CHAPTER 8

Solutions

Exercise 1

This exercise closely follows the example on page 246. We assume that the zero-coupon yield curve is flat at y = 5 %. The shifts in the yield curve are parallel. What positions and how many units of the 6-month bond and the 3-year bond are needed to create an approximate synthetic that will fund the 2-year bond ? Then, the values of the 2-year, 3-year, and 6-month bonds in terms of the corresponding yield *y*

will be given by

(t_0,T_2) = 90.70295

(t_0,T_3) = 86.38376

$(t_0,T_{0.5})$ = 97.59001

Using the "long" bond $B(t_0, T_3)$ and the "short" $B(t_0, T_{.5})$, we need to form a portfolio with initial cost 90.70295. This will equal the time-t_0 value of the target instrument, $B(t_0, T_2)$. We also want the sensitivities of this portfolio with respect to y to be the same as the sensitivity of the original instrument. We therefore need to solve the equations

$$\theta^1 B(t_0, T_3) + \theta^2 B(t_0, T_{.5}) = 90.70295 \qquad (1)$$

$$\theta^1 \frac{\partial B(t_0, T_3)}{\partial y} + \theta^2 \frac{\partial B(t_0, T_{.5})}{\partial y} = \frac{\partial B(t_0, T_2)}{\partial y} \qquad (2)$$

We can calculate the "current" values of the partials:

$$\frac{\partial B(t_0, T_{.5})}{\partial y} = \frac{-50}{(1+y)^{1.5}} = -46.4713 \tag{3}$$

$$\frac{\partial B(t_0, T_2)}{\partial y} = -172.768 \tag{4}$$

$$\frac{\partial B(t_0, T_3)}{\partial y} = -246.811 \tag{5}$$

Replacing these in equations (1) and (2) we get

$$\theta^1 86.38376 \ + \theta^2 \ \ 97.59001 = 90.70295 \tag{6}$$

$$\theta^1(-246.811) + \theta^2(-46.4713) = -172.768 \tag{7}$$

Solving

$$\theta^1 = 0.627202,\ \theta^2 = 0.372798 \tag{8}$$

Hence, we need to short 0.37 units of the 6-month bond and short 0.63 units of the 3-year bond to create an approximate synthetic that will fund the 2-year bond. This will generate the needed cash and has the same first-order sensitivities with respect to changes in y at time t_0. This is a simple example of immunizing a fixed-income portfolio.

Exercise 2

1. The option has a maturity of 200 days. In order to have 5 steps, the Δ must correspond to 40 days. But, we let a year be denoted by 1. In this case, using a days convention of act/365, the Δ becomes: Δ =40/365
2. In order to answer this question we need further assumptions about the tree structure. The question implies that the probability of the state denoted as "up" is constant. To obtain this probability we need to discretize the stochastic differential equation. One way to

proceed is the following. We discretize the continuous time dynamics using the Euler scheme and replace the μ with the risk-free rate. We obtain:

$$\Delta S_{t_i} \cong \left[0.06\Delta + \sigma\epsilon_{t_i}\right]S_{t_{i-1}}$$

Where ε is a two-state random variable, states being denoted by "up" and "down" respectively. We have the following two-state dynamics for S_t:

$$S_{t_i}^{up} = uS_{t_{i-1}}$$

$$S_{t_i}^{down} = dS_{t_{i-1}}$$

where $u_i = 1 + 0.06\Delta + \sigma\epsilon^{up}$

Let p be the probability of down state. And assume, a usual that $E_{t_{i-1}}^{P}[\epsilon] = 0$

Then we have the following equations:

$\epsilon^{up}p + \epsilon^{down}(1 - p) = 0$

which is equivalent to:

$(uS_{t_{i-1}})p + dS_{t_{i-1}}(1\text{-p})=(1+0.06\Delta)\, S_{t_{i-1}}$

where the $S_{t_{i-1}}$ can be eliminated. At this point we have three unknowns,

p, ϵ^{up}, ϵ^{down}. We need more conditions.

We can let the three to recombine, so that an "up" movement followed by a "down" movement results in the same value as a "down" followed by an "up":

$ud = 1$

We can now solve for p, ϵ^{up} or p, ϵ^{down}. Letting $\epsilon^{up} = 2\Delta$ gives,

$p = 0.663$

$\epsilon^{down} = -1.970\Delta.$

Note that the values of the random up and down movements, ϵ^j should depend on the Δ. This is needed since, the variance of this term will have to be proportional to Δ. This is one consequence of W_t being a Wiener process, in the continuous form of the dynamics.

Exercise 3

1. In this case we would follow the same steps as in Exercise 2, after replacing the drift of .06Δ by (.06 − .04)Δ.
2. The tree will be the same as in Exercise 1, until the third step. Then, all values are lowered by 5%.
3. The tree will shift down by $5 after the third step. This will complicate the tree since, after the third step the remainder of the tree will become non-recombining. The reason is that a constant dollar sum is deducted from the relevant S_{t_i} and this will correspond to a slightly different percentage change for the "up" and "down" states.

Exercise 4

We will solve the problem using the GBP/USD exchange rate of $1.85 USD interest rates of 1.5% and GBP interest rates of 5.5%

1. The new drift, which will be (0.015−0.055) Δ does not affect the choice of Δ. We still have Δ = 40/365.

2. Using the same method as in the answer for question 1, we find:

$$p=0.64$$

$$\epsilon^{up} = 1$$

$$\epsilon^{down} = -1.7789$$

which gives

$$u = 1.0154$$

$$d = 0.985$$

3. Using the initial point of $S_1 = 1.85$ recursively calculate:

$$S_i^{up} = uS_{i-1}$$

$$S_i^{down} = dS_{i-1}$$

4. In order to calculate the tree for the European Put we start from the last step, *i* = 5. Given that the exchange rate tree is recombining we will have a recombining tree for the Put option. There will be 5 nodes at step *i* = 5. These values will be $\{u^2, uS, S, dS, d^2S\}$

5. Using the *u, d* found in the previous question we can calculate the values of the exchange rate at step *i* = 5 and find the values of the put option denoted by P_i by letting:

$$P_5^j = max[1.50 - S_5^j, 0]$$

where j=1,…,5 represent the 5 possible states of the last step. We can then work backwards, using the relation:

$$P_{i-1} = \frac{1}{1 + 0.04\Delta}\left[pP_i^{up} + (1-p)P_i^{down}\right]$$

5. The American put requires working backwards in the tree as done in the previous question. However, at each node we need to check one more condition. The option holder can, at each node exercise the option. Thus the value of the relevant P_i should be compared with $1.50 - S_i$. If

the latter is greater, then the option will be exercised. This means that the subsequent values on the tree will have to be let equal to zero. Also, this exercise value will be discounted properly in obtaining the value of P_1.

Exercise 5

1. The existence of many Put writers suggest that, these players' positions will lose money when the market drops. To hedge this position, they have to sell short the underlying stock in the right amount. This can be shown by using a standard short Put payoff diagram.

2. A covered put position means, the trader has written a put and then shorted the underlying by selling delta units of the underlying. Now suppose the market drops further. the trader is hedged against this. But, the drop of the market implies a bigger delta in absolute value. So, more of the underlying needs to be shorted. Note that this means further sales hitting the market. It is this point that is being refuted by the traders in the first paragraph of the reading.

3. Short volatility positions are created by selling options, among others. The same trader can be long volatility somewhere else if they either own similar options or if they are long convexity.

4. This is quite possible. Note that cash markets are much smaller than the markets for derivatives on the same underlying. Thus the derivative market may be marginally short, but this may lead to significant short sales in the corresponding cash market.

5. The last paragraph of the reading suggests that most players try to cover their volatility positions. This may reduce the need for adjusting the corresponding delta hedge.

Exercise 6

In order to check whether or not the trees in Figure 8-7 are arbitrage free we would check whether the equality,

$$S_{i-1} = \frac{1}{1 + r^j \Delta}\left[S_i^{up} p + S_i^{down}(1-p)\right]$$

is satisfied at every node.

Exercise 7

<u>Calculation</u>

(For detailed calculation see also Excel file 'Exercise 8.7 Solution Excel Calculation' on book webpage.)

Boundary condition: Payoff of the European Call Option at expiration $C_T = \max[S_T - K, 0]$

And using the formula for self-financing replicating portfolio we calculate the stock and LIBOR position at time t using the portfolio value at time $t+1$.

$$\theta_t^{lend} = \frac{(\,C_{t+1}^{up} * S_{t+1}^{down} - \,C_{t+1}^{down} * S_{t+1}^{up})}{(\,L_{t+1}^{up} * S_{t+1}^{down} - \,L_{t+1}^{down} * S_{t+1}^{up})}$$

$$\theta_t^{stock} = \frac{(\,C_{t+1}^{up} * L_{t+1}^{down} - \,C_{t+1}^{down} * L_{t+1}^{up})}{(\,S_{t+1}^{up} * L_{t+1}^{down} - \,S_{t+1}^{down} * L_{t+1}^{up})}$$

And once we obtained the value of θ_t^{lend} and θ_t^{stock} we find the value of portfolio at time t.

$$C_t = S_t\theta_t^{stock} + L_t\theta_t^{lend}$$

We proceed backward until we obtain the portfolio value at time $t = 0$.

Observation

Using the replicating strategy to hedge the short European option position we obtain the initial price of the portfolio as the following:

European Call Price = 26.44 European Put Price = 11.72