CHAPTER 12

Solutions

Exercise 1

We can form the matrix equation and the solve for the corresponding state price Q^i

$$S_{5x1} = D_{5x4} * \ Q_{4x1}$$

$$Q^1 = 0.55, \qquad Q^2 = 0.49, \qquad Q^3 = 0.13, \qquad Q^4 = 0.06$$

$$S_T^5 = \sum_{i=1}^{4} Q^i S_T^i = 14.5$$

Exercise 2

1. If there exist positive state prices, Q^1,Q^2,Q^3 and Q^4, which satisfy the following matrix equation, then there are no arbitrage opportunities.

$$\begin{bmatrix} 1 \\ 0.91 \\ 0.86 \\ 0.77 \end{bmatrix} = \begin{bmatrix} 1.113 & 1.113 & 1.092 & 1.092 \\ 1 & 1 & 1 & 1 \\ 0.9 & 0.92 & 0.95 & 0.96 \\ 0.8 & 0.84 & 0.85 & 0.86 \end{bmatrix} \begin{bmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \end{bmatrix}$$

Solving the matrix equation above, we get:

$$\begin{bmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \end{bmatrix} = \begin{bmatrix} -0.69619 \\ 0.995238 \\ 1.55619 \\ -0.94524 \end{bmatrix}$$

2. As it can easily be seen, state prices, Q^1 and Q^4 are negative which implies that there are arbitrage opportunities in this market. However, such negative state-prices can result if the chosen model is incorrect. (If for example, the wrong number of states of the wrong set of assets is chosen.)

3. $1X2$ FRA rate is equal to current Libor rate of 5%. Remember that current time is $t = 1$, a $1X2$ FRA starts (and expires) at time 1 and settles at time 2 and finally the current Libor rate is 5%.

Exercise 3

1. $\Delta = \frac{200}{5} = 40\ days$

or

$$\Delta = \frac{40}{365} = 0.1096\ years$$

By using equations (12.100)–(12.102), we can write:

$$u = e^{\sigma\sqrt{\Delta}}$$

$$= e^{0.18\sqrt{0.1096}} = 1.061$$

$$d = 0.942$$

2. $p = \frac{e^{r\Delta} - d}{u - d} = \frac{e^{0.04\times0.1096} - 0.942}{1.061 - 0.942} = 0.52$

3. See the tree below for the binomial tree of stock prices.

$t=0$	$t=40$	$t=80$	$t=120$	$t=160$	$t=200$
					134.45
				126.72	
			119.44		119.37
		112.57		112.51	
	106.1		106.04		105.99
100		99.95		99.89	
	94.2		94.15		94.1
		88.74		88.69	
			83.59		83.54
				78.74	
					74.17

4. Using the binomial tree for stock prices in part *d*, we get the following binomial tree for the call premium:

$t=0$	$t=40$	$t=80$	$t=120$	$t=160$	$t=200$
					34.45
				27.093	
			20.187		19.37
		14.349		12.891	
	9.813		8.156		5.99
6.505		4.99		3.101	
	2.981		1.605		0
		0.831		0	
			0		0
				0	
					0

Exercise 4

1. In this case where the stock pays continuous dividends of 4%, there would be no changes in the binomial tree for the stock price. The tree is the same as in part (d) of the previous question. The only difference is in the calculation of up-probabilities.

$$p = \frac{e^{(r-q)\times\Delta} - d}{u - d} = \frac{e^{(0.04-0.04)\times 0.1096} - 0.942}{1.061 - 0.942} = 0.49$$

where, q is the continuous rate of dividends. By using the binomial formula to determine the call premium, we get:

$$c = e^{-0.04(5\times 0.1096)} \sum_{i=0}^{5} C_i \binom{5}{i} p^i (1-p)^{5-i}$$

Where c is the option premium at time zero and C_i's are the option values on the expiration date for i = 5, C_5 = 34.45, i = 4 and C_4 = 19.37, so on.

Computing the formula given above, we obtain, C_0 = 5.53.

2. If stock pays 5% of its value as a dividend at the third node (t = 120 days), this will result in the following binomial tree for the stock price:

$t=0$	$t=40$	$t=80$	$t=120$	$t=160$	$t=200$
					127.73
				120.39	
			113.47		113.40
		112.57		106.89	
	106.1		100.74		100.69
100		99.95		94.90	
	94.2		89.44		89.40
		88.74		84.26	
			79.41		79.36
				74.80	
					70.46

In order to compute call premium *c* at time zero, we can apply the binomial formula in part (b) with:

$$p = 0.52$$

and

$$C_5 = 27.73,\ C_4 = 13.40,\ C_3 = 0.69$$

and

$$C_2 = C_1 = C_0 = 0.$$

The result is:

$$c = 3.55.$$

3. See binomial tree. The third type of dividend payment creates a non-recombining tree.

Exercise 5

1. $\Delta = 0.1096$ years.
2. For this case we use equations (12.106)–(12.108), after adjusting them properly:

$$u = e^{(r-r_f)\Delta - \frac{1}{2}\sigma^2\Delta + \sigma\sqrt{\Delta}}$$

$$d = e^{(r-r_f)\Delta - \frac{1}{2}\sigma^2\Delta - \sigma\sqrt{\Delta}}$$

   Then, $u = 1.063$ and $d = 0.931$, $p = 0.5$.

3. Using the parameters in part one and two of this question, binomial tree for exchange-rate would be:

$t=0$	$t=40$	$t=80$	$t=120$	$t=160$	$t=200$
					1.9
				1.79	
			1.68		1.67
		1.58		1.56	
	1.49		1.47		1.46
1.4		1.39		1.37	
	1.3		1.29		1.27
		1.21		1.2	
			1.13		1.11
				1.05	
					0.98

4. The binomial tree for European put will be given as in the next page.

5. The binomial tree for American (put) option is somewhat more complicated than that of the European (put) option. The reason for that is that American options can be exercised earlier.

$t=0$	$t=40$	$t=80$	$t=120$	$t=160$	$t=200$
					0
				0	
			0.01		0
		0.031		0.02	
	0.065		0.052		0.04
0.11		0.099		0.085	
	0.156		0.147		0.13
		0.213		0.209	
			0.281		0.29
				0.354	
					0.42

At each node, value of an American option is equal to its intrinsic value or to the value which comes from holding it until next time (node) whichever is greater. This gives the binomial tree for the American put option, where the values are calculated backwards. The tree is shown below.

Exercise 6

1. Applying equations (12.100)–(12. 102), we determine u = 1.10, d = 0.91 and p = 0.51.

2. We can directly apply the binomial formula to determine the value of the call option. Alternatively we can use binomial tree methods. Both will lead to the same answer:

   C = $5.50

3. For the barrier option, it is instructive to have a look at the binomial tree. The value of the barrier call option along the paths which lead to $120 stock price becomes zero. In this case call premium will be: C_b = $0.041.

$t = 0$	$t = 40$	$t = 80$	$t = 120$	$t = 160$	$t = 200$
					$\max(-.50,0)$
				$\max(-.29,0)$	
			$\max(-.18,.01)$		$\max(-.27,0)$
		$\max(-.08,.04)$		$\max(-.06,.02)$	
	$\max(.01,.09)$		$\max(.03,.07)$		$\max(.04,.04)$
$\max(-.10,.15)$		$\max(.11,.14)$		$\max(.13,.08)$	
	$\max(.20,.21)$		$\max(.21,.21)$		$\max(.13,.13)$
		$\max(.29,.29)$		$\max(.3,.21)$	
			$\max(.37,.37)$		$\max(.29,.29)$
				$\max(.45,.35)$	
					$\max(.42,.42)$

4. Barrier Option will be cheaper

Exercise 7

(For detailed calculation see also Excel file 'Exercise 12.7 Solution Excel Calculation' on book webpage.)

Calculation

Using $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$ we can calculate the probability of stock price going up

$$p = \frac{e^{r\Delta} - d}{u - d}$$

And after calculating payoff at the expiration discount the call price at each node as the following:

$$c_t = \frac{p * c_{t+1}^{u} + (1-p) * c_{t+1}^{d}}{e^{r\Delta}}$$

Observation

As you increase the value of M the option price calculated from the binomial model gets closer to the actual value calculated from the *BSM* formula

European Call Price = 13.34   European Put Price = 10.26

No. of steps M	***European Call***	***European Put***
10	13.465	10.992
20	13.447	10.375
50	13.388	10.315
100	13.342	10.269

Exercise 8

(For detailed calculation see also Excel file 'Exercise 12.8 Solution Excel Calculation' on book webpage.)

Calculation

Using $u = e^{\sigma\sqrt{\Delta t}+(r-\sigma^2/2)\Delta t}$ and $d = e^{-\sigma\sqrt{\Delta t}+(r-\sigma^2/2)\Delta t}$ and the value of $p = 0.5$ proceed as following:

Once you calculate payoff at the expiration discount the call price at each node till time $t = 0$.

$$c_t = \frac{p * c_{t+1}^u + (1-p) * c_{t+1}^d}{e^{r\Delta}}$$

Observation

As you increase the value of M the option price calculated from the binomial model gets closer to the actual value calculated from the *BSM* formula

European Call Price = 13.34 European Put Price = 10.26

No. of steps M	*European Call*	*European Put*
5	*13.87*	*10.83*
7	*13.71*	*10.66*
10	*13.19*	*10.13*
15	*13.51*	*10.44*

Exercise 9

(For detailed calculation see also Excel file 'Exercise 12.9 Solution Excel Calculation' on book webpage.)

Calculation

Using $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$ we can calculate the probability of stock price going up

$$p = \frac{e^{r\Delta} - d}{u - d}$$

And after calculating payoff at the expiration discount the call price at each node as the following:

$$c_t = max\left[\frac{p * c_{t+1}^u + (1-p) * c_{t+1}^d}{e^{r\Delta}}, payoff\ at\ time\ 't'\right]$$

Observation

The value of American option as we increase the value of M is reported below:

No. of steps M	***American Call***	***American Put***
10	13.465	11.628
20	13.448	11.589
50	13.388	11.537
100	13.342	11.508

Exercise 10

(For detailed calculation see also Excel file 'Exercise 12.10 Solution Excel Calculation' on book webpage.)

Calculation

As you increase the value of M the option price calculated from the binomial model gets closer to the actual value calculated from the *BSM* formula

European Call Price = 9.12   European Put Price = 13.73

No. of steps M	***European Call***	***European Put***
10	9.257	13.873
20	9.236	13.851
50	9.174	13.789
100	9.127	13.743

Exercise 11

(For detailed calculation see also Excel file 'Exercise 12.11 Solution Excel Calculation' on book webpage.)

Calculation

Using $u = e^{\sigma\sqrt{\Delta t}+(r-div-\sigma^2/2)\Delta t}$ and $d = e^{-\sigma\sqrt{\Delta t}+(r-div-\sigma^2/2)\Delta t}$ and the value of $p = 0.5$ proceed as following: Once you calculate payoff at the expiration discount the call price at each node till time

$t = 0.$

$$c_t = \frac{p * c_{t+1}^u + (1-p) * c_{t+1}^d}{e^{r\Delta}}$$

Observation

As you increase the value of M the option price calculated from the binomial model gets closer to the actual value calculated from the *BSM* formula

European Call Price = 9.122 European Put Price = 13.737

No. of steps M	***European Call***	***European Put***
5	9.084	13.724
7	8.938	13.571
10	9.369	13.997
15	9.052	13.676

Exercise 12

(For detailed calculation see also Excel file 'Exercise 12.12 Solution Excel Calculation' on book webpage.)

Calculation

Using $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$ we can calculate the probability of stock price going up

$$p = \frac{e^{(r-r_f)\Delta} - d}{u - d}$$

And after calculating payoff at the expiration discount the call price at each node as the following:

$$c_t = \frac{p * c_{t+1}^{u} + (1-p) * c_{t+1}^{d}}{e^{r\Delta}}$$

$$BS\ Call = S_0 e^{-r_f(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$BS\ Put = K e^{-r(T-t)} N(-d_2) - S_0 e^{-r_f(T-t)} N(-d_1)$$

Observation

As you increase the value of M the option price calculated from the binomial model gets closer to the actual value calculated from the *BSM* formula

European Call Price = 0.1006 European Put Price = 0.1293

No. of steps M	***European Call***	***European Put***
10	0.097	0.126
20	0.099	0.127
50	0.1	0.128
100	0.1003	0.129

Exercise 13

(For detailed calculation see also Excel file 'Exercise 12.13 Solution Excel Calculation' on book webpage.)

Calculation

Using $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$ we can calculate the probability of stock price going up

$$p = \frac{e^{(r-r_f)\Delta} - d}{u - d}$$

And after calculating payoff at the expiration discount the call price at each node as the following:

$$c_t = max\left[\frac{p * c_{t+1}^u + (1-p) * c_{t+1}^d}{e^{r\Delta}}, \; payoff \; at \; time \; 't'\right]$$

Observation

The value of American option as we increase the value of M is reported below:

No. of steps M	***American Call***	***American Put***
10	0.103	0.126
20	0.104	0.128
50	0.105	0.129
100	0.105	0.129

Exercise 14

(For detailed calculation see also Excel file ‘Exercise 12.14 Solution Excel Calculation’ on book webpage.)

Calculation

Using of $u = e^{\sigma\sqrt{\Delta t}+(r-r_f-\sigma^2/2)\Delta t}$ and $d = e^{-\sigma\sqrt{\Delta t}+(r-r_f-\sigma_2/2)\Delta t}$ and the value of $p = 0.5$ proceed as following: Once you calculate payoff at the expiration discount the call price at each node till time

$t = 0.$

$$c_t = \frac{p * c_{t+1}^u + (1-p) * c_{t+1}^d}{e^{r\Delta}}$$

$$BS\ Call = S_0 e^{-r_f(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$BS\ Put = K e^{-r(T-t)} N(-d_2) - S_0 e^{-r_f(T-t)} N(-d_1)$$

Observation

As you increase the value of M the option price calculated from the binomial model gets closer to the actual value calculated from the *BSM* formula

European Call Price = 0.1006 European Put Price = 0.1293

No. of steps M	***European Call***	***European Put***
5	0.1038	0.1326
7	0.1022	0.131
10	0.1025	0.1312
15	0.1001	0.1288