Chapter 1

Chap1

INFINITE SERIES

This on-line chapter contains the material on infinite series, extracted from the printed version of the Seventh Edition and presented in much the same organization in which it appeared in the Sixth Edition. It is collected here for the convenience of instructors who wish to use it as introductory material in place of that in the printed book. It has been lightly edited to remove detailed discussions involving complex variable theory that would not be appropriate until later in a course of instruction. For Additional Readings, see the printed text.

Sec1.1

1.1 INTRODUCTION TO INFINITE SERIES

Perhaps the most widely used technique in the physicist's toolbox is the use of **infinite series** (i.e. sums consisting formally of an infinite number of terms) to represent functions, to bring them to forms facilitating further analysis, or even as a prelude to numerical evaluation. The acquisition of skill in creating and manipulating series expansions is therefore an absolutely essential part of the training of one who seeks competence in the mathematical methods of physics, and it is therefore the first topic in this text. An important part of this skill set is the ability to recognize the functions represented by commonly encountered expansions, and it is also of importance to understand issues related to the convergence of infinite series.

FUNDAMENTAL CONCEPTS

The usual way of assigning a meaning to the sum of an infinite number of terms is by introducing the notion of partial sums. If we have an infinite sequence of terms u_1, u_2, u_3, u_4, u_5,..., we define the i-th partial sum as

$$s_i = \sum_{n=1}^{i} u_n \ . \tag{1.1}$$ eq1.1

This is a finite summation and offers no difficulties. If the partial sums s_i converge to a finite limit as $i \to \infty$,

$$\lim_{i\to\infty} s_i = S \ , \tag{1.2}$$ eq1.2

the infinite series $\sum_{n=1}^{\infty} u_n$ is said to be **convergent** and to have the value S. Note that we **define** the infinite series as equal to S and that a necessary condition for convergence to a limit is that $\lim_{n\to\infty} u_n = 0$. This condition, however, is not sufficient to guarantee convergence.

Sometimes it is convenient to apply the condition in Eq. (1.2) in a form called the **Cauchy criterion**, namely that for each $\varepsilon > 0$ there is a fixed number N such that $|s_j - s_i| < \varepsilon$ for all i and j greater than N. This means that the partial sums must cluster together as we move far out in the sequence.

Some series **diverge**, meaning that the sequence of partial sums approaches $\pm\infty$; others may have partial sums that oscillate between two values, as for example

$$\sum_{n=1}^{\infty} u_n = 1 - 1 + 1 - 1 + 1 - \cdots - (-1)^n + \cdots \ .$$

This series does not converge to a limit, and can be called **oscillatory**. Often the term *divergent* is extended to include oscillatory series as well. It is important to be able to determine whether, or under what conditions, a series we would like to use is convergent.

Example 1.1.1. The Geometric Series

The geometric series, starting with $u_0 = 1$ and with a ratio of successive terms $r = u_{n+1}/u_n$, has the form

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} + \cdots \ .$$

Its n-th partial sum s_n(that of the first n terms) is[1]

$$s_n = \frac{1 - r^n}{1 - r} \ . \tag{1.3}$$

eq1.3

Restricting attention to $|r| < 1$, so that for large n, r^n approaches zero, s_n possesses the limit

$$\lim_{n\to\infty} s_n = \frac{1}{1 - r} \ , \tag{1.4}$$

eq1.4

showing that for $|r| < 1$, the geometric series converges. It clearly diverges (or is oscillatory) for $|r| \geq 1$, as the individual terms do not then approach zero at large n.

■

Example 1.1.2. The Harmonic Series

As a second and more involved example, we consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots \ . \tag{1.5}$$

eq1.5

The terms approach zero for large n, i.e. $\lim_{n\to\infty} 1/n = 0$, but this is not sufficient to guarantee convergence. If we group the terms (without changing their order) as

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \ ,$$

[1]Multiply and divide $s_n = \sum_{m=0}^{n-1} r^m$ by $1 - r$.

each pair of parentheses encloses p terms of the form

$$\frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{p+p} > \frac{p}{2p} = \frac{1}{2} \, .$$

Forming partial sums by adding the parenthetical groups one by one, we obtain

$$s_1 = 1, \quad s_2 = \frac{3}{2} \, , \quad s_3 > \frac{4}{2} \, , \quad s_4 > \frac{5}{2} \, , \ldots , \quad s_n > \frac{n+1}{2} \, ,$$

and we are forced to the conclusion that the harmonic series diverges.

Although the harmonic series diverges, its partial sums have relevance among other places in number theory, where $H_n = \sum_{m=1}^{n} m^{-1}$ are sometimes referred to as **harmonic numbers**.

■

We now turn to a more detailed study of the convergence and divergence of series, considering here series of positive terms. Series with terms of both signs are treated later.

COMPARISON TEST

If term by term a series of terms u_n satisfies $0 \le u_n \le a_n$, where the a_n form a convergent series, then the series $\sum_n u_n$ is also convergent. Letting s_i and s_j be partial sums of the u series, with $j > i$, the difference $s_j - s_i$ is $\sum_{n=i+1}^{j} u_n$, and this is smaller than the corresponding quantity for the a series, thereby proving convergence. A similar argument shows that if term by term a series of terms v_n satisfies $0 \le b_n \le v_n$, where the b_n form a divergent series, then $\sum_n v_n$ is also divergent.

For the convergent series a_n we already have the geometric series, whereas the harmonic series will serve as the divergent comparison series b_n. As other series are identified as either convergent or divergent, they may also be used as the known series for comparison tests.

Example 1.1.3. A Divergent Series

Test $\sum_{n=1}^{\infty} n^{-p}$, $p = 0.999$, for convergence. Since $n^{-0.999} > n^{-1}$ and $b_n = n^{-1}$ forms the divergent harmonic series, the comparison test shows that $\sum_n n^{-0.999}$ is divergent. Generalizing, $\sum_n n^{-p}$ is seen to be divergent for all $p \le 1$.

■

CAUCHY ROOT TEST

If $(a_n)^{1/n} \le r < 1$ for all sufficiently large n, with r independent of n, then $\sum_n a_n$ is convergent. If $(a_n)^{1/n} \ge 1$ for all sufficiently large n, then $\sum_n a_n$ is divergent.

The language of this test emphasizes an important point: the convergence or divergence of a series depends entirely upon what happens for large n. Relative to convergence, it is the behavior in the large-n limit that matters.

The first part of this test is verified easily by raising $(a_n)^{1/n}$ to the nth power. We get

$$a_n \le r^n < 1 \, .$$

Since r^n is just the nth term in a convergent geometric series, $\sum_n a_n$ is convergent by the comparison test. Conversely, if $(a_n)^{1/n} \geq 1$, then $a_n \geq 1$ and the series must diverge. This root test is particularly useful in establishing the properties of power series (Section 1.2).

D'ALEMBERT (OR CAUCHY) RATIO TEST

If $a_{n+1}/a_n \leq r < 1$ for all sufficiently large n and r is independent of n, then $\sum_n a_n$ is convergent. If $a_{n+1}/a_n \geq 1$ for all sufficiently large n, then $\sum_n a_n$ is divergent.

This test is established by direct comparison with the geometric series $(1+r+r^2+\cdots)$. In the second part, $a_{n+1} \geq a_n$ and divergence should be reasonably obvious. Although not quite as sensitive as the Cauchy root test, this D'Alembert ratio test is one of the easiest to apply and is widely used. An alternate statement of the ratio test is in the form of a limit: If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \begin{cases} < 1, & \text{convergence,} \\ > 1, & \text{divergence,} \\ = 1, & \text{indeterminate.} \end{cases} \tag{1.6}$$ eq1.6

Because of this final indeterminate possibility, the ratio test is likely to fail at crucial points, and more delicate, sensitive tests then become necessary. The alert reader may wonder how this indeterminacy arose. Actually it was concealed in the first statement, $a_{n+1}/a_n \leq r < 1$. We might encounter $a_{n+1}/a_n < 1$ for all **finite** n but be unable to choose an $r < 1$ **and independent of n** such that $a_{n+1}/a_n \leq r$ for all sufficiently large n. An example is provided by the harmonic series, for which

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} < 1 \ .$$

Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \ ,$$

no fixed ratio $r < 1$ exists and the test fails.

Example 1.1.4. D'Alembert Ratio Test

Test $\sum_n n/2^n$ for convergence. Applying the ratio test,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{1}{2}\,\frac{n+1}{n} \ .$$

Since

$$\frac{a_{n+1}}{a_n} \leq \frac{3}{4} \quad \text{for } n \geq 2,$$

we have convergence.

■

CAUCHY (OR MACLAURIN) INTEGRAL TEST

This is another sort of comparison test, in which we compare a series with an integral. Geometrically, we compare the area of a series of unit-width rectangles with the area under a curve.

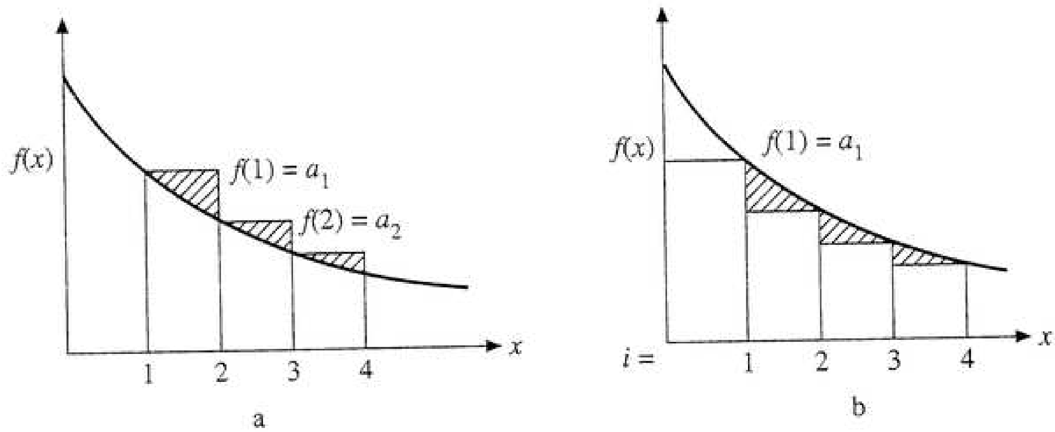

Figure 1.1: (a) Comparison of integral and sum-blocks leading. (b) Comparison of integral and sum-blocks lagging.

Fig1.1

Let $f(x)$ be a continuous, **monotonic decreasing function** in which $f(n) = a_n$. Then $\sum_n a_n$ converges if $\int_1^\infty f(x)dx$ is finite and diverges if the integral is infinite. The ith partial sum is

$$s_i = \sum_{n=1}^{i} a_n = \sum_{n=1}^{i} f(n) \; .$$

But, because $f(x)$ is monotonic decreasing, see Fig. 1.1(a),

$$s_i \geq \int_1^{i+1} f(x)dx \; .$$

On the other hand, as shown in Fig. 1.1(b),

$$s_i - a_1 \leq \int_1^{i} f(x)dx \; .$$

Taking the limit as $i \to \infty$, we have

$$\int_1^\infty f(x)dx \leq \sum_{n=1}^{\infty} a_n \leq \int_1^\infty f(x)dx + a_1 \; . \tag{1.7}$$

eq1.7

Hence the infinite series converges or diverges as the corresponding integral converges or diverges.

This integral test is particularly useful in setting upper and lower bounds on the remainder of a series after some number of initial terms have been summed. That is,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n \; , \tag{1.8}$$

eq1.8

and

$$\int_{N+1}^\infty f(x)\,dx \leq \sum_{n=N+1}^{\infty} a_n \leq \int_{N+1}^\infty f(x)\,dx + a_{N+1} \; . \tag{1.9}$$

eq1.9

To free the integral test from the quite restrictive requirement that the interpolating function $f(x)$ be positive and monotonic, we shall show that for any function $f(x)$ with a continuous derivative, the infinite series is exactly represented as a sum of two integrals:

$$\sum_{n=N1+1}^{N_2} f(n) = \int_{N_1}^{N_2} f(x)dx + \int_{N_1}^{N_2} (x - [x])f'(x)dx \ . \tag{1.10}$$

eq1.10

Here $[x]$ is the integral part of x, i.e. the largest integer $\le x$, so $x - [x]$ varies sawtoothlike between 0 and 1. Equation ((1.10) is useful because if both integrals in Eq. (1.10) converge, the infinite series also converges, while if one integral converges and the other does not, the infinite series diverges. If both integrals diverge, the test fails unless it can be shown whether the divergences of the integrals cancel against each other.

We need now to establish Eq. (1.10). We manipulate the contributions to the second integral as follows:

(1) Using integration by parts, we observe that

$$\int_{N_1}^{N_2} xf'(x)dx = N_2 f(N_2) - N_1 f(N_1) - \int_{N_1}^{N_2} f(x)dx \ .$$

(2) We evaluate

$$\int_{N_1}^{N_2} [x]f'(x)dx = \sum_{n=N_1}^{N_2-1} n \int_{n}^{n+1} f'(x)dx = \sum_{n=N_1}^{N_2-1} n\Big[f(n+1) - f(n)\Big]$$

$$= - \sum_{n=N_1+1}^{N_2} f(n) - N_1 f(N_1) + N_2 f(N_2) \ .$$

Subtracting the second of these equations from the first, we arrive at Eq. (1.10).

An alternative to Eq. (1.10) in which the second integral has its sawtooth shifted to be symmetrical about zero (and therefore perhaps smaller) can be derived by methods similar to those used above. The resulting formula is

$$\sum_{n=N1+1}^{N_2} f(n) = \int_{N_1}^{N_2} f(x)dx + \int_{N_1}^{N_2} (x - [x] - \tfrac{1}{2})f'(x)dx + \tfrac{1}{2}\Big[f(N_2) - f(N_1)\Big] \ . \tag{1.11}$$

eq1.11

Because they do not use a monotonicity requirement, Eqs. (1.10) and (1.11) can be applied to alternating series, and even those with irregular sign sequences.

Example 1.1.5. Riemann Zeta Function

The Riemann zeta function is defined by

$$\zeta(p) = \sum_{n=1}^{\infty} n^{-p} \ , \tag{1.12}$$

eq1.12

providing the series converges. We may take $f(x) = x^{-p}$, and then

$$\int_1^\infty x^{-p}\,dx = \left.\frac{x^{-p+1}}{-p+1}\right|_{x=1}^{\infty}, \qquad p \neq 1,$$

$$= \ln x\Big|_{x=1}^{\infty}, \qquad p = 1\,.$$

The integral and therefore the series are divergent for $p \leq 1$, and convergent for $p > 1$. Hence Eq. (1.12) should carry the condition $p > 1$. This, incidentally, is an independent proof that the harmonic series ($p = 1$) diverges logarithmically. The sum of the first million terms $\sum_{n=1}^{1{,}000{,}000} n^{-1}$ is only $14.392\,726\cdots$.

■

While the harmonic series diverges, the combination

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n} m^{-1} - \ln n\right) \tag{1.13}$$

eq1.12a

does converge, approaching a limit known as the **Euler-Mascheroni constant**.

Exam1.1.6

Example 1.1.6. A Slowly Diverging Series

Consider now the series

$$S = \sum_{n=2}^{\infty} \frac{1}{n \ln n}\,.$$

We form the integral

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \int_{x=2}^{\infty} \frac{d\ln x}{\ln x} = \ln\ln x\Big|_{x=2}^{\infty},$$

which diverges, indicating that S is divergent. Notice that the lower limit of the integral is in fact unimportant so long as it does not introduce any spurious singularities, as it is the large-x behavior that determines the convergence. Because $n \ln n > n$, the divergence is slower than that of the harmonic series. But because $\ln n$ increases more slowly than n^ε, where ε can have an arbitrarily small positive value, we have divergence even though the series $\sum_n n^{-(1+\varepsilon)}$ converges.

■

MORE SENSITIVE TESTS

Several tests more sensitive than those already examined are consequences of a theorem by Kummer. Kummer's theorem, which deals with two series of finite positive terms: u_n and a_n, states:

1. The series $\sum_n u_n$ converges if

$$\lim_{n\to\infty}\left(a_n \frac{u_n}{u_{n+1}} - a_{n+1}\right) \geq C > 0\,, \tag{1.14}$$

eq1.13

where C is a constant. This statement is equivalent to a simple comparison test if the series $\sum_n a_n^{-1}$ converges, and imparts new information only if that sum diverges. The more weakly $\sum_n a_n^{-1}$ diverges, the more powerful the Kummer test will be.

2. If $\sum_n a_n^{-1}$ diverges and

$$\lim_{n\to\infty}\left(a_n \frac{u_n}{u_{n+1}} - a_{n+1}\right) \le 0 \ , \tag{1.15}$$ eq1.14

then $\sum_n u_n$ diverges.

The proof of this powerful test is remarkably simple. Part 2 follows immediately from the comparison test. To prove Part 1, write cases of Eq. (1.14) for $n = N+1$ through any larger n, in the following form:

$$\begin{aligned} u_{N+1} &\le (a_N u_N - a_{N+1}u_{N+1})/C \ , \\ u_{N+2} &\le (a_{N+1}u_{N+1} - a_{N+2}u_{N+2})/C \ , \\ \cdots &\le \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ , \\ u_n &\le (a_{n-1}u_{n-1} - a_n u_n)/C \ . \end{aligned}$$

Adding, we get

$$\sum_{i=N+1}^{n} u_i \le \frac{a_N u_N}{C} - \frac{a_n u_n}{C} \tag{1.16}$$

$$< \frac{a_N u_N}{C} \ . \tag{1.17}$$ eq1.15

This shows that the tail of the series $\sum_n u_n$ is bounded, and that series is therefore proved convergent when Eq. (1.14) is satisfied for all sufficiently large n.

Gauss's Test is an application of Kummer's theorem to series $u_n > 0$ when the ratios of successive u_n approach unity and the tests previously discussed yield indeterminate results. If for large n

$$\frac{u_n}{u_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^2} \ , \tag{1.18}$$ eq1.16

where $B(n)$ is bounded for n sufficiently large, then the Gauss test states that $\sum_n u_n$ converges for $h > 1$ and diverges for $h \le 1$: There is no indeterminate case here.

The Gauss test is extremely sensitive, and will work for all troublesome series the physicist is likely to encounter. To confirm it using Kummer's theorem, we take $a_n = n \ln n$. The series $\sum_n a_n^{-1}$ is weakly divergent, as already established in Example 1.1.6.

Taking the limit on the left side of Eq. (1.14), we have

$$\begin{aligned} &\lim_{n\to\infty}\left[n \ln n \left(1 + \frac{h}{n} + \frac{B(n)}{n^2}\right) - (n+1)\ln(n+1)\right] \\ &= \lim_{n\to\infty}\left[(n+1)\ln n + (h-1)\ln n + \frac{B(n)\ln n}{n} - (n+1)\ln(n+1)\right] \\ &= \lim_{n\to\infty}\left[-(n+1)\ln\left(\frac{n+1}{n}\right) + (h-1)\ln n\right] \ . \end{aligned} \tag{1.19}$$ eq1.17

For $h < 1$, both terms of Eq. (1.19) are negative, thereby signalling a divergent case of Kummer's theorem; for $h > 1$, the second term of Eq. (1.19) dominates the first and is positive, indicating convergence. At $h = 1$, the second term vanishes, and the first is inherently negative, thereby indicating divergence.

Ex1.1.7

Example 1.1.7. Legendre Series

The series solution for the Legendre equation (encountered in Chapter 7 has successive terms whose ratio under certain conditions is

$$\frac{a_{2j+2}}{a_{2j}} = \frac{2j(2j+1)-\lambda}{(2j+1)(2j+2)} .$$

To place this in the form now being used, we define $u_j = a_{2j}$ and write

$$\frac{u_j}{u_{j+1}} = \frac{(2j+1)(2j+2)}{2j(2j+1)-\lambda} .$$

In the limit of large j, the constant λ becomes negligible (in the language of the Gauss test, it contributes to an extent $B(j)/j^2$, where $B(j)$ is bounded). We therefore have

$$\frac{u_j}{u_{j+1}} \to \frac{2j+2}{2j} + \frac{B(j)}{j^2} = 1 + \frac{1}{j} + \frac{B(j)}{j^2} . \qquad (1.20)$$ eq1.18

The Gauss test tells us that this series is divergent.

■

Exercises

1.1.1. (a) Prove that if $\lim_{n\to\infty} n^p u_n = A < \infty, \quad p > 1,$ the series $\sum_{n=1}^{\infty} u_n$ converges.

(b) Prove that if $\lim_{n\to\infty} n u_n = A > 0,$ the series diverges. (The test fails for $A = 0$.)

These two tests, known as **limit tests**, are often convenient for establishing the convergence of a series. They may be treated as comparison tests, comparing with

$$\sum_n n^{-q}, \quad 1 \le q < p.$$

1.1.2. If $\lim_{n\to\infty} \frac{b_n}{a_n} = K,$ a constant with $0 < K < \infty$, show that $\Sigma_n b_n$ converges or diverges with Σa_n.

Hint. If Σa_n converges, rescale b_n to $b'_n = \frac{b_n}{2K}$. If $\Sigma_n a_n$ diverges, rescale to $b''_n = \frac{2b_n}{K}$.

1.1.3. (a) Show that the series $\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^2}$ converges.

(b) By direct addition $\sum_2^{100,000}[n(\ln n)^2]^{-1} = 2.02288$. Use Eq. (1.9) to make a five-significant-figure estimate of the sum of this series.

1.1.4. Gauss's test is often given in the form of a test of the ratio

$$\frac{u_n}{u_{n+1}} = \frac{n^2 + a_1 n + a_0}{n^2 + b_1 n + b_0}.$$

For what values of the parameters a_1 and b_1 is there convergence? divergence?

ANS. Convergent for $a_1 - b_1 > 1$,
divergent for $a_1 - b_1 \leq 1$.

1.1.5. Test for convergence

(a) $\displaystyle\sum_{n=2}^{\infty}(\ln n)^{-1}$ (d) $\displaystyle\sum_{n=1}^{\infty}[n(n+1)]^{-1/2}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{10^n}$ (e) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2n+1}$.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n(2n+1)}$

1.1.6. Test for convergence

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ (d) $\displaystyle\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ (e) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\cdot n^{1/n}}$.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n2^n}$

1.1.7. For what values of p and q will $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^p(\ln n)^q}$ converge?

ANS. Convergent for $\begin{cases} p > 1, & \text{all } q, \\ p = 1, & q > 1, \end{cases}$ divergent for $\begin{cases} p < 1, & \text{all } q, \\ p = 1, & q \leq 1. \end{cases}$

1.1.8. Given $\sum_{n=1}^{1,000} n^{-1} = 7.485\ 470\ldots$ set upper and lower bounds on the Euler-Mascheroni constant.

ANS. $0.5767 < \gamma < 0.5778$.

1.1.9. (From **Olbers' paradox**.) Assume a static universe in which the stars are uniformly distributed. Divide all space into shells of constant thickness; the stars in any one shell by themselves subtend a solid angle of ω_0. **Allowing for the blocking out of distant stars by nearer stars**, show that the total net solid angle subtended by all stars, shells extending to infinity, is **exactly** 4π. [Therefore the night sky should be ablaze with light. For more details, see E. Harrison, *Darkness at Night: A Riddle of the Universe.* Cambridge, MA: Harvard University Press (1987).]

1.1.10. Test for convergence

$$\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^2 = \frac{1}{4}+\frac{9}{64}+\frac{25}{256}+\cdots .$$

ALTERNATING SERIES

In previous subsections we limited ourselves to series of positive terms. Now, in contrast, we consider infinite series in which the signs alternate. The partial cancellaton due to alternating signs makes converegence more rapid and much easier to identify. We shall prove the Leibniz criterion, a general condition for the convergence of an alternating series. For series with more irregular sign changes, the integral test of Eq. (1.10) is often helpful.

The **Leibniz criterion** applies to series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ with $a_n > 0$, and states that if a_n is *monotonically decreasing* (for sufficiently large n) and $\lim_{n\to\infty} a_n = 0$, then the series converges. To prove this theorem, notice that the remainder R_{2n} of the series beyond s_{2n}, the partial sum after $2n$ terms, can be written in two alternate ways:

$$\begin{aligned} R_{2n} &= (a_{2n+1}-a_{2n+2})+(a_{2n+3}-a_{2n+4})+\cdots \\ &= a_{2n+1}-(a_{2n+2}-a_{2n+3})-(a_{2n+4}-a_{2n+5})-\cdots . \end{aligned}$$

Since the a_n are decreasing, the first of these equations implies $R_{2n} > 0$, while the second implies $R_{2n} < a_{2n+1}$, so

$$0 < R_{2n} < a_{2n+1} .$$

Thus, R_{2n} is positive but bounded, and the bound can be made arbitrarily small by taking larger values of n. This demonstration also shows that the error from truncating an alternating series after a_{2n} results in an error that is negative (the omitted terms were shown to combine to a positive result) and bounded in magnitude by a_{2n+1}. An argument similar to that made above for the remainder after an odd number of terms, R_{2n+1}, would show that the error from truncation after a_{2n+1} is positive and bounded by a_{2n+2}. Thus, it is generally true that the error in truncating an alternating series with monotonically decreasing terms is of the same sign as the last term kept and smaller than the first term dropped.

The Leibniz criterion depends for its applicability on the presence of strict sign alternation. Less regular sign changes present more challenging problems for convergence determination.

Example 1.1.8. Series with Irregular Sign Changes

For $0 < x < 2\pi$ the series

$$S = \sum_{n=1}^{\infty}\frac{\cos(nx)}{n} = -\ln\left(2\sin\frac{x}{2}\right) \qquad (1.21)$$ eq1.19

converges, having coefficients that change sign often, but not so that the Leibniz criterion applies easily. To verify the convergence, we apply the integral test of

Eq. (1.10), inserting the explicit form for the derivative of $\cos(nx)/n$ (with respect to n) in the second integral:

$$S = \int_1^\infty \frac{\cos(nx)}{n}\,dn + \int_1^\infty \Big(n - [n]\Big)\left[-\frac{x}{n}\sin(nx) - \frac{\cos(nx)}{n^2}\right] dn\ . \qquad (1.22)$$ eq1.20

Using integration by parts, the first integral in Eq. (1.22) is rearranged to

$$\int_1^\infty \frac{\cos(nx)}{n}\,dn = \left[\frac{\sin(nx)}{nx}\right]_1^\infty + \frac{1}{x}\int_1^\infty \frac{\sin(nx)}{n^2}\,dn\ ,$$

and this integral converges because

$$\left|\int_1^\infty \frac{sin(nx)}{n^2}\,dn\right| < \int_1^\infty \frac{dn}{n^2} = 1\ .$$

Looking now at the second integral in Eq. (1.22), we note that its term $\cos(nx)/n^2$ also leads to a convergent integral, so we need only to examine the convergence of

$$\int_1^\infty \Big(n - [n]\Big)\frac{\sin(nx)}{n}\,dn\ .$$

Next, setting $(n-[n])\sin(nx) = g'(n)$, which is equivalent to defining $g(N) = \int_1^N (n-[n])\sin(nx)dn$, we write

$$\int_1^\infty \Big(n - [n]\Big)\frac{\sin(nx)}{n}\,dn = \int_1^\infty \frac{g'(n)}{n}\,dn = \left[\frac{g(n)}{n}\right]_{n=1}^\infty + \int_1^\infty \frac{g(n)}{n^2}\,dn\ ,$$

where the last equality was obtained using once again an integration by parts. We do not have an explicit expression for $g(n)$, but we do know that it is bounded because $\sin x$ oscillates with a period incommensurate with that of the sawtooth periodicity of $n - [n]$). This boundedness enables us to determine that the second integral in Eq. (1.22) converges, thus establishing the convergence of S.

■

ABSOLUTE AND CONDITIONAL CONVERGENCE

An infinite series is **absolutely** convergent if the absolute values of its terms form a convergent series. If it converges, but not absolutely, it is termed **conditionally** convergent. An example of a conditionally convergent series is the alternating harmonic series,

$$\sum_{n=1}^\infty (-1)^{n-1} n^{-1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots\ . \qquad (1.23)$$ eq1.21

This series is convergent, based on the Leibniz criterion. It is clearly not absolutely convergent; if all terms are taken with + signs, we have the harmonic series, which we already know to be divergent. The tests described earlier in this section for series of positive terms are, then, tests for absolute convergence.

Exercises

1.1.11. Determine whether each of these series is convergent, and if so, whether it is absolutely convergent:

(a) $\dfrac{\ln 2}{2} - \dfrac{\ln 3}{3} + \dfrac{\ln 4}{4} - \dfrac{\ln 5}{5} + \dfrac{\ln 6}{6} - \cdots$,

(b) $\dfrac{1}{1} + \dfrac{1}{2} - \dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6} - \dfrac{1}{7} - \dfrac{1}{8} + \cdots$,

(c) $1 - \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6} - \dfrac{1}{7} - \dfrac{1}{8} - \dfrac{1}{9} - \dfrac{1}{10} + \dfrac{1}{11} \cdots + \dfrac{1}{15} - \dfrac{1}{16} \cdots - \dfrac{1}{21} + \cdots$.

Ex1.1.12 **1.1.12. Catalan's constant** $\beta(2)$ is defined by

$$\beta(2) = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} \cdots .$$

Calculate $\beta(2)$ to six-digit accuracy.

Hint. The rate of convergence is enhanced by pairing the terms:

$$(4k-1)^{-2} - (4k+1)^{-2} = \frac{16k}{(16k^2-1)^2}.$$

If you have carried enough digits in your summation, $\sum_{1 \le k \le N} 16k/(16k^2-1)^2$, additional significant figures may be obtained by setting upper and lower bounds on the tail of the series, $\sum_{k=N+1}^{\infty}$. These bounds may be set by comparison with integrals, as in the Maclaurin integral test.

ANS. $\beta(2) = 0.9159\ 6559\ 4177 \cdots$.

OPERATIONS ON SERIES

We now investigate the operations that may be performed on infinite series. In this connection the establishment of absolute convergence is important, because it can be proved that the terms of an absolutely convergent series may be reordered according to the familiar rules of algebra or arithmetic:

- If an infinite series is absolutely convergent, the series sum is independent of the order in which the terms are added.
- An absolutely convergent series may be added termwise to, or subtracted termwise from, or multiplied termwise with another absolutely convergent series, and the resulting series will also be absolutely convergent.
- The series (as a whole) may be multiplied with another absolutely convergent series. The limit of the product will be the product of the individual series limits. The product series, a double series, will also converge absolutely.

No such guarantees can be given for conditionally convergent series, though some of the above properties remain true if only ne of the series to be combined is conditionally convergent.

Example 1.1.9. Rearrangement of Alternating Harmonic Series

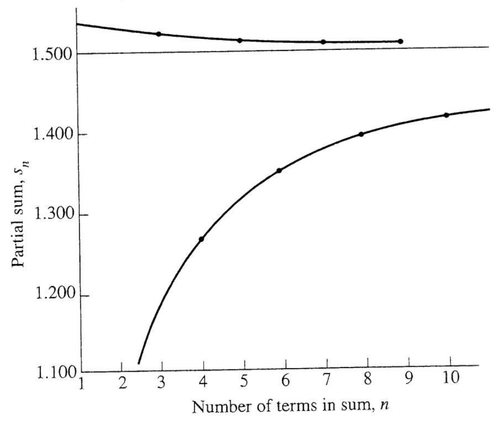

Figure 1.2: Alternating harmonic series—terms rearranged to give convergence to 1.5.

Fig1.2

Writing the alternating harmonic series as

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = 1 - (\tfrac{1}{2} - \tfrac{1}{3}) - (\tfrac{1}{4} - \tfrac{1}{5}) - \cdots \,, \tag{1.24}$$ eq1.22

it is clear that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1} < 1$. However, if we rearrange the order of the terms, we can make this series converge to $\frac{3}{2}$. We regroup the terms of Eq. (1.24), as

$$\begin{aligned}(1 + \tfrac{1}{3} + \tfrac{1}{5}) - (\tfrac{1}{2}) + (\tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{15}) - (\tfrac{1}{4})\\ + (\tfrac{1}{17} + \cdots + \tfrac{1}{25}) - (\tfrac{1}{6}) + (\tfrac{1}{27} + \cdots + \tfrac{1}{35}) - (\tfrac{1}{8}) + \cdots \,.\end{aligned} \tag{1.25}$$ eq1.23

Treating the terms grouped in parentheses as single terms for convenience, we obtain the partial sums

$$\begin{aligned} s_1 &= 1.5333 & s_2 &= 1.0333\\ s_3 &= 1.5218 & s_4 &= 1.2718\\ s_5 &= 1.5143 & s_6 &= 1.3476\\ s_7 &= 1.5103 & s_8 &= 1.3853\\ s_9 &= 1.5078 & s_{10} &= 1.4078\,. \end{aligned}$$

From this tabulation of s_n and the plot of s_n versus n in Fig. 1.2, the convergence to $\frac{3}{2}$ is fairly clear. Our rearrangement was to take positive terms until the partial sum was equal to or greater than $\frac{3}{2}$ and then to add negative terms until the partial sum just fell below $\frac{3}{2}$ and so on. As the series extends to infinity, all original terms will eventually appear, but the partial sums of this rearranged alternating harmonic series converge to $\frac{3}{2}$.

■

As the example shows, by a suitable rearrangement of terms, a conditionally convergent series may be made to converge to any desired value or even to diverge. This statement is sometimes called **Riemann's theorem**.

Another example shows the danger of multiplying conditionally convergent series.

Example 1.1.10. Square of a Conditionally Convergent Series May Diverge

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges, by the Leibniz criterion. Its square,

$$\left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}\right]^2 = \sum_n (-1)^n \left[\frac{1}{\sqrt{1}}\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n-2}} + \cdots + \frac{1}{\sqrt{n-1}}\frac{1}{\sqrt{1}}\right] ,$$

has a general term, in $[\ldots]$, consisting of $n-1$ additive terms, each of which is bigger than $\frac{1}{\sqrt{n-1}\sqrt{n-1}}$, so the entire $[\ldots]$ term is greater than $\frac{n-1}{n-1}$ and does not go to zero. Hence the general term of this product series does not approach zero in the limit of large n and the series diverges.

■

These examples show that conditionally convergent series must be treated with caution.

IMPROVEMENT OF CONVERGENCE

This section so far has been concerned with establishing convergence as an abstract mathematical property. In practice, the **rate** of convergence may be of considerable importance. A method for improving convergence, due to Kummer, is to form a linear combination of our slowly converging series and one or more series whose sum is known. For the known series the following collection is particularly useful:

$$\alpha_1 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 ,$$

$$\alpha_2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4} ,$$

$$\alpha_3 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{18} ,$$

....

$$\alpha_p = \sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+p)} = \frac{1}{p\,p!} . \qquad (1.26)$$ eq1.24

These sums can be evaluated via partial fraction expansions, and are the subject of Exercise 1.5.3.

The series we wish to sum and one or more known series (multiplied by coefficients) are combined term by term. The coefficients in the linear combination are chosen to cancel the most slowly converging terms.

Exam1.1.11 Example 1.1.11. Riemann Zeta Function $\zeta(3)$

From the definition in Eq. (1.12), we identify $\zeta(3)$ as $\sum_{n=1}^{\infty} n^{-3}$. Noticing that α_2 of Eq. (1.26) has a large-n dependence $\sim n^{-3}$, we consider the linear combination

$$\sum_{n=1}^{\infty} n^{-3} + a\alpha_2 = \zeta(3) + \frac{a}{4} \,. \tag{1.27}$$

eq1.25

We did not use α_1 because it converges more slowly than $\zeta(3)$. Combining the two series on the left-hand side termwise, we obtain

$$\sum_{n=1}^{\infty} \left[\frac{1}{n^3} + \frac{a}{n(n+1)(n+2)} \right] = \sum_{n=1}^{\infty} \frac{n^2(1+a) + 3n + 2}{n^3(n+1)(n+2)} \,.$$

If we choose $a = -1$, we remove the leading term from the numerator; then, setting this equal to the right-hand side of Eq. (1.27) and solving for $\zeta(3)$,

$$\zeta(3) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{3n+2}{n^3(n+1)(n+2)} \,. \tag{1.28}$$

eq1.26

The resulting series may not be beautiful but it does converge as n^{-4}, faster than n^{-3}. A more convenient form with even faster convergence is introduced in Exercise 1.1.16. There, the symmetry leads to convergence as n^{-5}.

■

Sometimes it is helpful to use the Riemann zeta function in a way similar to that illustrated for the α_p in the foregoing example. That approach is practical because the zeta function has been tabulated (see Table 1.1).

Example 1.1.12. Convergence Improvement

The problem is to evaluate the series $\sum_{n=1}^{\infty} 1/(1+n^2)$. Expanding $(1+n^2)^{-1} = n^{-2}(1+n^{-2})^{-1}$ by direct division, we have

$$\begin{aligned} (1+n^2)^{-1} &= n^{-2}\left(1 - n^{-2} + n^{-4} - \frac{n^{-6}}{1+n^{-2}}\right) \\ &= \frac{1}{n^2} - \frac{1}{n^4} + \frac{1}{n^6} - \frac{1}{n^8+n^6}. \end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \zeta(2) - \zeta(4) + \zeta(6) - \sum_{n=1}^{\infty} \frac{1}{n^8+n^6}.$$

The remainder series converges as n^{-8}. Clearly, the process can be continued as desired. You make a choice between how much algebra you will do and how much arithmetic the computer will do.

■

Tab1.1

Table 1.1: Riemann Zeta Function

s	$\zeta(s)$
2	1.64493 40668
3	1.20205 69032
4	1.08232 32337
5	1.03692 77551
6	1.01734 30620
7	1.00834 92774
8	1.00407 73562
9	1.00200 83928
10	1.00099 45751

REARRANGEMENT OF DOUBLE SERIES

An absolutely convergent double series (one whose terms are identified by two summation indices) presents interesting rearrangement opportunities. Consider

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{n,m} \ . \tag{1.29}$$

eq1.27

In addition to the obvious possibility of reversing the order of summation (i.e. doing the m sum first), we can make rearrangements that are more innovative. One reason for doing this is that we may be able to reduce the double sum to a single summation, or even evaluate the entire double sum in closed form.

As an example, suppose we make the following index substitutions in our double series: $m = q$, $n = p - q$. Then we will cover all $n \geq 0$, $m \geq 0$ by assigning p the range $(0, \infty)$, and q the range $(0, p)$, so our double series can be written

$$S = \sum_{p=0}^{\infty} \sum_{q=0}^{p} a_{p-q,q} \ . \tag{1.30}$$

eq1.28

In the nm plane our region of summation is the entire quadrant $m \geq 0$, $n \geq 0$; in the pq plane our summation is over the triangular region sketched in Fig. 1.3. This same pq region can be covered when the summations are carried out in the reverse order, but with limits

$$S = \sum_{q=0}^{\infty} \sum_{p=q}^{\infty} a_{p-q,q} \ .$$

The important thing to notice here is that these schemes all have in common that, by allowing the indices to run over their designated ranges, every $a_{n,m}$ is eventually encountered, and is encoutered exactly once.

Another possible index substitution is to set $n = s$, $m = r - 2s$. If we sum over s first, its range must be $(0, [r/2])$, where $[r/2]$ is the integer part of $r/2$, i.e. $[r/2] = r/2$ for r even and $(r-1)/2$ for r odd. The range of r is $(0, \infty)$. This situation corresponds to

$$S = \sum_{r=0}^{\infty} \sum_{s=0}^{[r/2]} a_{s,r-2s} \ . \tag{1.31}$$

eq1.29

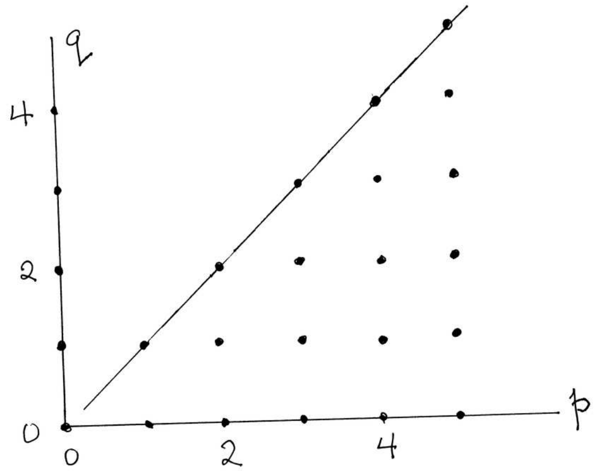

Fig1.3 Figure 1.3: The pq index space.

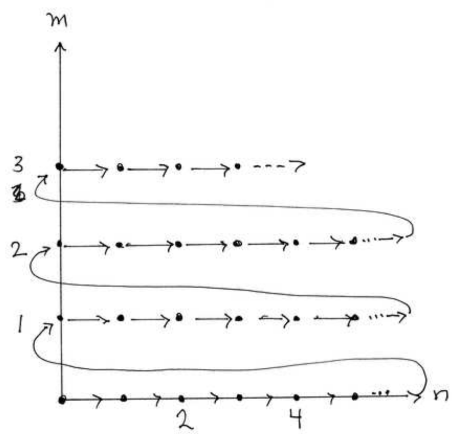

Fig1.4 Figure 1.4: Order in which terms are summed with m, n index set, Eq. (1.29).

The sketches in Figs. 1.4–1.6 show the order in which the $a_{n,m}$ are summed when using the forms given in Eqs. (1.29), (1.30), and (1.31).

If the double series introduced originally as Eq. (1.29) is absolutely convergent, then all these rearrangements will give the same ultimate result.

Exercises

Ex1.1.17 **1.1.13.** Show how to combine $\zeta(2) = \sum_{n=1}^{\infty} n^{-2}$ with α_1 and α_2 to obtain a series converging as n^{-4}.

Note. $\zeta(2)$ has the known value $\pi^2/6$. See Eq. (1.135).

1.1.14. Give a method of computing

$$\lambda(3) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}$$

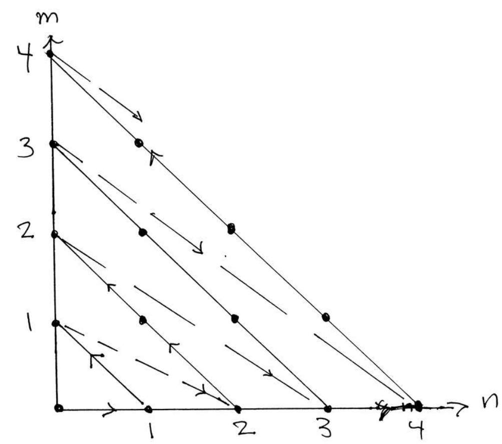

Fig1.5 Figure 1.5: Order in which terms are summed with p, q index set, Eq. (1.30).

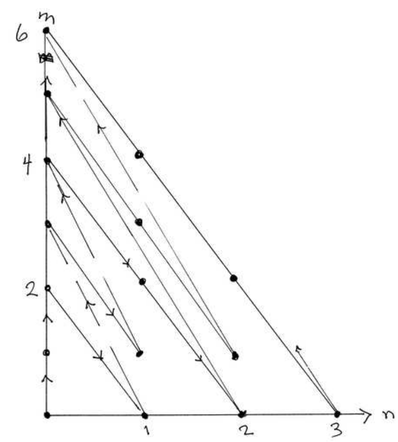

Fig1.6 Figure 1.6: Order in which terms are summed with r, s index set, Eq. (1.31).

that converges at least as fast as n^{-8} and obtain a result good to six decimal places.

ANS. $\lambda(3) = 1.051800.$

1.1.15. Show that (a) $\sum_{n=2}^{\infty}[\zeta(n)-1] = 1$, (b) $\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1] = \frac{1}{2}$,

where $\zeta(n)$ is the Riemann zeta function.

Ex1.1.16 **1.1.16.** The convergence improvement of Example 1.1.11 may be carried out more expediently (in this special case) by putting α_2, from Eq. (1.26), into a more symmetric form: Replacing n by $n-1$, we have

$$\alpha_2' = \sum_{n=2}^{\infty} \frac{1}{(n-1)n(n+1)} = \frac{1}{4}.$$

(a) Combine $\zeta(3)$ and α_2' to obtain convergence as n^{-5}.

(b) Let α_4' be α_4 with $n \to n-2$. Combine $\zeta(3), \alpha_2'$, and α_4' to obtain convergence as n^{-7}.

(c) If $\zeta(3)$ is to be calculated to six-decimal place accuracy (error 5×10^{-7}), how many terms are required for $\zeta(3)$ alone? combined as in part (a)? combined as in part (b)?

Note. The error may be estimated using the corresponding integral.

$$ANS. \quad (a)\ \zeta(3) = \frac{5}{4} - \sum_{n=2}^{\infty} \frac{1}{n^3(n^2-1)}.$$

Sec1.2

1.2 SERIES OF FUNCTIONS

We extend our concept of infinite series to include the possibility that each term u_n may be a function of some variable, $u_n = u_n(x)$. The partial sums become functions of the variable x,

$$s_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)\ , \tag{1.32}$$ eq1.30

as does the series sum, defined as the limit of the partial sums:

$$\sum_{n=1}^{\infty} u_n(x) = S(x) = \lim_{n\to\infty} s_n(x)\ . \tag{1.33}$$ eq1.31

So far we have concerned ourselves with the behavior of the partial sums as a function of n. Now we consider how the foregoing quantities depend on x. The key concept here is that of uniform convergence.

UNIFORM CONVERGENCE

If for any small $\varepsilon > 0$ there exists a number N, **independent of x** in the interval $[a, b]$ (that is, $a \le x \le b$) such that

$$\Big|S(x) - s_n(x)\Big| < \varepsilon\ , \qquad \text{for all } n \ge N\ , \tag{1.34}$$ eq1.32

then the series is said to be **uniformly convergent** in the interval $[a, b]$. This says that for our series to be uniformly convergent, it must be possible to find a finite N so that the absolute value of the tail of the infinite series, $|\sum_{i=N+1}^{\infty} u_i(x)|$, will be less than an arbitrary small ε for all x in the given interval, including the end points.

Exam1.2.1

Example 1.2.1. Nonuniform Convergence

Consider on the interval $[0, 1]$ the series

$$S(x) = \sum_{n=0}^{\infty} (1-x)x^n\ .$$

For $0 \le x < 1$, the geometric series $\sum_n x^n$ is convergent, with value $1/(1-x)$, so $S(x) = 1$ for these x values. But at $x = 1$, every term of the series will be zero, and therefore $S(1) = 0$. That is,

$$\sum_{n=0}^{\infty}(1-x)x^n = 1, \qquad 0 \le x < 1,$$
$$= 0, \qquad x = 1 \,. \tag{1.35}$$

eq1.33

So $S(x)$ is convergent for the entire interval $[0, 1]$, and because each term is nonnegative, it is also absolutely convergent. If $x \neq 0$, this is a series for which the partial sum s_N is $1 - x^N$, as can be seen by comparison with Eq. (1.3). Since $S(x) = 1$, the uniform convergence criterion is

$$\Big|1 - (1 - x^N)\Big| = x^N < \varepsilon \,.$$

No matter what the values of N and a sufficiently small ε may be, there will be an x value (close to 1) where this criterion is violated. The underlying problem is that $x = 1$ is the convergence limit of the geometric series, and it is not possible to have a convergence rate that is bounded independently of x in a range that includes $x = 1$.

We note also from this example that absolute and uniform convergence are independent concepts. The series in this example has absolute, but not uniform convergence. We will shortly present examples of series that are uniformly, but only conditionally convergent. And there are series that have neither or both of these properties.

■

WEIERSTRASS M (MAJORANT) TEST

The most commonly encountered test for uniform convergence is the Weierstrass M test. If we can construct a series of numbers $\sum_{i=1}^{\infty} M_i$, in which $M_i \ge |u_i(x)|$ for all x in the interval $[a, b]$ and $\sum_{i=1}^{\infty} M_i$ is convergent, our series $u_i(x)$ will be **uniformly** convergent in $[a, b]$.

The proof of this Weierstrass M test is direct and simple. Since $\sum_i M_i$ converges, some number N exists such that for $n + 1 \ge N$,

$$\sum_{i=n+1}^{\infty} M_i < \varepsilon \,.$$

This follows from our definition of convergence. Then, with $|u_i(x)| \le M_i$ for all x in the interval $a \le x \le b$,

$$\sum_{i=n+1}^{\infty} u_i(x) < \varepsilon \,.$$

Hence $S(x) = \sum_{n=1}^{\infty} u_i(x)$ satisfies

$$\Big|S(x) - s_n(x)\Big| = \Bigg|\sum_{i=n+1}^{\infty} u_i(x)\Bigg| < \varepsilon \,, \tag{1.36}$$

eq1.34

we see that $\sum_{n=1}^{\infty} u_i(x)$ is uniformly convergent in $[a, b]$. Since we have specified absolute values in the statement of the Weierstrass M test, the series $\sum_{n=1}^{\infty} u_i(x)$ is also seen to be absolutely convergent. As we have already observed in Example

Example 1.2.1, absolute and uniform convergence are different concepts, and one of the limitations of the Weierstrass M test is that it can only establish uniform convergence for series that are also absolutely convergent.

To further underscore the difference between absolute and uniform convergence, we provide another example.

Example 1.2.2. Uniformly Convergent Alternating Series

Consider the series

$$S(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n + x^2}\,, \qquad -\infty < x < \infty\,. \tag{1.37}$$ eq1.35

Applying the Leibniz criterion, this series is easily proven convergent for the entire interval $-\infty < x < \infty$, but it is **not** absolutely convergent, as the absolute values of its terms approach for large n those of the divergent harmonic series. The divergence of the absolute value series is obvious at $x = 0$, where we then exactly have the harmonic series. Nevertheless, this series is uniformly convergent on $-\infty < x < \infty$, as its convergence is for all x at least as fast as it is for $x = 0$. More formally,

$$\Big|S(x) - s_n(x)\Big| < \Big|u_{n+1}(x)\Big| \le \Big|u_{n+1}(0)\Big|\,.$$

Since $u_{n+1}(0)$ is independent of x, uniform convergence is confirmed.

■

ABEL'S TEST

A somewhat more delicate test for uniform convergence has been given by Abel. If $u_n(x)$ can be written in the form $a_n f_n(x)$, and

1. The a_n form a convergent series, $\sum_n a_n = A$,
2. For all x in $[a, b]$ the functions $f_n(x)$ are monotonically decreasing in n, i.e. $f_{n+1}(x) \le f_n(x)$,
3. For all x in $[a, b]$ all the $f(n)$ are bounded in the range $0 \le f_n(x) \le M$, where M is independent of x,

then $\sum_n u_n(x)$ converges uniformly in $[a, b]$.

This test is especially useful in analyzing the convergence of power series. Details of the proof of Abel's test and other tests for uniform convergence are given in the Additional Readings listed at the end of this chapter.

PROPERTIES OF UNIFORMLY CONVERGENT SERIES

Uniformly convergent series have three particularly useful properties. If a series $\sum_n u_n(x)$ is uniformly convergent in $[a, b]$ and the individual terms $u_n(x)$ are continuous,

1. The series sum $\quad S(x) = \sum_{n=1}^{\infty} u_n(x) \quad$ is also continuous.

2. The series may be integrated term by term. The sum of the integrals is equal to the integral of the sum.

$$\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx\ . \tag{1.38}$$ eq1.36

3. The derivative of the series sum $S(x)$ equals the sum of the individual-term derivatives:

$$\frac{d}{dx}\,S(x) = \sum_{n=1}^{\infty} \frac{d}{dx}\,u_n(x)\ , \tag{1.39}$$ eq1.37

provided the following additional conditions are satisfied:

$$\frac{du_n(x)}{dx} \text{ is continuous in } [a,b]\ ,$$

$$\sum_{n=1}^{\infty} \frac{du_n(x)}{dx} \text{ is uniformly convergent in } [a,b]\ .$$

Term-by-term integration of a uniformly convergent series requires only continuity of the individual terms. This condition is almost always satisfied in physical applications. Term-by-term differentiation of a series is often not valid because more restrictive conditions must be satisfied.

Exercises

1.2.1. Find the range of **uniform** convergence of the series

(a) $\eta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x}$ (b) $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$.

ANS. (a) $0 < s \le x < \infty$.
(b) $1 < s \le x < \infty$.

1.2.2. For what range of x is the geometric series $\sum_{n=0}^{\infty} x^n$ uniformly convergent?

ANS. $-1 < -s \le x \le s < 1$.

1.2.3. For what range of positive values of x is $\sum_{n=0}^{\infty} 1/(1+x^n)$

(a) convergent? (b) uniformly convergent?

1.2.4. If the series of the coefficients $\sum a_n$ and $\sum b_n$ are absolutely convergent, show that the Fourier series

$$\sum (a_n \cos nx + b_n \sin nx)$$

is **uniformly** convergent for $-\infty < x < \infty$.

old5.2.15 **1.2.5.** The Legendre series $\sum_{j \text{ even}} u_j(x)$ satisfies the recurrence relations

$$u_{j+2}(x) = \frac{(j+1)(j+2) - l(l+1)}{(j+2)(j+3)} x^2 u_j(x),$$

in which the index j is even and l is some constant (but, in this problem, **not** a nonnegative odd integer). Find the range of values of x for which this Legendre series is convergent. Test the endpoints.

ANS. $-1 < x < 1$.

old5.2.16 **1.2.6.** A series solution of the Chebyshev equation leads to successive terms having the ratio

$$\frac{u_{j+2}(x)}{u_j(x)} = \frac{(k+j)^2 - n^2}{(k+j+1)(k+j+2)} x^2,$$

with $k = 0$ and $k = 1$. Test for convergence at $x = \pm 1$.

ANS. Convergent.

1.2.7. A series solution for the ultraspherical (Gegenbauer) function $C_n^\alpha(x)$ leads to the recurrence

$$a_{j+2} = a_j \frac{(k+j)(k+j+2\alpha) - n(n+2\alpha)}{(k+j+1)(k+j+2)}.$$

Investigate the convergence of each of these series at $x = \pm 1$ as a function of the parameter α.

ANS. Convergent for $\alpha < 1$,
divergent for $\alpha \geq 1$.

TAYLOR'S EXPANSION

Taylor's expansion is a powerful tool for the generation of power series representations of functions. The derivation presented here provides not only the possibility of an expansion into a finite number of terms plus a remainder that may or may not be easy to evaluate, but also the possibility of the expression of a function as an infinite series of powers.

We assume that our function $f(x)$ has a continuous nth derivative[2] in the interval $a \leq x \leq b$. We integrate this nth derivative n times; the first three integrations yield

$$\int_a^x f^{(n)}(x_1)dx_1 = f^{(n-1)}(x_1)\Big|_a^x = f^{(n-1)}(x) - f^{(n-1)}(a) \,,$$

$$\int_a^x dx_2 \int_a^{x_2} f^{(n)}(x_1)dx_1 = \int_a^x dx_2 \left[f^{(n-1)}(x_2) - f^{(n-1)}(a)\right]$$

$$= f^{(n-2)}(x) - f^{(n-2)}(a) - (x-a)f^{(n-1)}(a) \,,$$

[2]Taylor's espansion may be derived under slightly less restrictive conditions; compare H. Jeffreys and B. S. Jeffreys, in the Additional Readings, Section 1.133.

$$\int_a^x dx_3 \int_a^{x_3} dx_2 \int_a^{x_2} f^{(n)}(x_1) dx_1 = f^{(n-3)}(x) - f^{(n-3)}(a) - (x-a) f^{(n-2)}(a) - \frac{(x-a)^2}{2!} f^{(n-1)}(a) \, .$$

Finally, after integrating for the nth time,

$$\int_a^x dx_n \cdots \int_a^{x_2} f^{(n)}(x_1) dx_1 = f(x) - f(a) - (x-a) f'(a) - \frac{(x-a)^2}{2!} f''(a) - \cdots - \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) \, .$$

Note that this expression is exact. No terms have been dropped, no approximations made. Now, solving for $f(x)$, we have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n \, , \quad (1.40)$$ eq1.38

where the remainder, R_n, is given by the n-fold integral

$$R_n = \int_a^x dx_n \cdots \int_a^{x_2} dx_1 \; f^{(n)}(x_1) \, . \quad (1.41)$$ eq1.39

We may convert R_n into a perhaps more practical form by using the **mean value theorem** of integral calculus:

$$\int_a^x g(x) \, dx = (x-a) \, g(\xi) \, , \quad (1.42)$$ eq1.40

with $a \le \xi \le x$. By integrating n times we get the Lagrangian form[3] of the remainder:

$$R_n = \frac{(x-a)^n}{n!} f^{(n)}(\xi) \, . \quad (1.43)$$ eq1.41

With Taylor's expansion in this form there are no questions of infinite series convergence. The series contains a finite number of terms, and the only questions concern the magnitude of the remainder.

When the function $f(x)$ is such that $\lim_{n\to\infty} R_n = 0$, Eq. (1.40) becomes Taylor's series:

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a) \, . \quad (1.44)$$ eq1.42

Here we encounter for the first time $n!$ with $n = 0$. Note that we define $0! = 1$.

[3] An alternate form derived by Cauchy is $R_n = \frac{(x-\xi)^{n-1}(x-a)}{(n-1)!} f^{(n)}(\xi)$.

Our Taylor series specifies the value of a function at one point, x, in terms of the value of the function and its derivatives at a reference point a. It is an expansion in powers of the **change** in the variable, namely $x - a$. This idea can be emphasized by writing Taylor's series in an alternate form in which we replace x by $x + h$ and a by x:

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x) . \qquad (1.45)$$

eq1.43

POWER SERIES

Taylor series are often used in situations where the reference point, a, is assigned the value zero. In that case the expansion is referred to as a **Maclaurin series**, and Eq. (1.40) becomes

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0) . \qquad (1.46)$$

eq1.44

An immediate application of the Maclaurin series is in the expansion of various transcendental functions into infinite (power) series.

Example 1.2.3. Exponential Function

Let $f(x) = e^x$. Differentiating, then setting $x = 0$, we have

$$f^{(n)}(0) = 1$$

for all n, $n = 1, 2, 3, \ldots$. Then, with Eq. (1.46), we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} . \qquad (1.47)$$

eq1.45

This is the series expansion of the exponential function. Some authors use this series to define the exponential function.

Although this series is clearly convergent for all x, as may be verified using the d'Alembert ratio test, it is instructive to check the remainder term, R_n. By Eq. (1.43) we have

$$R_n = \frac{x^n}{n!} f^{(n)}(\xi) = \frac{x^n}{n!} e^{\xi} ,$$

where ξ is between 0 and x. Irrespective of the sign of x,

$$|R_n| \le \frac{|x|^n e^{|x|}}{n!} ;$$

No matter how large $|x|$ may be, a sufficient increase in n will cause the denominator of this form for R_n to dominate over the numerator, and $\lim_{n\to\infty} R_n = 0$. Thus, the Maclaurin expansion of e^x converges absolutely over the entire range $-\infty < x < \infty$.

■

Now that we have an expansion for $\exp(x)$, we can return to Eq. (1.45), and rewrite that equation in a form that focuses on its differential operator characteristics. Defining D as the **operator** d/dx, we have

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n \mathsf{D}^n}{n!} f(x) = e^{h\mathsf{D}} f(x) . \qquad (1.48)$$

eq1.46

Exam1.2.4

Example 1.2.4. Logarithm

For a second Maclaurin expansion, let $f(x) = \ln(1+x)$. By differentiating, we obtain

$$f'(x) = (1+x)^{-1} \,,$$

$$f^{(n)}(x) = (-1)^{n-1}\,(n-1)!\,(1+x)^{-n} \,. \qquad (1.49)$$

eq1.47

Equation (1.46) yields

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + R_n$$

$$= \sum_{p=1}^{n} (-1)^{p-1}\,\frac{x^p}{p} + R_n \,. \qquad (1.50)$$

eq1.48

In this case, for $x > 0$ our remainder is given by

$$R_n = \frac{x^n}{n!}\,f^{(n)}(\xi) \,, \qquad 0 \le \xi \le x$$

$$\le \frac{x^n}{n} \,, \qquad 0 \le \xi \le x \le 1 \,. \qquad (1.51)$$

eq1.49

This result shows that the remainder approaches zero as n is increased indefinitely, providing that $0 \le x \le 1$. For $x < 0$, the mean value theorem is too crude a tool to establish a meaningful limit for R_n. As an infinite series,

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\,\frac{x^n}{n} \qquad (1.52)$$

eq1.50

converges for $-1 < x \le 1$. The range $-1 < x < 1$ is easily established by the d'Alembert ratio test. Convergence at $x = 1$ follows by the Leibniz criterion. In particular, at $x = 1$ we have the conditionally convergent alternating harmonic series, to which we can now put a value:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\,n^{-1} \,. \qquad (1.53)$$

eq1.51

At $x = -1$, the expansion becomes the harmonic series, which we well know to be divergent.

■

PROPERTIES OF POWER SERIES

The power series is a special and extremely useful type of infinite series, and as illustrated in the preceding subsection, may be constructed by the Maclaurin formula, Eq. (1.44). However obtained, it will be of the general form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n \,, \qquad (1.54)$$

eq1.52

where the coefficients a_i are constants, independent of x.

Equation (1.54) may readily be tested for convergence either by the Cauchy root test or the d'Alembert ratio test. If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = R^{-1} \ ,$$

the series converges for $-R < x < R$. This is the interval or **radius** of convergence. Since the root and ratio tests fail when x is at the limit points $\pm R$, these points require special attention.

For instance, if $a_n = n^{-1}$, then $R = 1$ and from Section 1.1 we can conclude that the series converges for $x = -1$ but diverges for $x = +1$. If $a_n = n!$, then $R = 0$ and the series diverges for all $x \neq 0$.

Suppose our power series has been found convergent for $-R < x < R$; then it will be uniformly and absolutely convergent in any **interior** interval $-S \le x \le S$, where $0 < S < R$. This may be proved directly by the Weierstrass M test.

Since each of the terms $u_n(x) = a_n x^n$ is a continuous function of x and $f(x) = \sum a_n x^n$ converges uniformly for $-S \le x \le S$, $f(x)$ must be a continuous function in the interval of uniform convergence. This behavior is to be contrasted with the strikingly different behavior of series in trigonometric functions, which are used frequently to represent discontinuous functions such as sawtooth and square waves.

With $u_n(x)$ continuous and $\sum a_n x^n$ uniformly convergent, we find that term by term differentiation or integration of a power series will yield a new power series with continous functions and the same radius of convergence as the original series. The new factors introduced by differentiation or integration do not affect either the root or the ratio test. Therefore our power series may be differentiated or integrated as ofen as desired within the interval of uniform convergence (Exercise 1.2.16). In view of the rather severe restriction placed on differentation of infinite series in general, this is a remarkable and valuable result.

UNIQUENESS THEOREM

We have already used the Maclaurin series to expand e^x and $\ln(1 + x)$ into power series. Throughout this book, we will encounter many situations in which functions are represented, or even defined by power series. We now establish that the power-series representation is unique.

We proceed by assuming we have two expansions of the same function whose intervals of convergence overlap in a region that includes the origin:

$$f(x) = \sum_{n=0}^{\infty} a_n \, x^n \ , \qquad -R_a < x < R_a$$

$$= \sum_{n=0}^{\infty} b_n \, x^n \ , \qquad -R_b < x < R_b \ . \qquad (1.55)$$

eq1.53

What we need to prove is that $a_n = b_n$ for all n.

Starting from

$$\sum_{n=0}^{\infty} a_n \, x^n = \sum_{n=0}^{\infty} b_n \, x^n \ , \qquad -R < x < R, \qquad (1.56)$$

eq1.54

where R is the smaller of R_a and R_b, we set $x = 0$ to eliminate all but the constant term of each series, obtaining

$$a_0 = b_0 \,.$$

Now, exploiting the differentiability of our power series, we differentiate Eq. (1.56), getting

$$\sum_{n=1}^{\infty} n a_n \, x^{n-1} = \sum_{n=1}^{\infty} n b_n \, x^{n-1} \,. \tag{1.57}$$

eq1.55

We again set $x = 0$, to isolate the new constant terms, and find

$$a_1 = b_1 \,.$$

By repeating this process n times, we get

$$a_n = b_n \,,$$

which shows that the two series coincide. Therefore our power series representation is unique.

This theorem will be a crucial point in our study of differential equations, in which we develop power series solutions. The uniqueness of power series appears frequently in theoretical physics. The establishment of perturbation theory in quantum mechanics is one example.

INDETERMINATE FORMS

The power-series representation of functions is often useful in evaluating indeterminate forms, and is the basis of **L'Hôpital's rule**, which states that if the ratio of two differentiable functions $f(x)$ and $g(x)$ becomes indeterminate, of the form $0/0$, at $x = x_0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}. \tag{1.58}$$

eq1.XXX

Proof of Eq. (1.58) is the subject of Exercise 1.2.12.

Sometimes it is easier just to introduce power-series expansions than to evaluate the derivatives that enter L'Hôpital's rule. For examples of this strategy, see the following Example and Exercise 1.2.15.

Example 1.2.5. Alternative to l'Hôpital's Rule

Evaluate

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} \,. \tag{1.59}$$

eq1.56

Replacing $\cos x$ by its Maclaurin-series expansion, Exercise 1.2.8, we obtain

$$\frac{1 - \cos x}{x^2} = \frac{1 - (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots)}{x^2} = \frac{1}{2!} - \frac{x^2}{4!} + \cdots \,.$$

Letting $x \to 0$, we have

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \,. \tag{1.60}$$

eq1.57

■

The uniquess of power series means that the coefficients a_n may be identified with the derivatives in a Maclaurin series. From

$$f(x) = \sum_{n=0}^{\infty} a_n \, x^n = \sum_{m=0}^{\infty} \frac{1}{n!} \, f^{(n)}(0) \, x^n$$

we have

$$a_n = \frac{1}{n!} \, f^{(n)}(0) \; .$$

INVERSION OF POWER SERIES

Suppose we are given a series

$$y - y_0 = a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=1}^{\infty} a_n \, (x - x_0)^n \; . \qquad (1.61)$$ eq1.58

This gives $(y - y_0)$ in terms of $(x - x_0)$. However, it may be desirable to have an explciit expression for $(x - x_0)$ in terms of $(y - y_0)$. That is, we want an expression of the form

$$x - x_0 = \sum_{n=1}^{\infty} b_n \, (y - y_0)^n \; , \qquad (1.62)$$ eq1.59

with the b_n to be determined in terms of the assumed known a_n. A brute-force approach, which is perfectly adequate for the first few coefficients, is simnply to substitute Eq. (1.61) into Eq. (1.62). By equating coefficients of $(x - x_0)^n$ on both sides of Eq. (1.62), and using the fact that the power series is unique, we find

$$\begin{aligned} b_1 &= \frac{1}{a_1} \; , \\ b_2 &= -\frac{a_2}{a_1^3} \; , \\ b_3 &= \frac{1}{a_1^5} \left(2a_2^2 - a_1 a_3\right) \; , \\ b_4 &= \frac{1}{a_1^7} \left(5a_1 a_2 a_3 - a_1^2 a_4 - 5a_2^3\right) \; , \qquad \text{and so on.} \end{aligned} \qquad (1.63)$$ eq1.60

Some of the higher coefficients are listed by Dwight.[4] A more general and much more elegant approach is developed by the use of complex variables in the first and second editions of *Mathematical Methods for Physicists.*

Exercises

Ex1.2.5 **1.2.8.** Show that

(a) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$,

[4]H. B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. New York: Macmillan (1961). (Compare Formula No. 50.)

(b) $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$.

1.2.9. Derive a series expansion of cot x in increasing powers of x by dividing the power series for $\cos x$ by that for $\sin x$.
Note. The resultant series that starts with $1/x$ is known as a **Laurent series** ($\cot x$ does not have a Taylor expansion about $x = 0$, although $\cot(x) - x^{-1}$ does). Although the two series for $\sin x$ and $\cos x$ were valid for all x, the convergence of the series for cot x is limited by the zeros of the denominator, $\sin x$.

1.2.10. Show by series expansion that

$$\frac{1}{2}\ln\frac{\eta_0 + 1}{\eta_0 - 1} = \coth^{-1}\eta_0, \quad |\eta_0| > 1.$$

This identity may be used to obtain a second solution for Legendre's equation.

1.2.11. Show that $f(x) = x^{1/2}$ (a) has no Maclaurin expansion but (b) has a Taylor expansion about any point $x_0 \neq 0$. Find the range of convergence of the Taylor expansion about $x = x_0$.

Ex1.2.1 **1.2.12.** Prove L'Hôpital's rule, Eq. (1.58).

old5.6.1 **1.2.13.** With $n > 1$, show that

$$\text{(a)}\ \frac{1}{n} - \ln\left(\frac{n}{n-1}\right) < 0\,, \qquad \text{(b)}\ \frac{1}{n} - \ln\left(\frac{n+1}{n}\right) > 0\,.$$

Use these inequalities to show that the limit defining the Euler-Mascheroni constant, Eq. (1.13), is finite.

1.2.14. In numerical analysis it is often convenient to approximate $d^2\psi(x)/dx^2$ by

$$\frac{d^2}{dx^2}\psi(x) \approx \frac{1}{h^2}[\psi(x+h) - 2\psi(x) + \psi(x-h)].$$

Find the error in this approximation.

ANS. Error $= \dfrac{h^2}{12}\psi^{(4)}(x)$.

Ex1.3.1 **1.2.15.** Evaluate $\displaystyle\lim_{x\to 0}\left[\frac{\sin(\tan x) - \tan(\sin x)}{x^7}\right]$.

ANS. $-\frac{1}{30}$.

Ex1.3.1 **1.2.16.** A power series converges for $-R < x < R$. Show that the differentiated series and the integrated series have the same interval of convergence. (Do not bother about the endpoints $x = \pm R$.)

1.3 BINOMIAL THEOREM

An extremely important application of the Maclaurin expansion is the derivation of the binomial theorem.

Let $f(x) = (1+x)^m$, in which m may be either positive or negative and is not limited to integral values. Direct application of Eq. (1.46) gives

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}\,x^2 + \cdots + R_n\ . \tag{1.64}$$ eq1.61

For this function the remainder is

$$R_n = \frac{x^n}{n!}\,(1+\xi)^{m-n}\, m(m-1)\cdots(m-n+1)\ , \tag{1.65}$$ eq1.62

with ξ between 0 and x. Restricting attention for now to $x \geq 0$, we note that for $n > m$, $(1+\xi)^{m-n}$ is a maximum for $\xi = 0$, so for positive x,

$$|R_n| \leq \frac{x^n}{n!}\,|m(m-1)\cdots(m-n+1)|\ , \tag{1.66}$$ eq1.63

with $\lim_{n\to\infty} R_n = 0$ when $0 \leq x < 1$. Because the radius of convergence of a power series is the same for positive and for negative x, the binomial series converges for $-1 < x < 1$. Convergence at the limit points ± 1 is not addressed by the present analysis, and depends upon m.

Summarizing, we have established the **binomial expansion**,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}\,x^2 + \frac{m(m-1)(m-2)}{3!}\,x^3 + \cdots\ , \tag{1.67}$$ eq1.64

convergent for $-1 < x < 1$. It is important to note that Eq. (1.67) applies whether or not m is integral, and for both positive and negative m. If m is a nonnegative integer, R_n for $n > m$ vanishes for all x, corresponding to the fact that under those conditions $(1+x)^m$ is a finite sum.

Because the binomial expansion is of frequent occurrence, the coefficients appearing in it, which are called **binomial coefficients**, are given the special symbol

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}\ , \tag{1.68}$$ eq1.65

and the binomial expansion assumes the general form

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n\ . \tag{1.69}$$ eq1.66

In evaluating Eq. (1.68), notice that when $n = 0$, the product in its numerator is empty (starting from m and **descending** to $m+1$); in that case the convention is to assign the product the value unity. We also remind the reader that 0! is defined to be unity.

In the special case that m is a positive integer, we may write our binomial coefficient in terms of factorials:

$$\binom{m}{n} = \frac{m!}{n!\,(m-n)!}\ . \tag{1.70}$$ eq1.67

Since $n!$ is undefined for negative integer n, the binomial expansion for positive integer m is understood to end with the term $n = m$, and will correspond to the coefficients in the polynomial resulting from the (finite) expansion of $(1 + x)^m$.

For positive integer m, the $\binom{m}{n}$ also arise in combinatorial theory, being the number of different ways n out of m objects can be selected. That, of course, is consistent with the coefficient set if $(1+x)^m$ is expanded. The term containing x^n has a coefficient that corresponds to the number of ways one can choose the "x" from n of the factors $(1 + x)$ and the 1 from the $m - n$ other $(1 + x)$ factors.

For negative integer m, we can still use the special notation for binomial coefficients, but their evaluation is more easily accomplished if we set $m = -p$, with p a positive integer, and write

$$\binom{-p}{n} = (-1)^n \, \frac{p(p+1)\cdots(p+n-1)}{n!} = \frac{(-1)^n \, (p+n-1)!}{n! \, (p-1)!} \, . \tag{1.71}$$

eq1.68

For nonintegral m, it is convenient to use the **Pochhammer symbol**, defined for general a and nonnegative integer n and given the notation $(a)_n$, as

$$(a)_0 = 1, \quad (a)_1 = a, \quad (a)_{n+1} = a(a+1)\cdots(a+n), \quad (n \geq 1) \, . \tag{1.72}$$

eq1.69

Both both integral and nonintegral m, the binomial coefficient formula can be written

$$\binom{m}{n} = \frac{(m-n+1)_n}{n!} \, . \tag{1.73}$$

eq1.70

There is a rich literature on binomial coefficients and relationships between them and on summations involving them. We mention here only one such formula that arises if we evaluate $1/\sqrt{1+x}$, i.e. $(1+x)^{-1/2}$. The binomial coefficient

$$\begin{aligned} \binom{-\frac{1}{2}}{n} &= \frac{1}{n!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right) \\ &= (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2^n \, n!} = (-1)^n \frac{(2n-1)!!}{(2n)!!} \, , \end{aligned} \tag{1.74}$$

eq1.71

where the "double factorial" notation indicates products of even or odd positive integers as follows:

$$\begin{aligned} 1\cdot 3\cdot 5\cdots(2n-1) &= (2n-1)!! \\ 2\cdot 4\cdot 6\cdots(2n) &= (2n)!! \, . \end{aligned} \tag{1.75}$$

eq1.72

These are related to the regular factorials by

$$(2n)!! = 2^n \, n! \qquad \text{and} \qquad (2n-1)!! = \frac{(2n)!}{2^n \, n!} \, . \tag{1.76}$$

eq1.73

Notice that these relations include the special cases $0!! = (-1)!! = 1$.

Example 1.3.1. Relativistic Energy

The total relativistic energy of a particle of mass m and velocity v is

$$E = mc^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \, , \tag{1.77}$$

eq1.74

where c is the velocity of light. Using Eq. (1.69) with $m = -1/2$ and $x = -v^2/c^2$, and evaluating the binomial coefficients using Eq. (1.74), we have

$$E = mc^2\left[1 - \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{3}{8}\left(-\frac{v^2}{c^2}\right)^2 - \frac{5}{16}\left(-\frac{v^2}{c^2}\right)^3 + \cdots\right]$$

$$= mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right) + \frac{5}{16}mv^2\left(-\frac{v^2}{c^2}\right)^2 + \cdots . \qquad (1.78)$$ eq1.75

The first term, mc^2, is identified as the rest-mass energy. Then

$$E_{\text{kinetic}} = \frac{1}{2}mv^2\left[1 + \frac{3}{4}\frac{v^2}{c^2} + \frac{5}{8}\left(-\frac{v^2}{c^2}\right)^2 + \cdots\right] . \qquad (1.79)$$ eq1.76

For particle velocity $v \ll c$, the expression in the brackets reduces to unity and we see that the kinetic portion of the total relativistic energy agrees with the classical result.

■

The binomial expansion can be generalized for positive integer n to polynomials:

$$(a_1 + a_2 + \cdots + a_m)^n = \sum \frac{n!}{n_1!n_2!\cdots n_m!} a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m} , \qquad (1.80)$$ eq1.76a

where the summation includes all different combinations of nonnegative integers $n_1, n_2, \ldots, n_m$ with $\sum_{i=1}^{m} n_i = n$. This generalization finds considerable use in statistical mechanics.

In everyday analysis, the combinatorial properties of the binomial coefficients make them appear often. For example, Leibniz's formula for the *nth* derivative of a product of two functions, $u(x)v(x)$, can be written

$$\left(\frac{d}{dx}\right)^n \Big(u(x)\,v(x)\Big) = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{d^i\,u(x)}{dx^i}\right)\left(\frac{d^{n-i}\,v(x)}{dx^{n-i}}\right) . \qquad (1.81)$$ eq1.77

Exercises

1.3.1. The classical Langevin theory of paramagnetism leads to an expression for the magnetic polarization,

$$P(x) = c\left(\frac{\cosh x}{\sinh x} - \frac{1}{x}\right).$$

Expand $P(x)$ as a power series for small x (low fields, high temperature).

new1.3.2 **1.3.2.** Given that

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^1 = \frac{\pi}{4},$$

expand the integrand into a series and integrate term by term obtaining[5]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + (-1)^n \frac{1}{2n+1} + \cdots ,$$

[5] The series expansion of $\tan^{-1} x$ (upper limit 1 replaced by x) was discovered by James Gregory in 1671, 3 years before Leibniz. See Peter Beckmann's entertaining book, *A History of Pi*, 2nd ed. Boulder, CO: Golem Press (1971) and L. Berggren, J. and P. Borwein, *Pi: A Source Book* , New York: Springer (1997).

which is Leibniz's formula for π. Compare the convergence of the integrand series and the integrated series at $x = 1$. Leibniz's formula converges so slowly that it is quite useless for numerical work.

old5.7.7 **1.3.3.** Expand the incomplete gamma function

$$\gamma(n+1, x) \equiv \int_0^x e^{-t} t^n dt$$

in a series of powers of x. What is the range of convergence of the resulting series?

$$ANS. \quad \int_0^x e^{-t} t^n dt = x^{n+1}\left[\frac{1}{n+1} - \frac{x}{n+2} + \frac{x^2}{2!(n+3)} - \cdots \frac{(-1)^p x^p}{p!(n+p+1)} + \cdots\right].$$

1.3.4. Develop a series expansion of $y = \sinh^{-1} x$ (that is, $\sinh y = x$) in powers of x by

(a) inversion of the series for $\sinh y$,
(b) a direct Maclaurin expansion.

new1.3.5 **1.3.5.** Show that for integral $n \geq 0$, $\displaystyle \frac{1}{(1-x)^{n+1}} = \sum_{m=n}^{\infty} \binom{m}{n} x^{m-n}$.

1.3.6. Show that $\displaystyle (1+x)^{-m/2} = \sum_{n=0}^{\infty} (-1)^n \frac{(m+2n-2)!!}{2^n n!(m-2)!!} x^n,$

for $m = 1,\ 2,\ 3, \ldots$.

1.3.7. Using binomial expansions, compare the three Doppler shift formulas:

(a) $\nu' = \nu \left(1 \mp \frac{v}{c}\right)^{-1}$ moving source;

(b) $\nu' = \nu \left(1 \pm \frac{v}{c}\right)$ moving observer;

(c) $\nu' = \nu \left(1 \pm \frac{v}{c}\right) \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ relativistic.

Note. The relativistic formula agrees with the classical formulas if terms of order v^2/c^2 can be neglected.

1.3.8. In the theory of general relativity there are various ways of relating (defining) a velocity of recession of a galaxy to its red shift, δ. Milne's model (kinematic relativity) gives

(a) $v_1 = c\delta(1 + \frac{1}{2}\delta)$,

(b) $v_2 = c\delta(1 + \frac{1}{2}\delta)(1+\delta)^{-2}$,

(c) $1 + \delta = \left[\frac{1 + v_3/c}{1 - v_3/c}\right]^{1/2}$.

1. Show that for $\delta \ll 1$ (and $v_3/c \ll 1$) all three formulas reduce to $v = c\delta$.
2. Compare the three velocities through terms of order δ^2.

Note. In special relativity (with δ replaced by z), the ratio of observed wavelength λ to emitted wavelength λ_0 is given by

$$\frac{\lambda}{\lambda_0} = 1 + z = \left(\frac{c+v}{c-v}\right)^{1/2}.$$

1.3.9. The relativistic sum w of two velocities u and v in the same direction is given by

$$\frac{w}{c} = \frac{u/c + v/c}{1 + uv/c^2}.$$

If

$$\frac{v}{c} = \frac{u}{c} = 1 - \alpha,$$

where $0 \le \alpha \le 1$, find w/c in powers of α through terms in α^3.

1.3.10. The displacement x of a particle of rest mass m_0, resulting from a constant force $m_0 g$ along the x-axis, is

$$x = \frac{c^2}{g}\left\{\left[1 + \left(g\frac{t}{c}\right)^2\right]^{1/2} - 1\right\},$$

including relativistic effects. Find the displacement x as a power series in time t. Compare with the classical result,

$$x = \tfrac{1}{2}gt^2.$$

1.3.11. By use of Dirac's relativistic theory, the fine structure formula of atomic spectroscopy is given by

$$E = mc^2\left[1 + \frac{\gamma^2}{(s + n - |k|)^2}\right]^{-1/2},$$

where

$$s = (|k|^2 - \gamma^2)^{1/2}, \quad k = \pm 1, \pm 2, \pm 3, \cdots.$$

Expand in powers of γ^2 through order γ^4 ($\gamma^2 = Ze^2/4\pi\epsilon_0\hbar c$, with Z the atomic number). This expansion is useful in comparing the predictions of the Dirac electron theory with those of a relativistic Schrödinger electron theory. Experimental results support the Dirac theory.

1.3.12. In a head-on proton-proton collision, the ratio of the kinetic energy in the center of mass system to the incident kinetic energy is

$$R = [\sqrt{2mc^2(E_k + 2mc^2)} - 2mc^2]/E_k.$$

Find the value of this ratio of kinetic energies for

(a) $E_k \ll mc^2$ (nonrelativistic)

(b) $E_k \gg mc^2$ (extreme-relativistic).

ANS. (a) $\frac{1}{2}$, (b) 0. The latter answer is a sort of law of diminishing returns for high-energy particle accelerators (with stationary targets).

1.3.13. With binomial expansions

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n, \quad \frac{x}{x-1} = \frac{1}{1-x^{-1}} = \sum_{n=0}^{\infty} x^{-n}.$$

Adding these two series yields $\sum_{n=-\infty}^{\infty} x^n = 0$.
Hopefully, we can agree that this is nonsense, but what has gone wrong?

1.3.14. (a) Planck's theory of quantized oscillators leads to an average energy

$$\langle \varepsilon \rangle = \frac{\sum\limits_{n=1}^{\infty} n\varepsilon_0 \exp(-n\varepsilon_0/kT)}{\sum\limits_{n=0}^{\infty} \exp(-n\varepsilon_0/kT)},$$

where ε_0 is a fixed energy. Identify the numerator and denominator as binomial expansions and show that the ratio is

$$\langle \varepsilon \rangle = \frac{\varepsilon_0}{\exp(\varepsilon_0/kT) - 1}.$$

(b) Show that the $\langle \varepsilon \rangle$ of part (a) reduces to kT, the classical result, for $kT \gg \varepsilon_0$.

1.3.15. Expand by the binomial theorem and integrate term by term to obtain the Gregory series for $y = \tan^{-1} x$ (note $\tan y = x$):

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \{1 - t^2 + t^4 - t^6 + \cdots\}\, dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 \le x \le 1.$$

1.3.16. The Klein-Nishina formula for the scattering of photons by electrons contains a term of the form

$$f(\varepsilon) = \frac{(1+\varepsilon)}{\varepsilon^2}\left[\frac{2+2\varepsilon}{1+2\varepsilon} - \frac{\ln(1+2\varepsilon)}{\varepsilon}\right].$$

Here $\varepsilon = h\nu/mc^2$, the ratio of the photon energy to the electron rest mass energy. Find $\lim\limits_{\varepsilon \to 0} f(\varepsilon)$.

ANS. $\frac{4}{3}$.

1.3.17. The behavior of a neutron losing energy by colliding elastically with nuclei of mass A is described by a parameter ξ_1,

$$\xi_1 = 1 + \frac{(A-1)^2}{2A} \ln \frac{A-1}{A+1}.$$

An approximation, good for large A, is

$$\xi_2 = \frac{2}{A + \frac{2}{3}}.$$

Expand ξ_1 and ξ_2 in powers of A^{-1}. Show that ξ_2 agrees with ξ_1 through $(A^{-1})^2$. Find the difference in the coefficients of the $(A^{-1})^3$ term.

1.3.18. Show that each of these two integrals equals Catalan's constant:

$$\text{(a)} \int_0^1 \arctan t \, \frac{dt}{t} \, , \quad \text{(b)} \; -\int_0^1 \ln x \frac{dx}{1+x^2} \, .$$

Note. The definition and numerical computation of Catalan's constant was addressed in Exercise 1.1.12.

Sec1.9

1.4 MATHEMATICAL INDUCTION

We are occasionally faced with the need to establish a relation which is valid for a set of integer values, in situations where it may not initially be obvious how to proceed. However, it may be possible to show that if the relation is valid for an arbitrary value of some index n, then it is also valid if n is replaced by $n+1$. If we can also show that the relation is unconditionally satisfied for some initial value n_0, we may then conclude (unconditionally) that the relation is also satisfied for $n_0 + 1$, $n_0 + 2$, $\cdots$. This method of proof is known as **mathematical induction**. It is ordinarily most useful when we know (or suspect) the validity of a relation, but lack a more direct method of proof.

Example 1.4.1. Sum of Integers

The sum of the integers from 1 through n, here denoted $S(n)$, is given by the formula $S(n) = n(n+1)/2$. An inductive proof of this formula proceeds as follows:

(1) Given the formula for $S(n)$, we calculate

$$S(n+1) = S(n)+(n+1) = \frac{n(n+1)}{2}+(n+1) = \left[\frac{n}{2}+1\right](n+1) = \frac{(n+1)(n+2)}{2} \, .$$

Thus, given $S(n)$, we can establish the validity of $S(n+1)$.

(2) It is obvious that $S(1) = 1(2)/2 = 1$, so our formula for $S(n)$ is valid for $n = 1$.

(3) The formula for $S(n)$ is therefore valid for all integer $n \geq 1$.

■

Exercises

1.4.1. Show that $\displaystyle\sum_{j=1}^{n} j^4 = \frac{n}{30}(2n+1)(n+1)(3n^2+3n-1)$.

new1.4.2 **1.4.2.** Prove the Leibniz formula for the repeated differentiation of a product:

$$\left(\frac{d}{dx}\right)^n \left[f(x)g(x)\right] = \sum_{j=0}^{n} \binom{n}{j} \left[\left(\frac{d}{dx}\right)^j f(x)\right] \left[\left(\frac{d}{dx}\right)^{n-j} g(x)\right] \, .$$

1.5 OPERATIONS ON SERIES EXPANSIONS OF FUNCTIONS

Sec1.4XX

There are a number of manipulations (tricks) that can be used to obtain series that represent a function or to manipulate such series to improve convergence. In addition to the procedures introduced in Section 1.1, there are others that to varying degrees make use of the fact that the expansion depends on a variable. A simple example of this is the expansion of $f(x) = \ln(1+x)$, which we obtained in Example 1.2.4 by direct use of the Maclaurin expansion and evaluation of the derivatives of $f(x)$. An even easier way to obtain this series would have been to integrate the power series for $1/(1+x)$ term by term from 0 to x:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad \Longrightarrow$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ .$$

A problem requiring somewhat more deviousness is given by the following example, in which we use the binomial theorem on a series which represents the derivative of the function whose expansion is sought.

Example 1.5.1. Application of Binomial Expansion

Sometimes the binomial expansion provides a convenient indirect route to the Maclaurin series when direct methods are difficult. We consider here the power series expansion

$$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\,\frac{x^{2n+1}}{(2n+1)} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \ . \qquad (1.82)$$ eq1.81

Starting from $\sin y = x$, we find $dy/dx = 1/\sqrt{1-x^2}$, and write the integral

$$\sin^{-1} x = y = \int_0^x \frac{dt}{(1-t^2)^{1/2}} \ .$$

We now introduce the binomial expansion of $(1-t^2)^{-1/2}$ and integrate term by term. The result is Eq. (1.82).

■

Another way of improving the convergence of a series is to multiply it by a polynomial in the variable, choosing the polynomial's coefficients to remove the least rapidly convergent part of the resulting series. Here is a simple example of this.

Example 1.5.2. Multiply Series by Polynomial

Returning to the series for $\ln(1+x)$, we form

$$\begin{aligned}(1+a_1x)\ln(1+x) &= \sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{x^n}{n} + a_1\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n+1}}{n} \\ &= x + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{1}{n}-\frac{a_1}{n-1}\right)x^n \\ &= x + \sum_{n=2}^{\infty}(-1)^{n-1}\,\frac{n(1-a_1)-1}{n(n-1)}\,x^n\,.\end{aligned}$$

If we take $a_1 = 1$, the n in the numerator disappears and our combined series converges as n^{-2}; the resulting series for $\ln(1+x)$ is

$$\ln(1+x) = \left(\frac{x}{1+x}\right)\left(1 - \sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}\,x^n\right)\,.$$

■

Another useful trick is to employ **partial fraction expansions**, which may convert a seemingly difficult series into others about which more may be known.

If $g(x)$ and $h(x)$ are polynomials in x, with $g(x)$ of lower degree than $h(x)$, and $h(x)$ has the factorization $h(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$, in the case that the factors of $h(x)$ are distinct (i.e., h has no multiple roots), then $g(x)/h(x)$ can be written in the form

$$\frac{g(x)}{h(x)} = \frac{c_1}{x-a_1} + \frac{c_2}{x-a_2} + \cdots + \frac{c_n}{x-a_n}\,. \qquad (1.83)$$ eq1.82

If we wish to leave one or more quadratic factors in $h(x)$, perhaps to avoid the introduction of imaginary quantities, the corresponding partial-fraction term will be of the form

$$\frac{ax+b}{x^2+px+q}\,.$$

If $h(x)$ has repeated linear factors, such as $(x-a_1)^m$, the partial fraction expansion for this power of $x-a_1$ takes the form

$$\frac{c_{1,m}}{(x-a_1)^m} + \frac{c_{1,m-1}}{(x-a_1)^{m-1}} + \cdots + \frac{c_{1,1}}{x-a_1}\,.$$

The coefficients in partial fraction expansions are usually found easily; sometimes it is useful to express them as limits, such as

$$c_i = \lim_{x\to a_i}(x-a_i)g(x)/h(x)\,. \qquad (1.84)$$ eq1.83

Exam1.4.3

Example 1.5.3. Partial Fraction Expansion

Let

$$f(x) = \frac{k^2}{x(x^2+k^2)} = \frac{c}{x} + \frac{ax+b}{x^2+k^2}\,.$$

We have written the form of the partial fraction expansion, but have not yet determined the values of a, b, and c. Putting the right side of the equation over a common denominator, we have

$$\frac{k^2}{x(x^2+k^2)} = \frac{c(x^2+k^2)+x(ax+b)}{x(x^2+k^2)} .$$

Expanding the right-side numerator and equating it to the left-side numerator, we get

$$0(x^2) + 0(x) + k^2 = (c+a)x^2 + bx + ck^2 ,$$

which we solve by requiring the coefficient of each power of x to have the same value on both sides of this equation. We get $b = 0$, $c = 1$, and then $a = -1$. The final result is therefore

$$f(x) = \frac{1}{x} - \frac{x}{x^2+k^2} . \qquad (1.85) \quad \boxed{\texttt{eq1.84}}$$

■

Still more cleverness is illustrated by the following procedure, due to Euler, for changing the expansion variable so as to improve the range over which an expansion converges. Euler's transformation, the proof of which (with hints) is deferred to Exercise 1.5.4, makes the conversion:

$$f(x) = \sum_{n=0}^{\infty} (-1)^n c_n x^n \qquad (1.86) \quad \boxed{\texttt{eq1.84a}}$$

$$= \frac{1}{1+x} \sum_{n=0}^{\infty} (-1)^n a_n \left(\frac{x}{1+x}\right)^n . \qquad (1.87) \quad \boxed{\texttt{eq1.84b}}$$

The coefficients a_n are repeated differences of the c_n:

$$a_0 = c_0, \quad a_1 = c_1 - c_0, \quad a_2 = c_2 - 2c_1 + c_0, \quad a_3 = c_3 - 3c_2 + 3c_1 - c_0, \quad \ldots \quad ;$$

their general formula is

$$a_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} c_{n-j} . \qquad (1.88) \quad \boxed{\texttt{eq1.84c}}$$

The series to which the Euler transformation is applied need not be alternating. The coefficients c_n can have a sign factor that cancels that in the definition.

Example 1.5.4. Euler Transformation

The Maclaurin series for $\ln(1+x)$ converges extremely slowly, with convergence only for $|x| < 1$. We consider the Euler transformation on the related series

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \cdots , \qquad (1.89) \quad \boxed{\texttt{eq1.84d}}$$

so, in Eq. (1.86), $c_n = 1/(n+1)$. The first few a_n are: $a_0 = 1$, $a_1 = \frac{1}{2} - 1 = -\frac{1}{2}$, $a_2 = \frac{1}{3} - 2(\frac{1}{2}) + 1 = \frac{1}{3}$, $a_3 = \frac{1}{4} - 3(\frac{1}{3}) + 3(\frac{1}{2}) - 1 = -\frac{1}{4}$, or in general

$$a_n = \frac{(-1)^n}{n+1} .$$

The converted series is then

$$\frac{\ln(1+x)}{x} = \frac{1}{1+x}\left[1 + \frac{1}{2}\left(\frac{x}{1+x}\right) + \frac{1}{3}\left(\frac{x}{1+x}\right)^2 + \cdots\right],$$

which rearranges to

$$\ln(1+x) = \left(\frac{x}{1+x}\right) + \frac{1}{2}\left(\frac{x}{1+x}\right)^2 + \frac{1}{3}\left(\frac{x}{1+x}\right)^3 + \cdots . \qquad (1.90)$$

This new series converges nicely at $x = 1$, and in fact is convergent for all $x < \infty$.

■

Exercises

1.5.1. Using a partial fraction expansion, show that for $0 < x < 1$,

$$\int_{-x}^{x} \frac{dt}{1-t^2} = \ln\left(\frac{1+x}{1-x}\right) .$$

Ex1.4.2 **1.5.2.** Prove the partial fraction expansion

$$\frac{1}{n(n+1)\cdots(n+p)} =$$

$$\frac{1}{p!}\left[\binom{p}{0}\frac{1}{n} - \binom{p}{1}\frac{1}{n+1} + \binom{p}{2}\frac{1}{n+2} - \cdots + (-1)^p\binom{p}{p}\frac{1}{n+p}\right],$$

where p is a positive integer.

Hint: Use mathematical induction. Two binomial coefficient formulas of use here are

$$\frac{p+1}{p+1-j}\binom{p}{j} = \binom{p+1}{j}, \qquad \sum_{j=1}^{p+1}(-1)^{j-1}\binom{p+1}{j} = 1 .$$

Ex1.4.3 **1.5.3.** The formula for α_p, Eq. (1.26), is a summation of the form $\sum_{n=1}^{\infty} u_n(p)$, with

$$u_n(p) = \frac{1}{n(n+1)\cdots(n+p)} .$$

Applying a partial fraction decomposition to the first and last factors of the denominator, i.e.,

$$\frac{1}{n(n+p)} = \frac{1}{p}\left[\frac{1}{n} - \frac{1}{n+p}\right],$$

show that $u_n(p) = \dfrac{u_n(p-1) - u_{n+1}(p-1)}{p}$ and that $\sum_{n=1}^{\infty} u_n(p) = \dfrac{1}{p\,p!}$.

Hint. It is useful to note that $u_1(p-1) = 1/p!$.

Ex1.4.4 **1.5.4.** Proof of Euler transformation: By substituting Eq. (1.88) into Eq. (1.87), verify that Eq. (1.86) is recovered.

Hint. It may help to rearrange the resultant double series so that both indices are summed on the range $(0, \infty)$. Then the summation not containing the coefficients c_j can be recognized as a binomial expansion.

1.5.5. Carry out the Euler transformation on the series for $\arctan(x)$:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots .$$

Check your work by computing $\arctan(1) = \pi/4$ and $\arctan(3^{-1/2}) = \pi/6$.

Sec1.4YY 1.6 SOME IMPORTANT SERIES

There are a few series that arise so often that all physicists should recognize them. Here is a short list that is worth committing to memory.

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \qquad -\infty < x < \infty, \qquad (1.91)$$ eq1.85

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, -\infty < x < \infty, \qquad (1.92)$$ eq1.86

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \qquad -\infty < x < \infty, \qquad (1.93)$$ eq1.87

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots, \qquad -\infty < x < \infty, \qquad (1.94)$$ eq1.87a

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots, \qquad -\infty < x < \infty, \qquad (1.95)$$ eq1.87b

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots, \qquad -1 \le x < 1, \qquad (1.96)$$ eq1.88

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \ -1 < x \le 1, \qquad (1.97)$$ eq1.89

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n = \sum_{n=0}^{\infty} \frac{(p-n+1)_n}{n!} x^n, \qquad -1 < x < 1 . \qquad (1.98)$$ eq1.90

Reminder: The notation $(a)_n$ is the Pochhammer symbol: $(a)_0 = 1$, $(a)_1 = a$, and for integers $n > 1$, $(a)_n = a(a+1)\cdots(a+n-1)$. It is not required that a, or p in Eq. (1.98), be positive or integral.

Exercises

old5.4.1 1.6.1. Show that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right), \qquad -1 < x < 1.$$

1.7 BERNOULLI NUMBERS

A generating-function approach is a convenient way to introduce the set of numbers first used in mathematics by Jacques (James, Jacob) Bernoulli. These quantities have been defined is a number of different ways, so extreme care must be taken in combining formulas from works by different authors. Our definition corresponds to that used in the reference work *Handbook of Mathematical Functions* (AMS-55). See Additional Readings.

Since the Bernoulli numbers, denoted B_n, do not depend upon a variable, their generating function depends only on a single (complex) variable, and the generating-function formula has the specific form

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n\, t^n}{n!}\,. \qquad (1.99)$$ eqCA.3

The inclusion of the factor $1/n!$ in the definition is just one of the ways some definitions of Bernoulli numbers differ. We defer for the moment the important question as to the range of convergence of the expansion in Eq. (1.99).

Since Eq. (1.99) is a Taylor series, we may identify the B_n as successive derivatives of the generating function:

$$B_n = \left[\frac{d^n}{dt^n}\left(\frac{t}{e^t - 1}\right)\right]_{t=0}. \qquad (1.100)$$ eqCA.4

To obtain B_0, we must take the limit of $t/(e^t - 1)$ as $t \to 0$, easily finding $B_0 = 1$. Applying Eq. (1.100), we also have

$$B_1 = \frac{d}{dt}\left(\frac{t}{e^t - 1}\right)\bigg|_{t=0} = \lim_{t\to 0}\left(\frac{1}{e^t - 1} - \frac{te^t}{(e^t - 1)^2}\right) = -\frac{1}{2}\,. \qquad (1.101)$$ eqCA.5

In principle we could continue to obtain further B_n, but it is more convenient to proceed in a more sophisticated fashion. Our starting point is to examine

$$\sum_{n=2}^{\infty} \frac{B_n\, t^n}{n!} = \frac{t}{e^t - 1} - B_0 - B_1\, t = \frac{t}{e^t - 1} - 1 + \frac{t}{2}$$

$$= \frac{-t}{e^{-t} - 1} - 1 - \frac{t}{2}\,, \qquad (1.102)$$ eqCA.6

where we have used the fact that

$$\frac{t}{e^t - 1} = \frac{-t}{e^{-t} - 1} - t\,. \qquad (1.103)$$ eqCA.7

Equation (1.102) shows that the summation on its left-hand side is an even function of t, leading to the conclusion that all B_n of odd n (other than B_1) must vanish.

TabCA.1

Table 1.2: Bernoulli Numbers

n	B_n	B_n
0	1	1.000000000
1	$-\frac{1}{2}$	−0.500000000
2	$\frac{1}{6}$	0.166666667
4	$-\frac{1}{30}$	−0.033333333
6	$\frac{1}{42}$	0.023809524
8	$-\frac{1}{30}$	−0.033333333
10	$\frac{5}{66}$	0.075757576

Note. Further values are given in AMS-55, see Abramowitz in Additional Readings.

We next use the generating function to obtain a recursion relation for the Bernoulli numbers. We form

$$\frac{e^t-1}{t}\,\frac{t}{e^t-1} = 1 = \left[\sum_{m=0}^{\infty}\frac{t^m}{(m+1)!}\right]\left[1-\frac{t}{2}+\sum_{n=1}^{\infty}B_{2n}\frac{t^{2n}}{(2n)!}\right]$$

$$= 1+\sum_{m=1}^{\infty}t^m\left[\frac{1}{(m+1)!}-\frac{1}{2m!}\right]$$

$$+\sum_{N=2}^{\infty}t^N\sum_{n=1}^{\leq N/2}\frac{B_{2n}}{(2n)!(N-2n+1)!}$$

$$= 1+\sum_{N=2}^{\infty}\frac{t^N}{(N+1)!}\left[-\frac{N-1}{2}+\sum_{n=1}^{\leq N/2}\binom{N+1}{2n}B_{2n}\right]. \tag{1.104}$$ eqCA.8

Since the coefficient of each power of t in the final summation of Eq. (1.104) must vanish, we may set to zero for each N the expression in its square brackets. Changing N, if even, to $2N$ and if odd, to $2N-1$, Eq. (1.104) leads to the pair of equations

$$\begin{aligned} N-\frac{1}{2} &= \sum_{n=1}^{N}\binom{2N+1}{2n}B_{2n}\,, \\ N-1 &= \sum_{n=1}^{N-1}\binom{2N}{2n}B_{2n}\,. \end{aligned} \tag{1.105}$$ eqCA.9

Either of these equations can be used to obtain the B_{2n} sequentially, starting from B_2. The first few B_n are listed in Table 1.2.

To obtain additional relations involving the Bernoulli numbers, we next consider the following representation of $\cot t$:

$$\cot t = \frac{\cos t}{\sin t} = i\left(\frac{e^{it}+e^{-it}}{e^{it}-e^{-it}}\right) = i\left(\frac{e^{2it}+1}{e^{2it}-1}\right) = i\left(1+\frac{2}{e^{2it}-1}\right)\,.$$

Multiplying by t and rearranging slightly,

$$t\cot t = \frac{2it}{2} + \frac{2it}{e^{2it}-1} = \sum_{n=0}^{\infty} B_{2n}\frac{(2it)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n B_{2n}\frac{(2t)^{2n}}{(2n)!}\ , \qquad (1.106)$$ eqCA.10

where the term $2it/2$ has canceled the B_1 term that would otherwise appear in the expansion, and we are left with an expression that contains no imaginary quantities.

Using the methods of complex variable theory (for details, see the section on Bernoulli numbers in Chapter 12 of the printed text) it can be shown that for $2n \geq 2$, we can start from Eq. (1.106) and reach the important result

$$B_{2n} = (-1)^{n+1}\frac{(2n)!}{(2\pi)^{2n}}\sum_{m=1}^{\infty}\frac{2}{m^{2n}} = (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n)\ ,$$
$$B_{2n+1} = 0\ . \qquad (1.107)$$ eqCA.13

Notice that this formula correctly shows that B_n of odd $n > 1$ vanish, and that the Bernoulli numbers of even $n > 0$ are identified as proportional to Riemann zeta functions, which first appeared in this book at Eq. (1.12). We repeat the definition:

$$\zeta(z) = \sum_{m=1}^{\infty}\frac{1}{m^z}\ .$$

Equation (1.107) is important because we already have a straightforward way to obtain values of the B_n, via Eq. (1.105), and Eq. (1.107) can be inverted to give a closed expression for $\zeta(2n)$, which otherwise was known only as a summation. This representation of the Bernoulli numbers was discovered by Euler.

It is readily seen from Eq. (1.107) that $|B_{2n}|$ increases without limit as $n \to \infty$. Numerical values have been calculated by Glaisher.[6] Illustrating the divergent behavior of the Bernoulli numbers, we have

$$B_{20} = -5.291 \times 10^2$$
$$B_{200} = -3.647 \times 10^{215}.$$

Some authors prefer to define the Bernoulli numbers with a modified version of Eq. (1.107) by using

$$\mathcal{B}_n = \frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n)\ , \qquad (1.108)$$ eqCA.14

the subscript being just half of our subscript and all signs positive. Again, when using other texts or references, you must check to see exactly how the Bernoulli numbers are defined.

The Bernoulli numbers occur frequently in number theory. The von Staudt-Clausen theorem states that

$$B_{2n} = A_n - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} - \cdots - \frac{1}{p_k}\ , \qquad (1.109)$$ eqCA.15

[6] J. W. L. Glaisher, table of the first 250 Bernoulli's numbers (to nine figures) and their logarithms (to ten figures). *Trans. Cambridge Philos. Soc.* **12**: 390 (1871-1879).

in which A_n is an integer and $p_1, p_2, \ldots, p_k$ are all the prime numbers such that $p_i - 1$ is a divisor of $2n$. It may readily be verified that this holds for

$$\begin{aligned} B_6 &\qquad (A_3 = 1, \quad p = 2, 3, 7), \\ B_8 &\qquad (A_4 = 1, \quad p = 2, 3, 5), \\ B_{10} &\qquad (A_5 = 1, \quad p = 2, 3, 11), \end{aligned}$$

and other special cases.

The Bernoulli numbers appear in the summation of integral powers of the integers,

$$\sum_{j=1}^{N} j^p, \qquad p \text{ integral},$$

and in numerous series expansions of the transcendental functions, including $\tan x$, $\cot x$, $\ln|\sin x|$, $(\sin x)^{-1}$, $\ln|\cos x|$, $\ln|\tan x|$, $(\cosh x)^{-1}$, $\tanh x$, and $\coth x$. For example,

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} + \cdots . \tag{1.110}$$ eqCA.16

The Bernoulli numbers are likely to come in such series expansions because of the definition, Eq. (1.99), the form of Eq. (1.106), and the relation to the Riemann zeta function, Eq. (1.107).

BERNOULLI POLYNOMIALS

If Eq. (1.99) is generalized slightly, we have

$$\frac{te^{ts}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(s)\,\frac{t^n}{n!} \tag{1.111}$$ eqCA.17

defining the **Bernoulli polynomials**, $B_n(s)$. It is clear that $B_n(s)$ will be a polynomial of degree n, since the Taylor expansion of the generating function will contain contributions in which each instance of t may (or may not) be accompanied by a factor s. The first seven Bernoulli polynomials are given in Table 1.3.

If we set $s = 0$ in the generating function formula, Eq. (1.111), we have

$$B_n(0) = B_n, \qquad n = 0, 1, 2, \ldots, \tag{1.112}$$ eqCA.18

showing that the Bernoulli polynomial evaluated at zero equals the corresponding Bernoulli number.

Two other important properties of the Bernoulli polynomials follow from the defining relation, Eq, (1.111). If we differentiate both sides of that equation with respect to s, we have

$$\begin{aligned} \frac{t^2 e^{ts}}{e^t - 1} &= \sum_{n=0}^{\infty} B_n'(s)\,\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} B_n(s)\,\frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} B_{n-1}(s)\,\frac{t^n}{(n-1)!}\;, \end{aligned} \tag{1.113}$$ eqCA.19

TabCA.2

Table 1.3: Bernoulli Polynomials

$B_0 = 1$
$B_1 = x - \frac{1}{2}$
$B_2 = x^2 - x + \frac{1}{6}$
$B_3 = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$
$B_4 = x^4 - 2x^3 + x^2 - \frac{1}{30}$
$B_5 = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$
$B_6 = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$

where the second line of Eq. (1.113) is obtained by rewriting its left-hand side using the generating-function formula. Equating the coefficients of equal powers of t in the two lines of Eq. (1.113), we obtain the differentiation formula

$$\frac{d}{ds} B_n(s) = nB_{n-1}(s), \quad n = 1, 2, 3, \ldots . \tag{1.114}$$ eqCA.20

We also have a symmetry relation, which we can obtain by setting $s = 1$ in Eq. (1.111). The left-hand side of that equation then becomes

$$\frac{te^t}{e^t - 1} = \frac{-t}{e^{-t} - 1} \ . \tag{1.115}$$ eqCA.21

Thus, equating Eq. (1.111) for $s = 1$ with the Bernoulli-number expansion (in $-t$) of the right-hand side of Eq. (1.115), we reach

$$\sum_{n=0}^{\infty} B_n(1)\, \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n \frac{(-t)^n}{n!} \ ,$$

which is equivalent to

$$B_n(1) = (-1)^n B_n(0) \ . \tag{1.116}$$ eqCA.22

These relations are used in the development of the Euler-Maclaurin integration formula.

Exercises

1.7.1. Verify the identities, Eqs. (1.103) and (1.115).

1.7.2. Show that the first Bernoulli polynomials are

$$B_0(s) = 1$$
$$B_1(s) = s - \tfrac{1}{2}$$
$$B_2(s) = s^2 - s + \tfrac{1}{6}.$$

Note that $B_n(0) = B_n$, the Bernoulli number.

1.7.3. Show that

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n}}{(2n)!} x^{2n-1}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Hint. $\tan x = \cot x - 2\cot 2x$.

1.8 EULER-MACLAURIN INTEGRATION FORMULA

SecCA.3

One use of the Bernoulli polynomials is in the derivation of the **Euler-Maclaurin integration formula**. This formula is used both to develop asymptotic expansions (treated later in this chapter) and to obtain approximate values for summations. An important application of the Euler-Maclaurin formula is its use to derive **Stirling's formula**, an asymptotic expression for the gamma function.

The technique we use to develop the Euler-Maclaurin formula is repeated integration by parts, using Eq. (1.114) to create new derivatives. We start with

$$\int_0^1 f(x)\,dx = \int_0^1 f(x) B_0(x)\,dx\,, \tag{1.117}$$

eqCA.27

where we have, for reasons that will shortly become apparent, inserted the redundant factor $B_0(x) = 1$. From Eq. (1.114), we note that

$$B_1'(x) = B_0(x)\,,$$

and we substitute $B_1'(x)$ for $B_0(x)$ in Eq. (1.117), integrate by parts, and identify $B_1(1) = -B_1(0) = \frac{1}{2}$, thereby obtaining

$$\int_0^1 f(x)\,dx = f(1)B_1(1) - f(0)B_1(0) - \int_0^1 f'(x)B_1(x)\,dx$$

$$= \frac{1}{2}\Big[f(1)+f(0)\Big] - \int_0^1 f'(x)B_1(x)\,dx\,. \tag{1.118}$$

eqCA.28

Again using Eq. (1.114), we have

$$B_1(x) = \frac{1}{2}\,B_2'(x)\,.$$

Inserting $B_2'(x)$ and integrating by parts again, we get

$$\int_0^1 f(x)\,dx = \frac{1}{2}\Big[f(1)+f(0)\Big] - \frac{1}{2}\Big[f'(1)B_2(1) - f'(0)B_2(0)\Big]$$

$$+ \frac{1}{2}\int_0^1 f^{(2)}(x)B_2(x)\,dx\,. \tag{1.119}$$

eqCA.29

Using the relation

$$B_{2n}(1) = B_{2n}(0) = B_{2n}, \quad n = 0, 1, 2, \ldots, \tag{1.120}$$

eqCA.30

Eq. (1.119) simplifies to

$$\int_0^1 f(x)\,dx = \frac{1}{2}\Big[f(1)+f(0)\Big] - \frac{B_2}{2}\Big[f'(1)-f'(0)\Big] + \frac{1}{2}\int_0^1 f^{(2)}(x)B_2(x)\,dx\,. \quad (1.121)$$ eqCA.31

Continuing, we replace $B_2(x)$ by $B_3'(x)/3$ and once again integrate by parts. Because

$$B_{2n+1}(1) = B_{2n+1}(0) = 0, \quad n = 1, 2, 3, \ldots, \quad (1.122)$$ eqCA.32

the integration by parts produces no integrated terms, and

$$\frac{1}{2}\int_0^1 f^{(2)}(x)B_2(x)\,dx = \frac{1}{2\cdot 3}\int_0^1 f^{(2)}(x)B_3'(x)\,dx = -\frac{1}{3!}\int_0^1 f^{(3)}(x)B_3(x)\,dx\,. \quad (1.123)$$ eqCA.33

Substituting $B_3(x) = B_4'(x)/4$ and carrying out one more partial integration, we get integrated terms containing $B_4(x)$, which simplify according to Eq. (1.120). The result is

$$-\frac{1}{3!}\int_0^1 f^{(3)}(x)B_3(x)\,dx = \frac{B_4}{4!}\Big[f^{(3)}(1)-f^{(3)}(0)\Big] + \frac{1}{4!}\int_0^1 f^{(4)}(x)B_4(x)\,dx\,. \quad (1.124)$$ eqCA.34

We may continue this process, with steps that are entirely analogous to those that led to Eqs. (1.123) and (1.124). After steps leading to derivatives of f of order $2q-1$, we have

$$\int_0^1 f(x)\,dx = \frac{1}{2}\Big[f(1)+f(0)\Big] - \sum_{p=1}^{q}\frac{1}{(2p)!}\,B_{2p}\left[f^{(2p-1)}(1) - f^{(2p-1)}(0)\right]$$

$$+ \frac{1}{(2q)!}\int_0^1 f^{(2q)}(x)B_{2q}(x)\;dx\,. \quad (1.125)$$ eqCA.35

This is the Euler-Maclaurin integration formula. It assumes that the function $f(x)$ has the required derivatives.

The range of integration in Eq. (1.125) may be shifted from $[0,1]$ to $[1,2]$ by replacing $f(x)$ by $f(x+1)$. Adding such results up to $[n-1,n]$, we obtain

$$\int_0^n f(x)\,dx = \tfrac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \tfrac{1}{2}f(n)$$

$$-\sum_{p=1}^{q}\frac{1}{(2p)!}\,B_{2p}\left[f^{(2p-1)}(n) - f^{(2p-1)}(0)\right]$$

$$+ \frac{1}{(2q)!}\int_0^1 B_{2q}(x)\sum_{\nu=0}^{n-1} f^{(2q)}(x+\nu)\,dx\,. \quad (1.126)$$ eqCA.36

Notice that the derivative terms at the intermediate integer arguments all cancel. However, the intermediate terms $f(j)$ do not, and $\frac{1}{2}f(0) + f(1) + \cdots + \frac{1}{2}f(n)$ appear exactly as in trapezoidal integration, or quadrature, so the summation over p may be interpreted as a correction to the trapezoidal approximation. Equation (1.126) may therefore be seen as a generalization of Eq. (1.10).

In many applications of Eq. (1.126) the final integral containing $f^{(2q)}$, though small, will not approach zero as q is increased without limit, and the Euler-Maclaurin formula then has an asymptotic, rather than convergent character. Such series, and the implications regarding their use, are the topic of a later section of this chapter.

One of the most important uses of the Euler-Maclaurin formula is in summing series by converting them to integrals plus correction terms.[7] Here is an illustration of the process.

ExamCA.3.1

Example 1.8.1. Estimation of $\zeta(3)$

A straightforward application of Eq. (1.126) to $\zeta(3)$ proceeds as follows (noting that all derivatives of $f(x) = 1/x^3$ vanish in the limit $x \to \infty$):

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{2}\,f(1) + \int_1^{\infty} \frac{dx}{x^3} - \sum_{p=1}^{q} \frac{B_{2p}}{(2p)!}\,f^{(2p-1)}(1) + \text{ remainder.} \qquad (1.127)$$ eqCA.37

Evaluating the integral, setting $f(1) = 1$, and inserting

$$f^{(2n-1)}(x) = -\frac{(2n+1)!}{2\,x^{2n+2}}$$

with $x = 1$, Eq. (1.127) becomes

$$\zeta(3) = \frac{1}{2} + \frac{1}{2} + \sum_{p=1}^{q} \frac{(2p+1)B_{2p}}{2\,x^{2p+2}} + \text{ remainder.} \qquad (1.128)$$ eqCA.38

To assess the quality of this result, we list, in the first data column of Table 1.4, the contributions to it. The line marked "explicit terms" consists presently of only the term $\frac{1}{2}f(1)$. We note that the individual terms start to increase after the B_4 term; since it our intention not to evaluate the remainder, the accuracy of the expansion is limited. As discussed more extensively in the section on asymptotic expansions, the best result available from these data is obtained by truncating the expansion before the terms start to increase; adding the contributions above the marker line in the table, we get the value listed as "Sum". For reference the accurate value of $\zeta(3)$ is 1.202057.

We can improve the result available from the Euler-Maclaurin formula by explicitly calculating some initial terms and applying the formula only to those that remain. This strategem causes the derivatives entering the formula to be smaller and diminishes the correction from the trapezoid-rule estimate. Simply starting the formula at $n = 2$ instead of $n = 1$ reduces the error markedly; see the second data column of Table 1.4. Now the "explicit terms" consist of $f(1) + \frac{1}{2}f(2)$. Starting the Euler-Maclaurin formula at $n = 4$ further improves the result, then reaching better than 7-figure accuracy.

■

When the Euler-Maclaurin formula is applied to sums whose summands have a finite number of nonzero derivatives, it can evaluate them exactly. See Exercise 1.8.1.

[7] See R. P. Boas and C. Stutz, Estimating sums with integrals. *Am. J. Phys.* **39**: 745 (1971), for a number of examples.

TabCA.3 Table 1.4: Contributions to $\zeta(3)$ of terms in Euler-Maclaurin formula. Left column: formula applied to entire summation; central column: formula applied starting from second term; right column: formula starting from fourth term.

	$n_0 = 1$	$n_0 = 2$	$n_0 = 4$
Explicit terms	0.500000	1.062500	1.169849
$\int_{n_0}^{\infty} x^{-3}\,dx$	0.500000	0.125000	0.031250
B_2 term	0.250000	0.015615	0.000977
B_4 term	−0.083333	−0.001302	−0.000020
B_6 term	0.083333	0.000326	0.000001
B_8 term	−0.150000	−0.000146	−0.000000
B_{10} term	0.416667	0.000102	0.000000
B_{12} term	−1.645238	−0.000100	−0.000000
B_{14} term	8.750000	0.000134	0.000000
Sum[a]	1.166667	1.201995	1.202057

a. Sums only include data above horizontal marker.

Exercises

old5.9.5 **1.8.1.** The Euler-Maclaurin integration formula may be used for the evaluation of finite series:

$$\sum_{m=1}^{n} f(m) = \int_1^n f(x)\,dx + \frac{1}{2}f(1) + \frac{1}{2}f(n) + \frac{B_2}{2!}\Big[f'(n) - f'(1)\Big] + \cdots .$$

Show that

(a) $\displaystyle\sum_{m=1}^{n} m = \tfrac{1}{2}\,n(n+1).$

(b) $\displaystyle\sum_{m=1}^{n} m^2 = \tfrac{1}{6}\,n(n+1)(2n+1).$

(c) $\displaystyle\sum_{m=1}^{n} m^3 = \tfrac{1}{4}\,n^2(n+1)^2.$

(d) $\displaystyle\sum_{m=1}^{n} m^4 = \tfrac{1}{30}\,n(n+1)(2n+1)(3n^2+3n-1).$

old5.10.11 **1.8.2.** The Euler-Maclaurin integration formula provides a way of calculating the Euler-Mascheroni constant γ to high accuracy. Using $f(x) = 1/x$ in Eq. (1.126) (with interval $[1, n]$) and the definition of γ, Eq. (1.13), we obtain

$$\gamma = \sum_{s=1}^{n} s^{-1} - \ln n - \frac{1}{2n} + \sum_{k=1}^{N} \frac{B_{2k}}{(2k)n^{2k}} .$$

Using double precision arithmetic, calculate γ for N =1,2,.. .

Note. See D. E. Knuth, Euler's constant to 1271 places. *Math. Comput.* **16**: 275 (1962).

ANS. For $n = 1000$, N=2
$\gamma = 0.5772\ 1566\ 4901.$

1.9 DIRICHLET SERIES

Series expansions of the general form

$$S(s) = \sum_n \frac{a_n}{n^s}$$

are known as **Dirichlet series**, and our knowledge of Bernoulli numbers enables us to evaluate a variety of expressions of this type. One of the most important Dirichlet series is that of the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ . \qquad (1.129)$$

eqCA.39

Example 1.9.1. Evaluation of $\zeta(2)$

Here is an alternative derivation of the formula for $\zeta(2)$. By methods of complex variable theory one can establish the summation formula

$$T(a) = \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi \coth \pi a}{2a} - \frac{1}{2a^2} \ .$$

Simply by taking the limit $a \to 0$, we have

$$\zeta(2) = \lim_{a\to 0} STa) = \lim_{a\to 0} \left[\frac{\pi}{2a} \left(\frac{1}{\pi a} + \frac{\pi a}{3} + \cdots \right) - \frac{1}{2a^2} \right] = \frac{\pi^2}{6} \ . \qquad (1.130)$$

eqCA.40

■

From the relation with the Bernoulli numbers, we find

$$\zeta(4) = \frac{\pi^4}{90} \ .$$

Values of $\zeta(2n)$ through $\zeta(10)$ are listed in Exercise 1.9.1. The zeta functions of odd integer argument seem unamenable to evaluation in closed form, but are easy to compute numerically (see Example 1.8.1).

Other useful Dirichlet series, in the notation of AMS-55 (see Additional Readings), include

$$\eta(s)=\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}=(1-2^{1-s})\,\zeta(s)\,, \qquad (1.131) \quad \boxed{\texttt{eqCA.41}}$$

$$\lambda(s)=\sum_{n=0}^{\infty}(2n-1)^{-s}=(1-2^{-s})\,\zeta(s)\,, \qquad (1.132) \quad \boxed{\texttt{eqCA.42}}$$

$$\beta(s)=\sum_{n=0}^{\infty}(-1)^{n}(2n+1)^{-s}\,. \qquad (1.133) \quad \boxed{\texttt{eqCA.43}}$$

Closed expressions are available (for integer $n \ge 1$) for $\zeta(2n)$, $\eta(2n)$, and $\lambda(2n)$, and for $\beta(2n-1)$. The sums with exponents of opposite parity cannot be reduced to $\zeta(2n)$ or performed by the contour-integral methods we discuss in Chapter 11. An important series that can only be evaluated numerically is that whose result is **Catalan's constant**, which is

$$\beta(2)=1-\frac{1}{3^2}+\frac{1}{5^2}-\cdots=0.91596559\ldots\,. \qquad (1.134) \quad \boxed{\texttt{eqCA.44}}$$

For reference, we list a few of these summable Dirichlet series:

$$\zeta(2)=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}\,, \qquad (1.135) \quad \boxed{\texttt{eqCA.45}}$$

$$\zeta(4)=1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}\,, \qquad (1.136) \quad \boxed{\texttt{eqCA.46}}$$

$$\eta(2)=1-\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{12}\,, \qquad (1.137) \quad \boxed{\texttt{eqCA.47}}$$

$$\eta(4)=1-\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{7\pi^4}{720}\,, \qquad (1.138) \quad \boxed{\texttt{eqCA.48}}$$

$$\lambda(2)=1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}\,, \qquad (1.139) \quad \boxed{\texttt{eqCA.49}}$$

$$\lambda(4)=1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots=\frac{\pi^4}{96}\,, \qquad (1.140) \quad \boxed{\texttt{eqCA.50}}$$

$$\beta(1)=1-\frac{1}{3}+\frac{1}{5}-\cdots=\frac{\pi}{4}\,, \qquad (1.141) \quad \boxed{\texttt{eqCA.51}}$$

$$\beta(3)=1-\frac{1}{3^3}+\frac{1}{5^3}-\cdots=\frac{\pi^3}{32}\,. \qquad (1.142) \quad \boxed{\texttt{eqCA.52}}$$

Exercises

ExCA.5.11 **1.9.1.** From

$$B_{2n}=(-1)^{n-1}\frac{2(2n)!}{(2\pi)^{2n}}\,\zeta(2n),$$

show that

(a) $\zeta(2) = \frac{\pi^2}{6}$, (d) $\zeta(8) = \frac{\pi^8}{9450}$,

(b) $\zeta(4) = \frac{\pi^4}{90}$, (e) $\zeta(10) = \frac{\pi^{10}}{93,555}$,

(c) $\zeta(6) = \frac{\pi^6}{945}$.

1.9.2. The integral

$$\int_0^1 [\ln(1-x)]^2 \frac{dx}{x}$$

appears in the fourth-order correction to the magnetic moment of the electron. Show that it equals $2\zeta(3)$.

Hint. Let $1 - x = e^{-t}$.

1.9.3. Show that

$$\int_0^\infty \frac{(\ln z)^2}{1+z^2}\, dz = 4\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\right).$$

Note. This series evaluates to $\pi^3/8$.

old5.9.15 **1.9.4.** Show that Catalan's constant, $\beta(2)$, may be written as

$$\beta(2) = 2\sum_{k=1}^{\infty}(4k-3)^{-2} - \frac{\pi^2}{8}.$$

Hint. $\pi^2 = 6\zeta(2)$.

1.9.5. Show that

(a) $\displaystyle\int_0^1 \frac{\ln(1+x)}{x}\, dx = \frac{1}{2}\zeta(2)$, (b) $\displaystyle\lim_{a\to 1}\int_0^a \frac{\ln(1-x)}{x}\, dx = \zeta(2)$.

Note that the integrand in part (b) diverges for $a = 1$ but that the integral is convergent.

1.9.6. (a) Show that the equation $\ln 2 = \sum_{s=1}^{\infty}(-1)^{s+1}s^{-1}$, Eq. (1.53), may be rewritten as

$$\ln 2 = \sum_{s=2}^{n} 2^{-s}\zeta(s) + \sum_{p=1}^{\infty}(2p)^{-n-1}\left[1 - \frac{1}{2p}\right]^{-1}.$$

Hint. Take the terms in pairs.

(b) Calculate $\ln 2$ to six significant figures.

1.9.7. (a) Show that the equation $\pi/4 = \sum_{s=1}^{\infty}(-1)^{s+1}(2s-1)^{-1}$, Eq. (1.141), may be rewritten as

$$\frac{\pi}{4} = 1 - 2\sum_{s=1}^{n} 4^{-2s}\zeta(2s) - 2\sum_{p=1}^{\infty}(4p)^{-2n-2}\left[1 - \frac{1}{(4p)^2}\right]^{-1}.$$

(b) Calculate $\pi/4$ to six significant figures.

SecCA.7

1.10 ASYMPTOTIC SERIES

Asymptotic series frequently occur in physics. In fact, one of the earliest and still important approximations of quantum mechanics, the **WKB expansion** (the initials stand for its originators, Wenzel, Kramers, and Brillouin), is an asymptotic series. In numerical computations, these series are employed for the accurate computation of a variety of functions. We consider here two types of integrals that lead to asymptotic series: first, integrals of the form

$$I_1(x) = \int_x^\infty e^{-u} f(u)\, du \,,$$

where the variable x appears as the lower limit of an integral. Second, we consider the form

$$I_2(x) = \int_0^\infty e^{-u} f\left(\frac{u}{x}\right) du \,,$$

with the function f to be expanded as a Taylor series (binomial series). Asymptotic series often occur as solutions of differential equations; we encounter many examples in later chapters of this book.

EXPONENTIAL INTEGRAL

The nature of an asymptotic series is perhaps best illustrated by a specific example. Suppose that we have the exponential integral function[8]

$$Ei(x) = \int_{-\infty}^x \frac{e^u}{u}\, du \,, \tag{1.143}$$ eqCA.65

which we find more convenient to write in the form

$$-Ei(-x) = \int_x^\infty \frac{e^{-u}}{u}\, du = E_1(x) \,, \tag{1.144}$$ eqCA.66

to be evaluated for large values of x. This function has a series expansion that converges for all x, namely

$$E_1(x) = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-1)^n x^n}{n\, n!} \,, \tag{1.145}$$ eqCA.67

which we derive in Chapter 13, but the series is totally useless for numerical evaluation when x is large. We need another approach, for which it is convenient to generalize Eq. (1.144) to

$$I(x,p) = \int_x^\infty \frac{e^{-u}}{u^p}\, du \,, \tag{1.146}$$ eqCA.68

where we restrict consideration to cases in which x and p are positive. As already stated, we seek an evaluation for large values of x.

[8]This function occurs frequently in astrophysical problems involving gas with a Maxwell-Boltzmann energy distribution.

Integrating by parts, we obtain

$$I(x,p) = \frac{e^{-x}}{x^p} - p\int_x^\infty \frac{e^{-u}}{u^{p+1}}\,du = \frac{e^{-x}}{x^p} - \frac{pe^{-x}}{x^{p+1}} + p(p+1)\int_x^\infty \frac{e^{-u}}{u^{p+2}}\,du\ .$$

Continuing to integrate by parts, we develop the series

$$I(x,p) = e^{-x}\left(\frac{1}{x^p} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}} - \cdots + (-1)^{n-1}\frac{(p+n-2)!}{(p-1)!x^{p+n-1}}\right)$$

$$+ (-1)^n \frac{(p+n-1)!}{(p-1)!}\int_x^\infty \frac{e^{-u}}{u^{p+n}}\,du\ . \qquad (1.147)$$ eqCA.69

This is a remarkable series. Checking the convergence by the d'Alembert ratio test, we find

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty}\frac{(p+n)!}{(p+n-1)!}\cdot\frac{1}{x} = \lim_{n\to\infty}\frac{p+n}{x} = \infty \qquad (1.148)$$ eqCA.70

for all finite values of x. Therefore our series as an infinite series diverges everywhere! Before discarding Eq. (1.148) as worthless, let us see how well a given partial sum approximates our function $I(x,p)$. Taking s_n as the partial sum of the series through n terms and R_n as the corresponding remainder,

$$I(x,p) - s_n(x,p) = (-1)^{n+1}\frac{(p+n)!}{(p-1)!}\int_x^\infty \frac{e^{-u}}{u^{p+n+1}}\,du = R_n(x,p)\ .$$

In absolute value

$$|R_n(x,p)| \le \frac{(p+n)!}{(p-1)!}\int_x^\infty \frac{e^{-u}}{u^{p+n+1}}\,du\ .$$

When we substitute $u = v + x$, the integral becomes

$$\int_x^\infty \frac{e^{-u}}{u^{p+n+1}}\,du = e^{-x}\int_0^\infty \frac{e^{-v}}{(v+x)^{p+n+1}}\,dv$$

$$= \frac{e^{-x}}{x^{p+n+1}}\int_0^\infty e^{-v}\left(1+\frac{v}{x}\right)^{-p-n-1}dv\ .$$

For large x the final integral approaches 1 and

$$|R_n(x,p)| \approx \frac{(p+n)!}{(p-1)!}\,\frac{e^{-x}}{x^{p+n+1}}\ . \qquad (1.149)$$

This means that if we take x large enough, our partial sum s_n will be an arbitrarily good approximation to the function $I(x,p)$. Our divergent series, Eq. (1.147), therefore is perfectly good for computations of partial sums. For this reason it is sometimes called a **semiconvergent series.** Note that the power of x in the denominator of the remainder, namely $p+n+1$, is higher than the power of x in the last term included in $s_n(x,p)$, namely $p+n$.

Thus, our asymptotic series for $E_1(x)$ assumes the form

$$e^x E_1(x) = e^x\int_x^\infty \frac{e^{-u}}{u}\,du$$

$$\approx s_n(x) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^2} - \frac{3!}{x^4} + \cdots + (-1)^n\frac{n!}{x^{n+1}}\ , \qquad (1.150)$$ eqCA.72

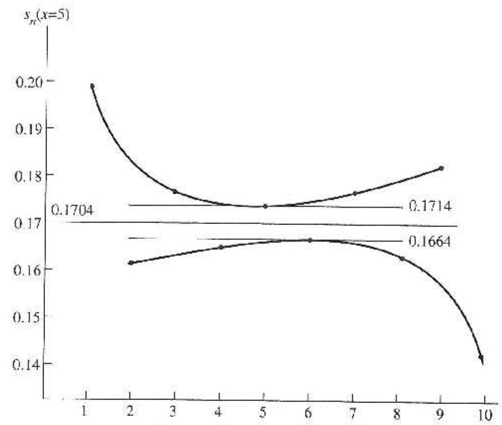

Figure 1.7: Partial sums of $e^x E_1(x)\,|_{x=5}$. FigCA.3

where we must choose to terminate the series after some n.

Since the remainder $R_n(x,p)$ alternates in sign, the successive partial sums give alternately upper and lower bounds for $I(x,p)$. The behavior of the series (with $p=1$) as a function of the number of terms included is shown in Fig. 1.7, where we have plotted partial sums of $e^x E_1(x)$ for the value $x=5$. The optimum determination of $e^x E_1(x)$ is given by the closest approach of the upper and lower bounds, that is, for $x=5$, between $s_6 = 0.1664$ and $s_5 = 0.1741$. Therefore

$$0.1664 \le e^x E_1(x)\Big|_{x=5} \le 0.1741\ . \qquad (1.151)$$ eqCA.73

Actually, from tables,

$$e^x E_1(x)\Big|_{x=5} = 0.1704\ , \qquad (1.152)$$ eqCA.74

within the limits established by our asymptotic expansion. Note that inclusion of additional terms in the series expansion beyond the optimum point reduces the accuracy of the representation. As x is increased, the spread between the lowest upper bound and the highest lower bound will diminish. By taking x large enough, one may compute $e^x E_1(x)$ to any desired degree of accuracy. Other properties of $E_1(x)$ are derived and discussed in Section 13.6.

COSINE AND SINE INTEGRALS

These integrals, defined by

$$\mathrm{Ci}(u) = -\int_u^\infty \frac{\cos t}{t}\,dt\ , \qquad (1.153)$$ eqCA.75

$$\mathrm{si}(u) = -\int_u^\infty \frac{\sin t}{t}\,dt\ , \qquad (1.154)$$ eqCA.76

have useful asymptotic expansions. By methods developed in detail in Chapter 12 of the printed text, we find the formulas

$$\mathrm{Ci}(u) \approx \frac{\sin u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n)!}{u^{2n}} - \frac{\cos u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n+1)!}{u^{2n+1}}\,, \qquad (1.155)$$ eqCA.80

$$\mathrm{si}(u) \approx -\frac{\cos u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n)!}{u^{2n}} - \frac{\sin u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n+1)!}{u^{2n+1}}\,. \qquad (1.156)$$ eqCA.81

DEFINITION OF ASYMPTOTIC SERIES

Poincaré has introduced a formal definition for an asymptotic series[9]. Following Poincaré, we consider a function $f(x)$ whose asymptotic expansion is sought, the partial sums s_n in its expansion, and the corresponding remainders $R_n(x)$. Though the expansion need not be a power series, we assume that form for simplicity in the present discussion. Thus,

$$x^n R_n(x) = x^n[f(x) - s_n(x)]\,, \qquad (1.157)$$ eqCA.82

where

$$s_n(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}\,. \qquad (1.158)$$ eqCA.83

The asymptotic expansion of $f(x)$ is defined to have the properties that

$$\lim_{x\to\infty} x^n R_n(x) = 0, \text{ for fixed } n, \qquad (1.159)$$ eqCA.84

and

$$\lim_{n\to\infty} x^n R_n(x) = \infty, \text{ for fixed } x. \qquad (1.160)$$ eqCA.85

These conditions were met for our examples, Eqs. (1.150), (1.155), and (1.156).[10]

For power series, as assumed in the form of $s_n(x), R_n(x) \approx x^{-n-1}$. With the conditions of Eqs. (1.159) and (1.160) satisfied, we write

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}\,. \qquad (1.161)$$ eqCA.86

Note the use of $\sim$ in place of $=$. The function $f(x)$ is equal to the series only in the limit as $x \to \infty$ and with the restriction to a finite number of terms in the series.

Asymptotic expansions of two functions may be multiplied together, and the result will be an asymptotic expansion of the product of the two functions. The asymptotic expansion of a given function $f(t)$ may be integrated term by term (just as in a uniformly convergent series of continuous functions) from $x \le t < \infty$, and the result will be an asymptotic expansion of $\int_x^\infty f(t)dt$. Term-by-term differentiation, however, is valid only under very special conditions.

[9]Poincaré's definition allows (or neglects) exponentially decreasing functions. The refinement of his definition is of considerable importance for the advanced theory of asymptotic expansions, particularly for extensions into the complex plane. However, for purposes of an introductory treatment and especially for numerical computation of expansions for which the variable is real and positive, Poincaré's approach is perfectly satisfactory.

[10]Some writers feel that the requirement of Eq. (1.160), which excludes convergent series of inverse powers of x, is artificial and unnecessary.

Some functions do not possess an asymptotic expansion; e^x is an example of such a function. However, if a function has an asymptotic expansion of the power-series form in Eq. (1.161), it has only one. The correspondence is not one to one; many functions may have the same asymptotic expansion.

One of the most useful and powerful methods of generating asymptotic expansions, the method of steepest descents, is developed in Chapter 12 of the printed text.

Exercises

old5.10.1 **1.10.1.** Integrating by parts, develop asymptotic expansions of the Fresnel integrals

$$\text{(a) } C(x) = \int_0^x \cos\frac{\pi u^2}{2}\,du \qquad \text{(b) } s(x) = \int_0^x \sin\frac{\pi u^2}{2}\,du.$$

These integrals appear in the analysis of a knife–edge diffraction pattern.

1.10.2. Rederive the asymptotic expansions of $\text{Ci}(x)$ and $\text{si}(x)$ by repeated integration by parts.

Hint. $\text{Ci}(x) + i\,\text{si}(x) = -\int_x^\infty \frac{e^{it}}{t}\,dt\,.$

old5.10.4 **1.10.3.** Derive the asymptotic expansion of the Gauss error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

$$\approx 1 - \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{2^2x^4} - \frac{1\cdot 3\cdot 5}{2^3x^6} + \cdots + (-1)^n\frac{(2n-1)!!}{2^n x^{2n}}\right)\,.$$

Hint: $\text{erf}(x) = 1 - \text{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt\,.$

Normalized so that $\text{erf}(\infty) = 1$, this function plays an important role in probability theory. It may be expressed in terms of the Fresnel integrals (Exercise 1.10.1), the incomplete gamma functions (Section 13.6), or the confluent hypergeometric functions (Section 18.6).

1.10.4. The asymptotic expressions for the various Bessel functions, Section 14.6, contain the series

$$P_\nu(z) \sim 1 + \sum_{n=1}^{\infty}(-1)^n\frac{\prod_{s=1}^{2n}[4\nu^2 - (2s-1)^2]}{(2n)!(8z)^{2n}}\,,$$

$$Q_\nu(z) \sim \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\prod_{s=1}^{2n-1}[4\nu^2 - (2s-1)^2]}{(2n-1)!(8z)^{2n-1}}\,.$$

Show that these two series are indeed asymptotic series.

1.10.5. For $x > 1$,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{x^{n+1}}.$$

Test this series to see if it is an asymptotic series.

1.10.6. Derive the following Bernoulli-number asymptotic series for the Euler-Mascheroni constant, defined in Eq. (1.13):

$$\gamma \sim \sum_{s=1}^{n} s^{-1} - \ln n - \frac{1}{2n} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)n^{2k}} \ .$$

Here n plays the role of x.

Hint. Apply the Euler-Maclaurin integration formula to $f(x) = x^{-1}$ over the interval $[1, n]$ for $N = 1, 2, \cdots$.

1.10.7. Develop an asymptotic series for

$$\int_0^\infty \frac{e^{-xv}}{(1+v^2)^2}\, dv \ .$$

Take x to be real and positive.

ANS. $\displaystyle \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \cdots + \frac{(-1)^n (2n)!}{x^{2n+1}}$.