APPENDIX D: Geometry of Lifting Surfaces

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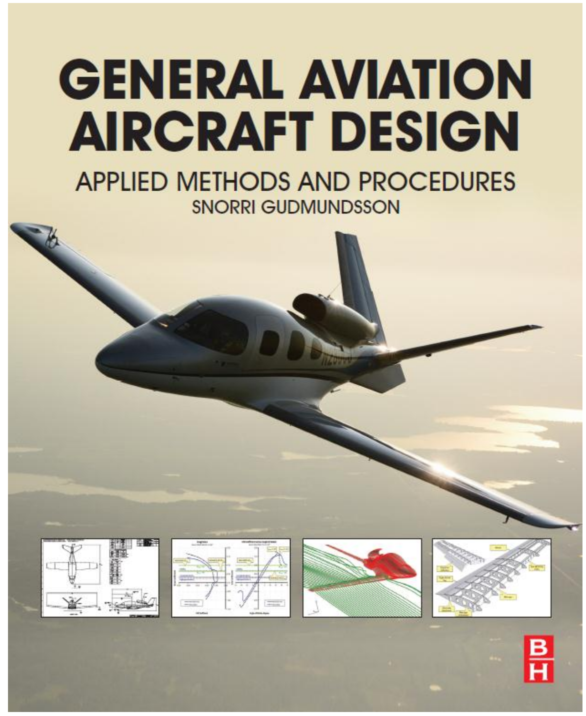


D.1 Introduction

This appendix focuses on the geometry generally used for aircraft lifting surfaces; typically the wings, as well the horizontal and vertical tails, but also specialty surfaces, such as winglets and ventral fins. Such geometry can range from the simple constant chord lifting surface sometimes chosen for the wing and then referred to as a "Hershey bar" wing, to the complicated ogee wing shape featured on the Concorde supersonic transport. Modern lifting surfaces often feature breaks in the leading or trailing edges (called "cranks"), which breaks it into a number of trapezoidal sections. The competent aircraft designer should know how to treat such lifting surfaces in the aerodynamic analysis of aircraft.

We will study the simplest to the most complex of lifting surfaces and develop formulation that allows the designer to evaluate important parameters that will be defined in the text, such as Mean Geometric Chord, Taper Ratio, and Aspect Ratio. The appendix does not consider characteristics such as a geometric or aerodynamic wing twist (washout). This discussion is limited to general analysis of flat, infinitely thin surfaces that resemble those that are commonly (and not so commonly) used to generate lift in aircraft. No treatment of internal structure or flight controls is considered here. This section is purely a mathematical treatment of 2-dimensional surface planforms geometries that are typically used for lifting.

The first type of surfaces presented in the section is the trapezoidal, but these cover the bulk of planform shapes used for aircraft. The discussion will be followed by a treatment of cranked surfaces, which are planform shapes of relatively regular polygons that can be broken down into trapezoidal subsections. Finally, this section will consider planform shapes, whose leading and trailing edges are best described with continuous curved functions.

D.2 Formulation for the Simple Trapezoidal Planform

During the early stages of the design process, there simply is not yet enough detail available to calculate the surface areas and volumes with a high degree of accuracy. Often the design consists of nothing more than some preliminary dimensions. For instance, we may have a reasonable idea about how long the fuselage might be, or its average diameter. It is often convenient to estimate the geometric properties of the airplane using some generic shapes that resemble the proposed form. This section will present a few such shapes and simple and handy formulas to estimate the areas of the surface and volume. We will start with the geometry of a few fundamental shapes that can be combined to form shapes that resemble that of a fuselage.

D.2.1 Approximation of an Airfoil Cross-Sectional Area

Often, the cross-sectional area (internal area) of an airfoil must be evaluated as it yields useful clues about the internal volume of a wing available for fuel storage. In the absence of precise airfoil data, the following approximation can be used to estimate the internal area of geometry resembling that of an airfoil:

Total area:

$$\boxed{A_{airfoil} = \frac{(k+3)C \cdot t}{6}} \tag{D-1}$$

Where: C = Airfoil chord, in ft or m
k = Location of the airfoil's maximum thickness as a fraction of C.
t = Airfoil thickness, in ft or m.

Sometimes it is more convenient to present the thickness using the thickness-to-chord ratio, denoted by (t/c). This way, the thickness is expressed using the product (t/c)·C. This way, Equation (D-1) is written as follows:

Total area:

$$\boxed{A_{airfoil} = \frac{(k+3)}{6}\left(\frac{t}{c}\right)C^2} \tag{D-2}$$

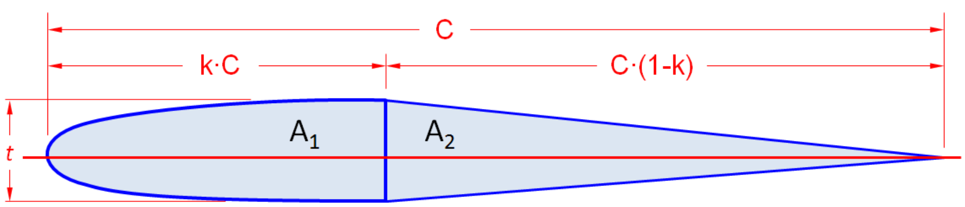


Figure D-1: An approximation of an airfoil using elementary geometry.

DERVIATION OF EQUATION (D-1):
Consider Figure D-1, which shows an airfoil of chord C approximated by a parabolic D-cell and a triangular section. It is assumed the two sections join at the chord station of maximum thickness, t, whose location is given by k·C, where k is the location of the airfoil's maximum thickness as a fraction of C. The cross-sectional areas of the two sections and the total area are given by the following expressions:

Parabolic section:

$$A_1 = \frac{2k \cdot C \cdot t}{3} \tag{i}$$

Triangular section:

$$A_2 = \frac{C \cdot (1-k)t}{2} \tag{ii}$$

Therefore, the internal area of the airfoil can be approximated by adding the two as shown below:

$$A_{airfoil} = A_1 + A_2 = \frac{2k \cdot C \cdot t}{3} + \frac{C \cdot (1-k)t}{2} = \frac{(k+3)C \cdot t}{6}$$

QED

D.2.2 Approximation of an Airfoil Perimeter

The perimeter of the airfoil is imperative when estimating the wetted area for drag estimation. In the absence of more accurate data the following approximation can be used to estimate the perimeter of an airfoil (assuming the geometry of Figure D-1 is a reasonable approximation of the airfoil):

$$\boxed{s_{airfoil} = \sqrt{\frac{t^2}{4} + 4(kC)^2} + \frac{t^2}{16kC}\sinh^{-1}\left(\frac{4kC}{t}\right) + \sqrt{t^2 + 4C^2(1-k)^2}} \tag{D-3}$$

Where: C = Airfoil chord, in ft or m
k = Location of the airfoil's maximum thickness as a fraction of C
t = Airfoil thickness, in ft or m

Again, it may be simpler to use the thickness-to-chord ratio, (t/c), rather than the exact thickness of the airfoil. If so, the perimeter of the airfoil is written as follows:

$$\boxed{s_{airfoil} = \frac{C}{2}\left[\sqrt{(t/c)^2 + 16k^2} + \frac{(t/c)^2}{8k}\sinh^{-1}\left(\frac{4k}{(t/c)}\right) + 2\sqrt{(t/c)^2 + 4(1-k)^2}\right]} \tag{D-4}$$

An evaluation of the above equations reveals that the term involving the inverse hyperbolic sine contributes less than 1.5% for a 25% thick airfoil and 0.12% for 5% thick airfoil. For this reason, it is possible to simplify Equations (D-3) and (D-4) as follows:

$$\begin{aligned} s_{airfoil} &\approx \sqrt{\frac{t^2}{4} + 4(kC)^2} + \sqrt{t^2 + 4C^2(1-k)^2} \\ s_{airfoil} &\approx \frac{C}{2}\left[\sqrt{(t/c)^2 + 16k^2} + 2\sqrt{(t/c)^2 + 4(1-k)^2}\right] \end{aligned} \tag{D-5}$$

This form improves the speed of computations with small error.

DERIAVATION OF EQUATION (D-3)

The perimeter along the upper and lower surface of the airfoil can be estimated by combining the length of the parabolic and triangular sections:

Parabolic section:

$$s_p = \sqrt{\frac{t^2}{4} + 4(kC)^2} + \frac{t^2}{16kC}\sinh^{-1}\left(\frac{4kC}{t}\right) \tag{i}$$

Triangular section:

$$s_t = 2\sqrt{\frac{t^2}{4} + C^2(1-k)^2} = \sqrt{t^2 + 4C^2(1-k)^2} \tag{ii}$$

Therefore, the total perimeter of the airfoil is:

$$s_{airfoil} = s_p + s_t = \sqrt{\frac{t^2}{4} + 4(kC)^2} + \frac{t^2}{16kC}\sinh^{-1}\left(\frac{4kC}{t}\right) + \sqrt{t^2 + 4C^2(1-k)^2}$$

QED

DERIVATION OF EQUATION (D-4)

The thickness, t, is given by (t/c)·C, where (t/c) is the thickness ratio and C is the airfoil chord. Replacing t in Equation (D-3) with this form yields:

$$s_{airfoil} = \sqrt{\frac{(t/c)^2 C^2}{4} + 4(kC)^2} + \frac{(t/c)^2 C^2}{16kC}\sinh^{-1}\left(\frac{4kC}{(t/c)C}\right) + \sqrt{(t/c)^2 C^2 + 4C^2(1-k)^2}$$

Algebraic manipulations lead to:

$$\begin{aligned} s_{airfoil} &= C\sqrt{\frac{(t/c)^2}{4} + 4k^2} + \frac{(t/c)^2 C}{16k}\sinh^{-1}\left(\frac{4k}{(t/c)}\right) + C\sqrt{(t/c)^2 + 4(1-k)^2} \\ &= \frac{C}{2}\sqrt{(t/c)^2 + 16k^2} + \left(\frac{(t/c)}{4}\right)^2 \frac{C}{k}\sinh^{-1}\left(\frac{4k}{(t/c)}\right) + C\sqrt{(t/c)^2 + 4(1-k)^2} \\ &= \frac{C}{2}\left[\sqrt{(t/c)^2 + 16k^2} + \frac{(t/c)^2}{8k}\sinh^{-1}\left(\frac{4k}{(t/c)}\right) + 2\sqrt{(t/c)^2 + 4(1-k)^2}\right] \end{aligned}$$

QED

D.2.3 Approximation Surface Areas and Volumes of a Generic Lifting Surface (without Fuselage)

The surface area and volume of a generic lifting surface like the one shown in the left image of Figure D-2 can be estimated assuming the above expressions for cross-sectional area and arc length are applicable.

Total wing volume:

$$\boxed{V = \frac{bC_r^2}{12}\left((k_r + 3)\left(\frac{t}{c}\right)_r + (k_t + 3)\left(\frac{t}{c}\right)_t \lambda^2\right)} \qquad \text{(D-6)}$$

Where: λ = Taper Ratio.
k = Location of the airfoil's maximum thickness as a fraction of C
The subscripts r and t refer to the root and tip airfoils, respectively (see Figure D-2).

The wetted wing area, excluding the fuselage can be estimated from:

$$\boxed{\begin{aligned} S_{wet} = \frac{bC_r}{4}\Bigg(&\left[\sqrt{(t/c)_r^2 + 16k_r^2} + 2\sqrt{(t/c)_r^2 + 4(1-k_r)^2}\right] + \\ &\lambda\left[\sqrt{(t/c)_t^2 + 16k_t^2} + 2\sqrt{(t/c)_t^2 + 4(1-k_t)^2}\right]\Bigg) \end{aligned}} \qquad \text{(D-7)}$$

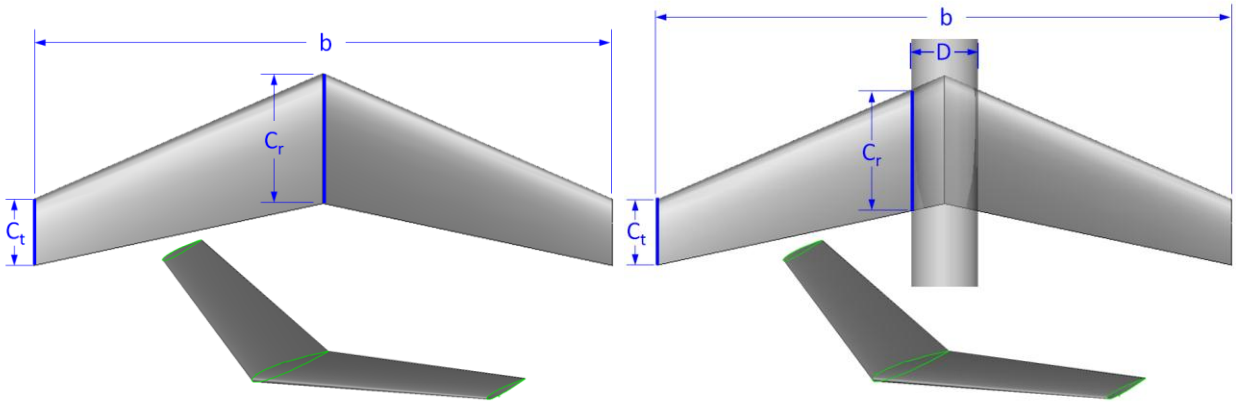


Figure D-2: An approximation of a tapered wing (left) and tapered wing and fuselage (right).

Since the wing is usually mounted on the fuselage, it is not out of the way to present a simple correction to Equation (D-7). Consider the fuselage of width D and wing shown in the right image of Figure D-2. Then Equation (D-7) can be rewritten to exclude the portion of the wing inside the fuselage. Note that the root chord, C_r, must refer to the chord along the side of the fuselage and this requires the Taper Ratio λ to be calculated (as C_t/C_r):

$$S_{wet} = \frac{(b-D)C_r}{4}\left(\left[\sqrt{(t/c)_r^2 + 16k_r^{\ 2}} + 2\sqrt{(t/c)_r^2 + 4(1-k_r)^2}\right] + \lambda\left[\sqrt{(t/c)_t^2 + 16k_t^{\ 2}} + 2\sqrt{(t/c)_t^2 + 4(1-k_t)^2}\right]\right) \qquad \text{(D-8)}$$

Where: b = Wing span
D = Fuselage width
t/c = Airfoil thickness ratio
λ = Taper Ratio.
k = Location of the airfoil's maximum thickness as a fraction of C
The subscripts r and t refer to the root and tip airfoils, respectively (see Figure D-2).

DERIVATION:
The cross-sectional area at the root and tip airfoils can be estimated using Equation (D-2) as follows (denoting the root and tip using the subscripts r and t):

$$A_r = \frac{(k_r+3)}{6}\left(\frac{t}{c}\right)_r C_r^2$$

$$A_t = \frac{(k_t+3)}{6}\left(\frac{t}{c}\right)_t C_t^2 = \frac{(k_t+3)}{6}\left(\frac{t}{c}\right)_t \lambda^2 C_r^2$$

Then, the total volume of the wing will be the average of these two times the wingspan b.

$$V = \frac{b}{2}\left(\frac{(k_r+3)}{6}\left(\frac{t}{c}\right)_r C_r^2 + \frac{(k_t+3)}{6}\left(\frac{t}{c}\right)_t \lambda^2 C_r^2\right) = \frac{bC_r^2}{12}\left((k_r+3)\left(\frac{t}{c}\right)_r + (k_t+3)\left(\frac{t}{c}\right)_t \lambda^2\right)$$

QED

DERIVATION:

The perimeter the root and tip airfoils can be estimated using Equation (D-5) as follows:

$$s_r = \frac{C_r}{2}\left[\sqrt{(t/c)_r^2 + 16k_r^{\;2}} + 2\sqrt{(t/c)_r^2 + 4(1-k_r)^2}\right]$$
$$s_t = \frac{C_t}{2}\left[\sqrt{(t/c)_t^2 + 16k_t^{\;2}} + 2\sqrt{(t/c)_t^2 + 4(1-k_t)^2}\right]$$

Then, the total volume of the wing will be the average of these two times the wingspan b (denoting the root and tip using the subscripts r and t):

$$S_{wet} = \frac{b}{2}\left(\frac{C_r}{2}\left[\sqrt{(t/c)_r^2 + 16k_r^{\;2}} + 2\sqrt{(t/c)_r^2 + 4(1-k_r)^2}\right] + \right.$$
$$\left. \frac{C_t}{2}\left[\sqrt{(t/c)_t^2 + 16k_t^{\;2}} + 2\sqrt{(t/c)_t^2 + 4(1-k_t)^2}\right]\right)$$
$$S_{wet} = \frac{b}{4}\left(C_r\left[\sqrt{(t/c)_r^2 + 16k_r^{\;2}} + 2\sqrt{(t/c)_r^2 + 4(1-k_r)^2}\right] + \right.$$
$$\left. \lambda C_r\left[\sqrt{(t/c)_t^2 + 16k_t^{\;2}} + 2\sqrt{(t/c)_t^2 + 4(1-k_t)^2}\right]\right)$$

QED

D.3 Equivalent Wing Planforms

Most aircraft do not feature perfectly trapezoidal wing planform shapes but rather ones that are "cranked" or broken along the leading or trailing edges. This is usually the result of some specific aerodynamic or structural requirements. A few examples are displayed in Figure D-3. It is prudent to ask how we define important geometric characteristics of such wing shapes. For instance, looking at the airplane in Figure D-3, where is the MGC? What is its length? How about Taper Ratio or leading edge sweep? At times it is necessary to treat such geometry to enable comparison between these more complicated planforms and their simpler counterparts. This section presents a method to treat such wings.

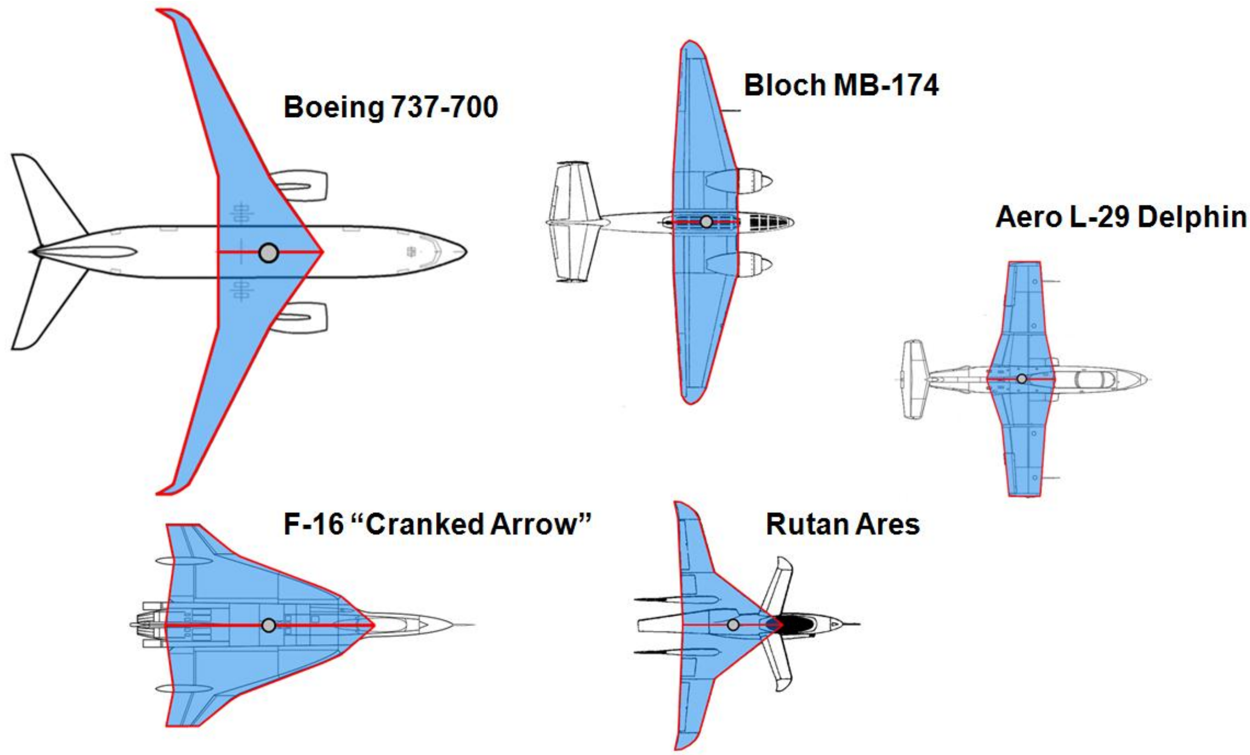


Figure D-3: Wing planforms that are more complicated than they look at first glance.

Consider the cranked half wing planform shape in Figure D-4, which could be the right wing of some aircraft (assuming we are looking from above). Many scientific documents that contain aerodynamic characteristics of 3-dimensional wings, such as the USAF DATCOM, require the user to determine aerodynamic properties based on the sweep of the quarter-chord or the leading edge. However, when considering the four panels the wing of Figure D-4 consists of, it is prudent we ask ourselves whose quarter-chord sweep is a suitable representation for the entire wing? The answer is none. Instead, it is appropriate to use a weighed approach that considers the "contribution" of each to a representative quarter-chord sweep of an equivalent wing. This weighing is based on the surface area of each panel. For instance, the figure suggests the sweep of the two inboard (or left) panels will contribute more significantly to the equivalent quarter chord sweep than the two outboard ones. The figure shows a single equivalent trapezoidal surface that has been superimposed on the original wing, allowing the aforementioned geometric properties to be determined and compared on an equivalent basis ("apples to apples" comparison). We shall now develop expressions to convert the wing into a simple trapezoidal planform.

The half wing in Figure D-4 consists of N-1 separate small trapezoidal sections that each can be considered as simple trapezoid. Note that this simplification does not extend to aerodynamic properties of the planform; airloads stall characteristics, drag, and similar properties should never be calculated for the simplified shape – the simplified shape is purely for geometric comparison.

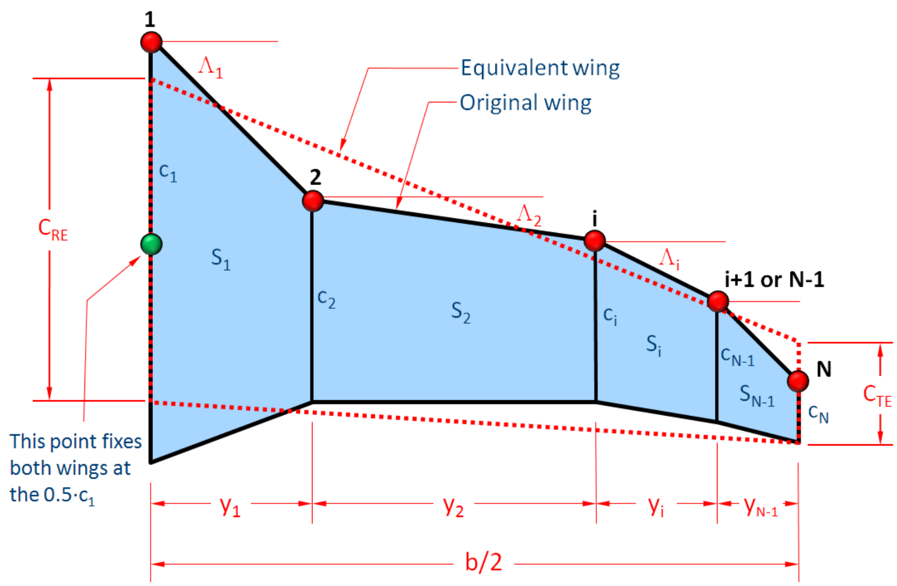


Figure D-4: Presenting the equivalent wing.

In order for the simplified trapezoidal planform to be considered geometrically equivalent to the original wing planform we want it to be of an equal span, planform area, and Aspect Ratio. Mathematically we write it as follows:

$$
\begin{aligned}
b_E &= b_O \\
S_E &= S_O \\
AR_E &= AR_O
\end{aligned}
$$

Where the subscripts E and O stand for "Equivalent" for "Original", respectively. The requirement for equivalent Aspect Ratio follows from these two requirements since it is defined as AR = b^2/S for each wing. Also note that the two wing planforms are placed such that their mid-chord points at the root are identical (green point in Figure D-4).

D.3.1 Geometry of the Equivalent Wing

The dimensions of the equivalent wing can be computed from the following expressions, which are based on a wing break-down as shown in Figure D-4.

Wing span:
$$\boxed{b_E = b_O} \tag{D-9}$$

Root chord:
$$\boxed{C_{RE} = K \times C_{WR} = \frac{2}{S_W}\sum_{i=1}^{N-1} c_i \cdot S_i} \tag{D-10}$$

Tip chord:

$$\boxed{C_{TE} = K \times C_{WT} = \frac{2}{S_W} \sum_{i=1}^{N-1} c_{i+1} \cdot S_i} \tag{D-11}$$

Leading edge sweep:

$$\boxed{\Lambda_{LE_E} = \frac{2}{S_O} \sum_{i=1}^{N-1} \Lambda_i \cdot S_i} \tag{D-12}$$

Quarter chord sweep:

$$\boxed{\tan \Lambda_{C/4_E} = \tan \Lambda_{LE_E} + \frac{C_{RE}}{2b}(\lambda_E - 1)} \tag{D-13}$$

Taper ratio:

$$\boxed{\lambda_E = \frac{C_{TE}}{C_{RE}}} \tag{D-14}$$

Where;

$$K = \frac{S_O}{S_W} \tag{D-15}$$

$$S_W = \frac{b_O}{S_O}\left(\sum_{i=1}^{N-1} c_i \cdot S_i + \sum_{i=1}^{N-1} c_{i+1} \cdot S_i \right) \tag{D-16}$$

$$C_{WR} = \frac{2}{S_O} \sum_{i=1}^{N-1} c_i \cdot S_i \tag{D-17}$$

$$C_{WT} = \frac{2}{S_O} \sum_{i=1}^{N-1} c_{i+1} \cdot S_i \tag{D-18}$$

DERIVATION:

Consider the cranked half wing planform shape in Figure D-4. We can define the geometry of each panel as follows:

Elemental area:

$$S_i = y_i \left(\frac{c_i + c_{i+1}}{2} \right) \tag{i}$$

Area of the half-span:

$$S_{half} = \frac{S_O}{2} = S_1 + S_2 + \ldots + S_N = \sum_{i=1}^{N-1} y_i \left(\frac{c_i + c_{i+1}}{2} \right) \tag{ii}$$

Therefore we can write the area of the original wing as follows:

$$S_O = 2(S_1 + S_2 + \ldots + S_N) = 2\sum_{i=1}^{N-1} y_i \left(\frac{c_i + c_{i+1}}{2} \right) = \sum_{i=1}^{N-1} y_i (c_i + c_{i+1}) \tag{iii}$$

Note that by definition the half-span area of the equivalent wing must be equal to this value. Then next thing we must do is to define the so-called *Weighted Root* and *Tip* chords, which is the contribution of each individual chord to the root and tip chord of the equivalent wing.

From Figure D-4 we can see that the Elemental Root Chords that comprise the weighted root chord are c_1, c_2, …, c_{N-1}. Similarly, the Elemental Tip Chords that comprise the weighted tip chord are c_2, c_3, …, c_N. Therefore, the *weighted root chord* can be defined as follows:

$$C_{WR} = \frac{1}{S_{half}} \sum_{i=1}^{N-1} c_i \cdot S_i = \frac{2}{S_O} \sum_{i=1}^{N-1} c_i \cdot S_i \qquad \text{(iv)}$$

Similarly, the *weighted tip chord* can be determined as follows:

$$C_{WT} = \frac{1}{S_{half}} \sum_{i=1}^{N-1} c_{i+1} \cdot S_i = \frac{2}{S_O} \sum_{i=1}^{N-1} c_{i+1} \cdot S_i \qquad \text{(iv)}$$

Let's define a *weighted wing area* for the entire wing:

$$S_W = b_O \left(\frac{C_{WR} + C_{WT}}{2} \right) = \frac{b_O \cdot (C_{WR} + C_{WT})}{2} \qquad \text{(v)}$$

An explicit form of Equation (v) is:

$$S_W = \frac{b_O \cdot (C_{WR} + C_{WT})}{2} = \frac{b_O}{S_O} \left(\sum_{i=1}^{N-1} c_i \cdot S_i + \sum_{i=1}^{N-1} c_{i+1} \cdot S_i \right) \qquad \text{(vi)}$$

Using these to calculate the root and tip chords of the equivalent wing will not necessarily give the true values and, consequently, would fails to provide matching S and AR. Therefore, we must introduce a special scaling factor, K, such that:

$$S_E = K \times S_W = K \times \frac{b_O \cdot (C_{WR} + C_{WT})}{2} = S_O \qquad \text{(vii)}$$

Therefore, the equivalent chords are given by:

$$C_{RE} = K \times C_{WR} \quad \text{and} \quad C_{TE} = K \times C_{WT} \qquad \text{(viii)}$$

Note that once the requirement for the S is met, then so is the requirement for the AR. The scaling factor, K, can be found from:

$$S_E = K \times S_W = S_O \quad \Leftrightarrow \quad K = \frac{S_O}{S_W} \qquad \text{(ix)}$$

Where; S_O = Original wing area and
S_W = Weighted wing area of both wing halfs (complete wing).

K can also be written as:

$$K = \frac{S_O}{S_W} = \frac{S_O}{\frac{b_O \cdot (C_{WR} + C_{WT})}{2}} = \frac{2 S_O}{b_O \cdot (C_{WR} + C_{WT})} \qquad \text{(x)}$$

Therefore, we can compute the equivalent root and tip chords as follows:

$$C_{RE} = K \times C_{WR} = \left(\frac{S_O}{S_W}\right)\left(\frac{2}{S_O}\sum_{i=1}^{N-1} c_i \cdot S_i\right) = \frac{2}{S_W}\sum_{i=1}^{N-1} c_i \cdot S_i \tag{D-10}$$

Similarly;

$$C_{TE} = K \times C_{WT} = \frac{2}{S_W}\sum_{i=1}^{N-1} c_{i+1} \cdot S_i \tag{D-11}$$

The leading edge angle for the equivalent wing is defined as the *weighted LE angle* as follows:

$$\Lambda_{LE_E} = \frac{1}{S_{half}}\sum_{i=1}^{N-1} \Lambda_i \cdot S_i = \frac{2}{S_O}\sum_{i=1}^{N-1} \Lambda_i \cdot S_i \tag{D-12}$$

The quarter chord angle for the equivalent wing can be found by applying Equation (D-9) in the form for the equivalent wing:

$$\tan\Lambda_{C/4_E} = \tan\Lambda_{LE_E} + \frac{C_{RE}}{2b}\left(\lambda_E - 1\right) \tag{D-13}$$

Where:

$$\lambda_E = \frac{C_{TE}}{C_{RE}} \tag{D-14}$$

QED

EXAMPLE D-3:
Evaluate the properties of the wing in Figure D-5.

SOLUTION:
Begin by calculating the elemental areas of the wing and then the wing area:

$$S_1 = y_1\left(\frac{c_1 + c_2}{2}\right) = 2.5\left(\frac{5+3}{2}\right) = 10 \text{ ft}^2$$

$$S_2 = 13.75 \text{ ft}^2$$

$$S_3 = 4.5 \text{ ft}^2$$

$$S_4 = 2.25 \text{ ft}^2$$

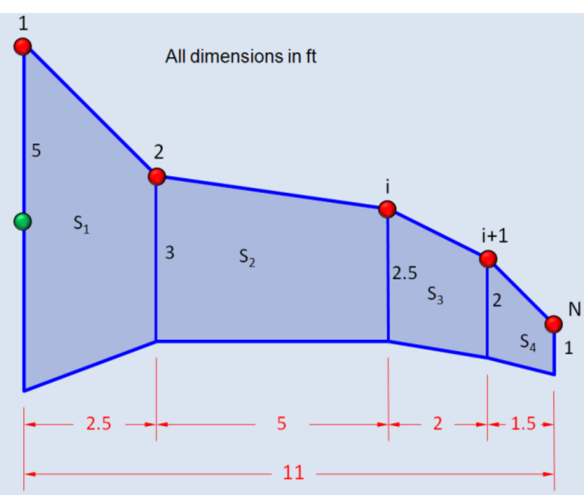


Figure D-5: The wing used in Example D-3.

Area of the half-span wing: $S_{half} = S_1 + S_2 + S_3 + S_4 = 30.5 \text{ ft}^2$

Area of the full-span wing: $S_O = 2S_{half} = 61.0 \text{ ft}^2$

Wing span of full-span wing: $b_O = 2(11) = 22 \text{ ft}$

AR of the full-span wing: $$AR_O = \frac{b_O^2}{S_O} = \frac{(22)^2}{61.0} = 7.934 = AR_E$$

Begin by calculating S_W using Equation (D-16):

$$S_W = \frac{b_O}{S_O}\left(\sum_{i=1}^{N-1} c_i \cdot S_i + \sum_{i=1}^{N-1} c_{i+1} \cdot S_i\right) = \frac{22}{61.0}\big((10\times 5 + 13.75\times 3 + 4.5\times 2.5 + 2.25\times 2) + \hookleftarrow$$

$$(10\times 3 + 13.75\times 2.5 + 4.5\times 2 + 2.25\times 1)\big) = 65.86 \text{ ft}^2$$

Then calculate K from Equation (D-15): $$K = \frac{S_O}{S_W} = \frac{61.0}{65.86} = 0.9261$$

Equivalent root and tip chords:

$$C_{RE} = \frac{2}{S_W}\sum_{i=1}^{N-1} c_i \cdot S_i = \frac{2}{65.86}(10\times 5 + 13.75\times 3 + 4.5\times 2.5 + 2.25\times 2) = 3.249 \text{ ft}$$

$$C_{TE} = \frac{2}{S_W}\sum_{i=1}^{N-1} c_{i+1} \cdot S_i = \frac{2}{65.86}(10\times 3 + 13.75\times 2.5 + 4.5\times 2 + 2.25\times 1) = 2.297 \text{ ft}$$

We can confirm that the equivalent wing has the same area and AR as the original wing.

$$S_E = b_E\left(\frac{C_{RE} + C_{TE}}{2}\right) = 22\left(\frac{3.249 + 2.297}{2}\right) = 61.0 \text{ ft}^2$$

$$AR_E = \frac{b_E^2}{S_E} = \frac{(22)^2}{61.0} = 7.934$$

D.4 Generalized Wing Planform (Advanced)

Some airplanes, such as the Anglo-French Concorde (Figure D-6) and the American General Dynamics F-16 "Cranked Arrow" feature wing planform shapes that are far more challenging to analyze, using even the equivalent wing method. The methodology presented in this section allows the designer to evaluate the properties of such planforms, provided the leading and trailing edges can be described as continuous mathematical functions.

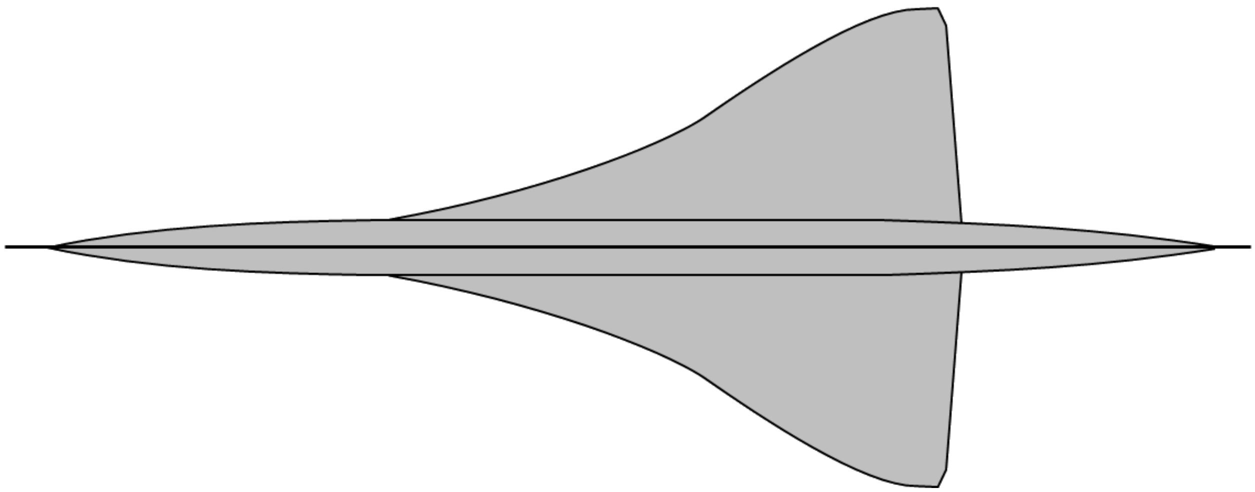


Figure D-6: A planform resembling the Anglo-French Concorde jetliner.

D.4.1 Geometric Description

Consider the wing planform in Figure D-7. It is enclosed between the two continuous curves, f(y) and g(y), and the vertical lines at y = 0 and y = b/2. Then we define the following properties:

Planform chord: $$c(y) = f(y) - g(y) \tag{D-19}$$

Elemental area: $$dS = c(y)dy \tag{D-20}$$

We also want to define a special parameter, called an *elemental weighing factor*:

$$dM = [\text{factor-of-interest}]\cdot c(y)dy \tag{D-21}$$

This factor is to be used to estimate a number of characteristics for the wing planform, such as the MGC, and its location, and others, as will be shown shortly. The method uses the product of the value of the "factor-of-interest" and the chord (as it varies along the span) to evaluate its "weight" of contribution to the overall property being evaluated. In fact, it determines the "centroid" of the property of interest. To better understand what this means consider Figure D-7 again. Let's say we are interested in knowing the average chord of the planform shown and how far from the plane of symmetry it is. In the former case the "factor-of-interest" is c(y), leading to $dM = c(y)^2 \cdot dy$ and in the latter case the "factor-of-interest" is y, so $dM = y \cdot c(y) \cdot dy$. Then, in order to get the actual average chord and its y-location, we have to integrate from y = 0 to b/2 and divide by the area itself. This will become clear shortly.

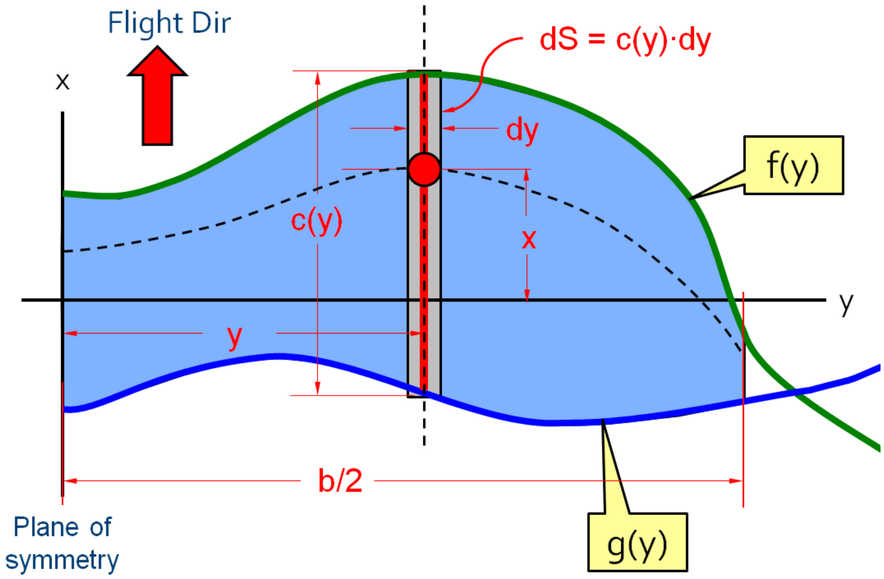


Figure D-7: A general wing planform (note the inverted coordinate system).

D.4.2 Elemental Weighing Factors for the General Planform

Below is the preliminary formulation of a number of important properties of wings. The first five are used to determine Mean Geometric Chord, the x-values of its LE- and TE- and quarter chord location, as well as the y-location (spanwise station):

Weighted chord:

$$\begin{aligned} dM_{CHORD} &= [\text{factor - of - interest}] \cdot c(y)dy \\ &= [c(y)] \cdot c(y)dy \\ &= c(y)^2\, dy \end{aligned} \tag{D-22}$$

Weighted X-location of LE:

$$\begin{aligned} dM_{X-LE} &= [\text{factor - of - interest}] \cdot c(y)dy \\ &= [f(y)] \cdot c(y)dy \\ &= f(y) \cdot c(y)dy \end{aligned} \tag{D-23}$$

Weighted X-location of TE:

$$\begin{aligned} dM_{X-TE} &= [\text{factor - of - interest}] \cdot c(y)dy \\ &= [g(y)] \cdot c(y)dy \\ &= g(y) \cdot c(y)dy \end{aligned} \tag{D-24}$$

Weighted X-location of C/4:

$$\begin{aligned} dM_{X-C/4} &= [\text{factor - of - interest}] \cdot c(y)dy \\ &= \left[f(y) - \frac{c(y)}{4} \right] \cdot c(y)dy \\ &= \left(f(y) - \frac{c(y)}{4} \right) \cdot c(y)dy \end{aligned} \tag{D-25}$$

Weighted Y-location of chord:

$$dM_Y = [\text{factor-of-interest}]\cdot c(y)dy$$
$$= [y]\cdot c(y)dy$$
$$= y\cdot c(y)dy \tag{D-26}$$

The next two formulations are used to determine the "average" sweep angles of the LE and the quarter-chord:

Weighted slope of LE sweep:

$$dM_{\Lambda_{LE}} = [\text{factor-of-interest}]\cdot c(y)dy$$
$$= [f'(y)]\cdot c(y)dy$$
$$= f'(y)\cdot c(y)dy \tag{D-27}$$

Weighted slope of C/4 sweep:

$$dM_{\Lambda_{C/4}} = [\text{factor-of-interest}]\cdot c(y)dy$$
$$= \left[\frac{d}{dy}\left(f(y) - \frac{c(y)}{4}\right)\right]\cdot c(y)dy \tag{D-28}$$

Finally, the area of the planform is given by:

Planform area:

$$S_{half} = \int_0^{b/2} c(y)dy \tag{D-29}$$

Symmetric surface area:

$$S = 2\cdot S_{half} \tag{D-30}$$

D.4.3 Properties of the General Planform

Using the concept of elemental weighing factors we can now develop formulation for the general wing planform shown in Figure D-7.

Mean Geometric Chord (MGC):

$$MGC = \frac{1}{S_{half}}\int_0^{b/2} dM_{CHORD} = \frac{1}{S_{half}}\int_0^{b/2} c(y)^2\, dy = \frac{2}{S}\int_0^{b/2} c(y)^2\, dy \tag{D-31}$$

X-location of LE of the MGC:

$$x_{MGC-LE} = \frac{1}{S_{half}}\int_0^{b/2} dM_{X-LE} = \frac{1}{S_{half}}\int_0^{b/2} f(y)\cdot c(y)dy = \frac{2}{S}\int_0^{b/2} f(y)\cdot c(y)dy \tag{D-32}$$

X-location of TE of the MGC:

$$x_{MGC-TE} = \frac{1}{S_{half}}\int_0^{b/2} dM_{X-TE} = \frac{1}{S_{half}}\int_0^{b/2} g(y)\cdot c(y)dy = \frac{2}{S}\int_0^{b/2} g(y)\cdot c(y)dy \tag{D-33}$$

X-location of C/4 of the MGC:

$$x_{MGC-C/4} = \frac{1}{S_{half}}\int_0^{b/2} dM_{X-C/4} = \frac{1}{S_{half}}\int_0^{b/2}\left[f(y) - \frac{c(y)}{4}\right]\cdot c(y)dy = \frac{2}{S}\int_0^{b/2}\left[f(y) - \frac{c(y)}{4}\right]\cdot c(y)dy \qquad \text{(D-34)}$$

Y-location of MGC:

$$y_{MGC} = \frac{1}{S_{half}}\int_0^{b/2} dM_Y = \frac{1}{S_{half}}\int_0^{b/2}[y]\cdot c(y)dy = \frac{2}{S}\int_0^{b/2} y\cdot c(y)dy \qquad \text{(D-35)}$$

Slope of LE at MGC:

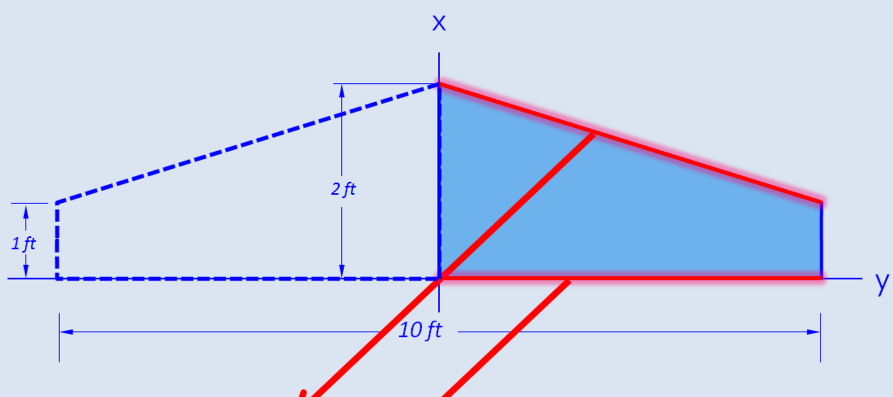
$$\Lambda_{MGC-LE} = \frac{1}{S_{half}}\int_0^{b/2} dM_{\Lambda LE} = \frac{1}{S_{half}}\int_0^{b/2}[f'(y)]\cdot c(y)dy = \frac{2}{S}\int_0^{b/2} f'(y)\cdot c(y)dy \qquad \text{(D-36)}$$

Slope of C/4 at MGC:

$$\Lambda_{MGC-C/4} = \frac{1}{S_{half}}\int_0^{b/2} dM_{\Lambda C/4} = \frac{1}{S_{half}}\int_0^{b/2}\left[\frac{d}{dy}\left(f(y) - \frac{c(y)}{4}\right)\right]\cdot c(y)dy \qquad \text{(D-37)}$$

EXAMPLE D-4: Planform Geometry

Determine the geometric properties for the simple trapezoidal wing planform shape below using the method developed so far. Compare the MGC to that using the "standard" formulation of Section D.1.



$$\left.\begin{array}{l} f(y) = 2 - \dfrac{y}{5} \\ \\ g(y) = 0 \end{array}\right\} \Rightarrow \quad c(y) = 2 - \frac{y}{5} = 2 - 0.2y$$

Figure D-8: The wing used in Example D-4.

SOLUTION:

The geometry is conveniently selected to allow us to compare the methodology to the already existing formulation, as clearly, its result must match for this the simplest case.

"Standard" Solution:

$$S_{half} = \frac{1}{2}(2+1)\cdot 5 = 7.5 \text{ ft}^2$$

$$y_c = \frac{5(3\cdot 1+1)}{3(2\cdot 1+1)} = 2.222 \text{ ft}$$

$$MGC = 2 - \frac{2.222}{5} = 1.5556 \text{ ft}$$

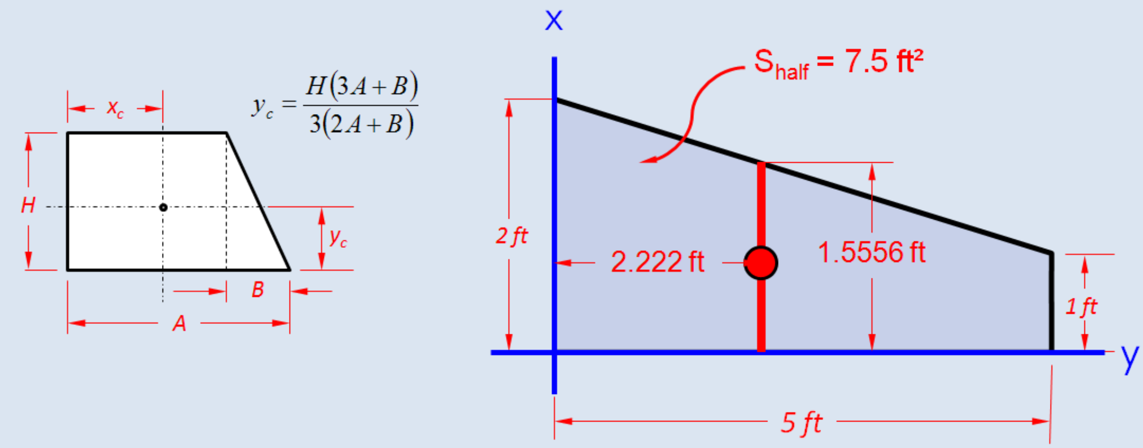


Figure D-9: Graphical representation of the solution.

Now, determine the properties using the method of this section:

Planform area: $$S_{half} = \int_0^{b/2} c(y)dy = \int_0^5 \left(2 - \frac{y}{5}\right)dy = \left[2y - \frac{y^2}{10}\right]_0^5 = 7.5 \text{ ft}^2$$

Symmetric surface area: $$S = 2\times 7.5 = 15 \text{ ft}^2$$

Mean Geometric Chord (MGC):

$$MGC = \frac{2}{S}\int_0^{b/2} c(y)^2\, dy = \frac{2}{15}\int_0^5 \left(2 - \frac{y}{5}\right)^2 dy$$

$$= \frac{2}{15}\int_0^5 \left(4 - \frac{4y}{5} + \frac{y^2}{25}\right)dy = \frac{2}{15}\left[4y - \frac{2y^2}{5} + \frac{y^3}{75}\right]_0^5 \approx 1.556 \text{ ft}$$

X-location of LE: $$x_{MGC-LE} = \frac{2}{S}\int_0^{b/2} f(y)\cdot c(y)dy = \frac{2}{15}\int_0^5 \left(2 - \frac{y}{5}\right)^2 dy \approx 1.556 \text{ ft}$$

X-location of TE: $$x_{MGC-TE} = \frac{2}{S}\int_0^{b/2} g(y)\cdot c(y)dy = 0$$

X-location of C/4:

$$x_{MGC-C/4} = \frac{2}{S}\int_0^{b/2}\left[f(y) - \frac{c(y)}{4}\right]\cdot c(y)dy = \frac{2}{15}\int_0^5\left[\left(2-\frac{y}{5}\right) - \frac{\left(2-\frac{y}{5}\right)}{4}\right]\cdot\left(2-\frac{y}{5}\right)dy$$

$$= \frac{2}{15}\int_0^5\left(2-\frac{y}{5}\right)\left[1-\frac{1}{4}\right]\cdot\left(2-\frac{y}{5}\right)dy = \frac{3}{4}\times\frac{2}{15}\int_0^5\left(2-\frac{y}{5}\right)\cdot\left(2-\frac{y}{5}\right)dy \approx 1.167 \text{ ft}$$

Y-location of MGC:

$$y_{MGC} = \frac{2}{S}\int_0^{b/2} y\cdot c(y)dy = \frac{2}{15}\int_0^5 y\cdot\left(2-\frac{y}{5}\right)dy = \frac{2}{15}\int_0^5\left(2y-\frac{y^2}{5}\right)dy = \frac{2}{15}\left[y^2 - \frac{y^3}{15}\right]_0^5 \approx 2.222 \text{ ft}$$

As a "Sanity check" consider that if the above value is correct, then MGC can also be calculated from c(y), i.e. c(2.222)=2-2.222/5=1.556 ft!

Slope of LE at MGC:

$$\Lambda_{MGC-LE} = \frac{2}{S}\int_0^{b/2} f'(y)\cdot c(y)dy = \frac{2}{15}\int_0^5\left(-\frac{1}{5}\right)\cdot\left(2-\frac{y}{5}\right)dy$$

$$\Lambda_{MGC-LE} = -\frac{2}{75}\left[2y - \frac{y^2}{10}\right]_0^5 = -0.2 \qquad (\approx -11.31°)$$

EXAMPLE D-4: Planform Geometry

A small airplane has a 20 ft wing span (b) and a planform enclosed between a leading edge described by the function f and trailing edge g, given by the following functions:

$$f(y) = \cos\left(\frac{4\pi y}{5b}\right)$$

$$g(y) = -\sin\left(\frac{2\pi y}{b}\right)$$

Determine the planform area and MGC of this wing.

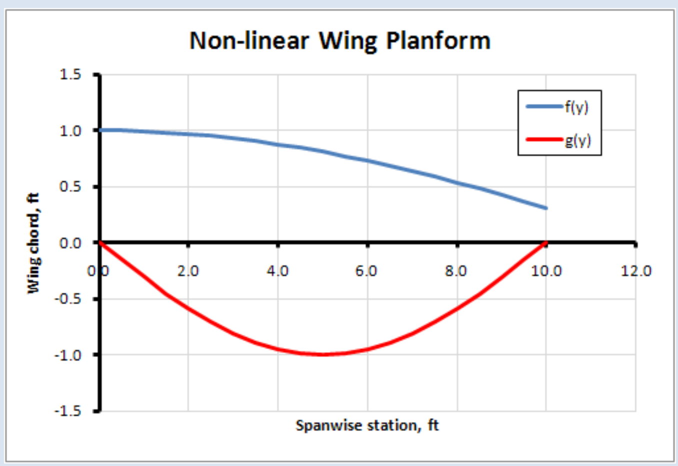


Figure D-10: The wing used in Example D-4.

SOLUTION:

We begin by defining the planform chord along the span of the wing:

$$c(y) = f(y) - g(y) = \cos\left(\frac{4\pi y}{5b}\right) + \sin\left(\frac{2\pi y}{b}\right)$$

This allows us to determine the planform area (note the factor of 2 two account for both wing halfs):

$$S = 2\int_0^{b/2} c(y)dy = 2\int_0^{b/2} \left[\cos\left(\frac{4\pi y}{5b}\right) + \sin\left(\frac{2\pi y}{b}\right)\right]dy$$

$$= 2\left[\frac{\sin\left(\frac{4\pi y}{5b}\right)}{\frac{4\pi}{5b}} - \frac{\cos\left(\frac{2\pi y}{b}\right)}{\frac{2\pi}{b}}\right]_0^{b/2} = 2\left[\frac{5b\sin\left(\frac{4\pi y}{5b}\right)}{4\pi} - \frac{b\cos\left(\frac{2\pi y}{b}\right)}{2\pi}\right]_0^{b/2}$$

$$= \frac{b}{\pi}\left[\frac{5}{2}\sin\left(\frac{4\pi y}{5b}\right) - \cos\left(\frac{2\pi y}{b}\right)\right]_0^{b/2} = \frac{20}{\pi}\left[\frac{5}{2}\sin\left(\frac{4\pi y}{100}\right) - \cos\left(\frac{2\pi y}{20}\right)\right]_0^{10}$$

$$= \frac{20}{\pi}\left[\left(\frac{5}{2}\sin\left(\frac{4\pi 10}{100}\right) - \cos\left(\frac{2\pi 10}{20}\right)\right) - \left(\frac{5}{2}\sin(0) - \cos(0)\right)\right]$$

$$\approx \frac{20}{\pi}\left[(2.378 + 1) - (0 - 1)\right] = 27.87 \text{ ft}^2$$

Mean Geometric Chord (MGC). Note that the closed form solution is omitted due to complexity and only the numerical result is presented:

$$MGC = \frac{2}{S}\int_0^{b/2} c(y)^2\, dy = \frac{2}{S}\int_0^{5} \left[\cos\left(\frac{4\pi y}{5b}\right) + \sin\left(\frac{2\pi y}{b}\right)\right]^2 dy \approx \left(\frac{2}{27.87}\right)(21.09) = 1.513 \text{ ft}$$

Other characteristics of this wing can be determined in a similar fashion.

D.5 Derivation of Some Standard Formulas

The methodology of Section D.4 is ideal to derive formulation for some "standard" planform shapes. In this section we will derive some well known standard expressions used with the simple trapezoidal planform in Section D.1, assuming the dimensions in Figure D-11.

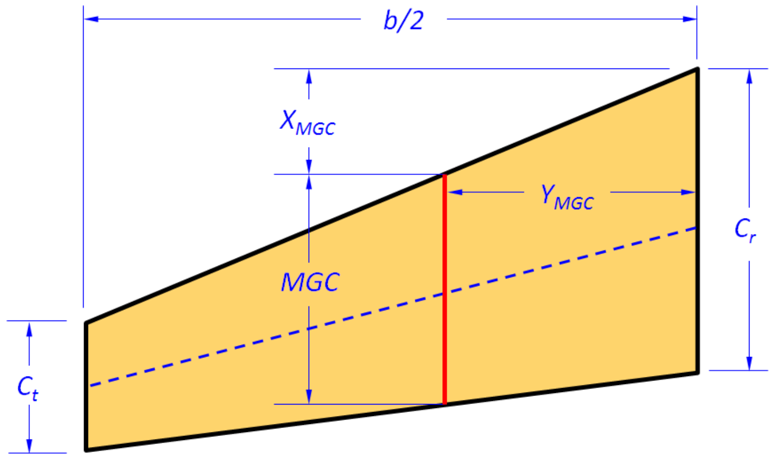


Figure D-11: A general trapezoidal wing planform.

D.5.1 Parametric Chord Function

Parametric Representation of c(y) for a simple tapered wing planform:

$$\boxed{c(y) = C_r\left[1 - \frac{2y}{b}(1-\lambda)\right]} \tag{D-38}$$

DERIVATION:

$$c(y) = C_r\left(1 - \frac{y}{b/2}\right) + C_t\left(\frac{y}{b/2}\right) = C_r\left(1 - \frac{2y}{b}\right) + \lambda C_r\left(\frac{2y}{b}\right)$$
$$= C_r\left[\left(1 - \frac{2y}{b}\right) + \lambda\left(\frac{2y}{b}\right)\right] = C_r\left[1 - \frac{2y}{b} + \lambda\left(\frac{2y}{b}\right)\right] = C_r\left[1 - \frac{2y}{b}(1-\lambda)\right]$$

Checks to confirm that the boundary conditions are satisfied:

$$\text{if } y = 0 \quad \Rightarrow \quad c(0) = C_r\left[1 - \frac{2(0)}{b}(1-\lambda)\right] = C_r$$
$$\text{if } y = \frac{b}{2} \quad \Rightarrow \quad c(b/2) = C_r\left[1 - \frac{2(b/2)}{b}(1-\lambda)\right] = C_r[1 - 1 + \lambda] = C_t$$

QED

D.5.2 Spanwise Location of the MGC

The spanwise location of the Mean Geometric Chord for the trapezoidal planform is determined using the familiar expression:

$$\boxed{y_{MGC} = \frac{b}{6}\frac{(1+2\lambda)}{(1+\lambda)}} \qquad \text{(D-39)}$$

DERIVATION:

Area:
$$S_{half} = \frac{C_r}{2}(1+\lambda)\frac{b}{2} = \frac{bC_r}{4}(1+\lambda)$$

Y-location of MGC:

$$y_{MGC} = \frac{1}{S_{half}}\int_0^{b/2} y \cdot c(y)dy = \frac{1}{S_{half}}\int_0^{b/2} y \cdot C_r\left[1-\frac{2y}{b}(1-\lambda)\right]dy$$
$$= \frac{C_r}{S_{half}}\int_0^{b/2}\left(y-\frac{2y^2}{b}(1-\lambda)\right)dy = \frac{C_r}{S_{half}}\left[\frac{y^2}{2}-\frac{2y^3}{3b}(1-\lambda)\right]_0^{b/2}$$
$$= \frac{C_r}{\frac{bC_r}{4}(1+\lambda)}\left[\frac{y^2}{2}-\frac{2y^3}{3b}(1-\lambda)\right]_0^{b/2}$$

Inserting the margins yields:

$$y_{MGC} = \frac{4}{b(1+\lambda)}\left[\frac{(b/2)^2}{2}-\frac{2(b/2)^3}{3b}(1-\lambda)\right]$$
$$= \frac{4}{b(1+\lambda)}\left[\frac{b^2}{8}-\frac{2b^3}{24b}(1-\lambda)\right]$$
$$= \frac{b}{(1+\lambda)}\left[\frac{1}{2}-\frac{1}{3}(1-\lambda)\right]$$
$$= \frac{b}{(1+\lambda)}\left[\frac{1}{2}\frac{(1+\lambda)}{(1+\lambda)}-\frac{1}{3}(1-\lambda)\frac{(1+\lambda)}{(1+\lambda)}\right]$$
$$y_{MGC} = \frac{b}{(1+\lambda)}\frac{1}{(1+\lambda)}\left[\frac{3(1+\lambda)}{6}-\frac{2(1-\lambda^2)}{6}\right]$$
$$= \frac{b}{6(1+\lambda)}\frac{\left[1+3\lambda+2\lambda^2\right]}{(1+\lambda)} = \frac{b}{6}\frac{(1+2\lambda)}{(1+\lambda)}$$

QED

D.5.3 Derivation of MGC

The Mean Aerodynamic Chord (MGC) is computed from:

$$\boxed{MGC = \frac{2C_r}{3}\frac{1+\lambda+\lambda^2}{(1+\lambda)}} \qquad \text{(D-40)}$$

DERIVATION:

Area:
$$S_{half} = \frac{C_r}{2}(1+\lambda)\frac{b}{2} = \frac{bC_r}{4}(1+\lambda)$$

MGC:
$$MGC = \frac{1}{S_{half}}\int_0^{b/2} c(y)^2\,dy = \frac{1}{S_{half}}\int_0^{b/2}\left(C_r\left[1-\frac{2y}{b}(1-\lambda)\right]\right)^2 dy$$
$$= \frac{C_r^2}{S_{half}}\int_0^{b/2}\left(1-\frac{2y}{b}(1-\lambda)\right)^2 dy$$
$$= \frac{C_r^2}{S_{half}}\int_0^{b/2}\left(1-\frac{4y}{b}(1-\lambda)+\frac{4y^2}{b^2}(1-\lambda)^2\right)dy$$
$$= \frac{C_r^2}{\frac{bC_r}{4}(1+\lambda)}\int_0^{b/2}\left(1-\frac{4y}{b}(1-\lambda)+\frac{4y^2}{b^2}(1-\lambda)^2\right)dy$$
$$= \frac{4C_r}{b(1+\lambda)}\left[y-\frac{2y^2}{b}(1-\lambda)+\frac{4y^3}{3b^2}(1-\lambda)^2\right]_0^{b/2}$$
$$= \frac{4C_r}{b(1+\lambda)}\left[\frac{b}{2}-\frac{2b^2}{4b}(1-\lambda)+\frac{4b^3}{24b^2}(1-\lambda)^2\right]$$
$$= \frac{2C_r}{3(1+\lambda)}\left[1+\lambda+\lambda^2\right] = \frac{2C_r}{3}\frac{1+\lambda+\lambda^2}{(1+\lambda)}$$

QED

LIST OF VARIABLES

Symbol	Description	Units (UK and SI)
AR	Aspect Ratio	
AR_E	Aspect Ratio of an equivalent wing	
AR_O	Aspect Ratio of a baseline wing used in equivalent wing analysis	
b	Wing span	ft or m
b_E	Wing span of an equivalent wing	ft or m
b_O	Wing span of a baseline wing used in equivalent wing analysis	ft or m
c(y)	An arbitrary function describing the chord of the planform	ft or m
C_{avg}	Average chord	ft or m
c_i	Chord index for equivalent wing analysis	ft or m
C_r	Chord at root	ft or m
C_{RE}	Chord at root of an equivalent wing	ft or m
C_t	Chord at tip	ft or m
C_{TE}	Chord at tip of an equivalent wing	ft or m
C_{WR}	Weighted chord at root	ft or m
C_{WT}	Weighted chord at tip	ft or m
dM	Elemental weighing factor	Various units
dM_{FOI}	Elemental weighing Factor-of-Interest (context dependent)	Various units

dS	Infinitesimal area	ft² or m²
f(y)	An arbitrary function describing the geometry of the LE	ft or m
g(y)	An arbitrary function describing the geometry of the TE	ft or m
K	Scaling factor for equivalent wing analysis	
LE	Leading Edge	
MAC	Mean Aerodynamic Chord	ft or m
MGC	Mean Geometric Chord	ft or m
N	Total number of individual chords in equivalent wing analysis	
S	Wing area	ft² or m²
S_E	Wing area of an equivalent wing	ft² or m²
S_E	Wing area of an equivalent wing	ft² or m²
S_{half}	Area of a one-half of a wing (either left or right half)	ft² or m²
S_i	Area index for equivalent wing analysis	ft² or m²
S_O	Wing area of a baseline wing used in equivalent wing analysis	ft² or m²
S_W	Weighted wing area	ft² or m²
TE	Trailing Edge	
X_{MGC}, x_{MGC}	Chordwise distance from root chord to the LE of MGC	ft or m
$x_{MGC-C/4}$	X-location of the quarter chord of the MGC	ft or m
x_{MGC-LE}	X-location of the LE of the MGC	ft or m
x_{MGC-TE}	X-location of the TE of the MGC	ft or m
y_i	Indexed span of elemental area S_i	ft or m
Y_{MGC}, y_{MGC}	Spanwise distance from root chord to the LE of MGC	ft or m
Γ_W	Dihedral	Degrees or radians
$\Lambda_{C/4}$	Quarter chord sweep angle	Degrees or radians
$\Lambda_{C/4\ E}$	Quarter chord sweep angle for an equivalent wing	Degrees or radians
Λ_i	Quarter chord sweep angle index for equivalent wing analysis	Degrees or radians
Λ_{LE}	Leading Edge sweep angle	Degrees or radians
$\Lambda_{LE\ E}$	Leading Edge sweep angle for an equivalent wing	Degrees or radians
$\Lambda_{MGC-C/4}$	Quarter chord sweep angle at the MGC	Degrees or radians
Λ_{MGC-LE}	LE sweep angle at the MGC	Degrees or radians
α_{ZL-R}	Zero-Lift angle at root	Degrees or radians
α_{ZL-T}	Zero-Lift angle at tip	Degrees or radians
λ	Taper ratio	
λ_E	Taper ratio for an equivalent wing	
λ_O	Taper ratio for an equivalent wing	