q < 0$ (the elliptical analog of the modified Bessel equation), the solutions of the radial Mathieu ODE are denoted by

$$\begin{array}{lll} \text{Ie}_n(\xi, q), & \text{Io}_n(\xi, q), & \text{regular or first kind,} \\ \text{Ke}_n(\xi, q), & \text{Ko}_n(\xi, q), & \text{irregular or second kind.} \end{array}$$

These functions, like the corresponding Bessel functions, are nonoscillatory and are known as the **evanescent modified Mathieu functions**.

Exercises

- 32.2.1.** Show that after setting $\alpha_2 = 0$ the particular integral of Eq. (32.32) is that given in Eq. (32.33).
- 32.2.2.** Obtain the terms through q^2 in the expansion in powers of q of $\text{ce}_0(q; \eta)$, the solution to Mathieu's PDE that for $q = 0$ is a constant. Obtain also the terms through q^2 in the expansion of the characteristic number $a_0(q)$.
- 32.2.3.** Since the eigenvalues $a_n(q)$ and $b_n(q)$ cannot cross, their relative ordering can be determined by comparing them at any convenient q value. Show that for small nonzero q (and therefore for all $q > 0$) $a_n > b_n$.
- 32.2.4.** Using the procedure of Example 32.2.1, obtain a value of $a_3(q)$ for $q = 8$ that is accurate to three decimal places.
Hint. Use Fig. 32.4 as a guide toward selection of an initial guess.
- 32.2.5.** Using the more accurate of the two approaches for determining coefficients that were described in Example 32.2.2, obtain the first several terms of the expansion of $\text{ce}_3(q, \eta)$ for $q = 8$ at a scale such that $A_1 = 1$.
- 32.2.6.** Write the first few terms of the expansion in powers of q of the modified Mathieu function $\text{Ce}_1(q; \xi)$.
Hint. Do not derive it by a process similar to that for ce_1 because its definition has nothing to do with periodicity.

32.3 PERIODICITY OF SOLUTIONS

There is a significant body of theory regarding the solutions of ODEs that are periodic in form, as is the Mathieu equation. The key result in this area is a theorem by Floquet. The present section considers the conditions needed in order that solutions to such an ODE be periodic. We have already found solutions, ce_m and se_m , that are periodic, but with different characteristic numbers λ (called a_m and b_m). Because $a_m \neq b_m$, ce_m and se_m are solutions to **different** Mathieu equations. That raises the question "Where is the second solution for the parameter values leading to ce_m (or se_m), and what are its properties?"

NONPERIODICITY OF SECOND SOLUTION

We now show that if ce_{2m} is a solution of the Mathieu ODE with given values of q and a_{2m} (here abbreviated to a), the second solution is nonperiodic.³ Since ce_{2m} is an even function with period π , the second solution, if also periodic, can be chosen to be an odd function of η and have an expansion of the form

$$y(\eta) = \sum_{n=0}^{\infty} C_{2n+2} \sin(2n+2)\eta. \quad (32.61)$$

The coefficients C_n in the expansion of y must satisfy the recurrence relations for a solution of its type, namely Eqs. (32.45), while the periodic solution ce_{2m} has an expansion whose coefficients $A_{2n}^{(2m)}(q)$ (here abbreviated A_{2n}) must satisfy Eqs. (32.42). In particular,

$$\begin{aligned} (a-4)A_2 - q(A_4 + 2A_0) &= 0, \\ (a-4)C_2 - qC_4 &= 0. \end{aligned}$$

Multiplying the first of these equations by C_2 , the second by A_2 , and subtracting, we get (assuming $q \neq 0$)

$$A_2C_4 - A_4C_2 = \begin{vmatrix} A_2 & A_4 \\ C_2 & C_4 \end{vmatrix} = 2A_0C_2. \quad (32.62)$$

For $n \geq 2$, from the recurrence relations

$$\begin{aligned} (a-4n^2)A_{2n} &= q(A_{2n+2} + A_{2n-2}), \\ (a-4n^2)C_{2n} &= q(C_{2n+2} + C_{2n-2}), \end{aligned}$$

division of the first by the second leads to a result that can be written

$$\begin{vmatrix} A_{2n} & A_{2n+2} \\ C_{2n} & C_{2n+2} \end{vmatrix} = \begin{vmatrix} A_{2n-2} & A_{2n} \\ C_{2n-2} & C_{2n} \end{vmatrix}. \quad (32.63)$$

Since Eq. (32.63) must be satisfied for all $n \geq 2$, we can combine it with Eq. (32.62) to reach

$$\begin{vmatrix} A_{2N} & A_{2N+2} \\ C_{2N} & C_{2N+2} \end{vmatrix} = 2A_0C_2,$$

where N can be as large as desired, and in particular can be chosen large enough that A_{2N} and A_{2N+2} approach zero (which they must do because the Fourier expansion of ce_{2m} converges). Since A_0 does not vanish, we conclude that $C_2 = 0$, and the recurrence relations show that all the other C_{2n} then vanish. There is thus no second solution of the form given in Eq. (32.61), and the second solution (which must exist) will be nonperiodic. A similar analysis can be carried out to show that if our Mathieu equation has periodic solution ce_{2m+1} or se_m , the corresponding second solution will be nonperiodic.

³This analysis is that of McLachlan, Additional Readings.

FLOQUET'S THEOREM

Floquet's theorem derives from the observation that an arbitrary solution $\Phi(\eta)$ of a linear homogeneous second-order ODE can at most be a linear combination of two linearly independent solutions $\Phi_1(\eta)$ and $\Phi_2(\eta)$, as in

$$\Phi(\eta) = c_1\Phi_1(\eta) + c_2\Phi_2(\eta) . \quad (32.64)$$

Because the Mathieu ODE has exactly the same form at $\eta + \pi$ as it has at η , the solution $\Phi(\eta + \pi)$ must be a linear combination of $\Phi_1(\eta)$ and $\Phi_2(\eta)$. Notice that we are not arguing that $\Phi(\eta + \pi) = \Phi(\eta)$; there is no inherent requirement that all solutions of the ODE have periodicity. We are only claiming that $\Phi(\eta + \pi)$ must be a solution to the same ODE as $\Phi(\eta)$ and therefore must be of the form

$$\Phi(\eta + \pi) = h_1\Phi_1(\eta) + h_2\Phi_2(\eta) . \quad (32.65)$$

But it is also true that

$$\begin{aligned} \Phi_1(\eta + \pi) &= a_1\Phi_1(\eta) + a_2\Phi_2(\eta) , \\ \Phi_2(\eta + \pi) &= b_1\Phi_1(\eta) + b_2\Phi_2(\eta) , \end{aligned} \quad (32.66)$$

and, from Eq. (32.64) evaluated for $\eta + \pi$,

$$\Phi(\eta + \pi) = c_1\Phi_1(\eta + \pi) + c_2\Phi_2(\eta + \pi) . \quad (32.67)$$

Combining Eqs. (32.66) and (32.67), we have

$$\Phi(\eta + \pi) = (a_1c_1 + b_1c_2)\Phi_1(\eta) + (a_2c_1 + b_2c_2)\Phi_2 . \quad (32.68)$$

From Eqs. (32.65) and (32.68), it is obvious that

$$\begin{aligned} a_1c_1 + b_1c_2 &= h_1 , \\ a_2c_1 + b_2c_2 &= h_2 . \end{aligned} \quad (32.69)$$

Floquet's analysis now proceeds by observing that we can choose c_1 and c_2 in such a way that $h_1 = \mu c_1$ and $h_2 = \mu c_2$, with the result that $\Phi(\eta + \pi) = \mu\Phi(\eta)$. This choice amounts to requiring c_1 and c_2 to be obtained from a solution of the eigenvalue problem

$$\left\{ \begin{array}{l} a_1c_1 + b_1c_2 = \mu c_1 \\ a_2c_1 + b_2c_2 = \mu c_2 \end{array} \right\} \quad \text{or} \quad \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mu \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} . \quad (32.70)$$

Summarizing, Floquet's theorem states that we can choose Φ so that it satisfies the condition $\Phi(\eta + \pi) = \mu\Phi(\eta)$, with μ a root of

$$\begin{vmatrix} a_1 - \mu & b_1 \\ a_2 & b_2 - \mu \end{vmatrix} = 0 \quad (32.71)$$

and with the c_i given as an eigenvector of Eqs. (32.70) corresponding to μ . A solution to Mathieu's ODE in the form given by Floquet's theorem is sometimes called a **Floquet solution**, and its qualitative behavior is associated with the eigenvalue μ .

Note that a Floquet solution will only be periodic for special values of μ ; a necessary condition for periodicity is that $|\mu| = 1$.

To gain more understanding of the Floquet solutions, let's assume that Φ is a solution corresponding to μ and define quantities ν and y such that $\mu = \exp(i\nu\pi)$ and $y(\eta) = \exp(-i\nu\eta)\Phi(\eta)$, meaning also that $\Phi(\eta + \pi) = \exp(i\nu\pi)\Phi(\eta)$. These definitions have the effect that

$$y(\eta + \pi) = e^{-i\nu(\eta+\pi)}\Phi(\eta + \pi) = e^{-i\nu\eta} \left[e^{-i\nu\pi}\Phi(\eta + \pi) \right] = e^{-i\nu\eta}\Phi(\eta) = y(\eta) , \quad (32.72)$$

showing that $y(\eta)$ is a periodic function of η with period π . Moreover,

$$\Phi(\eta) = e^{i\nu\eta}y(\eta) , \quad (32.73)$$

showing that a Floquet solution $\Phi(\eta)$ consists of a periodic function of η multiplied by a complex exponential in η . The quantity ν which controls the exponential behavior is referred to as the **characteristic exponent** of Φ .

Our present interest is in identifying periodic solutions to the Mathieu equation. We can distinguish various cases dependent upon the characteristic exponent ν . If $\Im\nu > 0$, our Floquet solution will asymptotically approach zero as η is increased, while if $\Im\nu < 0$, it will grow as η increases. Only if ν is real will Φ remain at the same magnitude through successive cycles of length π . Since Φ contains the factor y that is periodic with period π , Φ will be periodic with period

- (a) π if $\nu = 0, 2, \dots$ (corresponding to $\mu = 1$)
- (b) 2π if $\nu = 1, 3, \dots$, (corresponding to $\mu = -1$), or
- (c) $s\pi$ if $\nu = 2r/s$, where r and $s > 2$ are integers with no common divisor.

For the Mathieu equation, take now Φ_1 to be an even function of η , scaled such that $\Phi_1(0) = 1$ and $\Phi_1'(0) = 0$. Choose Φ_2 to be odd and scaled such that $\Phi_2(0) = 0$ and $\Phi_2'(0) = 1$. Then, from Eqs. (32.66) evaluated for $\eta = 0$, we have

$$\Phi_1(\pi) = a_1 , \quad \Phi_2(\pi) = b_1 ,$$

From the derivatives of Eqs. (32.66) we also have

$$\Phi_1'(\pi) = a_2 , \quad \Phi_2'(\pi) = b_2 .$$

Our eigenvalue equation, Eq. (32.71), then becomes

$$\begin{vmatrix} \Phi_1(\pi) - \mu & \Phi_2(\pi) \\ \Phi_1'(\pi) & \Phi_2'(\pi) - \mu \end{vmatrix} = \mu^2 - [\Phi_1(\pi) + \Phi_2'(\pi)]\mu + W = 0 , \quad (32.74)$$

where

$$W = \Phi_1(\pi)\Phi_2'(\pi) - \Phi_2(\pi)\Phi_1'(\pi) , \quad (32.75)$$

the Wronskian of the Φ_i , evaluated at $\eta = \pi$. But the Wronskian of the Mathieu ODE is a constant, see Eq. (7.62), and we can therefore evaluate W at $\eta = 0$, where clearly $W = 1$.

From the properties of quadratic equations, we now identify W as the product of the two roots, so we have

$$\mu_1 \mu_2 = 1 . \quad (32.76)$$

We can now enumerate the various possibilities for the μ_i . If $|\mu_1| < 1$, then $|\mu_2| > 1$ and vice versa. In these cases neither solution is periodic. If $|\mu_1| = |\mu_2| = 1$, we

write the roots in the form $e^{i\nu_i\pi}$ and see that Eq. (32.76) is only satisfied if $\nu_2 = -\nu_1$, meaning that $\mu_2 = \mu_1^*$. Thus, we can have solution(s) of period π if $\mu = 1$, solution(s) of period 2π if $\mu = -1$, and solution(s) of period $s\pi$ ($s > 2$) if the ν_i are rational fractions. In the cases $\mu = 1$ and $\mu = -1$ we will have a double root ($\mu_1 = \mu_2$); if the μ_i are complex, we will have distinct roots.

How many periodic solutions will we get? If the μ_i are distinct, we will get two periodic solutions, as there must be at least one eigenvector (i.e., periodic ODE solution) for each distinct eigenvalue. This means that periodic solutions of period $s\pi$, $s > 2$, must come in pairs. But if the μ_i are degenerate, corresponding to periodicity π or 2π , we will only get two eigenvectors (ODE solutions) if the matrix in Eq. (32.70) is not **defective**.⁴ While defectiveness may seem to be a far-fetched or unlikely possibility, it is exactly what happens when one of the Φ , say Φ_1 , is periodic with period π or 2π . For periodicity π , the coefficients b_1 and b_2 describing $\Phi_2(\pi)$ are both nonzero; our matrix from Eq. (32.70) has the form

$$\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix},$$

and its only eigenvector is Φ_1 .

For physical reasons the solutions of most importance are those with periods π or 2π , and these are the ce_m and se_m we analyzed in a previous section. We do not discuss further the solutions of periods $s\pi$ with $s > 2$.

32.4 BOUNDARY-VALUE PROBLEMS

We are now ready to proceed to a more complete discussion of boundary value problems such as the vibrating drumhead that was already introduced in Example 32.1.1.

After separating the equation for the drum into the elliptical coordinates ξ and η , we note the existence of an infinite number of solutions that satisfy the boundary conditions, each describing a particular mode of vibration of the drumhead. The angular solution will be a Mathieu function that must be periodic in η , with period 2π ; Mathieu functions with period π are acceptable solutions, as they also have 2π as an interval of periodicity. This means the angular part of a solution will be a function from the set $ce_n(q; \eta)$ and $se_n(q; \eta)$. The radial part of the same solution must be a function from the set $Ce_n(q; \xi)$ and $Se_n(q; \xi)$, where the values of q and n must be the same as those entering the η equation, as q and n together determine the separation constant λ .

We have one additional condition on our choice of Mathieu functions. They must be selected in a way that leads to continuity and differentiability across the line segment $\xi = 0$. If we have chosen the ξ dependence to be Ce_n , the solution will approach $\xi = 0$ from either side with nonzero amplitude but zero slope, and that behavior will lead to continuity only if the η dependence is even (so the contributions at η and $-\eta$ will match). Thus a choice of $Ce_n(q; \xi)$ must be accompanied by a choice of $ce_n(q; \eta)$. Likewise, a choice of $Se_n(q; \xi)$, causing $\xi = 0$ to be a nodal line which is approached with nonzero slope, must be accompanied with $se_n(q; \eta)$ so the solution will change sign and have a continuous slope as it passes through the node.

The possibilities for this problem are now clearer: we choose Ce_n and ce_n or Se_n and se_n , and then vary q as needed to obtain a node in ξ at the elliptical boundary

⁴A subsection of Section 6.5 discusses defective matrices and their eigenvectors.

ξ_0 of our drumhead. Since Ce and Se are oscillatory (but not periodic), there will be an infinite number of choices of q that can meet this condition. The fundamental frequency (that with no nodes within the drumhead) will be obtained from the choice of ce_0 and Ce_0 , with the first zero of Ce_0 at $\xi = \xi_0$. Other low-lying frequencies will correspond to the first few zeros of Ce_n or of Se_n with small values of n .

The finding of the q value placing a given zero of Ce_n or Se_n at ξ_0 can in principle be done by table look-up but is probably much easier to do on a computer. Once q has been determined, the vibration frequency is readily obtained using Eqs. (32.17).

If we have a problem with inner and outer radial boundaries, as for example the oscillations of a confocal annular elliptic lake (a possibility that would only seem reasonable to a mathematician), the ODE solutions have to include the Mathieu functions of the second kind. In addition, the standing wave solutions for a lake must obey Neumann boundary conditions; that is, the normal derivatives of the water surface vanish at each point of the boundaries. Satisfying these boundary conditions may be complicated and will not be pursued further here. However, numerical examples and plots, including some for traveling waves, are given by Gutiérrez-Vega *et al.* in the Additional Readings.

For zeros of Mathieu functions, their asymptotic expansions, and a more complete listing of formulas we refer to Abramowitz and Stegun (AMS-55), Jahnke and Emde, Gutiérrez-Vega *et al.*, and Gradshteyn and Ryzhik in the Additional Readings.

Example 32.4.1. Completion of Elliptical Drum Problem

We are now ready to complete the problem of the vibrational modes of the elliptical drum, introduced in Example 32.1.1. We take the boundary at $\xi_0 = 1$; for the purpose of this illustrative example we seek only the cosine-like solutions whose η dependence is given by ce_1 . There are an infinite number of solutions of this type that differ in the number of nodes they exhibit in the ξ (radial) direction. We shall find only a few solutions with small numbers of radial nodes.

The solutions we seek will have a spatial dependence $Ce_1(q; \xi)ce_1(q; \eta)$, and our present task is to find the values of q that are consistent with the boundary condition $Ce_1(q; \xi_0) = 0$. This problem would be quite difficult in the absence of computer support, so we assume the availability of a computer program that can calculate values of $Ce_1(q; \xi)$. We start our solution by making plots of Ce_1 to give us preliminary information about its nodal structure. Figure 32.6 plots Ce_1 for $0 \leq \xi \leq 1$ for the q values 4, 10, 16, and 32. The plots show that the oscillations become more rapid as q increases, and that somewhere between $q = 0$ and $q = 4$ there will be a q value that places the first zero of Ce_1 at the boundary $\xi = 1$. Between $q = 4$ and $q = 10$ there will be another value of q that places a zero of Ce_1 at the boundary, but with one node in ξ (an elliptical node) within the drumhead. Somewhere between $q = 10$ and $q = 16$ we will again have a q value that places a zero on the boundary and yields a solution with two interior elliptical nodes. A fourth mode of oscillation, with three interior elliptical nodes, will occur at a q value between 16 and 32.

We can find the q values that satisfy the boundary conditions (we can call them **roots**) by a successive approximation process; two reasonable approaches are successive bisection of an interval known to contain exactly one root, or the Newton-Raphson (NR) method, where we linearly interpolate (or extrapolate) from previously evaluated data points. Applying the NR method, we make the initial guess $q_1 = 2$ for the smallest root, and compute $f_1 = Ce_1(q_1; \xi_0)$ (remember, $\xi_0 = 1$). Our value for f_1 is negative, meaning that there is a node within the drumhead, so we try a

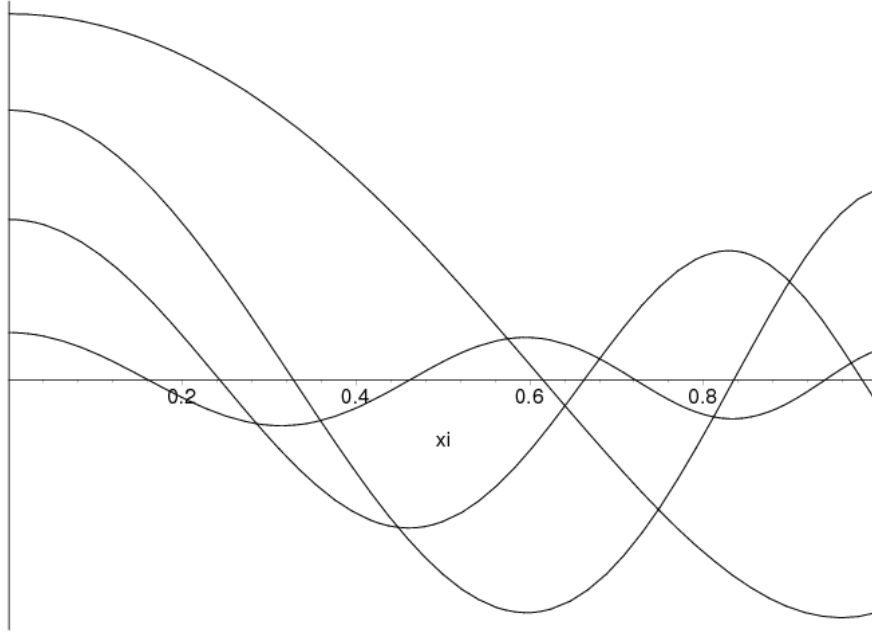


Figure 32.6: Plots of $Ce_1(q; \xi)$, showing how many nodes occur for $\xi < 1$ for $q = 4, 10, 16,$ and 32 (in decreasing order of their $\xi = 0$ intercepts on the graph). The vertical scales are arbitrary and different for each q .

smaller value of q , namely $q_2 = 1.8$, and compute the corresponding function value, f_2 . Our data at this point consist of

$$q_1 = 2, \quad f_1 = -0.074678, \quad q_2 = 1.8, \quad f_2 = +0.006504.$$

The NR interpolation gives the next trial value of q by the formula

$$q_{n+1} = q_n - \frac{(q_n - q_{n-1})f_n}{f_n - f_{n-1}}. \quad (32.77)$$

From the data for the first two points, we get $q_3 = 1.816$, yielding $f_3 = -0.000605$, and another iteration, using q_2 and q_3 , produces $q_4 = 1.8146$ and $f_4 = -0.000004$. This q value is a good approximation to that of the lowest-frequency mode of oscillation with ce_1 angular dependence.

Figure 32.6 indicates that the next ce_1 mode will have a q value between 4 and 10, and the plots suggest that the root will be near the middle of that range. Trying $q_1 = 7$, we get $f_1 = +0.010539$. For this root, the positive value for f_1 indicates that two nodes are within the drumhead, so the actual root is less than q_1 . We try $q_2 = 6.8$, getting $f_2 = +0.002326$. We haven't bracketed the root with q_1 and q_2 , but the NR method should still work. It leads to $q_3 = 6.743$ and $f_3 = -0.00024$, and on one more iteration, we get $q_4 = 6.7483$ and $f_4 = +0.000005$.

If this were an actual problem of practical interest, we could now use the physical data for the drumhead (mass, tension, and focal distance) to convert the q values into oscillation frequencies. The relevant formula is Eq. (32.17).

■

ORTHOGONALITY, SAME q

Both the Mathieu ODE and the 2-D PDE from which it arose in the elliptic drum problem are Sturm-Liouville systems, with the effect that these systems will possess orthogonal sets of solutions which can be used for the expansion of arbitrary functions. Consider here the functions $ce_m(q; \eta)$. The Mathieu ODE is self-adjoint, and if we take an interval of length 2π the periodic boundary conditions (irrespective of the parity of m) cause the ce_m of different m (and therefore different λ) to be orthogonal, with unit weight. Notice that the orthogonality of the ce_m is only established if both the functions involved are for the same value of q ; the Sturm-Liouville theory applies to eigenvalue equations for the same differential operator, which here means that the ce_m must be solutions for the same q . For a more detailed discussion of orthogonality conditions, see Eq. (8.13) and the discussion leading to that equation. A similar orthogonality condition applies to the functions se_m , and the fact that these functions have opposite parity guarantees that the ce_m are orthogonal to the se_m . These relations are summarized in the following equations:

$$\int_0^{2\pi} ce_m ce_n d\eta = \int_0^{2\pi} se_m se_n d\eta = 0, \text{ if } m \neq n; \quad (32.78)$$

$$\int_0^{2\pi} ce_m se_n d\eta = 0.$$

Because the functions $ce_m(q; \eta)$ and $se_m(q; \eta)$ are solutions to a homogeneous ODE, they can be defined to have any scale that proves convenient. When writing expansions in powers of q , we adopted a particular normalization by specifying that the term independent of q in the power series be, for ce_m , $\cos m\eta$, and for se_m , $\sin m\eta$. This normalization was convenient then, but becomes less so when making expansions in terms of these orthogonal functions, because integrals such as

$$\int_0^{2\pi} [ce_m(q; \eta)]^2 d\eta$$

then have values that are dependent upon q . A more convenient normalization for these functions is a choice that contains no q dependence. Various conventions exist; the most common, used by Whittaker and Watson and by Hochstadt (Additional Readings) is such that

$$\int_0^{2\pi} [ce_n]^2 d\eta = \int_0^{2\pi} [se_n]^2 d\eta = \pi, \text{ if } n \geq 1; \quad \int_0^{2\pi} [ce_0]^2 d\eta = 2\pi. \quad (32.79)$$

This normalization also agrees with that in AMS-55 (Additional Readings) except for ce_0 ; the normalization integral for that function in AMS-55 is set to π . The variety of choices used for normalization make it incumbent to use care when combining formulas from different sources.

If Mathieu functions are given by their Fourier series expansions at arbitrary scale, they can be given the normalization of Eq. (32.79) by multiplying the original coefficients by a factor $N^{-1/2}$, where N is given by the appropriate member of the following set of formulas:

$$N = \sum_{n=0}^{\infty} [A_{2n}^{(2m)}]^2 \quad \text{or} \quad \sum_{n=0}^{\infty} [A_{2n+1}^{(2m+1)}]^2 \quad \text{or} \quad \sum_{n=0}^{\infty} [B_{2n+1}^{(2m+1)}]^2 \quad \text{or} \quad \sum_{n=0}^{\infty} [B_{2n+2}^{(2m+2)}]^2. \quad (32.80)$$

The ce_n and the se_n separately form complete orthogonal sets on $(0, \pi)$, and each of the four subsets ce_{2n} , ce_{2n+1} , se_{2n} , and se_{2n+1} is complete and orthogonal on the smaller interval $(0, \pi/2)$. For example, the above definitions enable the following expansion (for any $q > 0$) of a function $f(\eta)$ which is periodic with period 2π :

$$f(\eta) = \frac{1}{2} a_0 ce_0(q; \eta) + \sum_{n=1}^{\infty} [\alpha_n ce_n(q; \eta) + \beta_n se_n(q; \eta)] . \quad (32.81)$$

The coefficients in Eq. (32.81) are given by

$$\begin{aligned} \alpha_n &= \frac{1}{\pi} \int_0^{2\pi} f(\eta) ce_n(q; \eta) d\eta, & n \geq 0; \\ \beta_n &= \frac{1}{\pi} \int_0^{2\pi} f(\eta) se_n(q; \eta) d\eta, & n \geq 1. \end{aligned} \quad (32.82)$$

Similar expansions exist for the ranges $(0, \pi)$ and $(0, \pi/2)$ using various subsets of ce_n and/or se_n .

The solutions of the radial Mathieu equation for the same q but different λ also exhibit orthogonality, with unit weight, on any interval for which the associated λ values cause them to satisfy Dirichlet or Neumann boundary conditions. This feature is usually not relevant when the radial and angular Mathieu functions are used together because the q values of the Ce_m or Se_m usually differ, but it does become relevant for the 2-D orthogonality to be discussed next.

ORTHOGONALITY, 2-D FUNCTIONS

In addition to the orthogonality of the ce_m and se_m for a given value of q , there are orthogonality relations over the two-dimensional space of the coordinates η and ξ . Looking back at the elliptical drum problem, Example 32.4.1, we note that its solutions could be identified as of the form

$$Z_{m,n}^e(\xi, \eta) = ce_m(q_{mn}^e; \eta) Ce_m(q_{mn}^e; \xi) \quad \text{or} \quad Z_{m,n}^o(\xi, \eta) = se_m(q_{mn}^o; \eta) Se_m(q_{mn}^o; \xi). \quad (32.83)$$

Here e and o stand respectively for “even” and “odd”; m labels the function of η and is equal to the number of oscillations of the angular functions, while q_{mn}^e (q_{mn}^o) denotes the value of q that causes Ce_m (Se_m) to have n interior radial nodes.

Considering now the PDE from which the separate ξ and η equations arose, we see that it is a two-dimensional self-adjoint equation subject to boundary conditions that make it a Sturm-Liouville system. Viewed from this perspective, the quantity q is an eigenvalue. Moreover, λ no longer corresponds to an eigenvalue; it is merely a separation constant which must have a value such that the η and ξ dependence of the solutions to the PDE are consistent with the boundary conditions. Noting also that the element of area for the scalar product is, in the elliptical coordinates, $\cosh^2 \xi - \cos^2 \eta$, which can also be written $\frac{1}{2}(\cosh 2\xi - \cos 2\eta)$, we see that there will be orthogonality conditions of the type

$$\int_0^{\xi_0} d\xi \int_0^{2\pi} d\eta Z_{mn}^e(\xi, \eta) Z_{rs}^e(\xi, \eta) (\cosh^2 \xi - \cos^2 \eta) = 0 \quad \text{when} \quad q_{mn}^e \neq q_{rs}^e. \quad (32.84)$$

We need to convert this fairly general observation into specific statements in which orthogonality is identified with the index values m, n, r, s .

Take first the case $m = r$. If $n = s$ the two functions are identical and have the same value of q . This is the normalization case and the integral of Eq. (32.84) must be nonzero. But if $n \neq s$ the functions will have different values of q and they will be orthogonal in the sense of Eq. (32.84). The other case is that $m \neq r$. If Z_{mn}^e and Z_{rs}^e have different values of q they will, in the sense of Eq. (32.84), be orthogonal. The only remaining possibility is that $m \neq r$ but n and s have values that cause Z_{mn}^e and Z_{rs}^e to have a common value of q . But in that case we know that the λ values of Z_{mn}^e and Z_{rs}^e must differ, because the equality of these λ values would indicate that we have two different periodic solutions to the same Mathieu equation.

Continuing with the case $m \neq r$, $n \neq s$, $q_{mn}^e = q_{rs}^e = q$ but $\lambda_{mn} \neq \lambda_{rs}$, we look further at the integral in Eq. (32.84). Expanding the integral into

$$\int_0^{\xi_0} \text{Ce}_m(q; \xi) \text{Ce}_r(q; \xi) \cosh^2 \xi \, d\xi \int_0^{2\pi} \text{ce}_m(q; \eta) \text{ce}_r(q; \eta) \, d\eta \\ - \int_0^{\xi_0} \text{Ce}_m(q; \xi) \text{Ce}_r(q; \xi) \, d\xi \int_0^{2\pi} \text{ce}_m(q; \eta) \text{ce}_r(q; \eta) \cos^2 \eta \, d\eta,$$

we note that the first term vanishes because of the orthogonality (with unit weight) of the η integral while the second term vanishes due to the orthogonality (again with unit weight) of the ξ integral. Summarizing,

$$\int_0^{\xi_0} d\xi \int_0^{2\pi} d\eta Z_{mn}^e(\xi, \eta) Z_{rs}^e(\xi, \eta) (\cosh^2 \xi - \cos^2 \eta) = 0 \quad \text{unless } m = r \text{ and } n = s. \quad (32.85)$$

A parallel result can be obtained for the odd 2-D functions, and the even-odd symmetry causes all the Z^e to be orthogonal to all the Z^o . Thus,

$$\int_0^{\xi_0} d\xi \int_0^{2\pi} d\eta Z_{mn}^o(\xi, \eta) Z_{rs}^o(\xi, \eta) (\cosh^2 \xi - \cos^2 \eta) = 0 \quad \text{unless } m = r \text{ and } n = s, \quad (32.86)$$

$$\int_0^{\xi_0} d\xi \int_0^{2\pi} d\eta Z_{mn}^e(\xi, \eta) Z_{rs}^o(\xi, \eta) (\cosh^2 \xi - \cos^2 \eta) = 0 \quad \text{all } m, n, r, s. \quad (32.87)$$

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