# APPENDIX E: Practical Mathematics for the Aircraft Designer

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GENERAL AVIATION AIRCRAFT DESIGN
APPLIED METHODS AND PROCEDURES
SNORRI GUDMUNDSSON
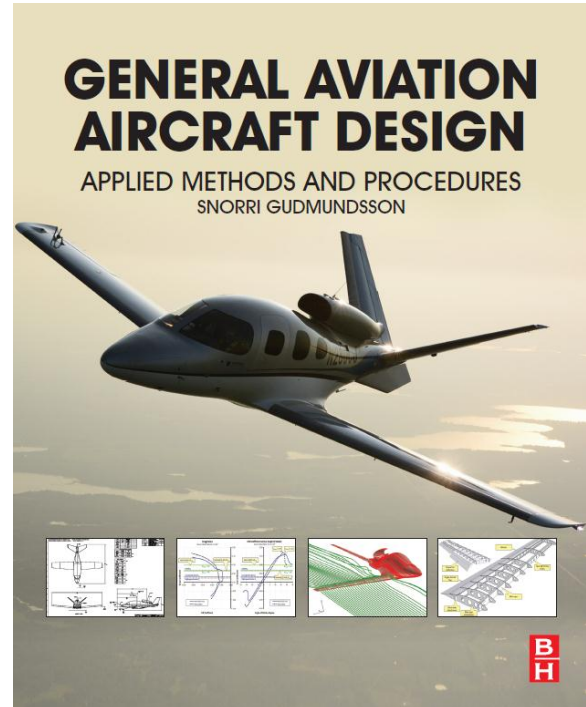
E.1 Algebra

E.1.1 Laws of Algebraic Operations

Commutative law:	a + b = b + a ab = ba
Associative law:	a + (b + c) = (a + b) + c a(bc) = (ab)c
Distributive law:	a(b + c) = ab + ac

E.1.2 Powers and Roots

$$a^0 = 1 \quad \text{if } a \neq 0 \qquad a^{-x} = \frac{1}{a^x} \qquad a^x a^y = a^{(x+y)} \qquad \frac{a^x}{a^y} = a^{(x-y)}$$

$$(ab)^x = a^x b^x \qquad (a^x)^y = a^{xy} \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$$

$$a^{\frac{1}{x}} = \sqrt[x]{a} \qquad \sqrt[x]{ab} = \sqrt[x]{a}\sqrt[x]{b} \qquad \sqrt[x]{\sqrt[y]{a}} = \sqrt[xy]{a} \qquad a^{\frac{x}{y}} = \sqrt[y]{a^x}$$

E.1.3 Proportions

If $\dfrac{a}{b} = \dfrac{c}{d}$ then the following are true; $\qquad \dfrac{a+b}{b} = \dfrac{c+d}{d}$, $\qquad \dfrac{a-b}{b} = \dfrac{c-d}{d}$, $\qquad \dfrac{a-b}{a+b} = \dfrac{c-d}{c+d}$

E.1.4 The Binomial Formula

The *Binomial Formula* for a positive integer *n*;

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + ... + y^n \qquad \text{(E-1)}$$

Special cases of the *binomial formula*:

$$
\begin{aligned}
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x-y)^2 &= x^2 - 2xy + y^2 \\
(ax-by)^2 &= a^2x^2 - 2abxy + b^2y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3 \\
(ax-by)^3 &= a^3x^3 - 3a^2bx^2y + 3ab^2xy^2 + b^3y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x-y)^4 &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \\
(x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \\
(x-y)^5 &= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5 \\
(x+y)^6 &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 \\
(x-y)^6 &= x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6
\end{aligned}
$$

Special Products;

$$
\begin{aligned}
x^2 - y^2 &= (x-y)(x+y) \\
x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \\
x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\
x^4 - y^4 &= (x-y)(x+y)(x^2 + y^2) \\
x^5 - y^5 &= (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \\
x^5 + y^5 &= (x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4) \\
x^6 - y^6 &= (x-y)(x+y)(x^2 + xy + y^2)(x^2 - xy + y^2) \\
x^4 + x^2y^2 + y^4 &= (x^2 + xy + y^2)(x^2 - xy + y^2) \\
x^4 + 4y^4 &= (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)
\end{aligned}
$$

$a \times 0 = 0$	$a \times \infty = \infty$	$0 \times \infty$	indeterminate
$\dfrac{0}{a} = 0$	$\dfrac{a}{0} = \infty$	$\dfrac{0}{0}$	indeterminate
$\dfrac{\infty}{a} = \infty$	$\dfrac{a}{\infty} = 0$	$\dfrac{\infty}{\infty}$	indeterminate
$a^0 = 1$	$0^a = 0$	0^0	indeterminate
$\infty^a = \infty$	$\infty - a = \infty$	∞^0	indeterminate
$a - a = 0$		$\infty - \infty$	indeterminate

If $a^2 > 1$ then;	$a^\infty = \infty$	$a^{-\infty} = 0$
If $a^2 = 1$ then;	$a^\infty = 1$	$a^{-\infty} = 1$
If $a^2 < 1$ then;	$a^\infty = 0$	$a^{-\infty} = \infty$

E.1.6 Definition of Imaginary and Complex Numbers

Roots of negative numbers have been used since the 1750's, when the concept *imaginary* was devised. Mathematicians of the 17th century used a book by Raffaele Bombelli, written in 1572, containing the theory of imaginary numbers. The theory was further advanced by the contributions of Johann Bernoulli (1667-1748), Leonhard Euler (1707-1783), and Carl Friedrich Gauss (1777-1783). The representation of complex numbers in the plane is attributed to Caspard Wessel (1745-1818) and Jean Robert Argand (1768-1822).
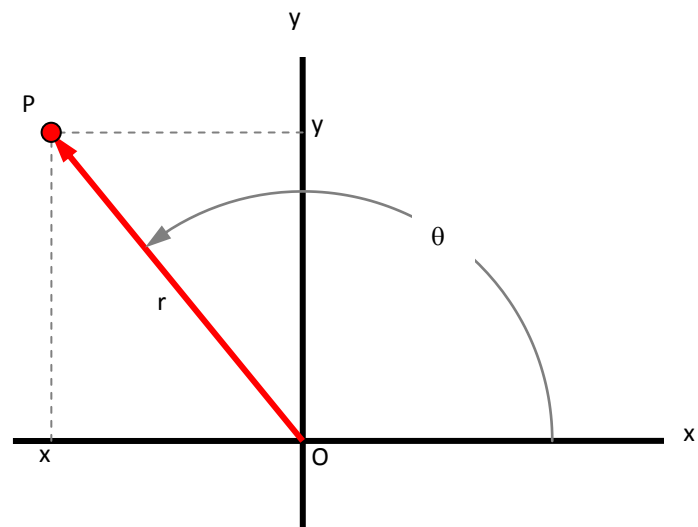

Figure E-1: The Gaussian Plane.

General Definition of Imaginary Numbers

A *complex number* is generally written as $a+ib$, where a and b are real numbers, and i, called the *imaginary unit*, has the property $i^2 = -1$. The real numbers a and b are called the *real* and *imaginary parts* of the complex number $a+ib$, respectively.

Complex Conjugates

The complex numbers $a+ib$ and $a-ib$ are called complex *conjugates* of each other.

Graphing a Complex Number

A complex number $a+ib$ can be plotted as a point (a, b) on the x-y plane, as shown in Figure E-1. The diagram is called the *Argand diagram* or the *Gaussian plane*. The imaginary number can thus be interpreted as the vector **OP**.

Polar Form of the Complex Number

The point **P** in Figure E-1 can also be represented in a polar form. From the figure we see that the terminal point of the vector **OP** is $a = r \cos \theta$ and $b = r \sin \theta$. Hence, we may write

$$a + ib = r(\cos \theta + i \sin \theta)$$

The *modulus, r*, and the *amplitude,* θ, or *argument* of $a+ib$ are given by

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1}(b/a)$$

Representation of Complex Numbers

The following representations are commonly used for the complex number $a+ib$ in Figure E-1:

$$(a,b) = a + ib = r(\cos \theta + i \sin \theta) = re^{i\theta} = (r,\theta) = r\angle\theta$$

E.1.7 Euler's Theorem of Complex Numbers

Refer to Figure E-1:
$$\boxed{\cos \theta + i \sin \theta = e^{i\theta}}$$
(E-2)

DERIVATION OF EQUATION (E-2):

Using Taylor expansion we can write each term in Equation (E-2) as follows:

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \dots$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

Combining those yields:

$$\cos \theta + i \sin \theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} - \dots = e^{i\theta}$$

QED

Note that the trigonometric parameters are also given by the following identity:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

E.1.8 Arithmetic Operations Using Complex Numbers[1]

The vectors **A** and **C** are defined as follows;

$$\mathbf{A} = a + ib = re^{i\theta} = r\angle\theta$$

$$\mathbf{C} = c + id = se^{i\phi} = r\angle\phi$$

Where; a, b, c, d, k, n, r = Constants.
θ, ϕ = Angles in radians.

Then, the following arithmetic operations can be derived for complex numbers.

Equality of $a + ib = c + id \iff a = c \quad \text{and} \quad b = d$

Addition of $\mathbf{A} + \mathbf{C} = (a + ib) + (c + id) = (a + c) + i(b + d)$

Subtraction of $\mathbf{A} - \mathbf{C} = (a + ib) - (c + id) = (a - c) + i(b - d)$

Multiplication of

$$\mathbf{A} \cdot \mathbf{C} = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

$$\mathbf{A} \cdot \mathbf{C} = (re^{i\theta})(se^{i\phi}) = rse^{i(\theta+\phi)} = rs\angle\theta + \phi$$

Division of:

$$\frac{\mathbf{A}}{\mathbf{C}} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}$$

$$\frac{\mathbf{A}}{\mathbf{C}} = \frac{re^{i\theta}}{se^{i\phi}} = \frac{r\angle\theta}{s\angle\phi} = \frac{r}{s}e^{i(\theta-\phi)} = \frac{r}{s}\angle\theta - \phi$$

The arithmetic operations are obtained by using conventional algebraic rules and by replacing i^2 with -1, wherever it occurs.

Powers of (De Moivre's Theorem)
If n is any real number, De Moivre's theorem states

$$\mathbf{A}^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$$\mathbf{A}^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n\angle n\theta$$

Roots of (De Moivre's Theorem)
If $1/n$ is any positive real number, De Moivre's can be written as follows

$$\mathbf{A}^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n} = r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$$

$$\mathbf{A}^{1/n} = (re^{i\theta})^{1/n} = r^{1/n}e^{i\theta/n} = r^{1/n}\angle\theta/n$$

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[1] Handbook of Mathematical, Scientific, and Engineering Formulas, Tables, Functions, Graphs, Transforms, 1984. Pg. 288-289.

Where k is any integer. From this there are n roots of a complex number, each being obtained by putting k = 0, 1, 2, ..., n-1 (see Example E-1).

EXAMPLE E-1:

Two complex numbers **A** and **C** are given. Use them to determine each of the above arithmetic operation, if $n = 3$ and...

$$\mathbf{A} = 3 - i2 \approx \sqrt{13}\angle - 33.69° \quad \text{and} \quad \mathbf{C} = 5 - i3 \approx \sqrt{34}\angle 30.96°$$

SOLUTION:

(a)   **A** + **C** = (3+5) + i(-2+3) = 8 + i

(b)   **A** - **C** = (3-5) + i(-2+(-3)) = -2 -5i

(c)   **A**·**C** = (3-i2)·(5+i3) = (3·5 - (-2)·3) + i(3·3 + (-2)·5) = 21 - i

(d) $$\frac{\mathbf{A}}{\mathbf{C}} \approx \frac{\sqrt{13}}{\sqrt{34}} \angle (-33.69° - 30.96°) = \sqrt{\frac{13}{34}} \angle - 64.65°$$

(e) $$\mathbf{A^3} \approx (\sqrt{13}\angle - 33.69°)^3 = 13^{3/2} \angle - 101.1°$$

(f)   For n = 0 we have; $$\mathbf{C}^{1/3} \approx (\sqrt{34}\angle 30.96°)^{1/3} = \sqrt[3]{34}\angle 10.32°$$

For n = 1 we have; $$\mathbf{C}^{1/3} \approx (\sqrt{34}\angle 30.96° + 360°)^{1/3} = \sqrt[3]{34}\angle 10.32° + 120°$$

For n = 2 we have; $$\mathbf{C}^{1/3} \approx (\sqrt{34}\angle 30.96° + 2\cdot 360°)^{1/3} = \sqrt[3]{34}\angle 10.32° + 240°$$
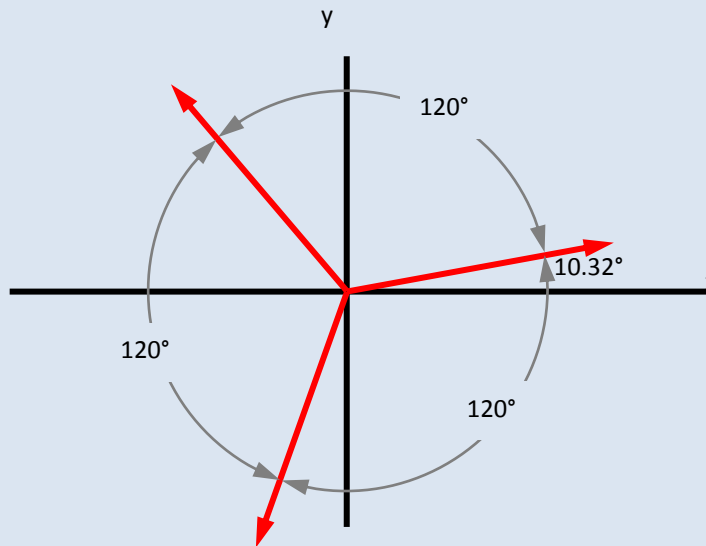
The three roots are depicted in Figure E-2:

Figure E-2: Graphical solution to part (f).

E.2 Logarithms and Exponentials

Definition of

$$\ln x = \int_1^x \frac{dt}{t}$$

(E-3)

The logarithm denoted with ln *x* is generally called the *natural*, *Naiperian*, or *hyperbolic* logarithm.

Standard Logarithmic Identities

$$\ln(ab) = \ln a + \ln b \qquad\qquad \ln a^r = r \ln a \quad (r \text{ is any real number})$$

$$\ln \frac{a}{b} = \ln a - \ln b \qquad\qquad \ln \frac{1}{a} = \ln a^{-1} = -\ln a$$

Logarithmic Identities for Imaginary Numbers

See E.1.6 through E.1.8 for discussion of imaginary (complex) numbers. Here $\mathbf{A} = re^{i\theta}$ and $\mathbf{C} = se^{i\gamma}$.

$$\ln(\mathbf{A} \cdot \mathbf{C}) = \ln \mathbf{A} + \ln \mathbf{C} \qquad\qquad \text{where;} \qquad -\pi < \theta + \gamma \le \pi$$

$$\ln \frac{\mathbf{A}}{\mathbf{C}} = \ln \mathbf{A} - \ln \mathbf{C} \qquad\qquad \text{where;} \qquad -\pi < \theta - \gamma \le \pi$$

$$\ln \mathbf{A}^n = n \ln \mathbf{A} \qquad\qquad \text{where } n \text{ is an integer and } -\pi < n \cdot \theta \le \pi$$

Special Values

$$\ln 1 = 0 \qquad\qquad\qquad\qquad \ln e = 1$$

$$\ln 0 = -\infty \qquad\qquad\qquad\quad \ln(-1) = \pi i$$

$$\ln(\pm i) = \pm \frac{\pi i}{2} \pi \qquad\qquad \ln e = 1, e \text{ is a real number such that; } \int_1^e \frac{dt}{t} = 1$$

Logarithms of Base *b*

$$\log_b x = \frac{\ln x}{\ln b} \qquad\qquad \log_b a = \frac{1}{\log_a b} \qquad\qquad \log_b x = \frac{\log_a x}{\log_a b} \qquad\qquad \log_e x = \ln x$$

The last three identities are derived from the first one.

Definition of
The exponential function, *e^x*, is a function such that;

$$\boxed{\frac{d}{dx}e^x = e^x}$$

(E-4)

Series Expansions

$$e^z = \exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \qquad \text{where } z = a + ib$$

$$e = \lim_{n\to\infty}(1 + \frac{1}{n})^n = 2.71828 \; 18284...$$

Fundamental Properties

$$x = \ln e^x \qquad\qquad x = e^{\ln x}$$

$$\ln(e^z) = z + 2k\pi i \qquad\qquad \text{where } k \text{ is any integer}$$

$$e^{\ln z} = z \qquad\qquad e^z = e^{z+2\pi ki} \qquad\qquad \text{periodic property}$$

Special Values

$$e = 2.71828 \; 18284... \qquad e^0 = 1 \qquad e^\infty = \infty$$

$$e^{-\infty} = 0 \qquad\qquad e^{\pm\pi i} = -1 \qquad\qquad e^{\pm\frac{\pi i}{2}} = \pm i$$

$$e^{2\pi ki} = 1 \qquad\qquad k \text{ is any integer}$$

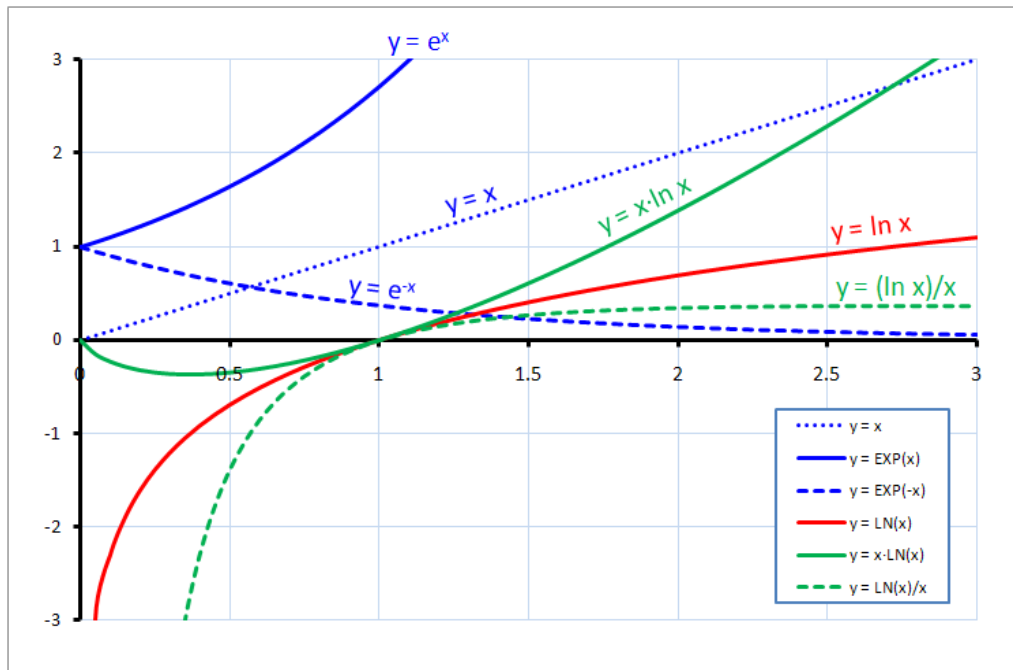## E.2.3 Graphing the Logarithms and Exponentials

Figure E-3: Graphing the logarithm and exponentials.

E.2.4 Hyperbolic Functions

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x}$$

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x}$$

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}$$

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x} + 1}$$

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} = \frac{2e^x}{e^{2x} - 1}$$

E.2.5 Inverse Hyperbolic Functions

Inverse hyperbolic sine:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Inverse hyperbolic cosine:

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \quad \text{where} \quad x \geq 1$$

Inverse hyperbolic tangent:

$$\tanh^{-1} x = \left(\frac{1}{2}\right) \ln \frac{1+x}{1-x} \quad \text{where} \quad |x| < 1$$

Inverse hyperbolic cotangent:

$$\coth^{-1} x = \left(\frac{1}{2}\right) \ln \frac{x+1}{x-1} \quad \text{where} \quad |x| > 1$$

Inverse hyperbolic secant:

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) \quad \text{where} \quad 0 < x \leq 1$$

Inverse hyperbolic cosecant:

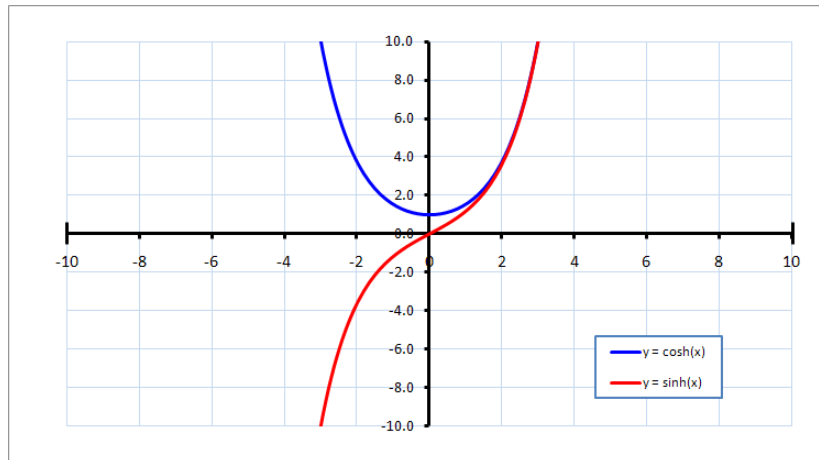$$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|}\right)$$
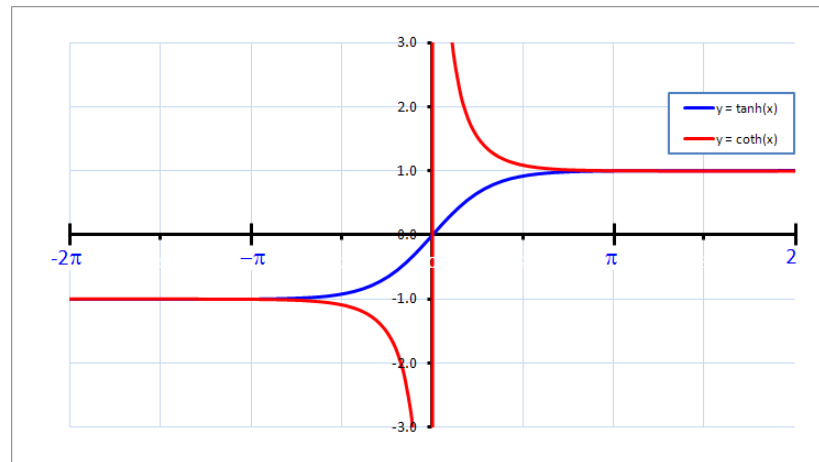
Figure E-4: Plotting sin x and cos x.

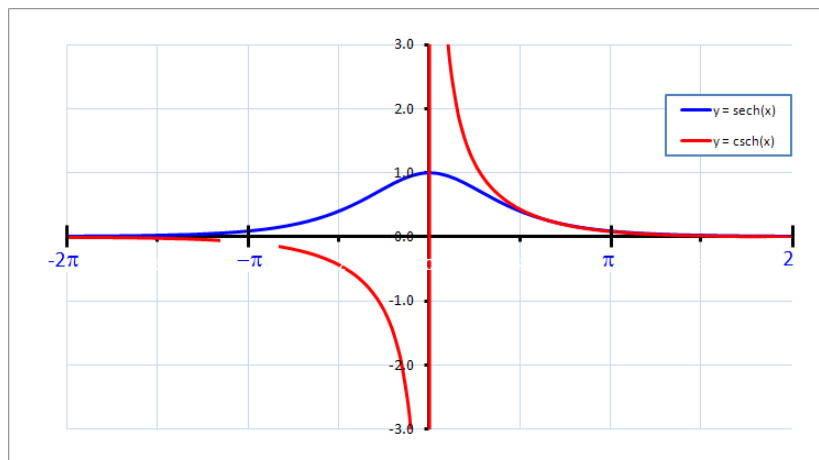**Figure E-5: Plotting sin x and cos x.**

Figure E-6: Plotting sin x and cos x.

E.3 Functions

E.3.1 Solving Two Linear Equations in a Canonical Form

Solving two linear equations is a common practice in algebra. Figure E-7 below shows the intersection of the two lines represented by the linear equations $Ax + By = C$ and $Dx + Ey = F$. The x and y values of their intersection is given by

$$x = \frac{BF - CE}{BD - AE} \quad and \quad y = \frac{CD - AF}{BD - AE}$$

(E-5)

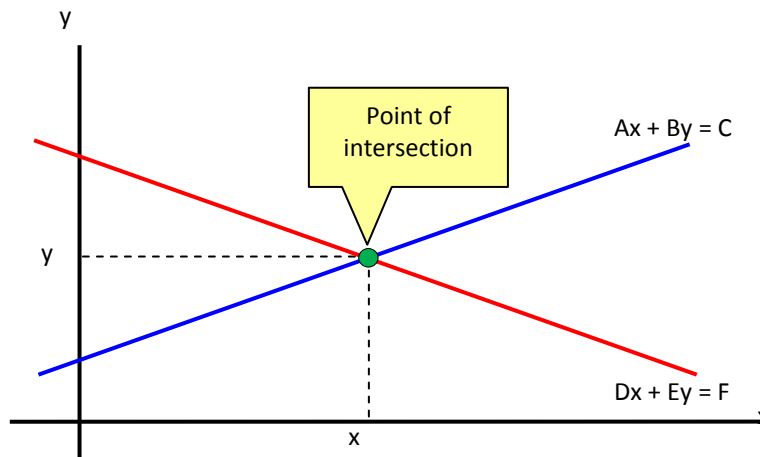Where the constants A, B, C, D, E, and F are known.

Figure E-7: Determining the point of intersection of two lines.

DERIVATION OF EQUATION (E-5):
Start by writing the expressions for the lines as shown below. Then, multiply the top equation by D and the lower by -A;

$$D \times (Ax + By = C) \Rightarrow \quad ADx + BDy = CD \qquad \text{(i)}$$
$$-A \times (Dx + Ey = F) \Rightarrow \quad -ADx - AEy = -AF \qquad \text{(ii)}$$

Then, add them together to yield: $(BD - AE)y = DC - AF$

When solved for y this equation gives the right part of Equation (E-5). Inserting this result into either (i) or (ii), here using (i), and dividing through by A (to solve for x) results in;

$$x = \frac{C}{A} - \frac{B}{A} \cdot \frac{(CD - AF)}{(BD - AE)}$$

Which can easily be expanded and manipulated to give the left part of Equation (E-5).

QED

E.3.2 Linear Equations in a Parametric Form

If the two linear equations are given in the form $y_1 = Ax + B$ and $y_2 = Cx + D$ as shown in Figure E-8, where the constants A, B, C, and D are known, then the x and y values of intersection of the two lines can be found using the equations;

$$x = \frac{D - B}{A - C} \quad \text{and} \quad y = \frac{AD - BC}{A - C} \tag{E-6}$$
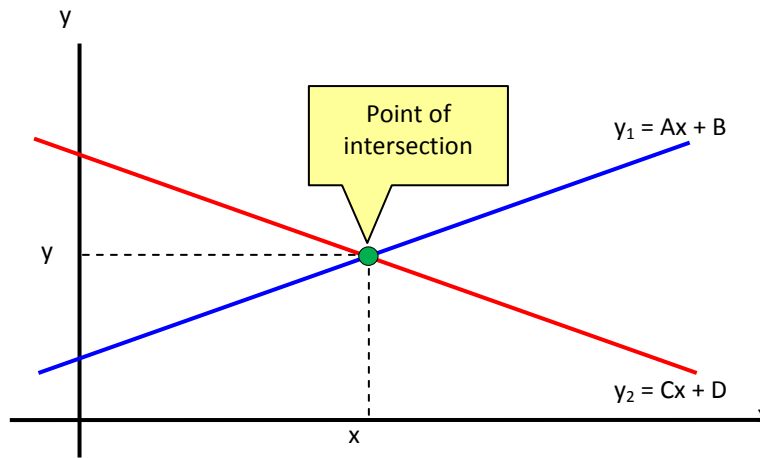

Figure E-8: Determining the point of intersection of two lines.

Also note the special formulation of a parametric line discussed in Article E.5.5, *Parametric Lines*.

DERIVATION OF EQUATION (E-6):
The two expressions must equal where they intersect, i.e. $y_1 = y_2$, or $Ax + B = Cx + D$. This can be solved for x yielding the left equation of (E-6). Consequently, by inserting the result for x, for instance, into y_1 and manipulating, yields

$$y = A\frac{(D - B)}{(A - C)} + B = \frac{A(D - B) + B(A - C)}{A - C} = \frac{AD - BC}{A - C}$$

which is the right side of Equation (E-6).

QED

EXAMPLE E-2:
Determine where the following two lines intersect;

$$y_1 = 2x - 2 \quad \text{and} \quad y_2 = -3x + 8$$

SOLUTION:
Using Equations (E-6) we find the x and y coordinates of their intersection.

$$x = \frac{8 - (-2)}{2 - (-3)} = 2 \quad \text{and} \quad y = \frac{(2)(8) - (-2)(-3)}{2 - (-3)} = 2$$

Alternatively, writing the two equations in the canonical form, we have:

$$2x - y = 2 \qquad \text{and} \qquad 3x + y = 8$$

Now, applying Equations (E-5), we get the same coordinates of the point of intersection.

$$x = \frac{(-1)(8) - (2)(1)}{(-1)(3) - (2)(1)} = 2 \qquad \text{and} \qquad y = \frac{(2)(3) - (2)(8)}{(-1)(3) - (2)(1)} = 2$$

E.3.3 Polynomial Fundamentals

Polynomials and their solutions are very practical in science and engineering. The solution of a polynomial is also called its *root*. An intuitive way to think of a solution to a polynomial is to think of the x_i that make *P(x)* = 0.

Definition of

Any equation of the form;

$$\boxed{P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0}$$

(E-7)

where a_0, a_1, $a_2,\ldots,a_n$ are constants where $a_0 \neq 0,$ is called *polynomial to a degree n*. Furthermore, the form of Equation (E-7) is called the *canonical form*.

The Roots of *P(x)* = 0

The x-values that make the polynomial *P(x)* equal to 0 are called the *roots* of *P(x).* The roots of the polynomial may be real and separate, or real and repeated, or complex conjugate and separate, and/or complex conjugate and repeated. If $x_1, x_2,\ldots, x_n$ are the roots of *P(x)* = 0, then;

$$P(x) = (x - x_1)(x - x_2)\ldots(x - x_n) = 0$$

(E-8)

Types of Equations and Their Solutions

Polynomials are very practical in science and engineering, so studying their solutions is extremely important. The solution is found once their roots (i.e. by finding the x_i's that make *P(x)* = 0) have been determined. Considering Equation (E-7), some specific forms of *P(x)* are well known and can be seen in the table below. The table depicts the form that the equations are most commonly found in, its most common name, and the third column indicates the article in which the solution can be found.

Polynomial	Name
$y = Ax + B$	Equation of a line Polynomial of degree one
$y = Ax^2 + Bx + C$	Equation of a parabola Quadratic equation
$y = Ax^3 + Bx^2 + Cx + D$	Cubic equation
$y = Ax^3 + Bx^2 + Bx + C$	Symmetrical cubic equation
$y = Ax^4 + Bx^3 + Cx^2 + Dx + E$	Quartic equation
$y = Ax^4 + Bx^3 + Cx^2 + Bx + A$	Symmetircal quartic equation
$y = Ax^4 + Bx^3 - Bx - A$	Antisymmetrical quartic equation
$y = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$	Quintic equation

$y = Ax^5 + Bx^4 + Cx^3 + Cx^2 + Bx + A$	Symmetrical quintic equation
$y = Ax^5 + Bx^4 + Cx^3 - Cx^2 - Bx - A$	Antisymmetrical quintic equation
$y = x^n - h$	Binomial equation, $h>0$
$y = Ax^2 + Bx^n + C$	Trinomial equation, ($n = 2, 3, ...$)

E.3.4 The Polynomial of Degree One

Is an equation in the canonical form; $A_0 + A_1 x = 0$

where the value of x can be found as follows, provided the constants A_0 and A_1 are known;

$$x = -\frac{A_0}{A_1}$$

(E-9)

E.3.5 The Quadratic Equation

The quadratic equation is an equation in the canonical form; $A_0 + A_1 x + A_2 x^2 = 0$

where the value(s) of x can be found with the following formula if the constants A_0, A_1, and A_2 are all known

$$x = \frac{-A_1 \pm \sqrt{A_1^2 - 4A_0 A_2}}{2A_2}$$

(E-10)

If the two values of x are nearly equal, then the numerical result can be determined more precisely using the following form of the equation:

$$x = \frac{2A_2}{-A_1 \pm \sqrt{A_1^2 - 4A_0 A_2}}$$

(E-11)

This equation is commonly presented using the coefficients A, B, and C: $A x^2 + B x + C = 0$

The solution is then given as: $$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$ (E-12)

DERIVATION OF EQUATION (E-12):
For convenience, start with the more common definition of a quadratic equation, i.e.:

$$Ax^2 + Bx + C = 0$$

The solution method starts with: $$x^2 + \frac{B}{A}x + \frac{C}{A} = 0$$

Adding supplemental terms:

$$x^2 + \frac{B}{A}x + \frac{B^2}{4A^2} - \frac{B^2}{4A^2} + \frac{C}{A} = 0$$

Which reduces to a pure quadratic form:

$$\left[x + \frac{B}{2A}\right]^2 - \frac{B^2}{4A^2} + \frac{C}{A} = 0$$

Which is solved as follows:

$$\left[x + \frac{B}{2A}\right]_{1,2} = \pm\sqrt{\frac{B^2}{4A^2} - \frac{C}{A}} \quad \Leftrightarrow \quad x_{1,2} = -\frac{B}{2A} \pm \sqrt{\frac{B^2}{4A^2} - \frac{C}{A}}$$

and can be easily manipulated to yield the two forms of the solution.

QED

EXAMPLE E-3:

(1) The roots of $x^2 - 2x - 8 = 0$ are: $\quad x_{1,2} = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-8)}}{2(1)} = 1 \pm 3$

(2) The roots of $x^2 - 8x + 16 = 0$ are: $\quad x_{1,2} = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(16)}}{2(1)} = 4$

(3) The roots of $x^2 - 2x + 8 = 0$ are: $\quad x_{1,2} = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(8)}}{2(1)} = 1 \pm \sqrt{7}i$

E.3.6 COMPUTER CODE: Solving the Quadratic Equation

The following Visual Basic for Applications function, intended for use with Microsoft Excel, implements the quadratic equation presented above. It returns a solution using both real and imaginary numbers, and even detects and solves if the entered coefficients (A, B, C) represent the equation of a line.

```
Function MAT_SolveQuadratic(A As Double, B As Double, C As Double, Mode As Byte) As
Variant
'This routine solves the quadratic equation Ax²+Bx+C=0 and returns the
'real and complex roots depending on the value of Mode. Solution is of
'the form:  x = ± real ± imaginary·i, meaning there are four numbers.
'
'Variables:     A, B, C = Constants of the quadratice equation
'               Mode    = 0 to return "real number 1"
'               Mode    = 1 to return "imaginary number 1"
'               Mode    = 2 to return "real number 2"
'               Mode    = 3 to return "imaginary number 2"
'               Mode    = 10 to return a string containing the solution
'
'Initialize
    Dim D As Double
    Dim R1 As Double, R2 As Double
    Dim I1 As Double, I2 As Double

'Presets
    D = B * B – 4 * A * C
```

```
'Explore alternative solutions
    If A = 0 Then          'We have case Bx+C=0     <=> x=-C/B
        If B = 0 Then      'We have case C=0          No solution
            SolutionString = "Error: A=0 in (-B±SQR[B²-4·A·C])/(2·A)"
            Exit Function
        Else
            R1 = -C / B
            R2 = 0
            SolutionString = Format$(R1) + "|" + Format$(R2)
        End If
    Else                   'Solve the quadratice
        If D >= 0 Then        'Real solution
            R1 = 0.5 * (-B + Sqr(D)) / A
            R2 = 0.5 * (-B - Sqr(D)) / A
            SolutionString = Format$(R1) + "|" + Format$(R2)
        Else               'Complex solution
            R1 = -0.5 * B / A
            R2 = R1
            D = Abs(D)
            I1 = 0.5 * Sqr(D) / A
            I2 = -0.5 * Sqr(D) / A
            SolutionString = Format$(R1) + "," + Format$(I1) + "|" + Format$(R2) + ","
+ Format$(I2)
        End If
    End If

'Present solution
    Select Case Mode
    Case 0
        MAT_SolveQuadratic = R1
    Case 1
        MAT_SolveQuadratic = I1
    Case 2
        MAT_SolveQuadratic = R2
    Case 3
        MAT_SolveQuadratic = I2
    Case 10
        MAT_SolveQuadratic = SolutionString
    End Select
End Function
```

E.3.7 Completing the Square

Completing the square is a handy method to solve the quadratic equation. Consider a quadratic equation of the form $ax^2 + bx + c = 0$. Then, completing the square involves putting it into the following form:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad a(x+d)^2 + e = 0$$

(E-13)

Where:

$$d = \frac{b}{2a} \quad \text{and} \quad e = c - \frac{b^2}{4a}$$

Note that completing the square is the most important step in deriving Equation (E-12). There are a few advantages to solving it this way. First, the form contains the x,y coordinates of the vertex of the parabola. Second, it is easy to solve the equation in the completed form. Third, sometimes, the original form $ax^2 + bx + c = 0$ is a part of a bigger problem (nonlinear dynamical control systems come to mind), and completing the square allows a

solution to be clearly represented because x only appears once. This is also helpful if the term "x" actually represents a function (e.g. tanh(r)) and this can yield a considerable simplification.

Note that Equation (E-13) can be explicitly written as follows:
$$a\left(x+\frac{b}{2a}\right)^2 + \left(c-\frac{b^2}{4a}\right) = 0$$

Therefore, the vertex of the parabola is given by:
$$(x,y) = \frac{1}{a}\left(-\frac{b}{2}, \frac{b^2}{4a}-c\right)$$

Consider the special case where a = 1:
$$x^2 + bx + c = 0 \quad\Rightarrow\quad \left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4} = 0 \qquad\text{(iii)}$$

Note that Equation (i) can also be written as follows:
$$a(x+d)^2 + e = 0 \quad\Leftrightarrow\quad (x+d)^2 = -\frac{e}{a}$$

And then the vertex of the curve (x,y) = (-d, e/a). Example: x²+4x+1=0 => d=4/2=2 and e=1-4²/4=-3. The vertex of this curve is at (x,y) = (-2, -3).

EXAMPLE E-4:

Complete the square for the following quadratic polynomials:

(1) The roots of x^2 - 2x - 8 = 0 are: $\Rightarrow$ $\left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4} = (x-1)^2 - 7 = 0$

(2) The roots of x^2 - 8x + 16 = 0 are: $\Rightarrow$ $\left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4} = (x-4)^2 = 0$

(3) The roots of x^2 - 2x + 8 = 0 are: $\Rightarrow$ $\left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4} = (x-1)^2 + 7 = 0$

E.4 Trigonometry

E.4.1 Definition of Trigonometric Functions for a Right Triangle

For the Right triangle (where angle C = 90°) in Figure E-9, side **AB** is called the "*hypotenuse*", side **AC** is the "*adjacent*", and side **BC** is the "*opposite*". With respect to this triangle, the trigonometric functions are defined as follows:

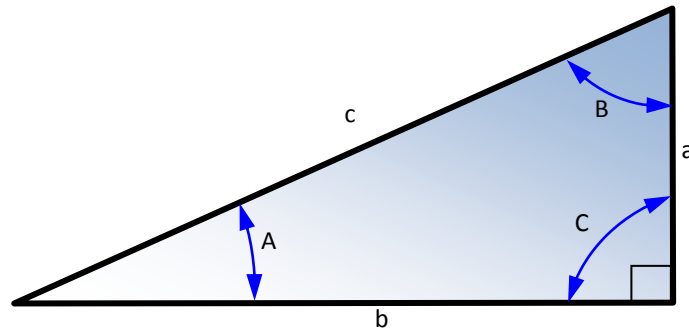|	
Sine of *A*:	sin *A* = *a/c* = opposite/hypotenuse
Cosine of *A*:	cos *A* = *b/c* = adjacent/hypotenuse
Tangent of *A*:	tan *A* = *a/b* = opposite/adjacent
Cotangent of *A*:	cot *A* = *b/a* = adjacent/opposite
Secant of *A*:	sec *A* = *c/b* = hypotenuse/adjacent
Cosecant of *A*:	csc *A* = *c/a* = hypotenuse/opposite

Figure E-9: The right triangle.

E.4.2 Trigonometric Functions of an Arbitrary Angle

Let α be any angle in standard position and let **P** = *P(x,y)* be any point on the terminal side of the angle as shown in Figure E-10. Denote the positive distance **OP** by *r*. Then;

$\sin \theta = y/r$	$\csc \theta = r/y$
$\cos \theta = x/r$	$\sec \theta = r/x$
$\tan \theta = y/x$	$\cot \theta = x/y$
$\text{exsec } \theta = \sec \theta - 1$	$\text{covers } \theta = 1 - \sin \theta$
$\text{vers } \theta = 1 - \cos \theta$	$\text{hav } \theta = \frac{1}{2} \text{ vers } \theta$
$\text{cis } \theta = \cos \theta + i \sin \theta = e^{i\theta}$	

Where; θ = Angle in radians
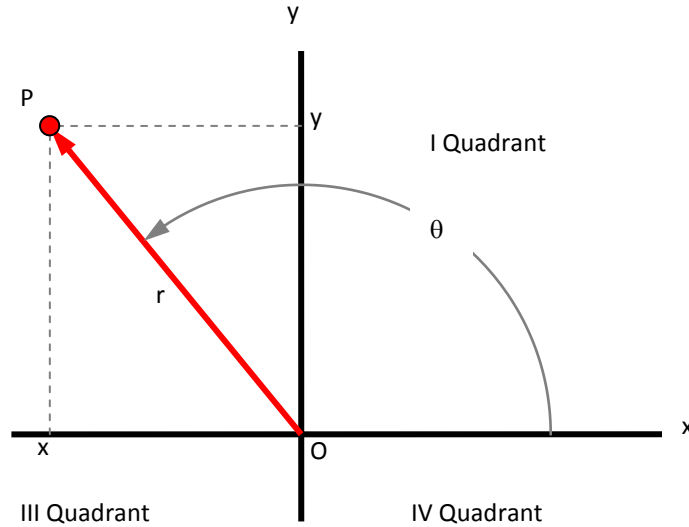
$$i = \sqrt{-1}$$

Figure E-10: Plane used to define trigonometric functions of an arbitrary angle.

E.4.3 Relationships between Sides and Angles of a Plane Triangle

Law of Sines

The following results hold for any plane triangle *ABC* with sides *a*, *b*, *c* and angles *A*, *B*, *C*, as shown in Figure E-11.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Law of Tangents

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}, \qquad \frac{a+c}{a-c} = \frac{\tan\frac{1}{2}(A+C)}{\tan\frac{1}{2}(A-C)}, \qquad \frac{b+c}{b-c} = \frac{\tan\frac{1}{2}(B+C)}{\tan\frac{1}{2}(B-C)}$$

Law of a Semiperimeter

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Where *s* = ½(*a* + *b* + *c*) is the _semiperimeter_ of the triangle. Similar relations can be obained for angles *B* and *C*.

A, B, C = Angles in radians
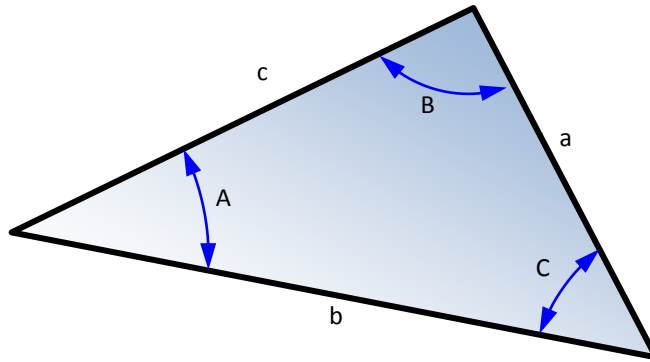a, b, c = Side lengths

Figure E-11: Plane triangle.

E.4.4 Small Angle Relations

If the angle ϕ is a small angle then the following approximations are particularly useful:

$$\cos \phi \approx 1$$
$$\sin \phi \approx \phi$$
$$\tan \phi \approx \phi$$

Where; ϕ = Angle in radians

EXAMPLE E-5:

Determine the error for the above relations for a 5° angle (i.e. 0.087266 radians).

$$\cos\left(5° \frac{\pi}{180}\right) \approx 0.996195 \quad \Rightarrow \quad \Delta = 1 - 0.996195 \approx 0.39\%$$

$$\sin\left(5° \frac{\pi}{180}\right) \approx 0.087156 \quad \Rightarrow \quad \Delta = 0.087266 - 0.087156 = 0.000111 \approx 0.13\%$$

$$\tan\left(5° \frac{\pi}{180}\right) \approx 0.087489 \quad \Rightarrow \quad \Delta = 0.087266 - 0.087489 = -0.000222 \approx -0.16\%$$

E.4.5 Angle-Radians Conversions

$$\theta_{degrees} = \theta_{radians} \frac{180}{\pi}$$

$$\theta_{radians} = \theta_{degrees} \frac{\pi}{180}$$

$$\theta_{grad} = \theta_{radians} \frac{200}{\pi}$$

$$\theta_{radians} = \theta_{grad} \frac{\pi}{200}$$

$$\theta_{degrees} = 0.9 \theta_{grad}$$

$$\theta_{grad} = \frac{\theta_{degrees}}{0.9}$$

E.4.6 Relations among Trigonometric Functions

Function of sin x

$$\sin x = \frac{1}{\cos x} \qquad \sin x = \pm\sqrt{1 - \cos^2 x} \qquad \begin{aligned} \sin x &= \cos(90° - x) \\ &= \sin(180° - x) \end{aligned}$$

Function of cos x

$$\cos x = \frac{1}{\sec x} \qquad \begin{aligned} \cos x &= \sin(90° - x) \\ &= -\cos(180° - x) \end{aligned} \qquad \cos x = \pm\sqrt{1 - \sin^2 x}$$

Functions of tan x

$$\tan x = \frac{1}{\cot x} = \frac{\sin x}{\cos x} \qquad \tan x = \pm\sqrt{\sec^2 x - 1} \qquad \begin{aligned} \tan x &= \cot(90° - x) \\ &= -\tan(180° - x) \end{aligned}$$

Functions of cot x

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x} \qquad \cot x = \pm\sqrt{\csc^2 x - 1} \qquad \begin{aligned} \cot x &= \tan(90° - x) \\ &= -\cot(180° - x) \end{aligned}$$

Functions of csc x

$$\csc x = \cot \frac{x}{2} - \cot x \qquad \csc x = \frac{1}{\sin x} \qquad \csc x = \pm\sqrt{\cot^2 x + 1}$$

Functions of sec x

$$\sec x = \frac{1}{\cos x} \qquad \sec x = \pm\sqrt{\tan^2 x + 1}$$

Other Relations

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x = \frac{1}{\sin^2 x}$$

E.4.7 Various Arithmetic Relations for Trigonometric Functions

Reciprocal Relations

$$\sin A = \frac{1}{\csc A}, \qquad \cos A = \frac{1}{\sec A}, \qquad \tan A = \frac{1}{\cot A}$$

$$\csc A = \frac{1}{\sin A}, \qquad \sec A = \frac{1}{\cos A}, \qquad \cot A = \frac{1}{\tan A}$$

Product Relations

$$\sin A = \tan A \cos A \qquad\qquad \cot A = \cos A \csc A$$

$$\cos A = \cot A \sin A \qquad\qquad \sec A = \csc A \tan A$$

$$\tan A = \sin A \sec A \qquad\qquad \csc A = \sec A \cot A$$

Double-Angle Relations

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos^2 A = \frac{1}{2}(1+\cos^2 2A)$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}, \qquad \cot 2A = \frac{\cot^2 A - 1}{2\cot A}$$

Quotient Relations

$$\sin A = \frac{\tan A}{\sec A}, \cos A = \frac{\cot A}{\csc A}, \tan A = \frac{\sin A}{\cos A}$$

$$\csc A = \frac{\sec A}{\tan A}, \sec A = \frac{\csc A}{\cot A}, \cot A = \frac{\cos A}{\sin A}$$

Pythagorean Relations

$$\sin^2 A + \cos^2 A = 1 \qquad 1+\tan^2 A = \sec^2 A \qquad 1+\cot^2 A = \csc^2 A$$

E.4.8 Angle-Sum and Angle-Product Relations

Angle-Sum and Angle-Difference Relations

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

$$\cot(A+B) = \frac{\cot B\cot A - 1}{\cot B + \cot A}$$

$$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$$
$$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$$

Multiple-Angle Relations

$$\cos 3A = 4\cos^3 A - 3\cos A \qquad\qquad \sin 3A = 3\sin A - 4\sin^3 A$$
$$\cos 4A = 8\cos^4 A - 8\cos^2 A + 1 \qquad \sin 4A = 4\sin A\cos A - 8\sin^3 A\cos A$$
$$\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A \qquad \sin 5A = 5\sin A - 20\sin^3 A + 16\sin^5 A$$
$$\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1 \qquad \sin 6A = 32\cos^5 A\sin A - 32\cos^3 A\sin A + 6\cos A\sin A$$
$$\cos nA = 2\cos(n-1)A\cos A - \cos(n-2)A \qquad \sin nA = 2\cos(n-1)A\cos A - \sin(n-2)A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$$

$$\tan nA = \frac{\tan(n-1)A + \tan A}{1 - \tan(n-1)A \tan A}$$

Function-Product Relations

$$\sin A \sin B = \frac{1}{2}\cos(A - B) - \frac{1}{2}\cos(A + B)$$

$$\cos A \cos B = \frac{1}{2}\cos(A - B) + \frac{1}{2}\cos(A + B)$$

$$\sin A \cos B = \frac{1}{2}\sin(A + B) + \frac{1}{2}\sin(A - B)$$

$$\cos A \sin B = \frac{1}{2}\sin(A + B) - \frac{1}{2}\sin(A - B)$$

Function-Sum and Function-Difference Relations

$$\sin A + \sin B = 2\sin\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B)$$

$$\cos A + \cos B = 2\cos\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B)$$

$$\sin A - \sin B = 2\cos\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$$

$$\cos A - \cos B = -2\sin\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$$

$$\tan A + \tan B = \frac{\sin(A + B)}{\cos A \cos B}$$

$$\cot A + \cot B = \frac{\sin(A + B)}{\sin A \sin B}$$

$$\tan A - \tan B = \frac{\sin(A - B)}{\cos A \cos B}$$

$$\cot A - \cot B = \frac{\sin(B - A)}{\sin A \sin B}$$

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan\frac{1}{2}(A + B)}{\tan\frac{1}{2}(A - B)}$$

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \tan\frac{1}{2}(A + B)$$

$$\frac{\sin A + \sin B}{\cos A - \cos B} = \cot\frac{1}{2}(B - A)$$

$$\frac{\sin A - \sin B}{\cos A + \cos B} = \tan\frac{1}{2}(A - B)$$

Half-Angle Relations

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$$

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}} = \frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$$

$$\cot\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{1 - \cos A}} = \frac{1 + \cos A}{\sin A} = \frac{\sin A}{1 - \cos A}$$

E.4.9 Power and Inverse Relations for Trigonometric Functions

Power Relations

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

$$\cos^3 A = \frac{1}{4}(3\cos A + \cos 3A)$$

$$\cos^4 A = \frac{1}{8}(3 + 4\cos 2A + \cos 4A)$$

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$$

$$\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$$

$$\sin^4 A = \frac{1}{8}(3 - 4\cos 2A + \cos 4A)$$

$$\cot^2 A = \frac{1 + \cos 2A}{1 - \cos 2A}$$

Relations between Inverse Trigonometric Functions

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$$

$$\cot^{-1}(-x) = \pi - \cot^{-1} x$$

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\sin^{-1}(-x) = -\sin^{-1} x$$

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

$$\tan^{-1}(-x) = -\tan^{-1} x$$

$$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$$

$$\csc^{-1}(-x) = \csc^{-1} x$$

$$\sec^{-1}(-x) = \pi - \sec^{-1} x$$
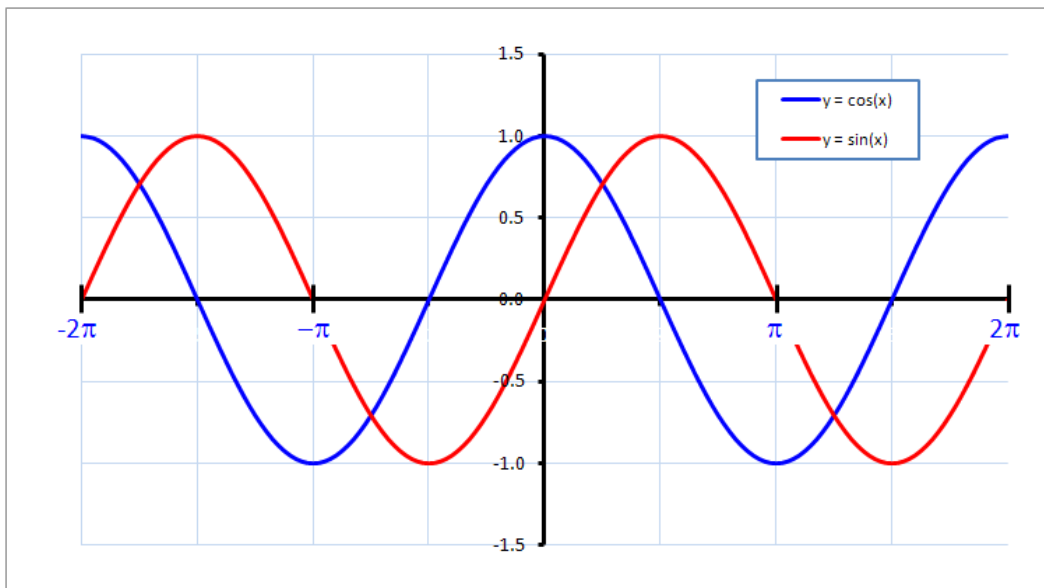
E.4.10 Plotting the Trigonometric Functions
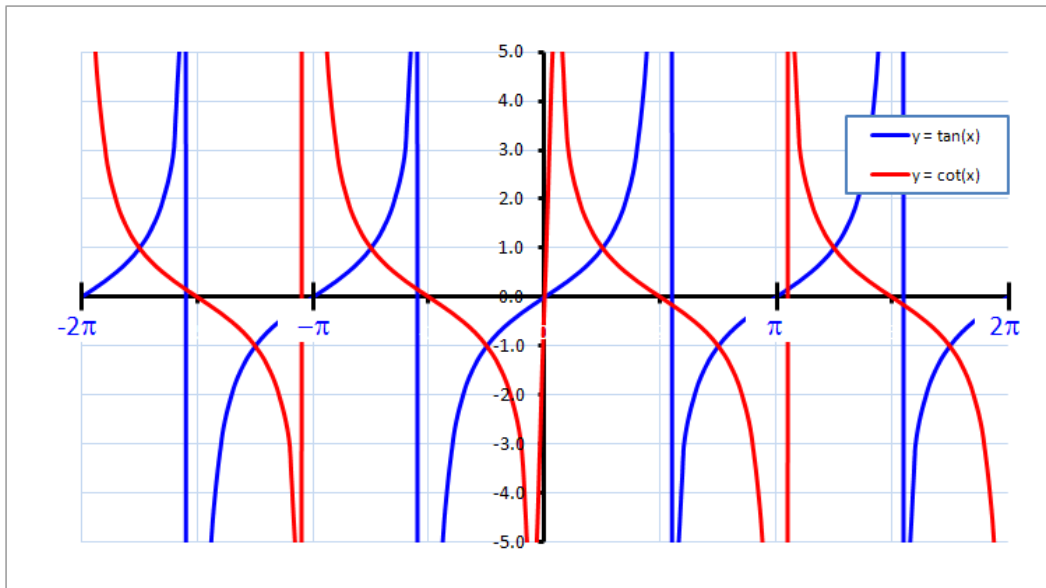

Figure E-12: Plotting sin x and cos x.
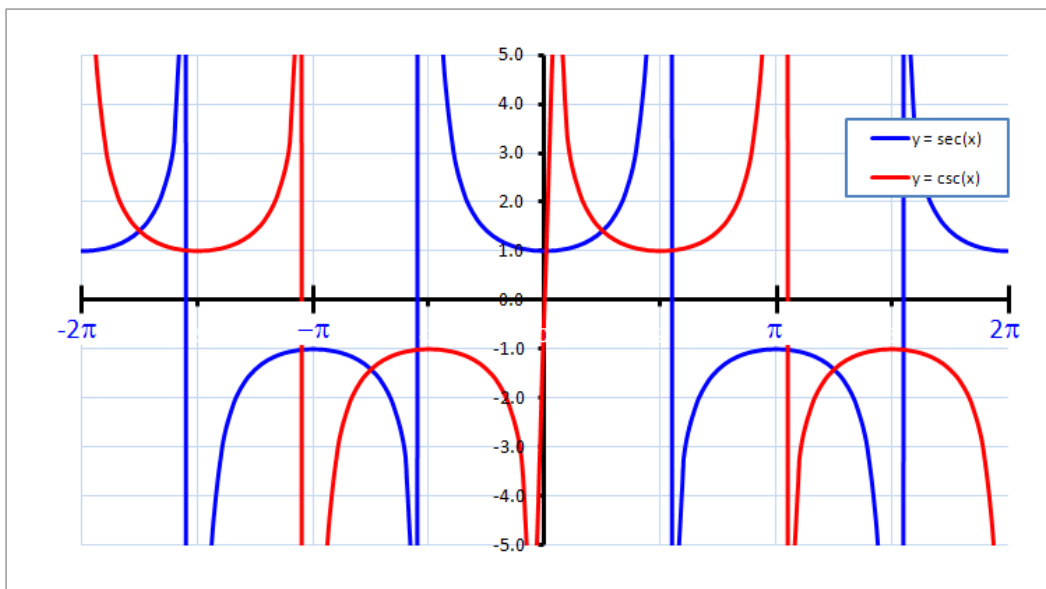
Figure E-13: Plotting tan x and cot x.

Figure E-14: Plotting csc x and sec x.
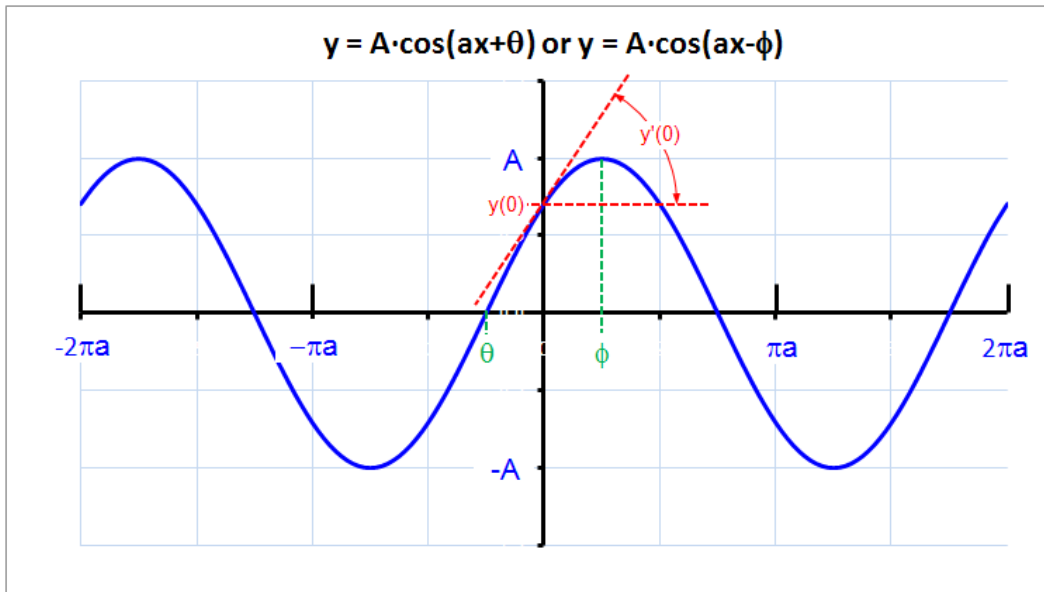
Figure E-15: General Plot of $y = A \cos(ax - \phi)$ or $y = A \cos(ax + \theta)$.

E.4.11 Great Circle, Small Circle, and Lunes

Consider the sphere in Figure E-16. A circle on a sphere having the same radius (and the same center) as the sphere is called a *great circle*. All other circles on the sphere are called *small circles*.

On a sphere of a sufficiently large radius, sufficiently small arcs of a great circle can be assumed to resemble a straight line.

Two great circles intersect in two points *A* and *B* that are the ends of a diameter of the sphere. These points, which are called *poles*, can be connected with a straight line that is bisected at the center of the sphere (see Figure E-16).

Two great circles have always a pair of diametrically opposite points in common, which divide the surface of the sphere into four *lunes*.

Figure E-16: Poles, great circles, and lunes.

The arc of a great circle going from point **P** to **Q** through an angle α in Figure E-16 has an arc length of:

$$PQ = R\alpha = R\pi \frac{a^\circ}{180^\circ}$$

where a° is the angle α in degrees. The area A of the lune **APBQA** through the angle α, is given by:
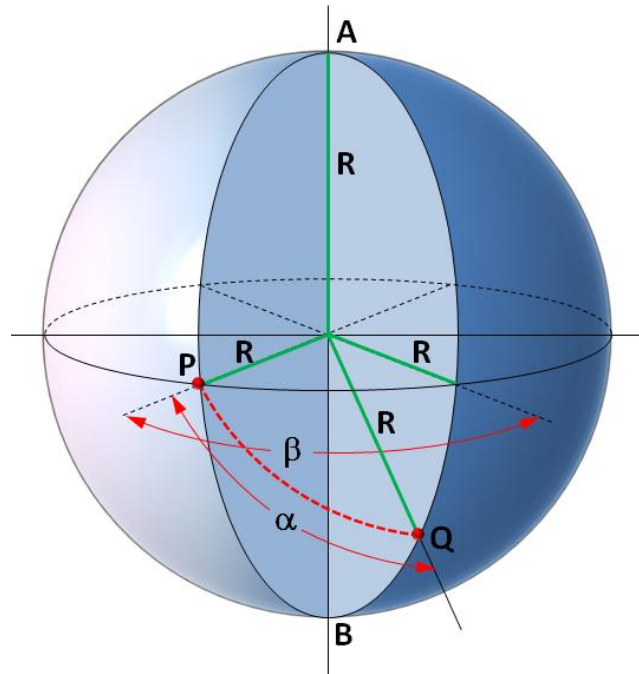
$$A = 2R^2\alpha = R^2\pi \frac{a^\circ}{90^\circ}$$

E.4.12 Spherical Trigonometric Relations

Law of Sines

Spherical triangle *ABC* is on the surface of a sphere is shown in Figure E-17. Sides *a*, *b*, and *c* (which are arcs of great circles) are measured by their angles subtended at center O of the sphere. α, β, γ are the angles opposite sides *a*, *b*, *c* respectively. Then the following results hold:

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$$
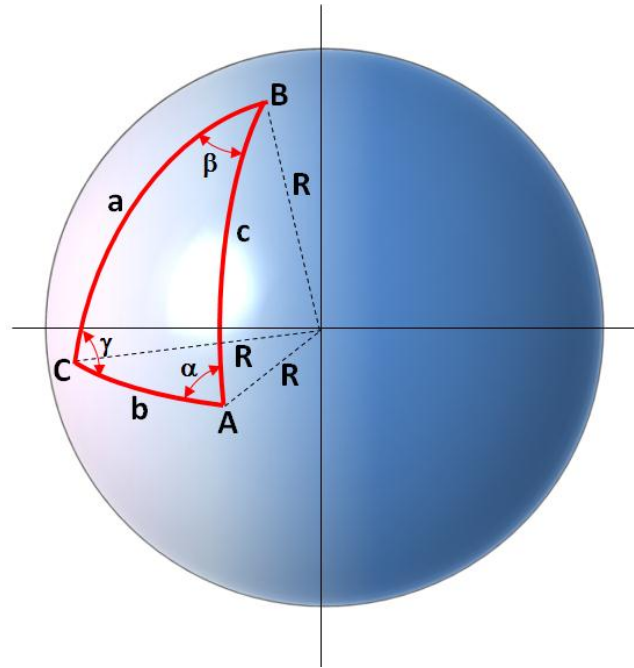
Figure E-17: A spherical triangle.

Law of Cosines for Sides

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos \beta$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$$

Law of Tangents

$$\frac{\tan\frac{1}{2}(\alpha + \beta)}{\tan\frac{1}{2}(\alpha - \beta)} = \frac{\tan\frac{1}{2}(a + b)}{\tan\frac{1}{2}(a - b)}$$

$$\frac{\tan\frac{1}{2}(\alpha + \gamma)}{\tan\frac{1}{2}(\alpha - \gamma)} = \frac{\tan\frac{1}{2}(a + c)}{\tan\frac{1}{2}(a - c)}$$

Law of Cosines for Angles

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a$$
$$\cos \beta = -\cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos b$$
$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c$$

Law of a Semiperimeter

$$\cos \frac{\alpha}{2} = \sqrt{\frac{\sin s \sin(s - c)}{\sin b \sin c}}$$

where $s = \frac{1}{2}(a + b + c)$ is the *semi-perimeter* of the triangle. Similar relations can be obtained for angles β

$$\frac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)} = \frac{\tan\frac{1}{2}(b+c)}{\tan\frac{1}{2}(b-c)}$$

and γ.

$$\cos\frac{a}{2} = \sqrt{\frac{\cos(S-\beta)\cos(S-\gamma)}{\sin\beta\sin\gamma}}$$

where $S = \frac{1}{2}(\alpha + \beta + \gamma)$. Similar relations can be obtained for side b and c.

Half-Angle and Half-Side Relations

$$\tan\frac{\alpha}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\,\sin(s-a)}}$$

$$\tan\frac{\beta}{2} = \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\,\sin(s-b)}}$$

$$\tan\frac{\gamma}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\,\sin(s-c)}}$$

where $s = \frac{1}{2}(a + b + c)$ is the *semiperimeter* of the triangle.

$$\tan\frac{a}{2} = \sqrt{\frac{-\cos S\cos(S-\alpha)}{\cos(S-\beta)\cos(S-\gamma)}}$$

$$\tan\frac{b}{2} = \sqrt{\frac{-\cos S\cos(S-\beta)}{\cos(S-\alpha)\cos(S-\gamma)}}$$

$$\tan\frac{c}{2} = \sqrt{\frac{-\cos S\cos(S-\gamma)}{\cos(S-\alpha)\cos(S-\beta)}}$$

Where; $S = \frac{1}{2}(\alpha + \beta + \gamma)$.

E.5 Analytic Geometry

E.5.1 Vector Properties of 2-D Lines

Consider the vector, **A**, in Figure E-18 extending from the point (x_a, y_a) to point (x_b, y_b) is written as follows:

$$\mathbf{A} = (x_b - x_a) \cdot \mathbf{i} + (y_b - y_a) \cdot \mathbf{j} = \begin{Bmatrix} x_b - x_a \\ y_b - y_a \end{Bmatrix}$$

Its length, L_A, is given by:

$$L_\mathbf{A} = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}$$
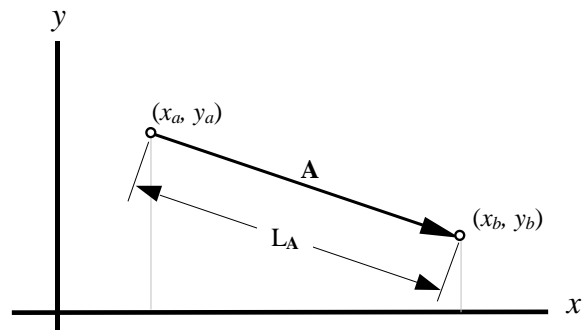
Figure E-18: Linear interpolation in 2-D space.

In polar form, the vector is written as follows:

$$\mathbf{A} = L_A \angle \theta$$

Where;

$$\theta = \tan^{-1}\left(\frac{y_b - y_a}{x_b - x_a}\right)$$

And

x_a, y_a	= coordinates of the starting point of the vector.
x_b, y_b	= coordinates of the end point of the vector.
L_A	= Length of the vector **A**.

E.5.2 Vector Properties of 3-D Lines

A vector, **A**, extending from the point (x_a, y_a, z_a) to point (x_b, y_b, z_a) is written as follows.

$$\mathbf{A} = (x_b - x_a) \cdot \mathbf{i} + (y_b - y_a) \cdot \mathbf{j} + (z_b - z_a) \cdot \mathbf{k} = \begin{Bmatrix} x_b - x_a \\ y_b - y_a \\ z_b - z_a \end{Bmatrix}$$

Its length, L_A, is given by: $\quad L_\mathbf{A} = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2}$

Where

x_a, y_a, z_a	= coordinates of the starting point of the vector.
x_b, y_b, z_b	= coordinates of the end point of the vector.
L_A	= Length of the vector **A**.

E.5.3 Linear Interpolation in 2-D

If x is known, y can be determined as follows:

$$y(x) = y_a + \left(\frac{y_b - y_a}{x_b - x_a}\right)(x - x_a) \qquad \text{(E-13)}$$

If y is known, x can be retrieved as follows:

$$x(y) = x_a + \left(\frac{x_b - x_a}{y_b - y_a}\right)(y - y_a) \qquad \text{(E-14)}$$
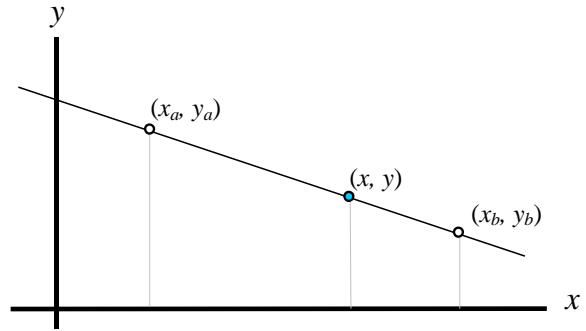
Figure E-19: Linear interpolation in 2-D space.

Where $\quad x_a, y_a$ = coordinates of one point on the line.
$\quad\quad\quad\quad x_b, y_b$ = coordinates of the other point on the line.
$\quad\quad\quad\quad x, y$ = coordinates of the unknown point on the line.

EXAMPLE E-6:
Two points on a line are given by (5, 1) and (1, 5). Find the value of y for x = 2.5. Using Equation (E-13) the solution is given by:

$$y(2.5) = 1 + \left(\frac{5-1}{1-5}\right)(2.5 - 5) = 3.5$$

E.5.4 Linear Interpolation in 3-D

If x is known, y and z can be retrieved as follows:

$$y(x) = y_a + \left(\frac{y_b - y_a}{x_b - x_a}\right)(x - x_a)$$

$$z(x) = z_a + \left(\frac{z_b - z_a}{x_b - x_a}\right)(x - x_a) \qquad \text{(E-14)}$$

If y is known, x and z can be retrieved as follows:

$$x(y) = x_a + \left(\frac{x_b - x_a}{y_b - y_a}\right)(y - y_a)$$

$$z(y) = z_a + \left(\frac{z_b - z_a}{y_b - y_a}\right)(y - y_a) \qquad \text{(E-15)}$$

If z is known, x and y can be retrieved as follows:

$$x(z) = x_a + \left(\frac{x_b - x_a}{z_b - z_a}\right)(z - z_a)$$

$$y(z) = y_a + \left(\frac{y_b - y_a}{z_b - z_a}\right)(z - z_a)$$

(E-16)

Where x_a, y_a, z_a = coordinates of one point on the line.
x_b, y_b, z_b = coordinates of the other point on the line.
x, y, z = coordinates of the unknown point on the line.
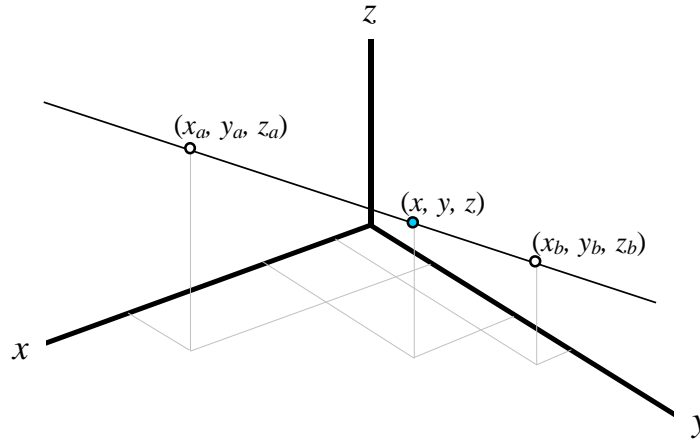
Figure E-20: Linear interpolation in 3-D space.

EXAMPLE E-7:
Two points on a line are given by (5, 1, 4) and (1, 5, 2). Find the value of y and z for x = 2.5. Using Equation (E-16) the solution is given by:

$$y(2.5) = 1 + \left(\frac{5-1}{1-5}\right)(2.5-5) = 3.5$$

$$z(2.5) = 4 + \left(\frac{2-4}{1-5}\right)(2.5-5) = 3.25$$

E.5.5 Parametric Lines

Parametric lines are extremely useful in software codes as they will not cause singularities that may cause a code to crash. For instance, when calculating the slope of a line using Equation (E-13), a singularity will occur when $x_b = x_a$. A parametric line circumvents this problem by never calculating the slope of the line. The line requires a parameter t to be calculated, which can be thought of as the fraction of the distance between two points.

$$x(t) = x_a(1 - t) + x_b t$$
$$y(t) = y_a(1 - t) + y_b t$$
$$z(t) = z_a(1 - t) + z_b t$$

(E-17)

EXAMPLE E-8:

Two points on a line are given by the initial point (5, 1, 0) and final point (1, 5, 0). Find the value of x and y for t = 0, 0.5, and 1.

SOLUTION:

Using Equation (E-17) with t = 0 we get:

$$x(t) = x_a(1-t) + x_b t = 5 \cdot (1-0) + 1 \cdot 0 = 5$$
$$y(t) = y_a(1-t) + y_b t = 1 \cdot (1-0) + 5 \cdot 0 = 1$$

This is the initial point. Using Equation (E-18) with t = 0.5 we get:

$$x(t) = x_a(1-t) + x_b t = 5 \cdot (1-0.5) + 1 \cdot 0.5 = 3$$
$$y(t) = y_a(1-t) + y_b t = 1 \cdot (1-0.5) + 5 \cdot 0.5 = 3$$

This is the midpoint on the line. Using Equation (E-18) with t = 1 we get:

$$x(t) = x_a(1-t) + x_b t = 5 \cdot (1-1) + 1 \cdot 1 = 1$$
$$y(t) = y_a(1-t) + y_b t = 1 \cdot (1-1) + 5 \cdot 1 = 5$$

This is the end point.

E.5.6 COMPUTER CODE: Intersection of Two Parametric Lines

The following code, written in Visual Basic for Applications and intended for use with Microsoft Excel will determine the Y-value of the input value, Xin, on the interval [X1, X2] onto which the linear relation [Y1, Y2] is mapped, using the parametric formulation for a 2-dimensional case.

```
Function ParametricInterpolation(X1 As Single, Y1 As Single, X2 As Single, Y2 As
Single, Xin As Single) As Single
'This function calculates the Y-value corresponding to the value of Xin, using
'a parametric formulation. It is assumed that X1, Y1 represent the lower extreme
'of the intervale (e.g. where t = 0) and X2, Y2 represent the higher extreme (t = 1).
'
'Initialize
    Dim t As Single, L As Single

'Presets
    L = X2 - X1
    If L = 0 Then    'Flag an input error
        ParametricInterpolation = -9.9E+100
        Exit Function
    End If
    t = (Xin - X1) / L

'Process
    ParametricInterpolation = Y1 * (1 - t) + Y2 * t
End Function
```

E.5.7 Intersection of Two Parametric Lines

Determining the intersection of two parametric lines is often necessary, in particular when conditions of optimization must be determined. Consider the parametric lines $A = P_1 \cdot (1-t_A) + P_2 \cdot t_A$ and $B = P_3 \cdot (1-t_B) + P_4 \cdot t_B$ in Figure E-21. The intersection of these two lines occurs at the point (x_0, y_0) the when the two parameters take the following values:

$$t_A = \frac{(x_3 - x_1)(y_3 - y_4) - (y_3 - y_1)(x_3 - x_4)}{(x_2 - x_1)(y_3 - y_4) - (x_3 - x_4)(y_2 - y_1)}$$

$$t_B = \frac{(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)}{(x_2 - x_1)(y_3 - y_4) - (x_3 - x_4)(y_2 - y_1)}$$

(E-18)
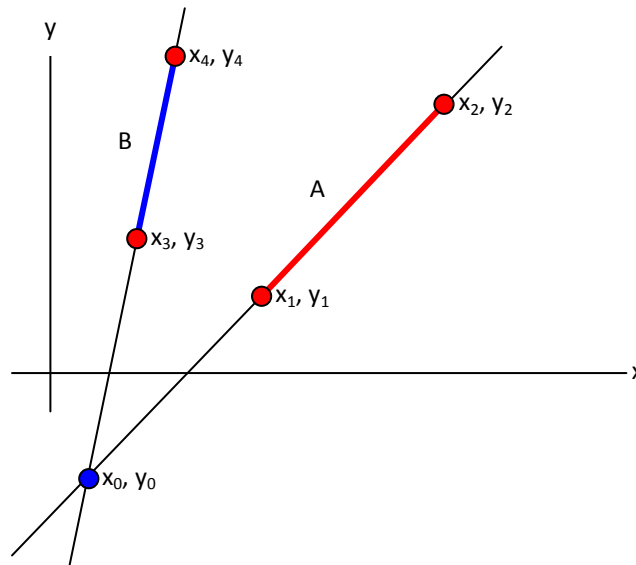
Figure E-21: Determining the point of intersection of two lines.

DERIVATION OF EQUATION (E-18):

Point on line A is given by:

$$x = x_1(1-t_A) + x_2 t_A$$
$$y = y_1(1-t_A) + y_2 t_A$$

Point on line B is given by:

$$x = x_3(1-t_B) + x_4 t_B$$
$$y = y_3(1-t_B) + y_4 t_B$$

We need to find t_A and t_B such that $x_A = x_B = x_0$ and $y_A = y_B = y_0$.

We write

$$x_0 = x_1(1-t_A) + x_2 t_A = x_3(1-t_B) + x_4 t_B$$
$$y_0 = y_1(1-t_A) + y_2 t_A = y_3(1-t_B) + y_4 t_B$$

Which can be rewritten as:

$$(x_2 - x_1)t_A + (x_3 - x_4)t_B = x_3 - x_1$$
$$(y_2 - y_1)t_A + (y_3 - y_4)t_B = y_3 - y_1$$

So now we have two equations with two unknowns, and we determine t_A and t_B, for instance using Cramer's rule or by solving directly:

$$t_A = \frac{(x_3 - x_1)(y_3 - y_4) - (y_3 - y_1)(x_3 - x_4)}{(x_2 - x_1)(y_3 - y_4) - (x_3 - x_4)(y_2 - y_1)}$$

$$t_B = \frac{(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)}{(x_2 - x_1)(y_3 - y_4) - (x_3 - x_4)(y_2 - y_1)}$$

QED

E.5.8 Quadratic Curve-Fitting

Sections E.5.3 through E.5.5 are simple linear curve fitting methods. The ability to fit curves to data is a priceless tool for the airplane designer. All sciences attempt to provide tools to determine various physical characteristics of some natural phenomenon using pure physical formulation. This can be called the *prediction approach*. However, many processes in nature are so complex and nonlinear that the only choice toward understanding is to conduct experiments and fit curves to the data. This is the *empirical approach*. This section presents a powerful and simple, albeit limited, method to fit a quadratic curve through three data points.

Consider the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) in Figure E-22. We can now fit a quadratic polynomial y = Ax² + Bx + C through the points. The coefficient A, B, and C are obtained from:

$$\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

(E-19)

DERIVATION OF EQUATION (E-19):

If the function **y = Ax² + Bx + C** goes through each of the three points, we can write for each point:

Point 1:
$$y_1 = A \cdot x_1^2 + B \cdot x_1 + C$$

Point 2:
$$y_2 = A \cdot x_2^2 + B \cdot x_2 + C$$

Point 3:
$$y_3 = A \cdot x_3^2 + B \cdot x_3 + C$$

This can be written in a matrix format as follows:

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

So it follows that the coefficients A, B, and C for the polynomial can be determined from:

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

<div align="right">**QED**</div>
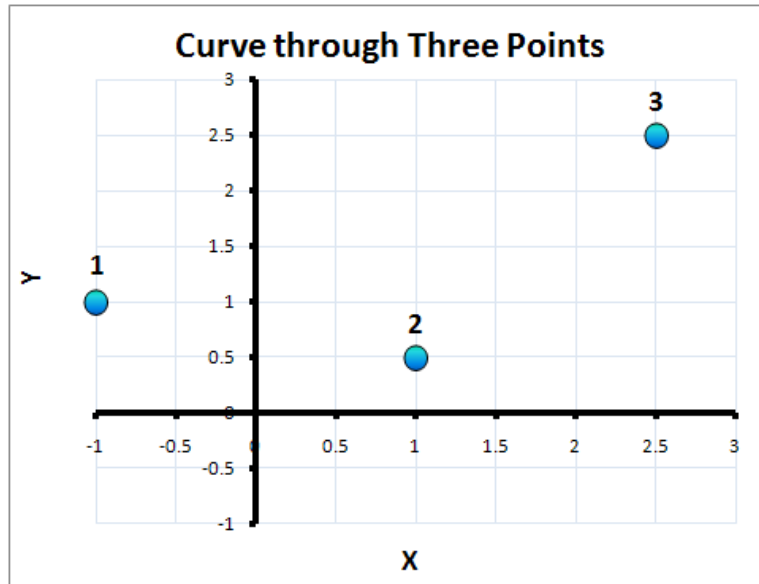
Figure E-22: Determination of a function y = Ax² + Bx + C that intersects the points 1, 2, and 3.

EXAMPLE E-9:
Determine the coefficient A, B, and C for a quadratic function going through the points shown in Figure E-22.

SOLUTION:
Begin by setting up the matrix:

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 6.25 & 2.5 & 1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.5 \\ 2.5 \end{Bmatrix}$$

Inverting the matrix yields:

$$\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{bmatrix} 0.14286 & -0.33333 & 0.19048 \\ -0.5 & 0.5 & 0 \\ 0.35714 & 0.83333 & -0.19048 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.5 \\ 2.5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.5 \\ 2.5 \end{Bmatrix}$$

This yields the following values for the constants A, B, and C:

<div align="center">A = 0.45238   B = -0.25   C = 0.29762</div>

The solution can now be plotted, showing each point is indeed intersected:
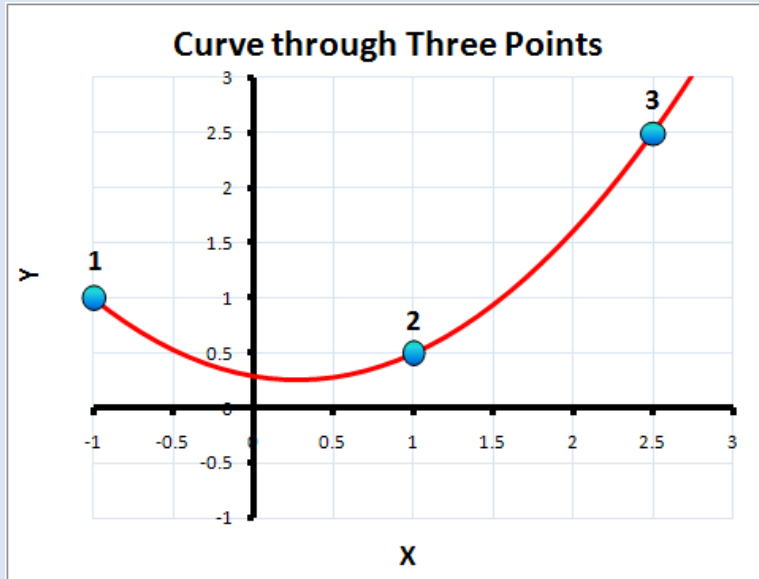
Figure E-23: The function y = Ax² + Bx + C has been defined such it intersects the points 1, 2, and 3.

E.5.9 Cubic Curve-Fitting

This section extends the method of the previous section to a curve fit through four data points. Consider the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) in Figure E-24. We can now fit a quadratic polynomial $y = Ax^2 + Bx + C$ through the points. The coefficient A, B, and C are obtained from:

$$\begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

(E-20)

DERIVATION OF EQUATION (E-20):
Identical to that of Article E.5.7, *Quadratic Curve-Fitting*, using four variables rather than three.

QED
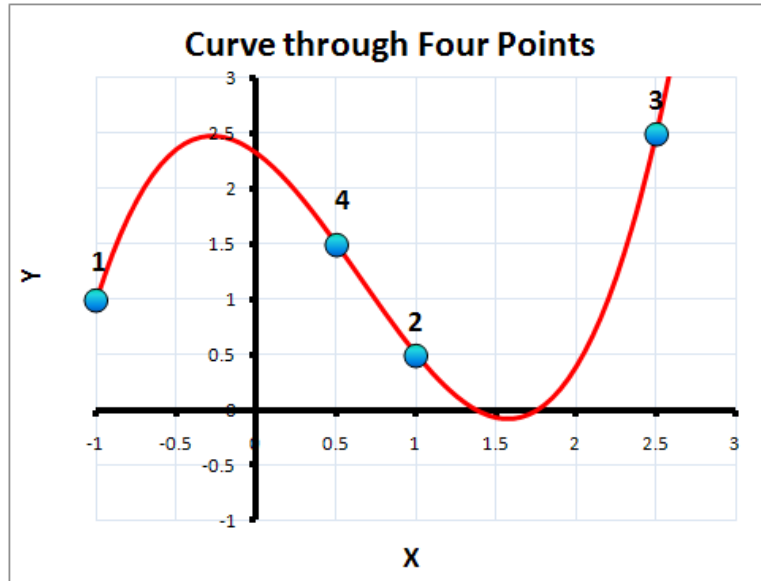
Figure E-24: Determination of a function y = Ax3 + Bx² + Cx + D that intersects the points 1, 2, 3, and 4.

E.5.10 Linear Least-Squares Curve Fitting - y = Ax + B

For a data domain of the form $(x_1, y_1), (x_2, y_2), ...(x_i, y_i), ...(x_n, y_n)$, a best fit line of the same form as equation A.8.4.1, i.e.;

$$y(x) = Ax + B$$

(E-21)

can be found such that the standard deviation (see A.9) between the y_i's and the $y(x_i)$'s found from the resulting line is minimized. Then, the Coefficients *A* and *B* are given with;

$$y(x) = Ax + B$$

(E-22)

E.5.11 STEP-BY-STEP: Rapid Interpolation of 2D Lookup Tables

Consider the sample 2D lookup table shown in Figure E-25. It is a frequent task for the practicing engineer to interpolate values in such tables using an argument that fall between the values in the table. For instance, consider the task of estimating the value for A = 4.5 and B = 6.25 in the table below. The resulting value will require a double interpolation inside the dark region. This section introduces a simple method to allow for a quick interpolation of such values, using parametric line interpolation.

A/B	1	2	3	4	5	6	7	8	9
1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
3	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
4	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
5	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
6	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9

7	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
8	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9
9	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
10	10.1	10.2	10.3	10.4	10.5	10.6	10.7	10.8	10.9
11	11.1	11.2	11.3	11.4	11.5	11.6	11.7	11.8	11.9

Figure E-25: A sample 2D lookup table.

STEP 1:
Determine the four values (h) enclosing the solution:

A/B	1	2	...	C_j	C_{j+1}	...	C_{N-2}	C_{N-1}	C_N
1	h_{11}	h_{12}							
2	h_{21}	h_{22}							
...									
R_i				h_{ij}	$h_{i\,j+1}$				
R_{i+1}				$h_{i+1\,j}$	$h_{i+1\,j+1}$				
...									
R_{N-1}								$h_{M-1\,N-1}$	$h_{M-1\,N}$
R_N								$h_{M\,N-1}$	$h_{M\,N}$

STEP 2:
Calculate the parameters s and t from:

$$s = \frac{R - R_i}{R_{i+1} - R_i} \qquad and \qquad t = \frac{C - C_j}{C_{j+1} - C_j}$$

STEP 3:
Calculate the interpolated value h:

$$h = \left[h_{ij}(1-s) + h_{(i+1)j}s \right](1-t) + \left[h_{i(j+1)}(1-s) + h_{(i+1)(j+1)}s \right]t \qquad \text{(E-22)}$$

E.5.12 COMPUTER CODE: Rapid Interpolation of 2D Lookup Tables

The following Visual Basic routine implements the 2-dimensional interpolation presented above:

```
Function MATH_Lookup2DTable(Table() As Single, RowValue As Single, ColValue As Single)
As Single
'This routine calculates the value of the table using the RowValue and ColValue,
'assuming the following setup of the array Table()
'
'      Table(i,j) = [0       ColVal1  ColVal2  ColVal3 ...   ColValn]
'                   [RowVal1  x11      x12      x13     ...   x1n    ]
'                   [RowVal2  x21      x22      x23     ...   x2n    ]
'                   ....
'                   [RowValm  xm1      xm2      xm3     ...   xmn    ]
'
'So the row values to be interpolated are the first column (Column 0) and col values
'in the top row (Row 0). IT IS ASSUMED THAT RowVals AND ColVals ARE SORTED FROM LOWEST
'TO HIGHEST.
'
```

```
'Initialize
    'Indexes
    Dim i As Integer, j As Integer
    Dim M As Integer, N As Integer
    Dim Ri As Integer, Ci As Integer
    'Parameters
    Dim t As Single, s As Single
    'Interrim values
    Dim HA As Single, HB As Single

'Presets
    M = UBound(Table, 1)     'Number of rows
    N = UBound(Table, 2)     'Number of cols
'Make sure the RowValue is inside the bounds of Table(i,j)
    If RowValue < Table(1, 0) Or RowValue > Table(M, 0) Then
        Table2DInterpolate = -1
        Exit Function
    End If

'Make sure the ColValue is inside the bounds of Table(i,j)
    If ColValue < Table(0, 1) Or ColValue > Table(0, N) Then
        Table2DInterpolate = -2
        Exit Function
    End If

'Look up the index for RowValue
    Ri = 1
    For i = 1 To M
        If RowValue <= Table(i, 0) Then
            Exit For
        End If
        Ri = i
    Next i

'Look up the index for ColValue
    Ci = 1
    For j = 1 To N
        If ColValue <= Table(0, j) Then
            Exit For
        End If
        Ci = j
    Next j

'Perform interpolation
    t = (RowValue - Table(Ri, 0)) / (Table(Ri + 1, 0) - Table(Ri, 0))
    s = (ColValue - Table(0, Ci)) / (Table(0, Ci + 1) - Table(0, Ci))
    HA = Table(Ri, Ci) * (1 - t) + Table(Ri + 1, Ci) * t
    HB = Table(Ri, Ci + 1) * (1 - t) + Table(Ri + 1, Ci + 1) * t
    Table2DInterpolate = HA * (1 - s) + HB * s
End Function
```

E.5.13 Constructing a Body of Revolution by Splicing Curves

Sometimes it is desirable to achieve greater accuracy in the estimation of surface area of a body of revolution than possible using the assembly of elementary solids (e.g. paraboloid, cylinder, cone, and frustum). This can be achieved by assembling one using splicing curves. This leads to a solid that consists of smooth segmented mathematical curves that are joined at the ends where their tangents are guaranteed to be equal. The resulting curve is thus consistent in space and first derivative, although it is also possible to extend the method to guarantee their curvature is continuous as well.

Refer to Article E.5.7, *Quadratic Curve-Fitting* for introductory comments about fitting curves through specific points. This tool can be extended by fitting curves to points that represent specific shapes of great importance to the airplane designer. In this section, polynomials are used to generate a shape that can be used to represent a fuselage (or an external fuel tank) as a body of revolution. Among advantages this offers is it allows the designer to define a parametric fuselage during space claim studies and fuselage drag analysis. Among properties that can be extracted from the method is the wetted surface area (for drag analysis) and volume of the body.
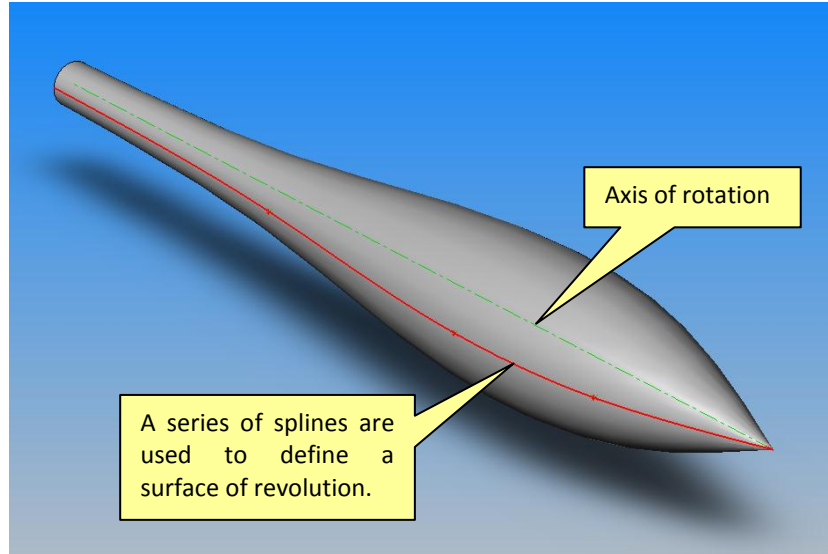
Figure E-26: A hypothetical tadpole surface constructed by splines.

Figure E-26 shows a hypothetical tadpole surface used to do initial drag studies of a powered sailplane. The reader should note that even though the final surface has a nose shape that differs such the forward part is asymmetric, the difference in volume and surface area is usually very small or even negligible.

Piecewise Splicing of Curves to Represent a Geometric Entity

Consider the geometry in Figure E-27, which represents a typical tadpole fuselage. The OML can be "constructed" by tying together a number of splines. This must be accomplished by ensuring the continuity of selected characteristics, such as y, y', and/or y''. Here, the fuselage will be constructed using three splines; one representing the nose, one representing the tailboom, and one tying those two together. These are called the *nose spline*, *tailboom spline* and *tie spline*, respectively.
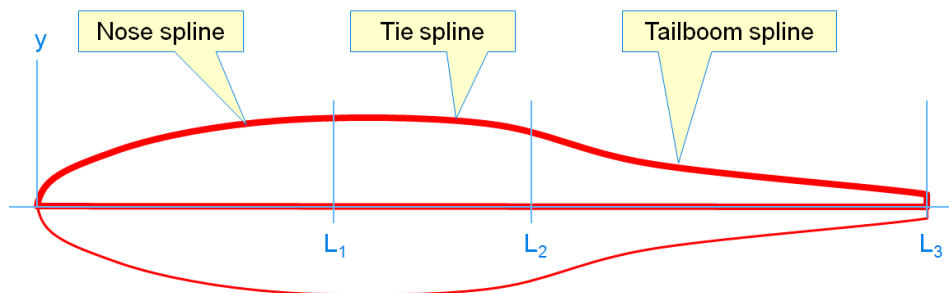
Figure E-27: An outline of the side of a fuselage formed using 3 splines.

Since this is a body of revolution it suffices to consider only one side as shown in Figure E-28. The nose spline is denoted by Y_1, the tie spline by Y_2, and the tailboom spline by Y_3. The dimensions L_1, L_2, and L_3 represent the physical location along the x-axis where they are joined and d_1, d_2, and d_3 represents the height (or width) at those points. The splines can be modeled using any smooth and differentiable curve found to suit the shape. Typically, polynomials and trigonometric curves are helpful.
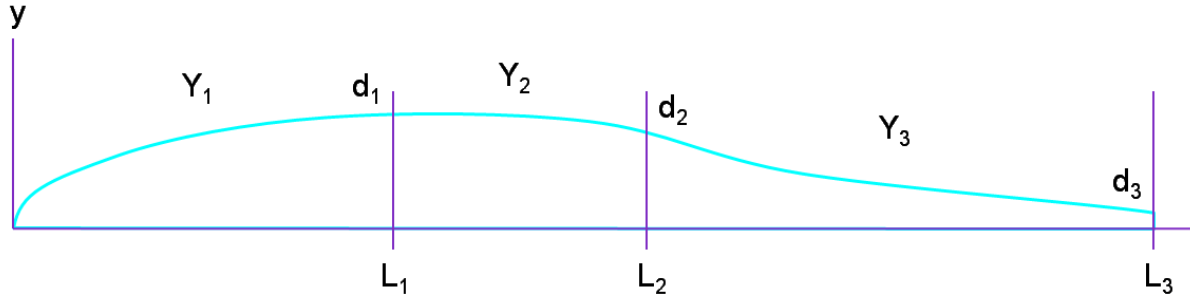
Figure E-28: Defining important parameters for the 3 splines.

Mathematically, the splines Y_1, Y_2, and Y_3, are defined on the following intervals:

$$\text{if } 0 \le x \le L_1 \quad \Rightarrow \quad y = Y_1$$
$$\text{if } L_1 < x \le L_2 \quad \Rightarrow \quad y = Y_2$$
$$\text{if } L_2 < x \le L_3 \quad \Rightarrow \quad y = Y_3$$

Their boundary conditions can be determined based on desired magnitude, slope, and curvature as shown below:

Magnitude requirement:	Slope requirement:	Curvature requirement:
at $x = 0 \Rightarrow Y_1 = 0$ at $x = L_1 \Rightarrow Y_1 = Y_2 = d_1$ at $x = L_2 \Rightarrow Y_2 = Y_3 = d_2$ at $x = L_3 \Rightarrow Y_3 = d_3$	at $x = L_1 \Rightarrow Y_1' = Y_2'$ at $x = L_2 \Rightarrow Y_2' = Y_3'$ at $x = L_3 \Rightarrow Y_3' = 0$	at $x = L_1 \Rightarrow Y_1'' = Y_2''$ at $x = L_2 \Rightarrow Y_2'' = Y_3''$

Let's define 3 splines that can be spliced to form a fuselage (note that any spline formulation can be picked, although boundary conditions will change the resulting matrix from what is shown here). Here we will define two splines for the forward fuselage to demonstrate the impact on splines 2 and 3.

Spline 1: $\qquad Y_1 = A_0 \cdot x^{2/3} \qquad \text{or} \qquad Y_1 = A_0 \cdot \sin\left(\dfrac{\pi x}{2L_1}\right)$

Spline 2: $\qquad Y_2 = A_1 \cdot x^3 + A_2 \cdot x^2 + A_3 \cdot x + A_4$

Spline 3: $\qquad Y_3 = A_5 \cdot x^2 + A_6 \cdot x + A_7$

Note that using the first of the two options given for Y_1 will be presented here. The 3 splines contain 8 unkowns (A_0 through A_7). We have to set up formulation to obtain these. To solve, let's construct an 8 x 8 matrix to accommodate the eight unknowns, as follows:

$$[k][A] = [C]$$

Where the matrices [k], [A], and [C] are defined as follows:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & & & & & \\ k_{21} & k_{22} & k_{23} & & & & & \\ k_{31} & k_{32} & k_{33} & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & k_{66} & k_{67} & k_{68} & \\ & & & k_{76} & k_{77} & k_{78} & \\ & & & k_{86} & k_{87} & k_{88} & \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \end{bmatrix}$$

Then, we will populate it with the appropriate coefficients k_{ij}, which we'll obtain by investigating the characteristics of the splines. Let's write possible rules:

At x = L₁:	$Y_1 = A_0 \cdot x^{2/3} = d_1$ $Y_2 = A_1 \cdot x^3 + A_2 \cdot x^2 + A_3 \cdot x + A_4 = d_1$ $Y_2' = Y_1' \quad \Leftrightarrow \quad -\dfrac{2}{3} A_0 \cdot x^{-1/3} + 3A_1 \cdot x^2 + 2A_2 \cdot x + A_3 = 0$ $Y_2'' = Y_1'' \quad \Leftrightarrow \quad \dfrac{2}{9} A_0 \cdot x^{-4/3} + 6A_1 \cdot x + 2A_2 = 0$
At x = L₂:	$Y_2 = d_2 \quad \Leftrightarrow \quad A_1 \cdot x^3 + A_2 \cdot x^2 + A_3 \cdot x + A_4 = d_2$ $Y_3 = d_2 \quad \Leftrightarrow \quad A_5 \cdot x^2 + A_6 \cdot x + A_7 = d_2$ $Y_2' = Y_3' \quad \Leftrightarrow \quad 3A_1 \cdot x^2 + 2A_2 \cdot x + A_3 - 2A_5 \cdot x - A_6 = 0$
At x = L₃:	$Y_3 = d_3 \quad \Leftrightarrow \quad A_5 \cdot x^2 + A_6 \cdot x + A_7 = d_3$ $Y_3' = 0 \quad \Leftrightarrow \quad 2A_5 \cdot x + A_6 = 0$

Populating the Matrix
Let's start populating the matrix by considering our "rules". For instance, we start with the 'k' matrix blank as shown below:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \end{bmatrix}$$

First, let's consider the requirement that $Y_1 = d_1$ at $x = L_1$: $\qquad Y_1(L_1) = A_0 \cdot L_1^{2/3} = d_1$

Adding this to the matrix results in (note the C-matrix as well):

$$
\begin{bmatrix}
L_1^{2/3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7
\end{bmatrix}
=
\begin{bmatrix}
d_1 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7
\end{bmatrix}
$$

Second, consider the requirement that $Y_2 = d_1$ at $x = L_1$:

$$Y_2(L_1) = A_1 \cdot L_1^3 + A_2 \cdot L_1^2 + A_3 \cdot L_1 + A_4 = d_1$$

Adding this to the matrix yields (note the C-matrix as well):

$$
\begin{bmatrix}
L_1^{2/3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_1^3 & L_1^2 & L_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7
\end{bmatrix}
=
\begin{bmatrix}
d_1 \\ d_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7
\end{bmatrix}
$$

Third, consider the requirement that $Y_2' = Y_1'$ at $x = L_1$:

$$Y_2' = Y_1' \iff 3A_1 \cdot x^2 + 2A_2 \cdot x + A_3 = \frac{2}{3} A_0 \cdot x^{-1/3}$$

$$\iff -\frac{2}{3} A_0 \cdot x^{-1/3} + 3A_1 \cdot x^2 + 2A_2 \cdot x + A_3 = 0$$

$$\Rightarrow -\frac{2}{3} A_0 \cdot L_1^{-1/3} + 3A_1 \cdot L_1^2 + 2A_2 \cdot L_1 + A_3 = 0$$

Adding this to the matrix yields (note the C-matrix as well):

$$
\begin{bmatrix}
L_1^{2/3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_1^3 & L_1^2 & L_1 & 1 & 0 & 0 & 0 \\
-\dfrac{2}{3}L_1^{-1/3} & 3L_1^2 & 2L_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7
\end{bmatrix}
=
\begin{bmatrix}
d_1 \\ d_1 \\ 0 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7
\end{bmatrix}
$$

Continuing in a similar fashion, we complete populating the matrices, eventually yielding:

$$
\begin{bmatrix}
L_1^{2/3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_1^3 & L_1^2 & L_1 & 1 & 0 & 0 & 0 \\
-\dfrac{2}{3}L_1^{-1/3} & 3L_1^2 & 2L_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2L_3 & 1 & 0 \\
0 & L_2^3 & L_2^2 & L_2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & L_2^2 & L_2 & 1 \\
0 & 3L_2^2 & 2L_2 & 1 & 0 & -2L_2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & L_3^2 & L_3 & 1
\end{bmatrix}
\begin{bmatrix}
A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7
\end{bmatrix}
=
\begin{bmatrix}
d_1 \\ d_1 \\ 0 \\ 0 \\ d_2 \\ d_2 \\ 0 \\ d_3
\end{bmatrix}
$$

The goal is to determine the coefficients A_0 through A_7, which can then be used to accurately plot the fuselage. The solution is obtained by solving:

$$[A] = [k]^{-1}[C]$$

EXAMPLE E-10: Fuselage Curve Splicing
Determine a fuselage OML with the dimensions shown in Figure E-29:
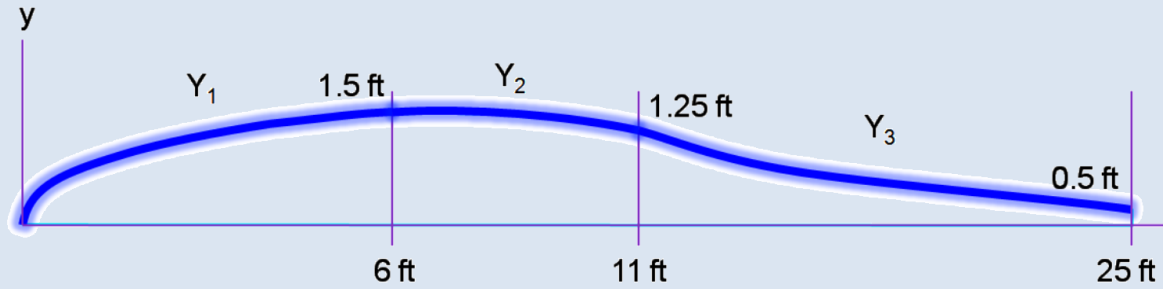

Figure E-29: Outside Mold Line of a desired fuselage shape.

Resulting matrix representation:

$$
\begin{bmatrix}
3.302 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 216 & 36 & 6 & 1 & 0 & 0 & 0 \\
-0.3669 & 108 & 12 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 50 & 1 & 0 \\
0 & 1331 & 121 & 11 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 121 & 11 & 1 \\
0 & 363 & 22 & 1 & 0 & -22 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 625 & 25 & 1
\end{bmatrix}
\begin{bmatrix}
A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7
\end{bmatrix}
=
\begin{bmatrix}
1.50 \\ 1.50 \\ 0 \\ 0 \\ 1.25 \\ 1.25 \\ 0 \\ 0.50
\end{bmatrix}
$$

Inverting the matrix [k] and multiplying with the [C] yields the coefficients for all three splines:

$$
\begin{bmatrix}
A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7
\end{bmatrix}
=
\begin{bmatrix}
0.3029 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.004444 & 0.016 & 0.04 & -0.04 & -0.016 & -0.00571 & 0.04 & 0.005714 \\
-0.1244 & -0.408 & -1.12 & 0.92 & 0.408 & 0.1314 & -0.92 & -0.1314 \\
1.124 & 3.168 & 10.12 & -6.72 & -3.168 & -0.96 & 6.72 & 0.96 \\
-3.227 & -6.776 & -29.04 & 15.84 & 7.776 & 2.263 & -15.84 & -2.263 \\
0 & 0 & 0 & 0.07143 & 0 & 0.005102 & 0 & -0.0051 \\
0 & 0 & 0 & -2.571 & 0 & -0.2551 & 0 & 0.2551 \\
0 & 0 & 0 & 19.643 & 0 & 3.189 & 0 & -2.189
\end{bmatrix}
\begin{bmatrix}
1.50 \\ 1.50 \\ 0 \\ 0 \\ 1.25 \\ 1.25 \\ 0 \\ 0.50
\end{bmatrix}
$$

Yielding the following coefficients:

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{bmatrix} = \begin{bmatrix} 0.4543 \\ 0.0064 \\ -0.1901 \\ 1.7587 \\ -3.5869 \\ 0.0038 \\ -0.1913 \\ 2.8916 \end{bmatrix}$$

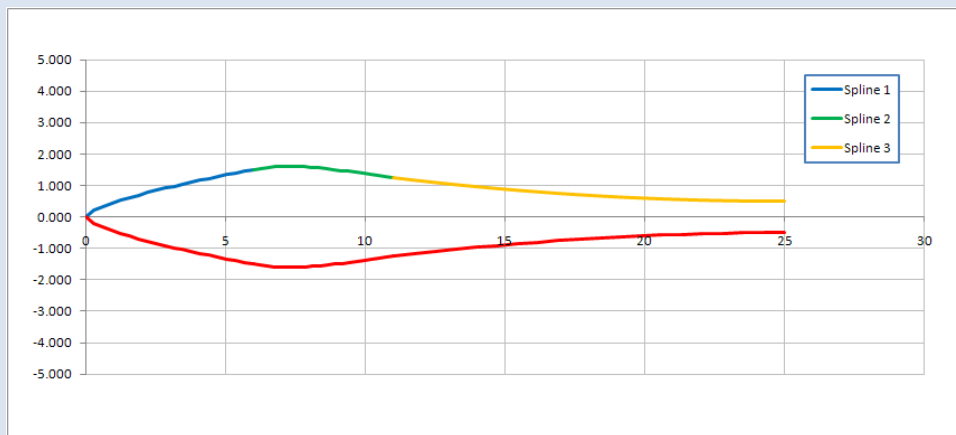Now we can plot the fuselage fully defined as shown in Figure E-30:

Figure E-30: The fuselage consisting of 3 splines.

E.6 Calculus

Consider a function $f(x)$. If f is continuous and smooth on an interval $[a, b]$, we say that f is differentiable and continuous.

If $y = f(x)$ is differentiable at a, then f must also be continuous at a. On the other hand, a function may be continuous at a point but not be differentiable there (see Figure E-31).

Let f be a differentiable function, and let $f'(x)$ be its derivative.

The derivative of $f'(x)$ (if it has one) is written $f''(x)$ and is called the second derivative of f. Similarly, the derivative of a second derivative, if it exists, is written $f'''(x)$ and is called the third derivative of f.

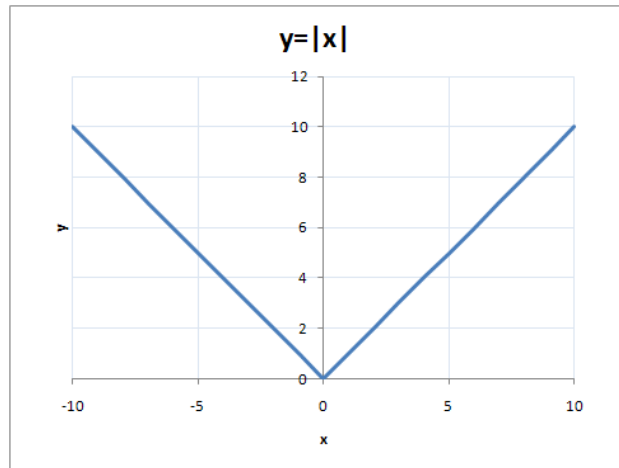These repeated derivatives are called higher-order derivatives.

Figure E-31: An example of a continuous curve, non-differentiable at x=0.

Linearity:

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}$$

Product Rule:

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

Reciprocal Rule:

$$\frac{d}{dx}\left(\frac{1}{f}\right) = -\frac{1}{f^2}\frac{df}{dx} \qquad f \neq 0$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2} \qquad g \neq 0$$

Chain Rule:

$$\frac{d}{dx}f(g(x)) = \frac{df}{dg}\left(\frac{dg}{dx}\right)$$

Derivative of an Inverse Function:
$$\frac{d}{dx} f^{-1} = \frac{1}{df\left(f^{-1}\right)/dx}$$

Generalize Power Rule:
$$\frac{d}{dx} f^{g} = f^{g}\left(\frac{dg}{dx}\ln f + \frac{g}{f}\frac{df}{dx}\right)$$

Derivative of an Implicit[2] Function:
$$y'_{x} = -\frac{F'_{x}}{F'_{y}} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Derivative of a Parametric Function[3]:
$$y'_{x} = \frac{g'\left(f^{-1}(x)\right)}{f'\left(f^{-1}(x)\right)}$$

E.6.3 Derivatives of Simple Functions

$$\frac{dc}{dx} = 0 \qquad c \text{ is a constant}$$

$$\frac{dx}{dx} = 1$$

$$\frac{d(cx)}{dx} = c \qquad c \text{ is a constant}$$

$$\frac{d|x|}{dx} = \frac{x}{|x|} = \operatorname{sgn} x \qquad x \neq 0$$

$$\frac{d(x^{n})}{dx} = nx^{n-1} \qquad \text{where both } x^{n} \text{ and } nx^{n-1} \text{ are defined}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = -x^{-2} = -\frac{1}{x^{2}}$$

$$\frac{d}{dx}\left(\frac{1}{x^{n}}\right) = \frac{d}{dx}\left(x^{-n}\right) = -nx^{-(n+1)} = -\frac{n}{x^{n+1}}$$

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \tfrac{1}{2}x^{-\frac{1}{2}} = -\frac{1}{2\sqrt{x}} \qquad x > 0$$

$$\frac{d}{dx}\left(e^{x}\right) = e^{x}$$

$$\frac{d}{dx}\left(a^{x}\right) = a^{x}\ln a$$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\log_{a} x\right) = \frac{1}{x\ln a}$$

---

[2] Here the implicit function y(x) is defined as F(x,y(x))=0.
[3] Here the function y(x) is defined parametrically as x = f(t) and y = g(t), where t is some parameter.

E.6.4 Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x \qquad\qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad\qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad\qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

E.6.5 Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad\qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \qquad\qquad \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2} \qquad\qquad \frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$$

E.6.6 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}\sinh x = \cosh x \qquad\qquad \frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = \sec h^2 x \qquad\qquad \frac{d}{dx}\sec hx = -\sec hx \tanh x$$

$$\frac{d}{dx}\coth x = -\csc h^2 x \qquad\qquad \frac{d}{dx}\csc hx = -\csc hx \coth x$$

E.6.7 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}} \qquad\qquad \frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2} \qquad\qquad \frac{d}{dx}\sec h^{-1} x = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\coth x = \frac{1}{1-x^2} \qquad\qquad \frac{d}{dx}\csc h^{-1} x = -\frac{1}{|x|\sqrt{1+x^2}}$$

E.6.8 Numerical Derivatives of Functions of the Form f(x)

Assume *f(x)* to be a continuous and differentiable on an interval [a, b]. Let h be a small change in *x*. Then, the derivatives of this function can be calculated using the following expressions:

First Order Derivatives

Newton's difference quotient:
$$\frac{df}{dx} = f' = \frac{f(x+h)-f(x)}{h}$$

Three point method:
$$\frac{df}{dx} = f' = \frac{f(x+h)-f(x-h)}{2h}$$

Second Order Derivatives

Three point method:
$$\frac{d^2 f}{dx^2} = f'' = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

Third Order Derivatives

Five point method:
$$\frac{d^3 f}{dx^3} = f''' = \frac{f(x+2h)-2f(x+h)+2f(x-h)-f(x-2h)}{2h^3}$$

Fourth Order Derivatives

Five point method:
$$\frac{d^4 f}{dx^4} = f^{(4)} = \frac{f(x+2h)-4f(x+h)+6f(x)-4f(x-h)+f(x-2h)}{h^4}$$

EXAMPLE E-11:

Compare the numerical and exact value for the first four derivatives of $f(x) = x^4 - 2x^2$ at x = 1.

SOLUTION:

Let h = 0.01. This way we begin by computing the following values:

$f(x-2h) = f(0.98) = (0.98)^4 - 2(0.98)^2 = -0.99843184$
$f(x-h) = f(0.99) = (0.99)^4 - 2(0.99)^2 = -0.99960399$
$f(x) = f(1) = (1)^4 - 2(1)^2 = -1$
$f(x+h) = f(1.01) = (1.01)^4 - 2(1.01)^2 = -0.99959599$
$f(x+2h) = f(1.02) = (1.02)^4 - 2(1.02)^2 = -0.99836784$

Inserting these into the appropriate formulas leads to:

Exact	Numerical
$f' = 4x^3 - 4x = 4(1)^3 - 4(1) = 0$	$f' = \dfrac{f(1.01)-f(0.99)}{2(0.01)} = 0.0004$
$f'' = 12x^2 - 4 = 12(1)^3 - 4 = 8$	$f'' = \dfrac{f(1.01)-2f(1)+f(0.99)}{(0.01)^2} = 8.0002$

$f''' = 24x = 24(1) = 24$	$f''' = \dfrac{f(1.02) - 2f(1.01) + 2f(0.99) - f(0.98)}{2(0.01)^3} = 24.0000$
$f^{(4)} = 24$	$f^{(4)} = \dfrac{f(1.02) - 4f(1.01) + 6f(1) - 4f(0.99) + f(0.98)}{(0.01)^4} = 24.0000$

E.6.9 Numerical Derivatives of a Point Set of the Form (x_i, y_i)

Often the engineer has to determine the derivative of a set of discrete points rather than a continuous differentiable function. An example of such a differentiation is the analysis of the quality of geometry representing an airfoil. Taylor series expansion can be used to evaluate such derivatives numerically. The Taylor series expansion of a function f(x) is given by:

$$f(x) = f(x_0) + \sum_{k=1}^{\infty} \frac{(x - x_0)^k}{k!} f^{(k)}(x_0)$$
(E-23)

Assume {x_i, y_i} to be a list of coordinates on an interval [a, b]. The above expression is used to derive the first derivative is given by:

$$y' = \frac{y_{i-1} - y_{i+1}}{x_{i+1} - x_{i-1}}$$
(E-24)

The second order derivative can be found from:

$$y'' = \frac{4(y_{i-1} - 2y_i + y_{i+1})}{(x_{i+1} - x_{i-1})^2}$$
(E-25)

This scheme is called a finite difference scheme.

E.6.10 Curvature for a Parametric Formulation

For a *plane parametric curve* c(t) = (x(t), y(t)) the curvature is given by:

$$\kappa = \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{3/2}}$$
(E-26)

EXAMPLE E-12:
Determine the curvature for the function c(t) = (t, t^3).

SOLUTION:

$$\left. \begin{array}{cc} x' = 1 & x'' = 0 \\ y' = 3t^2 & y'' = 6t \end{array} \right\} \quad \Rightarrow \quad \kappa = \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{3/2}} = \frac{6t}{\left(1 + 9t^4\right)^{3/2}}$$
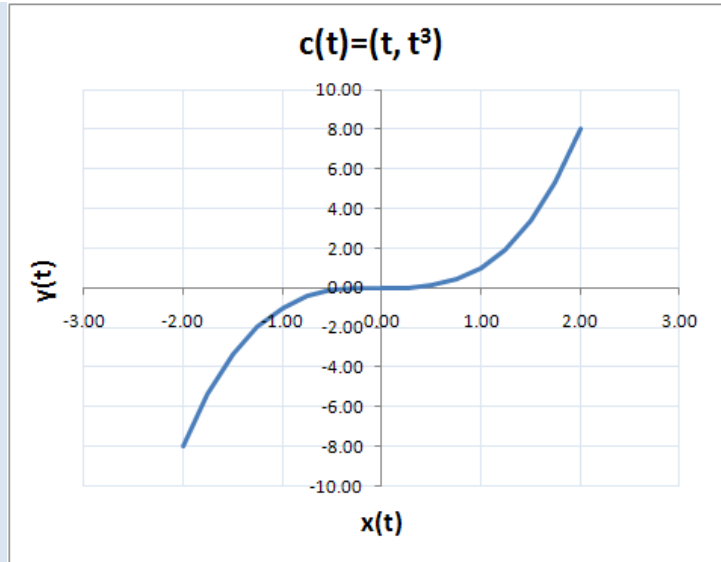
Figure E-32: A parametric representation of the function y = x³.

E.6.11 Curvature for an Explicit Formulation

For a *plane explicit curve* y = y(x) the curvature is given by:

$$\kappa = \frac{y''}{\left(1 + y'^2\right)^{3/2}}$$

(E-27)

EXAMPLE E-13:

Determine the curvature for the function y(x) = x³.

SOLUTION:

$$\left.\begin{array}{l} y' = 3x^2 \\ y'' = 6x \end{array}\right\} \quad \Rightarrow \quad \kappa = \frac{y''}{\left(1 + y'^2\right)^{3/2}} = \frac{6x}{\left(1 + 9x^4\right)^{3/2}}$$

E.6.12 Radius of Curvature

Once curvature has been determined, the *radius of curvature* can be determined from:
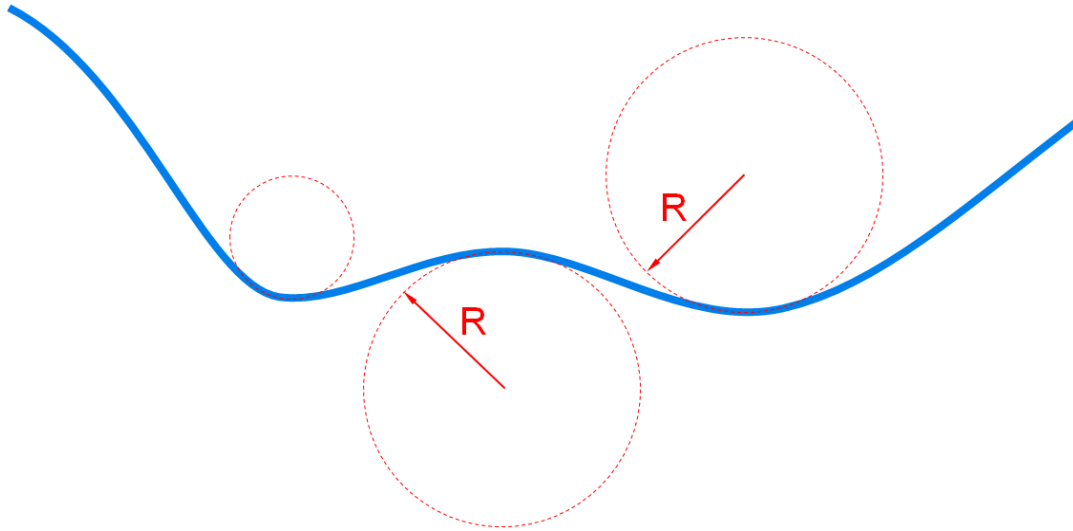
$$\kappa = \frac{1}{R}$$

(E-28)

Figure E-33: A graphic representation of radius of curvature of a function.

E.6.13 Numerical Estimation of Derivatives

Often data is not given as a smooth continuous function, but rather in a tabular format or by some other means, such as discrete sensory data. Then it is desirable to be able to determine derivatives using such format. This is accomplished by writing the derivatives using a discrete numerical format. The basics of the methodology to determine the value of a function using numerical derivatives is shown below. Assuming that a table of x values and derivatives are given, i.e. x_1, y'_1, x_2, y'_2, ..., x_n, y'_n. Then the value of the function can be calculated from:

$$y_i = y_{i-1} + y'(x_i - x_{i-1})$$

Such an expression might be utilized to convert speed into a distance (assuming x represents time). More sophisticates schemes to accomplish this exist, but ultimately, the accuracy of the estimation will depend on the quality of the derivative, y'. To show this

Here, three different numerical schemes to evaluate the derivative of a function (and ultimately curvature) will be presented, using the function $y = x^3$ evaluated on the interval [0, 3]. The schemes include a forward and backward stepping schemes and a Taylor series derivative. First, let's evaluate the actual derivative at x=0.50.

$$y'(0.5) = 3(0.5)^2 = 0.75$$

CASE 1: Forward stepping finite difference scheme

$$y'_{FWD} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{0.422 - 0.125}{0.75 - 0.50} = 1.188$$

CASE 2: Backward stepping finite difference scheme

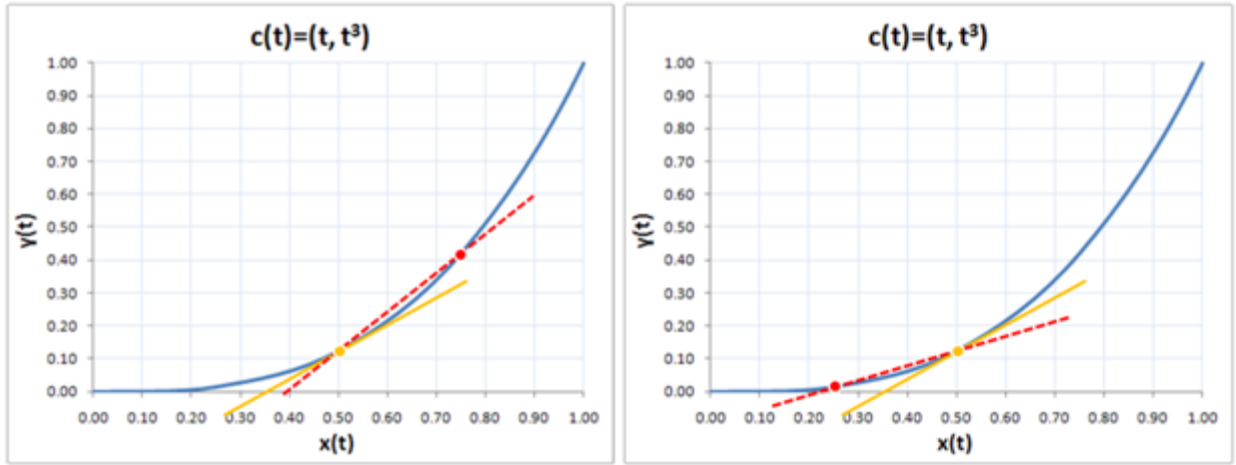$$y'_{BWD} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} = \frac{0.125 - 0.016}{0.50 - 0.25} = 0.436$$

Figure E-34: Graphical representation of forward and backward stepping finite difference schemes.

CASE 3: Taylor Series Representation of Derivatives

Let's apply a Taylor series finite difference scheme using Equation (E-24) for the first derivative at x=0.50.

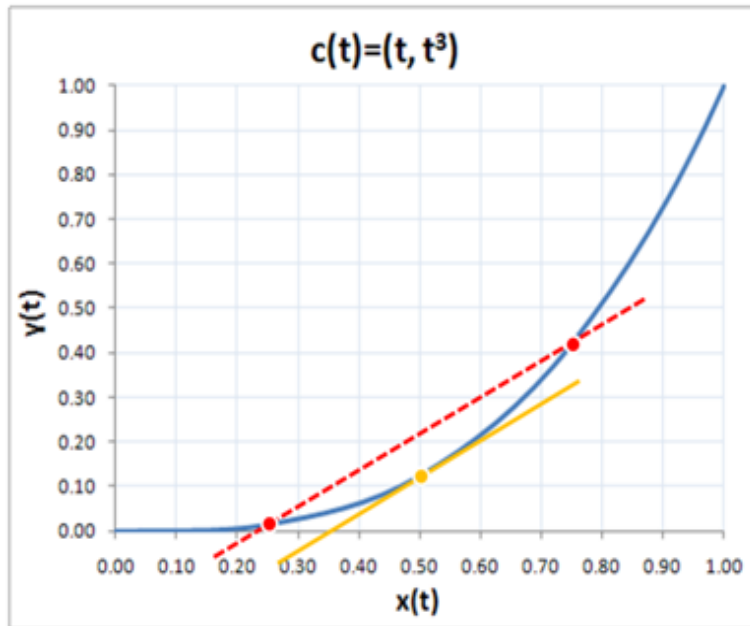$$y'_{TAYLOR} = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}} = \frac{0.422 - 0.016}{0.75 - 0.25} = 0.813$$

Figure E-35: Graphical representation of Taylor series finite difference scheme.

A table displaying the exact and approximated derivatives:

X	$Y = X^3$	$Y' = 3X^2$	$Y'' = 6X$	Y'_{FWD}	Y'_{BWD}	Y'_{TAYLOR}
0.00	0.000	0.000	0.000	0.0625		
0.25	0.016	0.188	1.500	0.4375	0.0625	0.2500
0.50	0.125	0.750	3.000	1.1875	0.4375	0.8125
0.75	0.422	1.688	4.500	2.3125	1.1875	1.7500
1.00	1.000	3.000	6.000	3.8125	2.3125	3.0625
1.25	1.953	4.688	7.500	5.6875	3.8125	4.7500
1.50	3.375	6.750	9.000	7.9375	5.6875	6.8125
1.75	5.359	9.188	10.500	10.5625	7.9375	9.2500
2.00	8.000	12.000	12.000	13.5625	10.5625	12.0625
2.25	11.391	15.188	13.500	16.9375	13.5625	15.2500
2.50	15.625	18.750	15.000	20.6875	16.9375	18.8125
2.75	20.797	22.688	16.500	24.8125	20.6875	22.7500
3.00	27.000	27.000	18.000		24.8125	

If the finite differences are very small, say $\Delta x = 0.01$, these schemes will result in the following accuracy:

0.49	0.118	0.720	2.940	0.7351	0.7057	0.7204
0.50	0.125	0.750	3.000	0.7651	0.7351	0.7501
0.51	0.133	0.780	3.060	0.7957	0.7651	0.7804

Conclusion: A Taylor series finite difference scheme is reasonably accurate and simple to implement.
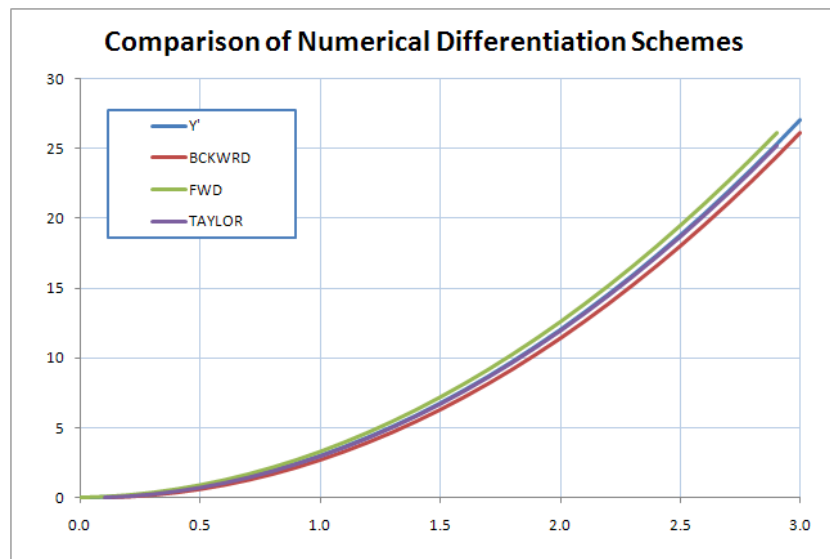

Figure E-36: Comparing the accuracy of the three finite difference schemes. The Taylor scheme is clearly the most accurate of the three.

E.6.14 Numerical Estimation of Curvature

We can now compute the curvature in a spreadsheet as follows:

$$\kappa = \frac{y''}{\left(1 + y'^2\right)^{3/2}} \qquad y' = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}} \qquad y'' = \frac{4\left(y_{i-1} - 2y_i + y_{i+1}\right)}{\left(x_{i+1} - x_{i-1}\right)^2}$$

X	$Y = X^3$	$Y' = 3X^2$	$Y'' = 6X$	κ	Y'_{TAYLOR}	Y''_{TAYLOR}	κ	R
0.00	0.000	0.000	0.000	0.00000				
0.25	0.016	0.188	1.500	1.42424	0.2500	1.500	1.36961	0.730
0.50	0.125	0.750	3.000	1.53600	0.8125	3.000	1.40248	0.713
0.75	0.422	1.688	4.500	0.59624	1.7500	4.500	0.54957	1.820
1.00	1.000	3.000	6.000	0.18974	3.0625	6.000	0.17944	5.573
1.25	1.953	4.688	7.500	0.06812	4.7500	7.500	0.06557	15.250
1.50	3.375	6.750	9.000	0.02833	6.8125	9.000	0.02757	36.271
1.75	5.359	9.188	10.500	0.01330	9.2500	10.500	0.01304	76.702
2.00	8.000	12.000	12.000	0.00687	12.0625	12.000	0.00677	147.772
2.25	11.391	15.188	13.500	0.00383	15.2500	13.500	0.00378	264.406
2.50	15.625	18.750	15.000	0.00227	18.8125	15.000	0.00224	445.745
2.75	20.797	22.688	16.500	0.00141	22.7500	16.500	0.00140	715.678
3.00	27.000	27.000	18.000	0.00091				

A comparison between the exact and numerical curvatures is plotted below. It reveals the Taylor scheme is indeed highly accurate, in spite of being applied here on a crude grid.
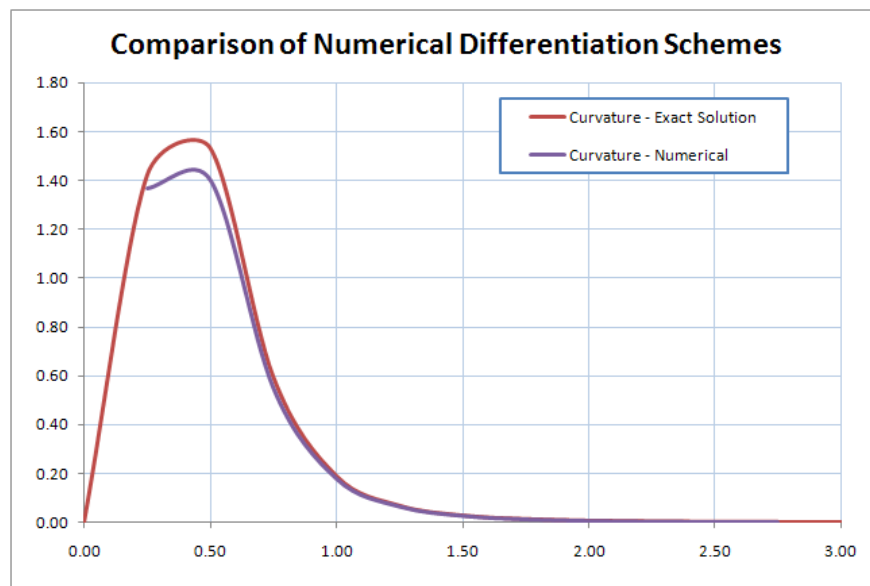

Figure E-37: Comparing the accuracy of the Taylor series finite difference schemes.

E.6.15 Numerical Partial Derivatives

Assume f(x,y) to be a continuous and differentiable on an interval [a, b] along x and [c,d] along y. Let h be a small change in x and k to be a small change in y. Then, the first order partial derivatives can be calculated as follows:

$$\frac{\partial f}{\partial x} = f_x(x, y) = \frac{f(x+h, y) - f(x-h, y)}{2h}$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \frac{f(x, y+k) - f(x, y-k)}{2k}$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}(x, y) = \frac{f(x+h, y) - 2f(x, y) + f(x-h, y)}{h^2}$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy}(x, y) = \frac{f(x, y+k) - 2f(x, y) + f(x, y-k)}{k^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}(x, y) = \frac{f(x+h, y+k) - f(x+h, y-k) - f(x-h, y+k) + f(x-h, y-k)}{4hk}$$

E.6.16 Common Integrals

$$\int a\,dx = ax$$

$$\int af(x)\,dx = a\int f(x)\,dx$$

$$\int (u \pm v \pm w \pm \cdots)\,dx = \int u\,dx \pm \int v\,dx \pm \int w\,dx \pm \cdots$$

$$\int u\,dv = uv - \int v\,du$$

$$\int f(ax)\,dx = \frac{1}{a}\int f(u)\,du$$

$$\int F\{f(x)\}\,dx = \int F(u)\frac{dx}{du}\,du = \int \frac{F(u)}{f'(x)}\,du$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1}, n \neq -1$$

$$\int \frac{du}{u} = \ln|u| = \begin{matrix} \ln u & if\ u > 0 \\ \ln(-u) & if\ u < 0 \end{matrix}$$

$$\int \ln x\,dx = x\ln|x| - x$$

$$\int e^x dx = e^x$$

$$\int e^{ax} dx = \frac{e^{ax}}{a}$$

$$\int a^x dx = \int e^{x \ln a} dx = \frac{e^{x \ln a}}{\ln a} = \frac{a^x}{\ln a}, a > 0, a \neq 1$$

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\int \tan x \, dx = \ln \sec x = -\ln \cos x$$

$$\int \cot x \, dx = \ln \sin x$$

$$\int \sec x \, dx = \ln(\sec x + \tan x) = \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$$

$$\int \csc x \, dx = \ln(\csc x - \cot x) = \ln \tan \frac{x}{2}$$

$$\int \sec^2 x \, dx = \tan x$$

$$\int \csc^2 x \, dx = -\cot x$$

$$\int \tan^2 x \, dx = \tan x - x$$

$$\int \cot^2 x \, dx = -\cot x - x$$

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{2}(x - \sin x \cos x)$$

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}(x + \sin x \cos x)$$

$$\int \sec x \tan x \, dx = \sec x$$

$$\int \csc x \cot x \, dx = -\csc x$$

$$\int \sinh x \, dx = \cosh x$$

$$\int \cosh x \, dx = \sinh x$$

$$\int \tanh x \, dx = \ln(\cosh x)$$

$$\int \coth x \, dx = \ln(\sinh x)$$

$$\int \operatorname{sech} x \, dx = \sin^{-1}(\tanh x) \quad or \quad 2\tan^{-1} e^x$$

$$\int \operatorname{csch} x \, dx = \ln(\tanh \frac{x}{2}) \quad or - \coth^{-1} e^x$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x$$

$$\int \operatorname{csch}^2 x \, dx = -\coth x$$

$$\int \tanh^2 x \, dx = x - \tanh x$$

$$\int \coth^2 x \, dx = x - \coth x$$

$$\int \sinh^2 x \, dx = \frac{\sinh 2x}{4} - \frac{x}{2} = \frac{1}{2}(\sinh x \cosh x - x)$$

$$\int \cosh^2 x \, dx = \frac{\sinh 2x}{4} + \frac{x}{2} = \frac{1}{2}(\sinh x \cosh x + x)$$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x$$

$$\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) = -\frac{1}{a}\coth^{-1}\frac{x}{a}, \text{where } x^2 > a^2$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) = \frac{1}{a}\tanh^{-1}\frac{x}{a}, \text{where } x^2 < a^2$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) \quad or \quad \sinh^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

$$\int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a}\ln\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right)$$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\ln\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right)$$

$$\int f^{(n)} g\, dx = f^{(n-1)}g - f^{(n-2)}g' + f^{(n-3)}g'' - \cdots (-1)^n \int f g^{(n)} dx$$

EXAMPLE E-14: Integration by Parts:

Integration by parts, given by the formula: $\int u\, dv = uv - \int v\, du$ is a powerful method to solve complicated integrals. Use this method to solve the integral $\int te^{-iat}dt$, where i is the imaginary operator $i^2 = -1$.

SOLUTION:
Begin by selecting which part of the integral will represent u and dv. Here this is done by placing them in parentheses:

$$\int te^{-iat}dt = \int \underbrace{(t)}_{u}\underbrace{\left(e^{-iat}dt\right)}_{dv}$$

$$u = t \qquad \Rightarrow \quad du = dt$$

Then solve for du and v:

$$dv = e^{-iat}dt \qquad \Rightarrow \quad v = \int e^{-iat}dt = -\frac{e^{-iat}}{ia}$$

Inserting this into the original integral leads to:

$$\int te^{-iat}dt = \int (t)\left(e^{-iat}dt\right) = (t)\left(-\frac{e^{-iat}}{ia}\right) - \int\left(-\frac{e^{-iat}}{ia}\right)(dt) = -\frac{te^{-iat}}{ia} + \frac{1}{ia}\int e^{-iat}dt$$

Simplify:

$$-\frac{te^{-iat}}{ia} + \frac{1}{ia}\int e^{-iat}dt = -\frac{te^{-iat}}{ia} + \left(\frac{1}{ia}\right)\frac{e^{-iat}}{-ia} = -\frac{te^{-iat}}{ia} - \frac{e^{-iat}}{(ia)^2} = \left[-\frac{t\cdot ia}{(ia)^2} - \frac{1}{(ia)^2}\right]e^{-iat} = \left(\frac{iat+1}{a^2}\right)e^{-iat}$$

Therefore:

$$\int te^{-iat}dt = \left(\frac{iat+1}{a^2}\right)e^{-iat}$$

E.6.17 STEP-BY-STEP: Numerical Integration

Following are two methods commonly used to evaluate integrals numerically; the Trapezoidal Rule and Simpson's Rule.

Trapezoidal Rule

Consider the graph in Figure E-38, whose area between x = a and x = b we want to find. The trapezoidal rule allows the area to be determined by adding up the area of a number of trapezoids, whose shape is governed by the function. The following outlines a procedure to calculate the area.
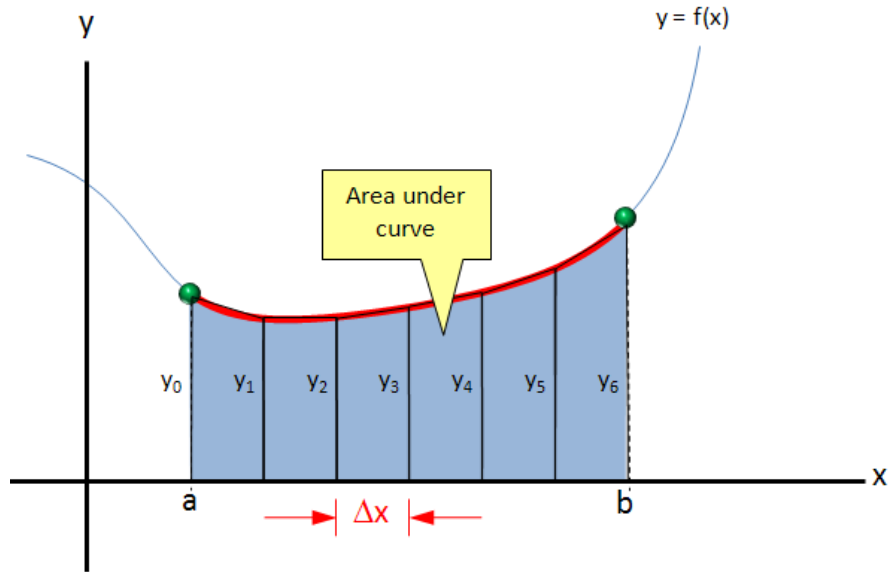

Figure E-38: The Trapezoidal Rule uses trapezoids to approximate the area.

STEP 1:

Decide the number of segments, n, to include in the approximation. The more segments the more accurate.

STEP 2:

Calculate the width of the trapezoids $\qquad \Delta x = \dfrac{b-a}{n}$

STEP 3:

Tabulate the value of f(x) as shown below.

i	x	$y_i = f(x+i\cdot\Delta x)$
0	a	$y_0 = f(a)$
1	$x+\Delta x$	$y_1 = f(x+\Delta x)$
2	$x+2\cdot\Delta x$	$y_2 = f(x+2\cdot\Delta x)$
...	...	...
n-1	$x+(n-1)\cdot\Delta x$	$y_{n-1} = f(x+(n-1)\cdot\Delta x)$
n	b	$y_n = f(b)$

STEP 4:

Compute the area using the following expression:

$$\int_a^b f(x)dx \approx \Delta x \left[\frac{y_0}{2} + y_1 + y_2 + ... + \frac{y_n}{2} \right]$$

(E-29)

Simpson's Rule
Consider the graph in Figure E-39. Simpson's rule is an improvement over the trapezoidal rule as it uses segments of a parabola to simulate the area. These approach the true shape of the curve more accurately. The following outlines a procedure to calculate the area. A drawback of the method is that the number of segments, n, must be an even number.
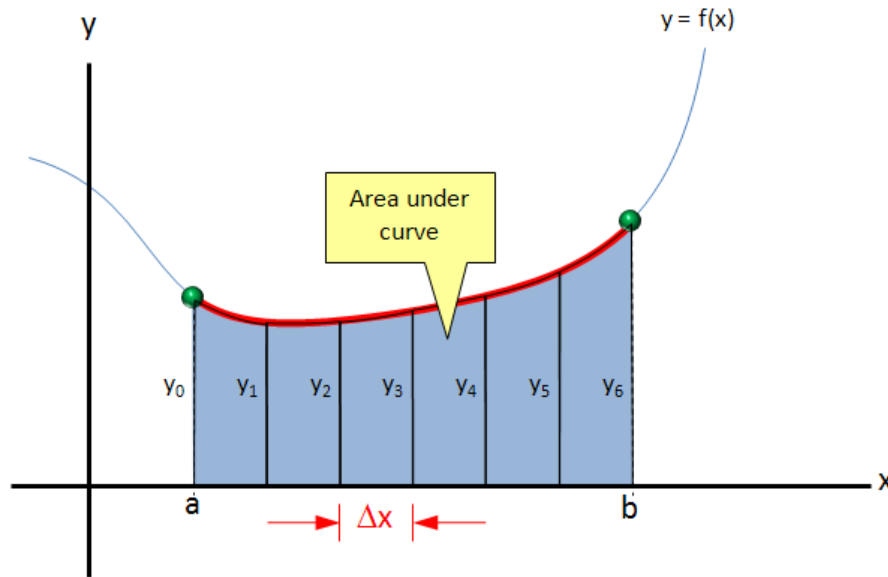

Figure E-39: Simpson's Rule uses parabolas to approximate the area.

STEP 1:
Decide the number of segments, n, to include in the approximation. The value of n must be an even number.

STEP 2:
Calculate the width of the parabolic segments $\quad \Delta x = \dfrac{b - a}{n}$

STEP 3:
Tabulate the value of *f(x)* as shown in STEP 3 for the Trapezoidal Rule.

STEP 4:
Compute the area using the following expression:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3} \left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + ... + 4y_{n-1} + y_n \right]$$

(E-30)

E.6.18 Length of a Curve

Consider the continuous function y = f(x), defined on an interval [a,b]. The length of the curve on the interval can be calculated from:

$$L = \int_a^b \sqrt{1+\left(f'\right)^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{(E-31)}$$
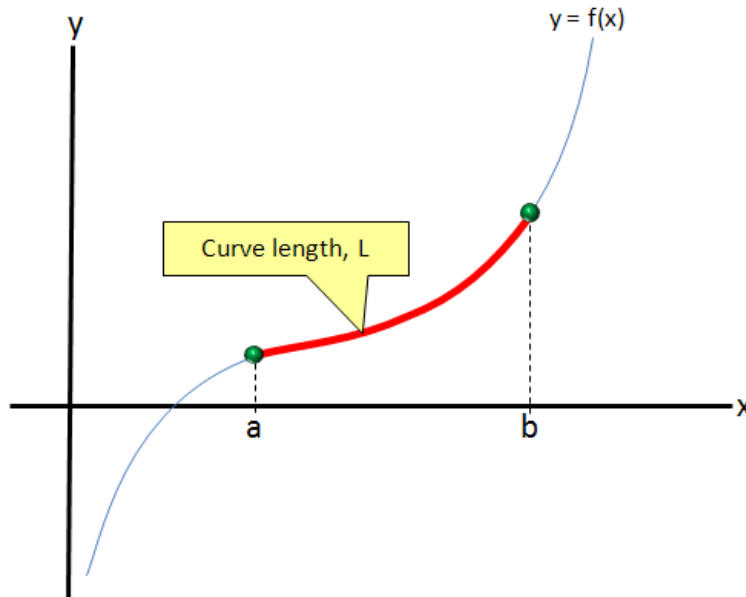

Figure E-40: Determination of the length of a function on an interval.

E.6.19 STEP-BY-STEP: Finding Roots Using the Newton-Raphson Method

It is a common task for the engineer to determine the roots of a function numerically. There are a number of algorithms that can be resorted to for this purpose. The most elementary of those will now be detailed, but their derivation is omitted: For the below methods it is assumed a function y = f(x) has been defined in the interval [a, b] and is continuous and differentiable on the interval. The method is implemented as follows:

STEP 1:
Start with an initial guess which is reasonably close to the true root. Call this value x_0.

STEP 2:
Calculate the next approximation to the root; call it x_1, using the following expression:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

STEP 3:
If $f(x_1) \neq 0$ then set $x_0 = x_1$ and go back to **STEP 1**. Repeat until desired accuracy is achieved.

EXAMPLE E-15:
Determine where the function y = 2-x² becomes zero, such that x>0, using the Newton-Raphson scheme. Of course we can easily show that the exact answer is $\sqrt{2} \approx 1.41421356...$.

SOLUTION:
The derivative for the function is given by y' = -2x.

STEP 1:   Let $x_0 = 1$.

STEP 2: $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{2 - x_0^2}{-2x_0} = 1 - \frac{2-1}{-2} = 1.5$$

STEP 3:   Let $x_0 = 1.5$.

STEP 4: $$x_1 = x_0 - \frac{2 - x_0^2}{-2x_0} = 1.5 - \frac{2 - (1.5)^2}{-2(1.5)} = 1.4166667$$

STEP 5:   Let $x_0 = 1.4166667$.

STEP 6: $$x_1 = x_0 - \frac{2 - x_0^2}{-2x_0} = 1.4166667 - \frac{2 - (1.4166667)^2}{-2(1.4166667)} = 1.414216$$

After only three iterations we have already achieved accuracy to the sixth significant digit.

E.6.20 STEP-BY-STEP: Finding Roots Using the Bisection Method

The *Bisection Method* is perhaps the simplest algorithm to determine the roots of a function. Its limitation is that it converges slowly, but its advantage is that it is inherently easy to understand. The method is implemented on an interval [a, c] as follows, assuming some continuous and smooth target function f(x) defined on the interval. It is required that the root exists on the interval.

STEP 1:
Start with an initial guess which brackets the true root. Call the left and right values of the interval a and c, respectively.

STEP 2:
Calculate the value of f at point a. Call it f(a).

STEP 3:

Calculate the midpoint on the interval: $$b = \frac{a+c}{2}$$

STEP 4:
Calculate the value of f at point b. Call it f(b).

STEP 3:
Check the signs of f(a) and f(b). Now, one of two things may happen:

    (1) If the signs are opposite, then the root is found on the interval [a,b].
    (2) If the signs are equal, then the root is found on the interval [b, c].

Redefine the interval based on (1) or (2):

(1) If the signs are opposite, then let c = b.
(2) If the signs are equal, then let a = b and f(a) = f(b).

STEP 4:
If |c-a|> tolerance go back to **STEP 3**. Repeat until desired accuracy is achieved.

E.6.21 Vector Calculus

Dot Products

Let **a** = $(a_1, a_2, ..., a_n)$ and **b** = $(b_1, b_2, ..., b_n)$ be two vectors. Then the dot product is defined as:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + ... + a_n b_n \qquad \text{(E-32)}$$

Del Operator

In vector calculus, the symbol ∇ is called the del operator (the symbol is called *nabla*) and it works as a vector operator by converting a scalar field into a vector.

Cartesian coordinates:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \qquad \text{(E-33)}$$

Spherical coordinates:

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi} \qquad \text{(E-34)}$$

Gradient

Let f be a scalar field. In Cartesian coordinate system let f = f(x,y,z) and in a spherical coordinate system f = f(r,θ, ϕ). Then the gradient of f is given by:

Cartesian coordinates:

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \qquad \text{(E-35)}$$

Spherical coordinates:

$$\nabla f(r, \theta, \phi) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi} \qquad \text{(E-36)}$$

The gradient of a product of functions, f and g is given by:

$$\nabla(fg) = f\nabla g + g\nabla f \qquad \text{(E-37)}$$

The dot product of the vectors **a** and **b** is given by:

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \qquad \text{(E-38)}$$

Divergence

Divergence indicates whether a vector field converges toward or diverges from a point in space. Let $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$ be a vector field in a Cartesian coordinate system or $\mathbf{V} = V_r\hat{r} + V_y\hat{\theta} + V_z\hat{\phi}$ in a spherical coordinate system. Then the divergence of the vector field **V** is given by:

Cartesian coordinates:

$$\text{div } \mathbf{V} = \nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x}\mathbf{i} + \frac{\partial V_y}{\partial y}\mathbf{j} + \frac{\partial V_z}{\partial z}\mathbf{k} \qquad \text{(E-39)}$$

Spherical coordinates:

$$\text{div } \mathbf{V} = \frac{\partial V_r}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}\hat{\phi} \qquad \text{(E-40)}$$

The divergence of a product of a function, f, and vector, **a**, is given by:

$$\nabla \cdot (f\mathbf{a}) = f\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla f \qquad \text{(E-41)}$$

The vector product of the vectors **a** and **b** is given by:

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b} \qquad \text{(E-42)}$$

Curl

The curl at a point describes the infinitesimal rotation of a 3D vector field at that point. Let $\mathbf{V}(x, y, z) = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$ be a vector field in a Cartesian coordinate system. The curl is defined as:

Cartesian coordinates:

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \qquad \text{(E-43)}$$

Spherical coordinates:

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{1}{r}\dfrac{\partial}{\partial \theta} & \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \phi} \\ V_r & V_\theta & V_\phi \end{vmatrix} \qquad \text{(E-44)}$$

The curl of a product of a function, f, and vector, **a**, is given by:

$$\nabla \times (f\mathbf{a}) = (\nabla f) \times \mathbf{a} + f\nabla \times \mathbf{a} \qquad \text{(E-45)}$$

The curl of a vector product of the vectors **a** and **b** is given by:

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} \qquad \text{(E-46)}$$

EXAMPLE E-16:

Determine the gradient of the function $f(x,y,z) = 2xy-x^2yz+5z^3$ and the divergence of the vector $\mathbf{V}(x,y,z) = 2xy\mathbf{i}-x^2yz\mathbf{j}+5z^3\mathbf{k}$.

SOLUTION:

The gradient is determined from:

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = \left(2y - 2xyz\right)\mathbf{i} + \left(2y - x^2 z\right)\mathbf{j} + \left(-x^2 y + 15z^2\right)\mathbf{k}$$

The divergence is determined from:

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x}\mathbf{i} + \frac{\partial V_y}{\partial y}\mathbf{j} + \frac{\partial V_z}{\partial z}\mathbf{k} = \left(2y - 2xyz\right)\mathbf{i} + \left(2y - x^2 z\right)\mathbf{j} + \left(-x^2 y + 15z^2\right)\mathbf{k}$$

E.7 Linear Algebra

E.7.1 Cramer's Rule

Cramer's Rule is a theorem of linear algebra that allows a system of linear equations to be solved, provided a unique solution exists. It is named after the Swiss mathematician Gabriel Cramer (1704–1752). Consider the system of linear equations presented in a conventional form as follows:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{Bmatrix} \tag{E-47}$$

The solution method involves the calculation of the determinants of the square matrix of coefficients [A] and that of the matrix formed using the members of the right matrix {b}. Cramer's rule states that the unknown members of the {x} array can be determined using the expression:

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})} \quad \text{for } i = 1, 2, \ldots, N \tag{E-48}$$

This rule can be used for complex numbers as well as real ones, however, it is computationally slow for large matrices compared to other solution methods. In spite of that, the rule is practical for small matrices, as it allows the solution to be presented in explicit formulas that lend themselves well for use in computer codes. Here, only two such cases will be presented; when N = 2 and N = 3.

Consider the 2x2 system:
$$\begin{array}{l} A_{11}x + A_{12}y = B_1 \\ A_{21}x + A_{22}y = B_2 \end{array} \quad \text{or} \quad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} \tag{E-49}$$

Then x and y can be found from:

$$x = \frac{\begin{vmatrix} B_1 & A_{12} \\ B_2 & A_{22} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}} = \frac{B_1 A_{22} - A_{12}B_2}{A_{11}A_{22} - A_{12}A_{21}} \quad \text{and} \quad y = \frac{\begin{vmatrix} A_{11} & B_1 \\ A_{21} & B_2 \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}} = \frac{A_{11}B_2 - B_1 A_{21}}{A_{11}A_{22} - A_{12}A_{21}} \tag{E-50}$$

Consider the 3x3 system:
$$\begin{array}{l} A_{11}x + A_{12}y + A_{13}z = B_1 \\ A_{21}x + A_{22}y + A_{23}z = B_2 \\ A_{31}x + A_{32}y + A_{33}z = B_3 \end{array} \quad \text{or} \quad \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \end{Bmatrix} \tag{E-51}$$

Then x, y, and z can be found from:

$$x = \frac{\begin{vmatrix} B_1 & A_{12} & A_{13} \\ B_2 & A_{22} & A_{23} \\ B_3 & A_{32} & A_{33} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} A_{11} & B_1 & A_{13} \\ A_{21} & B_2 & A_{23} \\ A_{31} & B_3 & A_{33} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ A_{31} & A_{32} & B_3 \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}} \qquad \text{(E-52)}$$

This way the denominator would be calculated from:

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = A_{11}\left(A_{22}A_{33} - A_{23}A_{32}\right) - A_{12}\left(A_{21}A_{33} - A_{23}A_{31}\right) + A_{13}\left(A_{21}A_{32} - A_{22}A_{31}\right) \qquad \text{(E-53)}$$

The numerators are calculated in an identical fashion.

E.7.2 COMPUTER CODE: Cramer's Rule

The following Visual Basic routine solves a 2x2 system of equations using Cramer's rule:

```
Function Cramer2x2(A11 As Single, A12 As Single, A21 As Single, A22 As Single, B1 As
Single, B2 As Single, Mode As Byte) As Single
'This routine calculates the x or y of the 2x2 matrix [A]{x}={B} using Cramer's Rule.
'Note that if Mode = 1, then the x-value is returned.
'Note that if Mode = 2, then the y-value is returned.
'
'Initialize
    Dim Denominator As Single

'Presets
    Denominator = A11 * A22 - A12 * A21

'Calculate
    If Denominator <> 0 Then
        If Mode = 1 Then        'Calculate the x
            Cramer2x2 = (B1 * A22 - A12 * B2) / Denominator
        ElseIf Mode = 2 Then    'Calculate the y
            Cramer2x2 = (A11 * B2 - B1 * A21) / Denominator
        End If
    Else
        Cramer2x2 = 0
    End If
End Function
```

The following Visual Basic routine solves a 3x3 system of equations using Cramer's rule:

```
Function Cramer3x3(A11 As Single, A12 As Single, A13 As Single, A21 As Single, A22 As Single, A23
As Single, A31 As Single, A32 As Single, A33 As Single, B1 As Single, B2 As Single, B3 As Single,
Mode As Byte) As Single
'This routine calculates the x or y of the 3x3 matrix [A]{x}={B} using Cramer's Rule.
'Note that if Mode = 1, then the x-value is returned.
'Note that if Mode = 2, then the y-value is returned.
'Note that if Mode = 3, then the z-value is returned.
'
```

```
'Initialize
    Dim Denominator As Single, Numerator As Single

'Presets
    Denominator = A11 * (A22 * A33 – A23 * A32) – A12 * (A21 * A33 – A23 * A31) + A13 * (A21 *
A32 – A22 * A31)

'Calculate
    If Denominator <> 0 Then
        If Mode = 1 Then        'Calculate the x
            Numerator = B1 * (A22 * A33 – A23 * A32) – A12 * (B2 * A33 – A23 * B3) + A13 * (B2 *
A32 – A22 * B3)
        ElseIf Mode = 2 Then    'Calculate the y
            Numerator = A11 * (B2 * A33 – A23 * B3) – B1 * (A21 * A33 – A23 * A31) + A13 * (A21 *
B3 – B2 * A31)
        ElseIf Mode = 3 Then    'Calculate the z
            Numerator = A11 * (A22 * B3 – B2 * A32) – A12 * (A21 * B3 – B2 * A31) + B1 * (A21 *
A32 – A22 * A31)
        End If
        Cramer3x3 = Numerator / Denominator
    Else
        Cramer3x3 = 0
    End If
End Function
```