CHAPTER 10

Solutions

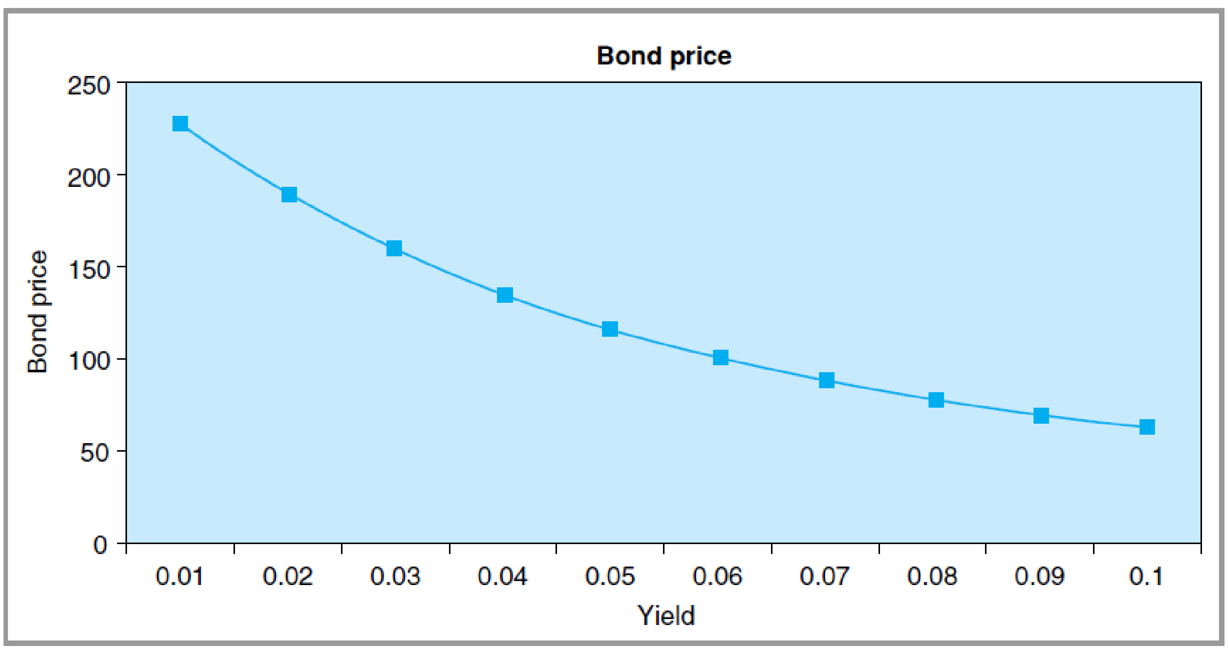
Exercise 1

1. By definition, the price of a coupon bond will be given by,

$$P(0,t_n) = \left[\sum_{i=1}^{n} \frac{c}{\prod_{j=1}^{i}(1+y)^j}\right] + \frac{100}{\prod_{j=1}^{n}(1+y)^j}$$

Since the yield curve is flat, $yk = y1$ for all k. The forward rate, fi, is defined as

$$f_i = \frac{\prod_{j=1}^{i}(1+y)^j}{\prod_{j=1}^{i-1}(1+y)^j} - 1 = y.$$



2. P(0,30) = \$87.59, y = 7%. Bond delta, using the price equation in part a, can be expressed as: where n is years to maturity, and c is the coupon payment. So initially, bond delta is −1116 and forward contract delta is −100. In order to be delta neutral, we must short 11 forward contracts for each long coupon bond in the portfolio.

   In order to construct a zero cost portfolio, we need, in addition, to borrow \$87.59 at the ongoing interest rate 7%. (Remember that value of the forward contract is initially zero.)

$$\frac{dP}{dy} = -\frac{c}{y^2} \times \left[1 - (1+y)^{(-n)}\right] + \frac{c}{y} \times \frac{n}{(1+y)^{n+1}} - \frac{100n}{(1+y)^{n+1}}$$

   where n is years to maturity, and c is the coupon payment. So initially, bond delta is −1116 and forward contract delta is −100. In order to be delta neutral, we must short 11 forward contracts for each long coupon bond position in the portfolio.

   According to this result, if we want to construct a zero cost portfolio, we need to borrow \$87.59 at on going interest rate (7%) (Remember that value of the forward contract is initially zero).

3. See the table below. Sum of the entries on the last column is the total convexity gain.

Yield	Bond Delta	Forward Delta	# of Forwards	Price	Mark-to-Market
0.07	-1116	-100	11	93	0
0.09	-754	-100	8	91	22
0.07	-1116	-100	11	93	-15
0.09	-754	-100	8	91	22

0.07	-1116	-100	11	93	-15
0.09	-754	-100	8	91	22
0.07	-1116	-100	11	93	-15
				Gains:	22

4. Other costs are funding cost and other operational costs, fees and commission paid.

Exercise 2

1. Price of 30 year bond is,

$$B(0,30) = \frac{100}{(1+y)^{30}}$$

and it is equal to \$23.14 when y = 5%. In order to meet the zero-cost condition, we borrow \$23.14 at a rate of 5%. Bond's delta is given by,

$$\frac{dB}{dy} = -\frac{3000}{(1+y)^{31}}$$

So initial bond delta is -661 and euro dollar contract delta is -25. That means for each long bond position, we must short 661/25 euro contracts to achieve delta neutrality, initially.

2. The solution to this problem is very similar to solution given for question 1, part (d) above. The sum of the entries on the last column is the convexity gains.

Yield	Bond Delta	ED Delta	# of Contracts	Price of ED	Mark-to-Market
0.05	-661	-25	26	98.75	0

0.06	-493	-25	19	98.5	6.5
0.04	-889	-25	35	99	-9.5
0.06	-493	-25	19	98.5	17.5
0.04	-889	-25	35	99	-9.5
0.06	-493	25	19	98.5	17.5
0.04	-889	-25	35	99	-9.5
				Convexity Gains:	13

Exercise 3

Interest rate fluctuations are wider in this question compared to previous question. That means higher volatility which implies higher convexity gains. So, the total rate of return on this bond will be higher, while interest rate for 30 year bond decreases.

Exercise 4

Let *f* be forward rate on Libor-on-arrears FRA. And *F* be the forward rate on market traded FRA. Then, existence of convexity requires the following adjustment between these two forward rates:

$$f = F + \sigma^2$$

So the spread is equal to σ^2, $(0.02)^2$.

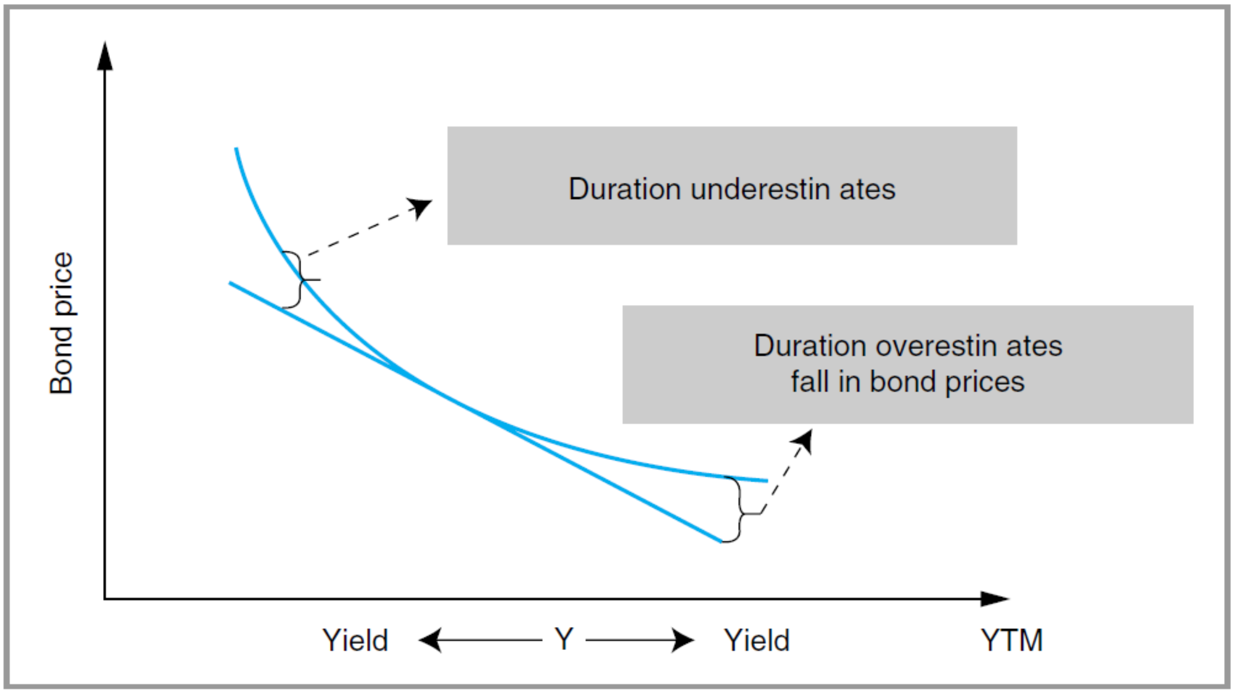
Exercise 5

(Engineering Convexity Positions; Case Study: Convexity of Long bonds, Swaps and Arbitrage)

1. We now, explain the notion of convexity of long bonds. For a given decrease in the yield, bond prices will increase more compared to the linear approximation of the price-yield relationship. For a given increase in the yield, bond prices will decrease less compared to the linear approximation of the price-yield relationship.

The Figure 1 shows this.

Figure 1:



We can also give an example.

Let us find the bond sensitivities for a 3-year bond and a 30-year bond given the following conditions:

(a) y_0 = 6.5% and $\Delta\, y_0$ = 0.3

(b) y_1 = 6.9% and $\Delta\, y_1$ = 0.6

where, y_0, y_1 are the yields of 3 year and 30 year default-free discount bonds, whose prices are denoted by B_3 and B_{30}. These prices will be given by:

$$B_3 = \frac{100}{(1+y_0)^3}$$

$$B_{30} = \frac{100}{(1+y_0)^{30}}$$

The first order sensitivities are related to these bonds *duration*. For the short bond this will be given by:

$$\frac{\partial B_3}{\partial y_0} \ \frac{1}{B_3} = (-3)\frac{100}{(1+y_0)^{3+1}}$$

$$= \frac{-300}{(1+y_0)^4}$$

$$=3\left(\frac{1}{(1+y_0)} \frac{100}{(1+y_0)^3}\right)$$

$$=3\left(\frac{1}{(1+y_0)} B_3\right)$$

This means that the percentage in the bond price will be:

$$\frac{\partial B_3}{\partial y_0} = (-3)\frac{100}{(1+y_0)^{3+1}}$$

$$= \frac{-300}{(1+y_0)^4}$$

$$= 3\left(\frac{1}{\big((1+y_0)\big)} \frac{100}{(1+y_0)^3}\right)$$

$$= 3\frac{1}{(1+y_0)} B_3$$

$$\frac{\partial B_3}{\partial y_0}\frac{1}{B_3} = 3\frac{1}{(1+y_0)}$$

Hence the term *modified duration*. The right hand side in this expression gives the slightly modified maturity of the cash payments associated with this security.

For the long bond we get:

$$\frac{\partial B_{30}}{\partial y_1}\frac{1}{B_{30}} = 30\frac{1}{(1+y_1)}$$

We can use this in order to get approximate measures of bond price sensitivities. For example:

$$\frac{\Delta B_3}{B_3} \cong 3\frac{1}{(1+y_0)}\Delta y_0$$

$$\frac{\Delta B_{30}}{B_{30}} \cong 30\frac{1}{(1+y_1)}\Delta y_1$$

These measures indicate that, the 30-year bond will be about 10 times more sensitive to an interest rate change than a 3-year bond, for the same amount of yield movement.

We can also calculate second order sensitivities. These convexity or "Gamma" effects will show how the first order sensitivities change as the yield moves.

$$\frac{\partial^2 B_3}{B\partial y_0^2}\frac{1}{B_3} = 12\frac{1}{(1+y)^2}$$

and

$$\frac{\partial^2 B_{30}}{B\partial y_0^2}\frac{1}{B_{30}} = 930\frac{1}{(1+y)^2}$$

Thus, the long bond duration will be about 80 times more sensitive to changes in the yield when compared with the short bond.

2. Suppose we short the 3-year bond and go long on the 30-year bond. Then, consider three cases where interest rates move up +0.3, stay same or move down −0.3. How will all these affect the bond portfolio?

- If interest rates rise we will gain more on the short position of less convex bonds than the amount we would lose on the long position of more convex bonds;

• If interest rates fall we will gain more on the long position of more convex bonds than the amount we would lose on the short position of less convex bonds;

• If interest rates remain unchanged, portfolio's value will remain the same.

According to this we are short the lesser convex bond and long the more convex bond. As yields fall the price of this latter rises higher than the less convex bond and as yields rise its price falls less.

3. Swap convexity will be similar to coupon-bond convexity analysis. Consider the following terminology:

(a) s_t = Swap rate at time *t* on a swap that starts at *t*.

(b) *n* = Number of swap settlements.

(c) δ = Tenor of the floating leg. δ = 1/4 corresponding to a 3-month floating rate leg.

(d) *N* = Notional amount.

(e) L_{t_i} = Libor rate to settle in-arrears at time t_{i+1}.

(f) F_{t_i} = The FRA rate that corresponds to the floating Libor rate L_{t_i} .

(g) fixed and floating day basis, both 30/360.

Under these conditions the value of the swap at time, t_0, will be given by:

$$V_{t_0} = \sum_{i=1}^{n} \left[\frac{(F_{t_{i-1}} - s_{t_0}) N\delta}{\prod_{j=1}^{i} (1 + F_{t_{i-1}}\delta)} \right]$$

Note that this is a convex(concave) function.

Note that this gives the discounts B_i as of t_0 as:

$$B_i = \frac{\mathbf{1}}{\prod_{\boldsymbol{j=1}}^{\boldsymbol{i}} (\mathbf{1} + \boldsymbol{F}_{\boldsymbol{t_{i-1}}}\delta)}$$

Now suppose the we consider two swaps with n = 3 and n = 5 respectively, with δ = 1. The Swap notional is 10m. The yield curve is flat at 4%. This makes all forward rates equal 4%. Then we can calculate the following numbers using the formula above.

Changes in the value of the swap	**Scenario 1**	**Scenario 2**
Type of Swap	Parallel shift down 1%	Parallel shift up 1%
3 yr swap 5 yr swap	- \$193,887.49 - \$376,497.28	\$ 188,338.21 (value 0 at t=0) \$ 358,940.77 (value 0 at t=0)

From these numbers we see that the 5-year swap is more sensitive to the changes in the interest rates than 3-year swap; therefore, the trader might gain more by trading long-dated swaps.

4. We let a Libor-in-arrears instrument pay according to the following function:

$$V_t = 100\left(1 - L_{t_i}\delta\right)$$

A eurodollar futures has this pricing function. If a position is taken at time t_0 with the forward rate f_{t_0} the net payoff at t_i will be:

$$V_{t_i} - V_{t_0} = \left(f_{t_0} - L_{t_i}\right)N\delta$$

A market traded FRA on the other will have the time t_i payoff

$$W_{t_i} = \frac{\left(F_{t_0} - L_{t_i}\right)N\delta}{\left(1 + L_{t_i}\delta\right)}$$

Note that one payoff is Linear in L_{t_i} whereas the other is non-linear.

The ϵ in the relationship between f_{t_0} and F_{t_0} gives the convexity adjustment:

$$F_{t_0} = f_{t_0} + \epsilon$$

Under these conditions the two forward rates would not be the same due to the convexity.

5. The position taken by the knowledgeable professionals can be summarized as follows:

(a) Receive Libor-in-arrears with a Libor-in-arrears FRA

(b) Pay Libor at the start of the period using a market traded FRA.

(c) Sell caps against the Libor-in-arrears being received.

(d) Delta hedge the swap

In this environment, swaps are more convex as they are equal to a series of FRAs.

6. Knowledgeable market professionals take their position using swaps. We discuss the answer using FRAs. Swaps can be reconstructed from FRAs and hence our approach can be duplicated for swaps as well. Essentially, the less competent professionals are using the same f_{t_0} in valuing both FRAs, the paid-in-arrears and the Libor-in-arrears. Thus by buying the Libor is arrears FRA and selling the paid arrears FRA, one would end up with the net convexity adjustment factor ϵ. This factor is equal to

$$\epsilon = F_{t_i}^2 \sigma^2 \Delta \frac{1}{1 + F_{t_i}\delta}$$

7. At this point the position taken is not true arbitrage, because the gains depend on the level of volatility, although they are always positive. But, once the transaction costs are taken into account, the position may lose money if the volatility goes down significantly. This is due to the fact that the gains are a function of the σ^2.

   Besides, there are the usual counterparty risks.

8. A cap is a series of European interest rate call options covering a different forward time periods each with the same strike price. You can think of it as a series of call options on

FRAs. Caps are priced using Black's formula, which assumes a forward rate that moves as a Martingale.

9. Smart Traders can lock-in their potential convexity and volatility gains by selling a cap on the forward yields. Premium from the cap includes implied volatility expectation for the remaining time to the next period as well as the expected convexity gains.

Exercise 6

(For detailed calculation see also Excel file 'Exercise 10.6 Solution Matlab Calculation' on book webpage.)

Solution:

The bond price formula is given by $B(t,T) = A(t,T)e^{-C(t,T)}$

where,

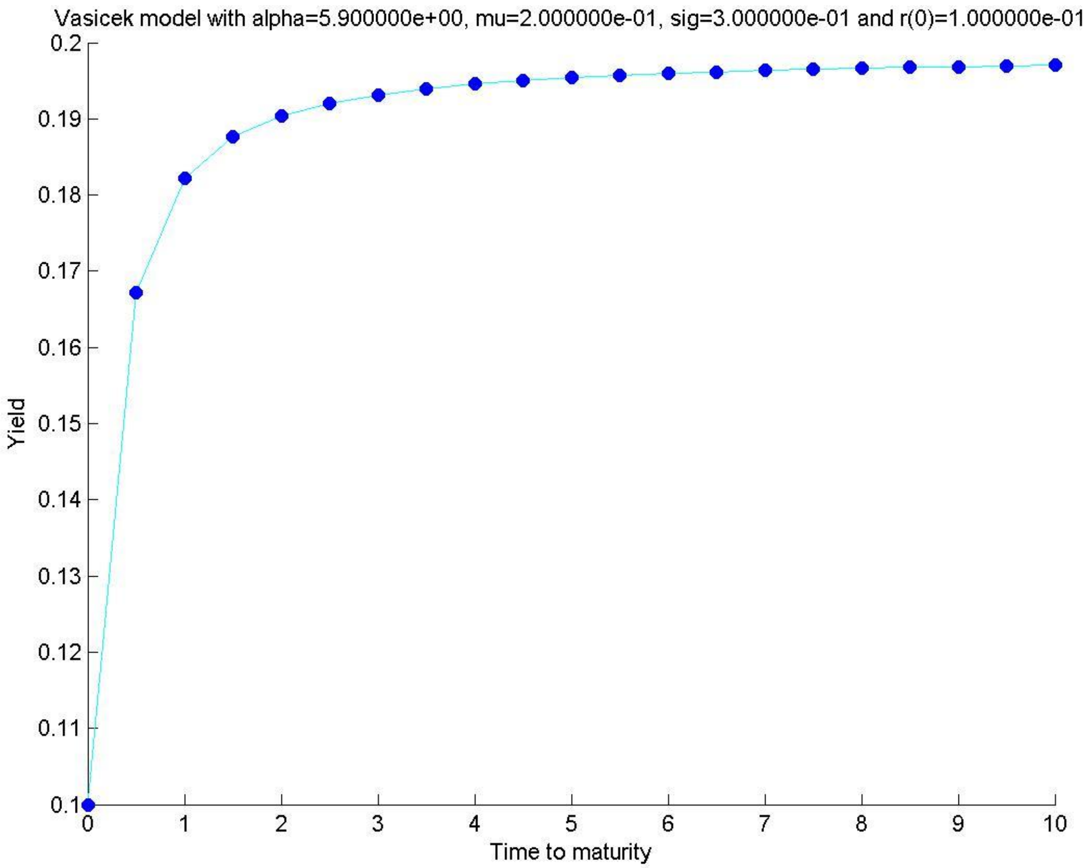
$$C(t,T) = \frac{(1 - e^{-\alpha(T-t)})}{\alpha}$$

$$A(t,T) = e^{\frac{(C(t,T) - (T-t))(\alpha^2\mu - \frac{\sigma^2}{2})}{\alpha^2} - \frac{\sigma^2 C(t,T)^2}{4\alpha}}$$

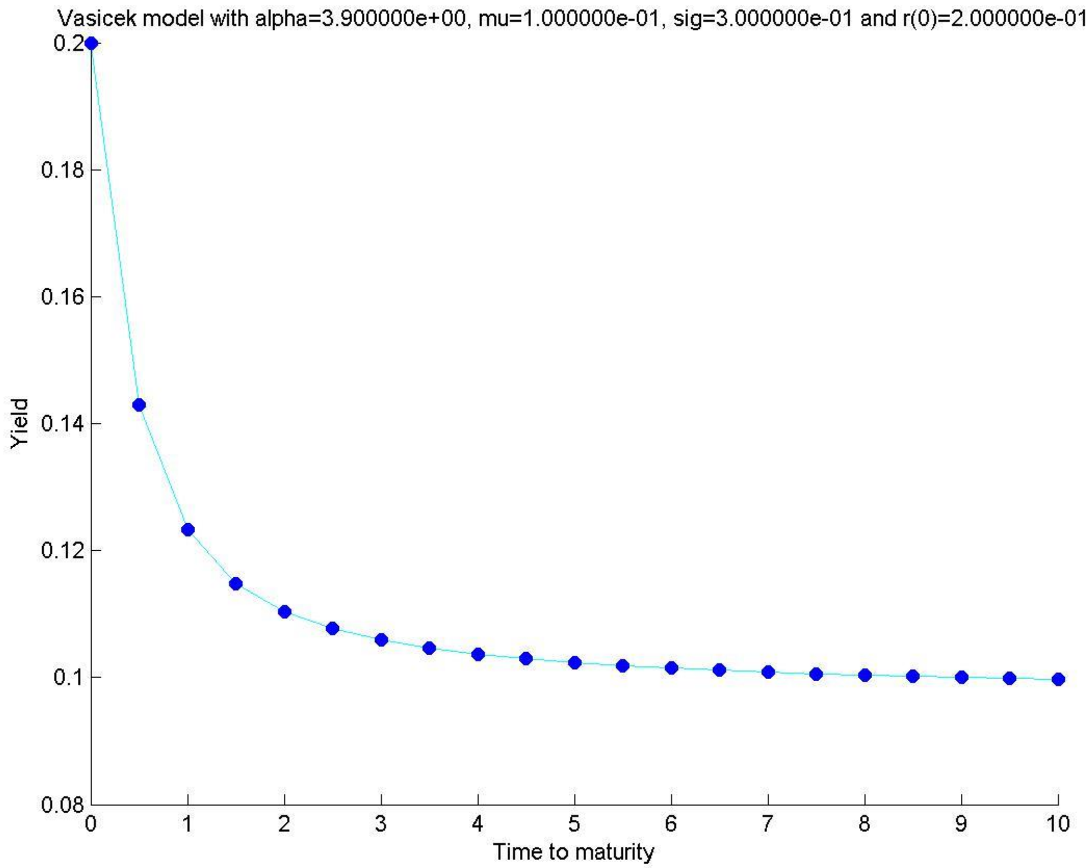
And finally we obtain the yield of the bond by $y(t,T) = -\frac{\log(T-t)}{(T-t)}$

Plots

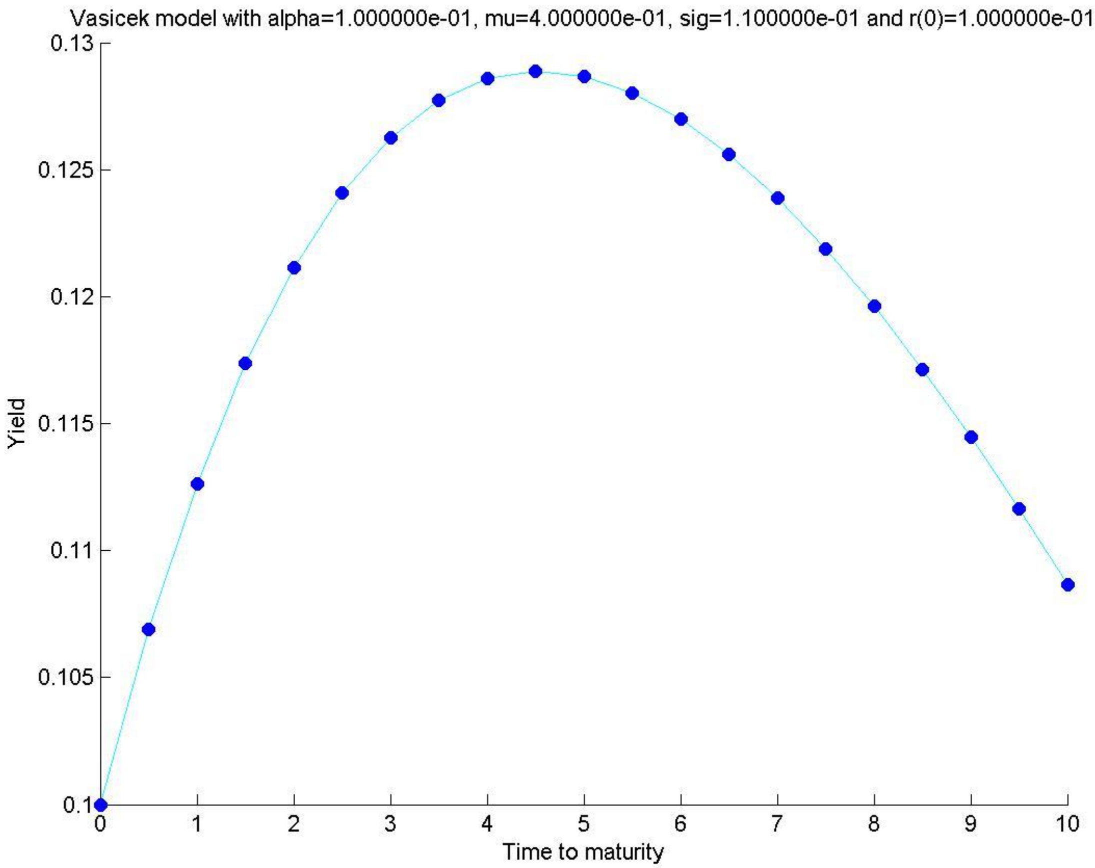
Term structure plot for the first set of values i.e. [5.9, 0.3, 0.2, 0.1]



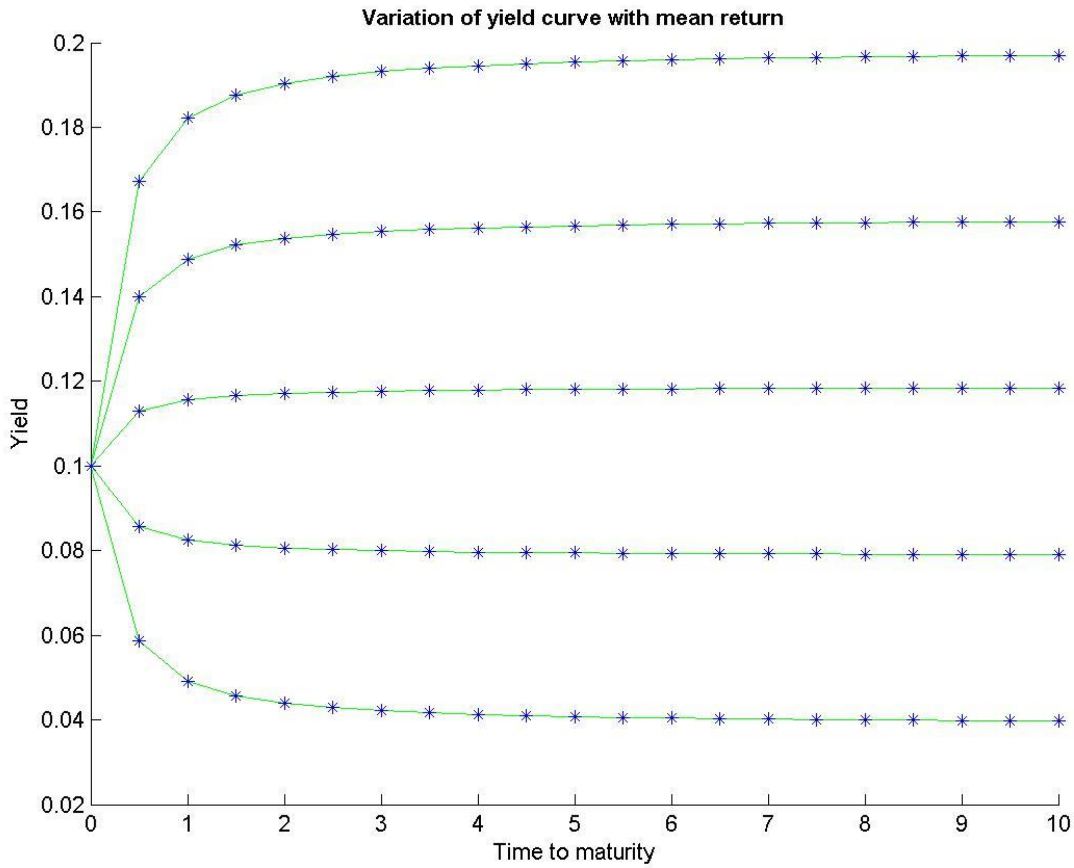
Term structure plot for the second set of values i.e. [3.9, 0.1, 0.3, 0.2]



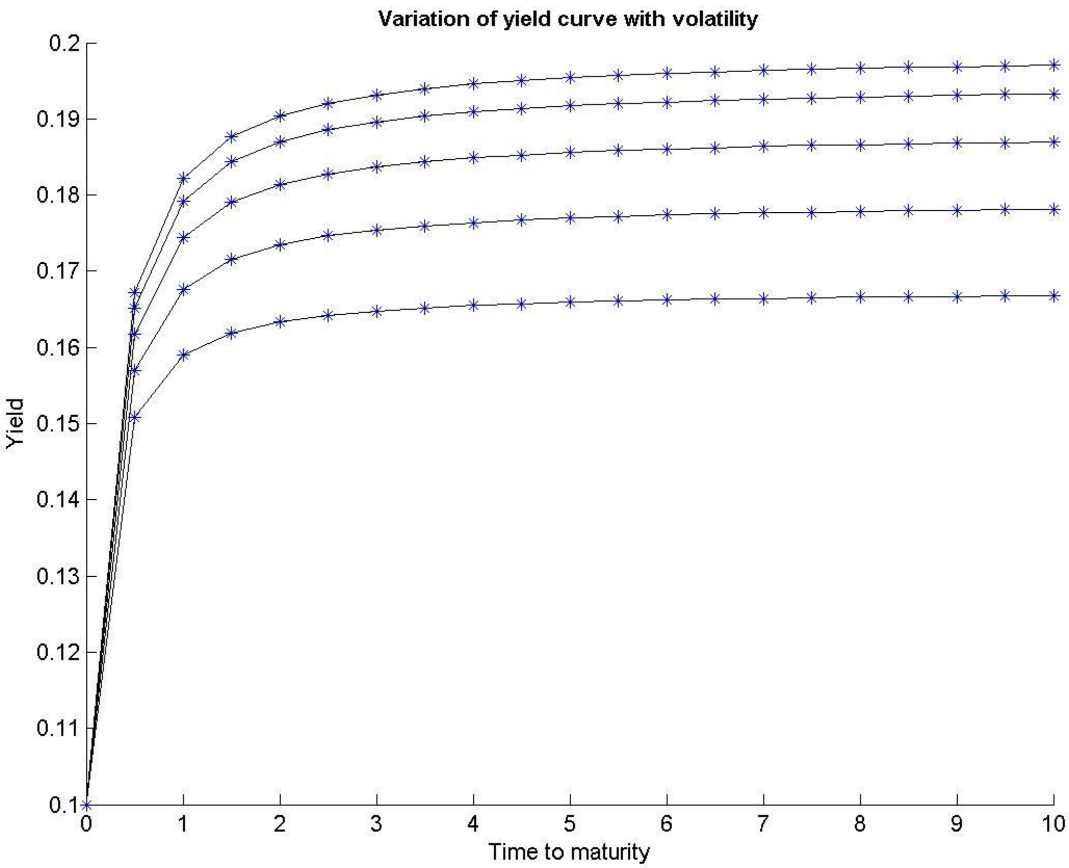
Term structure plot for the third set of values i.e. [0.1, 0.4, 0.11, 0.1]



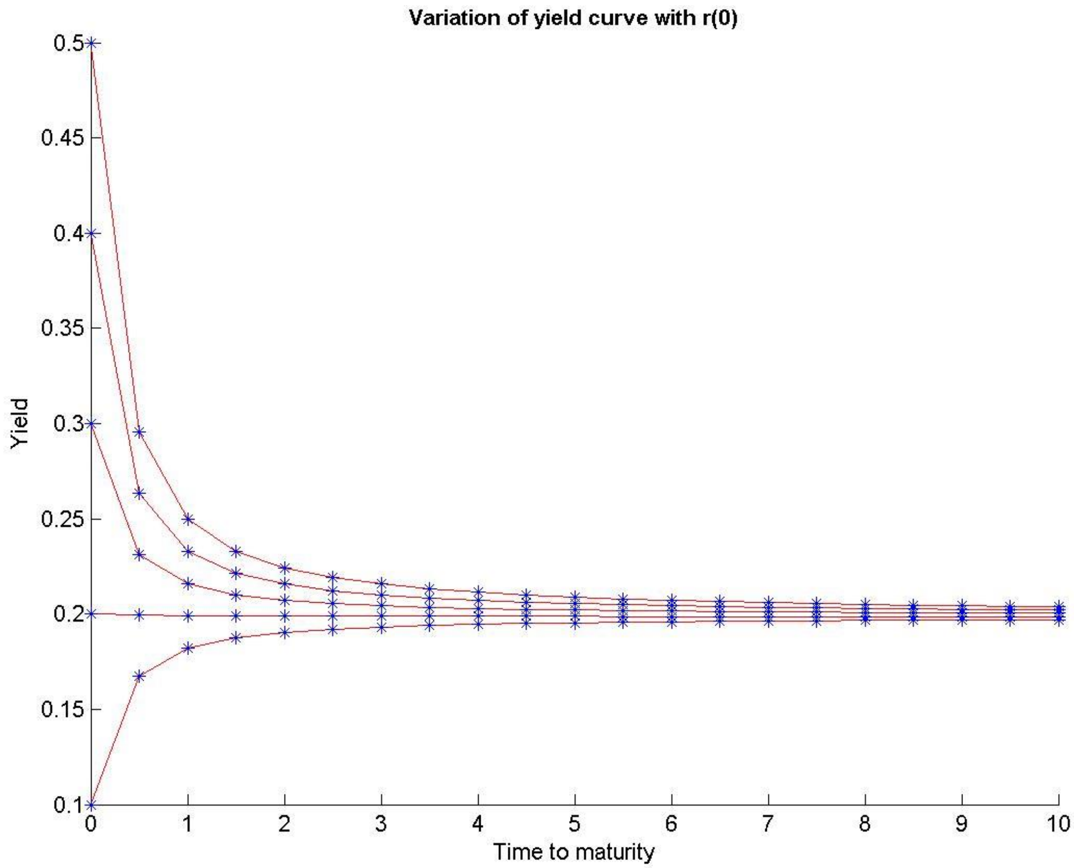
Variation of term structure with the change in mean return μ



Variation of term structure with the change in volatility σ



Variation of term structure with the change in initial return r(0)



Exercise 7

(For detailed calculation see also Excel file 'Exercise 10.7 Solution Matlab Calculation' on book webpage.)

<u>Solution:</u>

The bond price formula is given by $B(t,T) = A(t,T)e^{-C(t,T)}$

where the functions $A(t,T)$ and $C(t,T)$ are given by,

$$A(t,T) = \left(2\frac{\gamma e^{1/2(\alpha+\gamma)(T-t)}}{(\alpha+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right)^{2\frac{\alpha\mu}{\sigma^2}}$$

$$C(t,T) = \left(2\frac{e^{\gamma(T-t)} - 1}{(\alpha + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}\right)$$

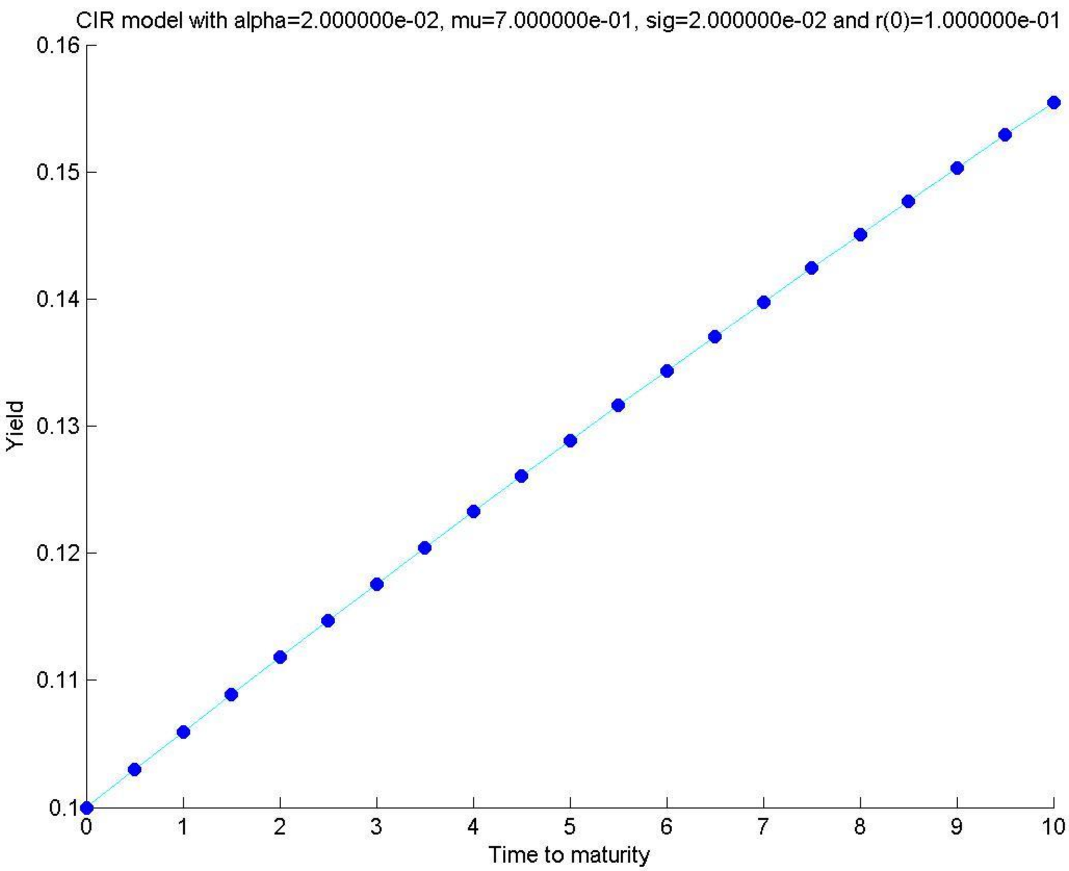
And where γ is defined as

$$\gamma = \sqrt{\alpha^2 + 2\sigma^2}$$

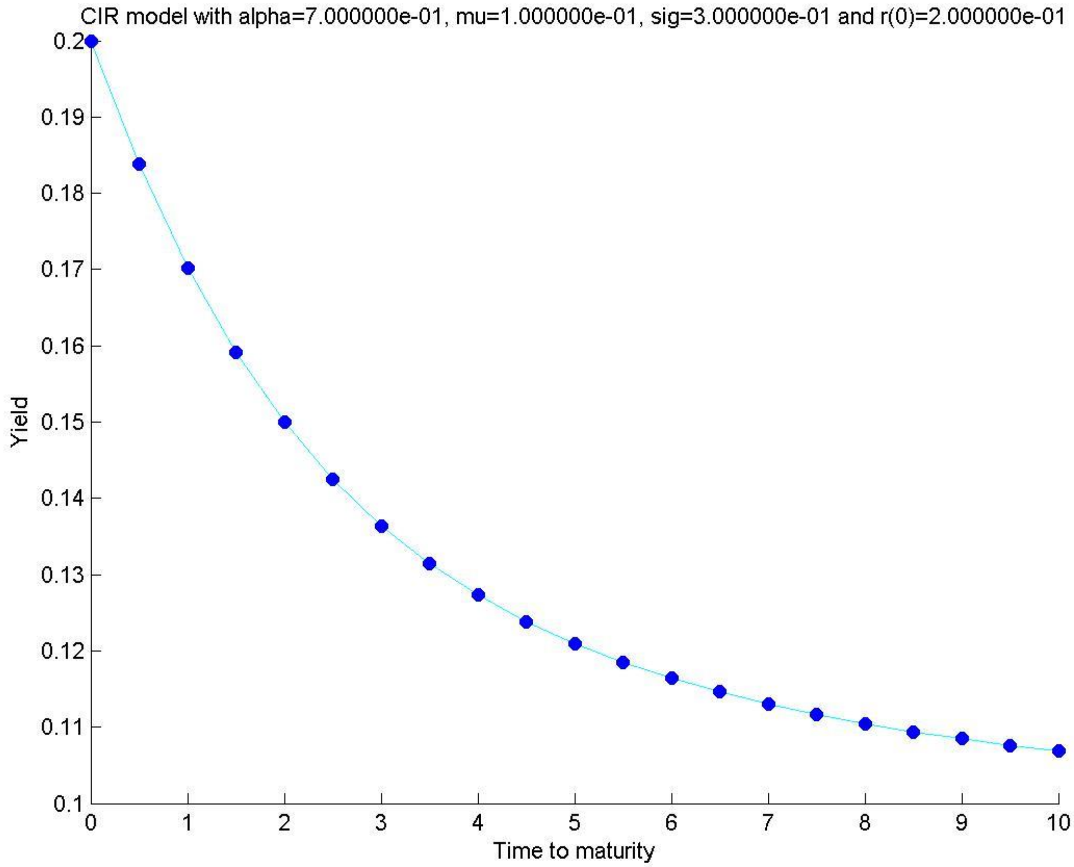
And finally we obtain the yield of the bond by $y(t,T) = -\frac{\log(T-t)}{(T-t)}$

Plots

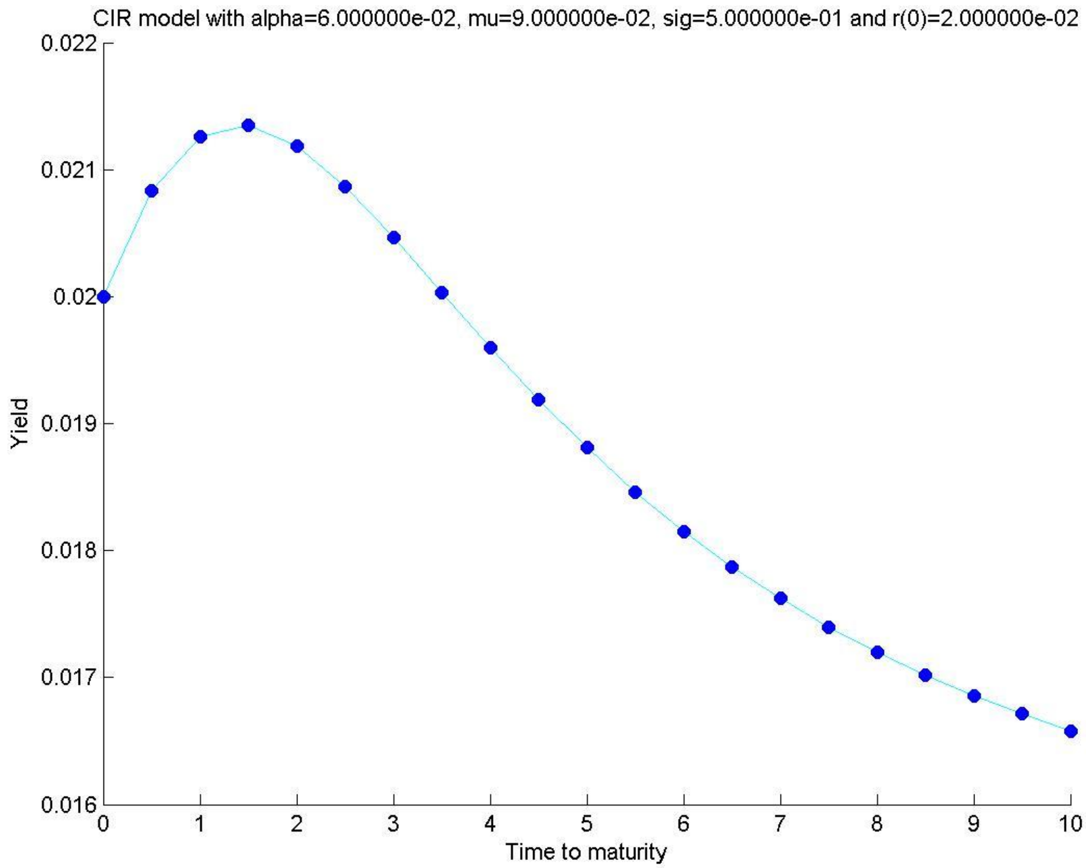
Term structure plot for the first set of values i.e. [0.02, 0.7, 0.02, 0.1]



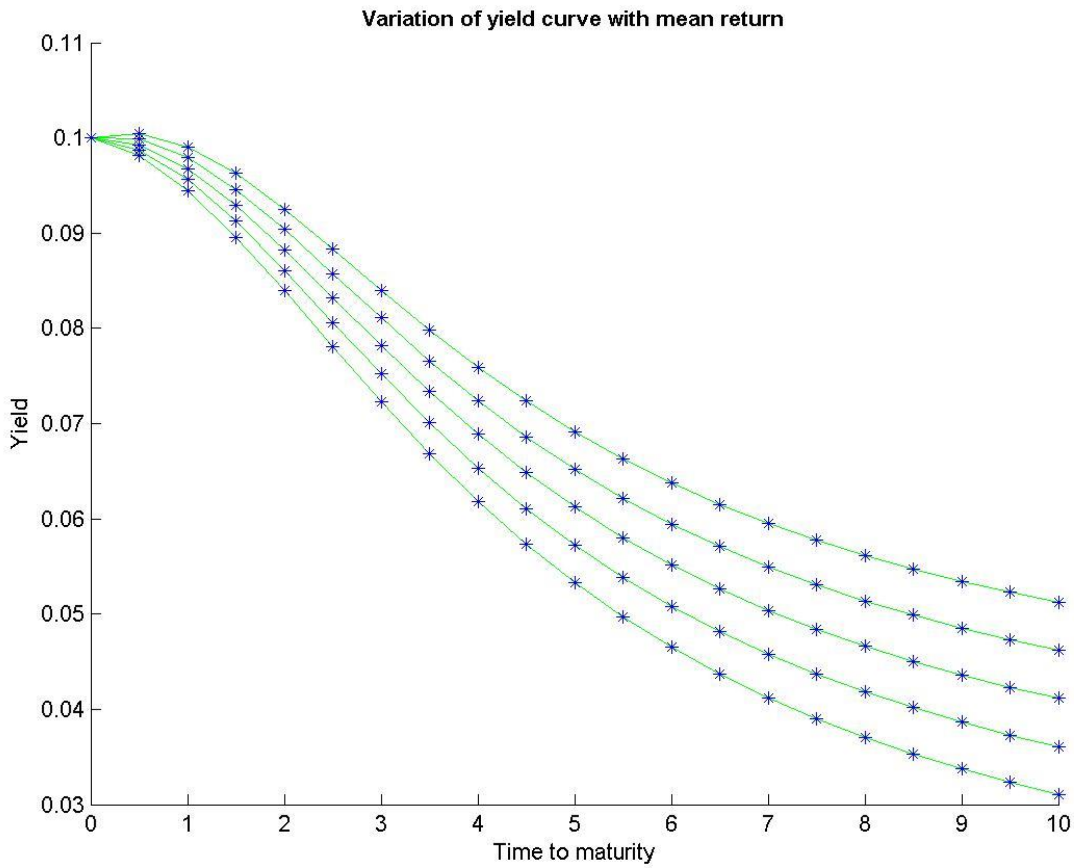
Term structure plot for the second set of values i.e. [0.7, 0.1, 0.3, 0.2]



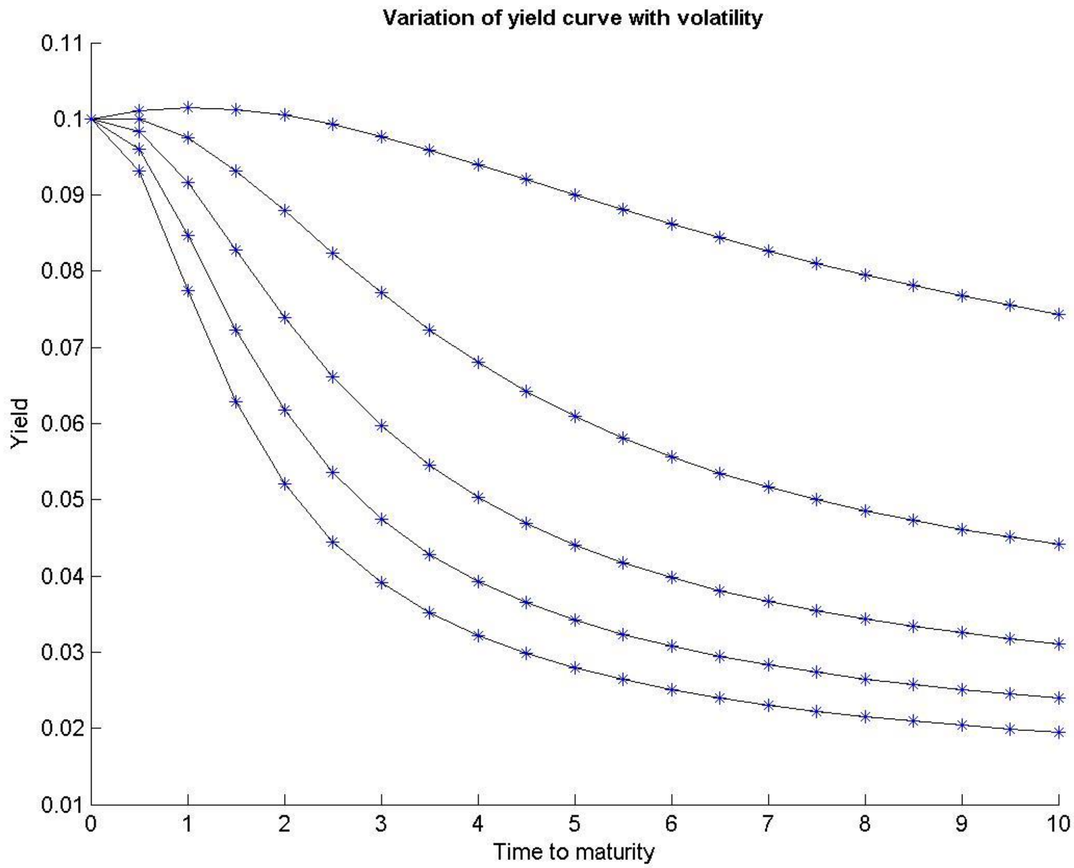
Term structure plot for the third set of values i.e. [0.06, 0.09, 0.5 0.02]



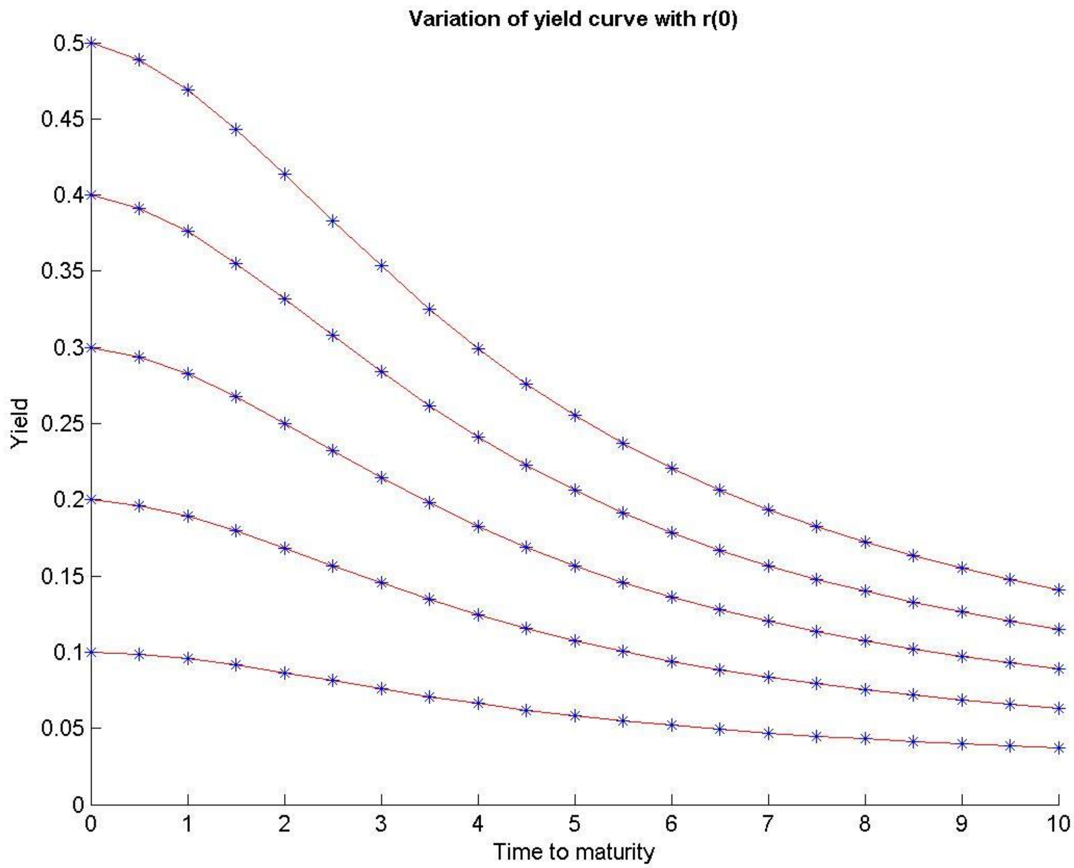
Variation of term structure with the change in mean return μ



Variation of term structure with the change in volatility σ



Variation of term structure with the change in initial return r(0)



Variation of yield curve with r(0)
Yield
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
9
10
Time to maturity