Problems and exercises on the theory of shape

To get a feel for what the software will be doing for you, do these problems and exercises using pencil, paper and a scientific calculator. The numbers of landmarks are small, to keep the level of tedium to a minimum!

1. Suppose that the configuration matrix for a given shape is:
   a. How many landmarks are there in this configuration? How many dimensions does it have?

   \[
   A = \begin{bmatrix}
   0.0 & -1.0 \\
   0.0 & 0.5 \\
   0.7 & -0.2
   \end{bmatrix}
   \]

   b. Sketch the shape representing this configuration (you may want to use graph paper, if it helps). Number the landmarks.

   c. Write out the row vector form of this landmark configuration.

   d. Find the centroid position of this landmark configuration. How many coordinates are in the centroid position? Sketch the location of the centroid on your picture from (b) above.

   e. Write out the centered form of this configuration matrix, by subtracting the value of the \(X\)-coordinate of the centroid from each of the values in the first column, and subtracting the value of the \(Y\)-coordinate of the centroid from each of the values in the second column.

   f. Find the centroid size of this landmark configuration.

   g. Now form the pre-shape configuration for \(A\). Do this by dividing the centered form
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of this matrix (solution to (e) above) by the centroid size (solution to (f) above). Remember that when you divide a matrix by a scalar (an ordinary number, like centroid size), you must divide each value in the matrix by the scalar divisor (which is centroid size in this case).

2. Suppose a configuration of dimensional landmarks is given by:

\[ B = \{0.3, -1.0, 0.25, -0.4, 0.0, 0.75, -0.2, 0.35\} \]

a. How many landmarks are in this configuration?
b. Write out the configuration matrix for this configuration.
c. Find the centroid for this configuration.
d. Find the centroid size for this configuration.

3. Given the landmark configuration:

\[ C = \{0.1, 0.1, 0.1, 0.3, -1.0, 1.1, -0.6, -0.3, 0.2, 0.3, -0.1, 0.15\} \]

a. Can you determine what \( K \) and \( M \) are?

4. Suppose we have the configuration matrix:

\[ \mathbf{X} = \begin{bmatrix} 0.5 & 0.5 \\ -0.2 & 0.3 \\ 0.1 & 0.3 \end{bmatrix} \]

a. Compute a configuration matrix that would represent \( \mathbf{X} \) in pre-shape space.
b. Now, for the truly stout of heart, suppose we have a second configuration matrix in pre-shape space:

\[ \mathbf{Y} = \begin{bmatrix} 0.6864 & 0.1961 \\ -0.6824 & -0.0981 \\ 0.0 & -0.0981 \end{bmatrix} \]

Determine the angle that you would have to rotate the pre-shape space matrix form of \( \mathbf{X} \) (from (a) above) to produce a partial Procrustes superposition of \( \mathbf{X} \) on the reference form \( \mathbf{Y} \).

5. Given two matrices:
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\[ \begin{bmatrix} 0.7146 & 0.2150 \\ -0.6438 & -0.0913 \\ -0.0709 & -0.1237 \end{bmatrix} \]

\[ \begin{bmatrix} 0.6864 & 0.1961 \\ -0.6824 & -0.0981 \\ 0.0 & -0.0981 \end{bmatrix} \]

a. where \( X \) is in partial Procrustes superposition with \( Y \):

b. Find the partial Procrustes distance between the two.

c. Use the partial Procrustes distance to find the Procrustes distance between the two.

d. Use the Procrustes distance to calculate the full Procrustes distance between the two.

**Answers to problems and exercises**

(A full solution is given if the calculation has not been seen before.)

1. Looking at the configuration matrix:
   
   a. There are three landmarks \((K = 3 \text{ rows})\) and each is in two dimensions \((M = 2 \text{ columns})\).
   
   b. See Figure 4.19.
   
   c. In row form, \( A = \{0.0, -1.0, 0.0, 0.5, 0.7, -0.2\} \).
   
   d. The centroid is located at \((0.2333, -0.2333)\), or \( X = 0.2333, Y = -0.233 \).

   The centroid position is calculated:
   
   \[
   X_c = \frac{(0 + 0 + 0.7)}{3} = 0.2333 \\
   Y_c = \frac{(-1 + 0.5 - 0.2)}{3} = -0.2333
   \]

   Figure 4.20 shows the location of the centroid.

   e. We simply subtract the \( X \)-coordinate of the centroid \((0.2333)\) from the first column of \( A \), and subtract the \( Y \)-coordinate of the centroid \((-0.2333)\) from the second column. This leaves us with:

\[
\begin{bmatrix} -0.2333 & -0.7667 \\ -0.2333 & 0.7333 \\ 0.4667 & 0.0333 \end{bmatrix}
\]

   Note that if you add up the values in the first column you get zero, which is also true for the second column. Thus the centroid position of the centered matrix is \((0, 0)\).
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f. The centroid size is 1.2055. The centroid size is the square root of the summed squared distances of the landmarks from the centroid, which is:

\[
CS = \{(0 – 0.2333)^2 + (-1 – (-0.2333))^2 + (0 – 0.2333)^2 \\
+ (0.5 – (-0.2333))^2 + (0.7 – 0.2333)^2 \\
+ (-0.2 – (-0.2333))^2\}^{1/2} = 1.2055
\]

An easier approach is to use the centered form of the configuration matrix (with the centroid set to zero). With this form, we can take the square root of the summed squared coordinates of the landmarks:

\[
CS = \{(-0.2333)^2 + (-0.7667)^2 + (-0.2333)^2 \\
+ (0.7333)^2 + (0.4667)^2 + (0.0333)^2\}^{1/2} = 1.2055
\]

g. The resulting pre-shape space configuration is

\[
A_{pre-shape} = \begin{bmatrix}
-0.1936 & -0.0630 \\
-0.1936 & 0.6083 \\
0.3871 & 0.0277
\end{bmatrix}
\]

Note that the entries are identical to the centered matrix values (see (e) above) divided by 1.2055.

2. a. There are four landmarks.
   b. The configuration matrix is:

\[
\begin{bmatrix}
0.3 & -1.0 \\
0.25 & -0.4 \\
0.0 & 0.75 \\
-0.2 & 0.35
\end{bmatrix}
\]

c. The centroid is located at \(X_C = 0.0875, Y_C = -0.075\).

d. The centroid size \(CS = 1.4087\).

3. No! This might be \(K = 6\) and \(M = 2\) (a two-dimensional system), or \(K = 4, M = 3\) (a three-dimensional system). If the data are in a row format, you cannot tell the value of \(K\) or \(M\) from looking at it.

4. Looking at the configuration matrix:
   a. The pre-shape space form of \(X\) is
b. The triangles can be iteratively rotated, or Equation 4.13 can be used:

\[
\theta = \arctan \left( \frac{\sum_{j=1}^{K} Y_{Rj} X_{Tj} - X_{Rj} Y_{Tj}}{\sum_{j=1}^{K} X_{Rj} X_{Tj} + Y_{Rj} Y_{Tj}} \right)
\] (4.13)

Substituting the appropriate values of \(X\) and \(Y\) for the reference (R) and target (T) yields:

\[
\theta = \tan^{-1} \left( (0.1961 \times 0.7013 + (-0.0981) \times (-0.6376) + (-0.0981) \\
\times (-0.0638) + (-0.6864) \times 0.2550 + (-0.6864) \times (-0.1275) \\
+ 0 \times (-0.1275)) / (0.6864 \times 0.7013 + (-0.6864) \times (-0.6376) \\
+ 0 \times (0.0638) + 0.1961 \times 0.2550 + (-0.0981) \times (-0.1275) \\
+ (-0.0981) \times (-0.1275)) \right)
\]

\[
\theta = -0.0562 \text{ radians} = -3.2175^\circ
\]

5. Looking at the matrices:

a. To find the partial Procrustes distance between the two, we take the square root of the summed squared differences in the landmark coordinates:

\[
D_p = \sqrt{(0.7146 - 0.6864)^2 + (0.2150 - 0.1962)^2 \\
+ (-0.6438 - (-0.6864))^2 + (-0.0913 - (-0.0981))^2 \\
+ (-0.0709 - 0)^2 + (-0.1237 - (-0.0981))^2}
\] = 0.0933

b. Because \(\rho = 2 \arcsin (D_p)\), \(\rho = 2 \arcsin(0.0933/2) = 0.0933\) radians; the two are equal through three decimal places.

c. \(D_F = \sin(\rho)\), so \(D_F = \sin(0.0933) = 0.0932\).