Exercises 3.1

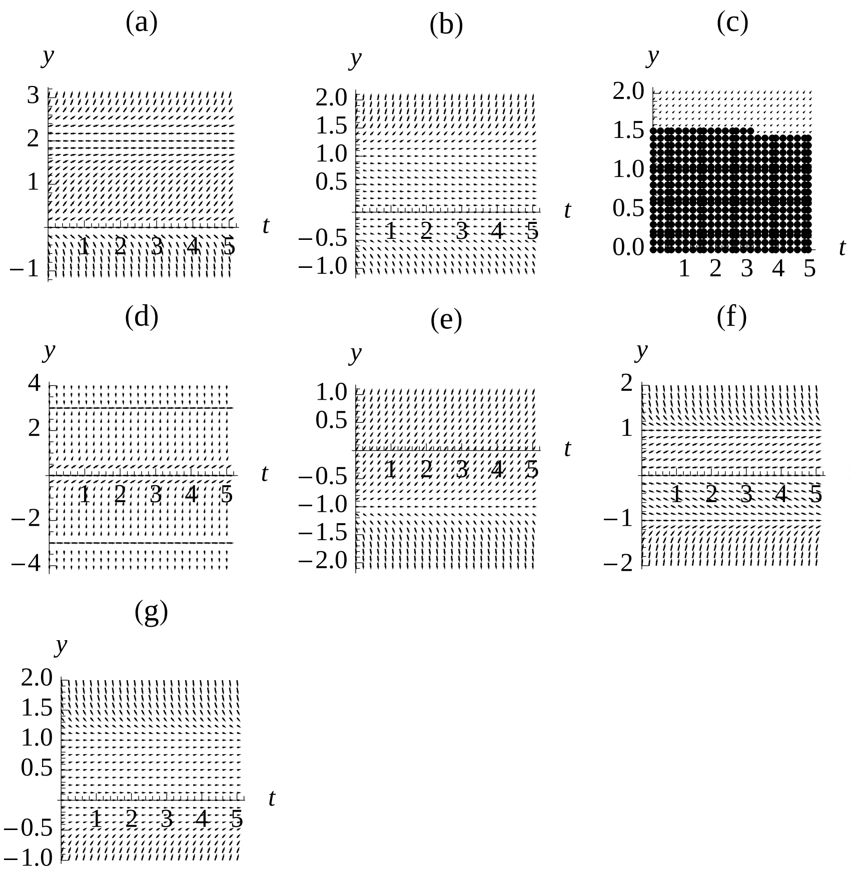
1. $y(t) = 100e^{kt}$. With $t = 3$, $y(3) = 100e^{3k} = 200$ so $e^{3k} = 2$ which means that $e^k = 2^{1/3}$ and $y(t) = 2^{t/3}$. $y(30) = 2^{10} = 1024$; while $y(t) = 2^{t/3} = 4250$ when $t = 3\ln(4250)/\ln(2) \approx 36.16$.
3. $y(t) = e^{kt}$ and $y(1) = e^k = 2/3$ so $y(t) = (3/2)^{-t}$. $y(t) = (3/2)^{-t} = 1/3$ when $t = \ln(3)/\ln(3/2) \approx 2.71$.
5. $y(1000) = 100e^{1000k} = 50$ so $e^k = 2^{-1/1000}$ and $y(t) = 100 \cdot 2^{-t/1000}$. $y(1) = 100 \cdot 2^{-1/1000} \approx 99.93$ and $y(500) \approx 70.71$.
7. $y(t) = 500e^{kt}$ and $y(6) = 500e^{6k} = 600$ so $e^k = (6/5)^{1/6}$ and $y(t) = 500 \cdot (6/5)^{t/6}$. $y(24) = 500 \cdot (6/5)^4 = 5184/5 \approx 1036.8$. $500 \cdot (6/5)^{t/6} = 1000$ when $t = 6\ln(2)/\ln(6/5) \approx 22.81$.
9. $y(5) = e^{5k} = 1/2$ so $e^k = 2^{-1/5}$ and $y(t) = 2^{-t/5}$. Then $y(t) = 2^{-t/5} = 1/6$ when $t = 5\ln(6)/\ln(2) \approx 12.92$ and $y(15) = 2^{-3} = 1/8$.
11. Let H denote the half-life of the radioactive substance. Then, $y(t) = e^{Hk} = 1/2$ gives us $y(t) = (1/2)^{t/H}$. Thus, $y(t_1) = (1/2)^{t_1/H}$ so $\ln y(t_1) = -\dfrac{t_1}{H}\ln 2$ and $y(t_2) = (1/2)^{t_2/H}$ so $\ln y(t_2) = -\dfrac{t_2}{H}\ln 2$. Subtracting these two equations and solving for H gives the result.
13. We set $y(0) = 800$. Then, $y(5) = 800e^{5k} = 560 \Rightarrow e^k = (7/10)^{1/5} \Rightarrow y(t) = 800 \cdot (7/10)^{t/5}$. To find the half-life solve $800 \cdot (7/10)^{t/5} = 400$ for t: $t = \dfrac{5\ln(1/2)}{\ln(7/10)} \approx 9.72$ days
15. The half-life of ^{226}Ra is approximately 1700 years. Then, $y(1700) = e^{1700k} = 1/2 \Rightarrow e^k = (1/2)^{1/1700}$ so $y(t) = (1/2)^{t/1700}$, $y(100) \approx 0.96$ and approximately 96% of the original amount remains.
17. We use the results of Example 2: $y(t) = 2^{-t/5730}$. Then, $t_{tool} \approx 3561.13$ years old, $t_{fossil} \approx 4222.81$ years old, $t_{fossil} - t_{t00} \approx 661.68$ years. No.
19. First, we rewrite the equation as $dy/dt - ry = -ay^2$. Now, we let $w = y^{1-2} = y^{-1}$. Then, $dw/dt = -y^{-2}dy/dt$ so $-y^2 dw/dt = dy/dt$. Then,

$$\begin{aligned}
-y^2\frac{dw}{dt} - ry &= -ay^2 \\
\frac{dw}{dt} + rw &= a \qquad \text{(Divide by } -y^2\text{)} \\
e^{rt}\frac{dw}{dt} + re^{rt}w &= ae^{rt} \qquad \text{(Multiply by the integrating factor } e^{rt}\text{)} \\
\frac{d}{dt}\left(e^{rt}w\right) &= ae^{rt} \\
e^{rt}w &= \frac{a}{r}e^{rt} + C \qquad \text{(Integrate)} \\
w &= \frac{a}{r} + Ce^{-rt} \qquad \text{(Solve for } w\text{)} \\
y &= \frac{1}{a/r + Ce^{-rt}} \qquad \text{(Solve for } y\text{)}
\end{aligned}$$

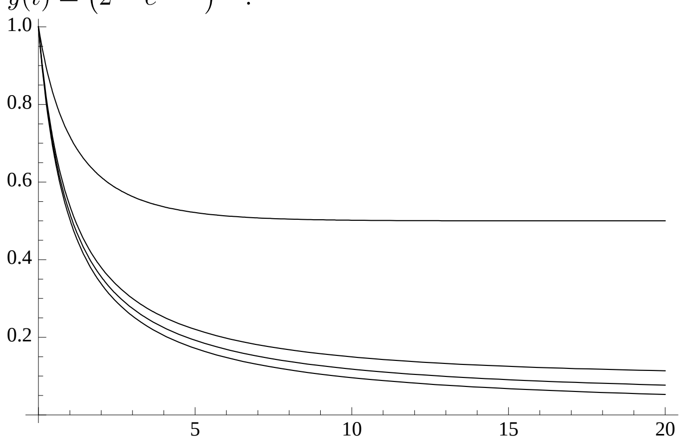
Now apply the initial condition $y(0) = y_0$ to obtain $C = (ay_0 + r)/(ry_0)$. Thus,

$$y = \frac{1}{\frac{a}{r} + \frac{ay_0 + r}{ry_0} e^{-rt}} = \frac{ry_0}{ay_0 + (ay_0 + r)e^{-rt}}.$$

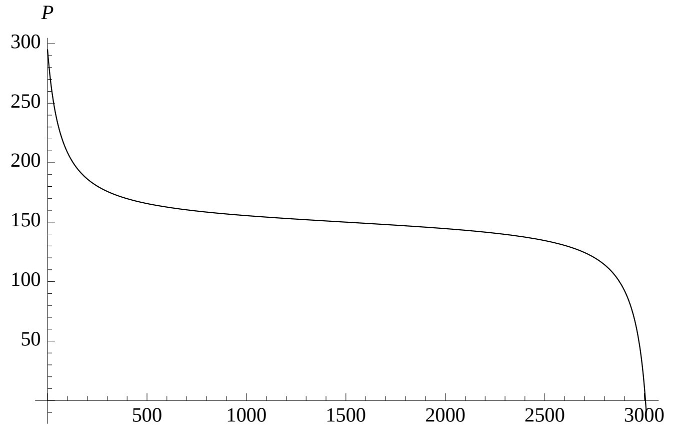
21. $y = 100000\left(1 + 9e^{-t/100}\right)^{-1}$; $y(25) \approx 1.25 \times 10^5$; $\lim_{t\to\infty} y(t) = 1{,}000{,}000$.
23. $y = 200\left(1 + 199\left(\frac{3}{199}\right)^t\right)^{-1}$; $y(2) \approx 191$; because there is no t so that $y = 200$, all students do not theoretically learn of the rumor.
25. $y(t) = 1 + 499e^{-5t}$; $y(20) \approx 1$; quickly.
27. $\lim_{m\to\infty} S_0\left(1 + k/m\right)^{mt} = S_0 e^{kt}$
29. (a) $-r\left(1 - \frac{1}{A}y\right)y = 0 \Rightarrow y = 0,\ A$; (c) $y(t) = 2\left(1 + e^{rt}\right)^{-1}$, $\lim_{t\to\infty} y(t) = 0$; (d) $y(t) = 6\left(3 - e^{rt}\right)^{-1}$, $\lim_{t\to\infty} y(t) = 0$
31. (a) $y = 2$, semistable; $y = 0$ unstable; (b) $y = 0$, unstable; $y = 1$, unstable; (c) $y = 0$, semistable; $y = 1$, unstable; (d) $y = 3$, stable; $y = 0$, semistable, $y = -3$, unstable; (e) $y = -1$, unstable; $y = 0$, semistable; (f) $y = -1$, stable; $y = 0$, unstable; $y = 1$ stable.



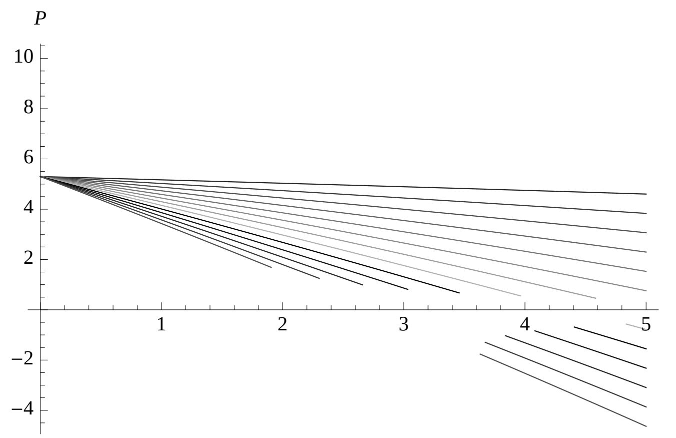
33. $y(t) = \left(100 - 99e^{-t/100}\right)^{-1}$, $y(t) = \left(20 - 19e^{-t/20}\right)^{-1}$, $y(t) = \left(10 - 9e^{-t/10}\right)^{-1}$, $y(t) = \left(2 - e^{-t/2}\right)^{-1}$.



35. (a) $P(t) = \dfrac{\sqrt{4ah - r^2}\tan\left(\frac{1}{2}\left(C\sqrt{4ah - r^2} - t\sqrt{4ah - r^2}\right)\right) + r}{2a}$. From the graph we see that $P(t) = 0$ when $t \approx 3006$



With $P(t) = 50.\sqrt{4.h - 9.}\,\tan\left(0.005\left(200.\cos^{-1}\left(-\frac{500.\sqrt{4.h-9.}}{\sqrt{1.\times 10^6 h - 156191.}}\right) - 1.\sqrt{4.h-9.}t\right.\right.$
150, h must be slightly smaller than $h \approx 0.1667$



37. $y' = (r - ay)y \Rightarrow y'' = y(ay - r)(2ay - r)$. The zeros are 0, r/a, and $r/(2a)$. Now make a sign chart to see when y'' is positive and negative to determine when y is concave up (y'' positive) and concave down (y'' negative).

39. $y(t) = \dfrac{Ay_0}{y_0 + (A - y_0)e^{rt}}$. The denominator is 0 if $t = \dfrac{1}{r}\ln\left(\dfrac{y_0}{y_0 - A}\right)$.

Exercises 3.2

1. $T = 70e^{kt} + 30$ and $T(15) = 70e^{15k} + 30 = 80$ so $70e^{15k} = 50$ and then $e^k = (5/7)^{1/15}$. Then $T(t) = 70 \cdot (5/7)^{t/15} + 30$. Then,

$$\begin{aligned} 70\left(\frac{5}{7}\right)^{t/15} + 30 &= 50 \\ \left(\frac{5}{7}\right)^{t/15} &= \frac{2}{7} \\ t &= \frac{15\ln(2/7)}{\ln(5/7)} \approx 55.85\,\mathrm{min} \end{aligned}$$

3. With $T(t) = -35e^k + 75$, we find e^k:

$$\begin{aligned} T(5) &= -35e^{5k} + 75 = 50 \\ -35e^{5k} &= -25 \\ e^{5k} &= \frac{5}{7}\,\mathrm{so}\, e^k = \left(\frac{5}{7}\right)^{1/5} \\ T(t) &= -35\left(\frac{5}{7}\right)^{t/5} + 75. \end{aligned}$$

Next, we solve $-35\left(\frac{5}{7}\right)^{t/5} + 75 = 60$ resulting in $t = \dfrac{5\ln(3/7)}{\ln(5/7)} \approx 12.59$ min.

5. Using Newton's Law of Cooling, $T(t) = (T_0 - T_s)e^{-kt} + T_s$, observe that $t = 0$ corresponds to 3:00 p.m. With this convention, $T_0 = 79$, $T(3) = 68$, and $T_s = 60$. This means that $T(t) = (79 - 60)e^{-kt} + 60 = 19(e^{-k})^t = 19e^{-kt} + 60$. Because $T(3) = 68$,

$$\begin{aligned} T(3) = 19e^{-3k} + 60 &= 68 \\ 19e^{-3k} &= 8 \\ e^{-k} &= \left(\frac{8}{19}\right)^{1/3} \end{aligned}$$

so $T(t) = 19\left(\dfrac{8}{19}\right)^{t/3} + 60$. Solving $T(t) = 19\left(\dfrac{8}{19}\right)^{t/3} + 60 = 98.6$ yields $t \approx -2.45$ hr, 12:30 p.m.

7. Using Newton's Law of Cooling, $T(t) = (T_0 - T_s)e^{-kt} + T_s$, observe that $T(t) = (90 - 70)e^{-kt} + 70 = 20e^{-kt} + 70$. Because $T(3) = 20e^{-3k} + 70 = 80$, $20e^{-3k} = 10 \Rightarrow e^{-3k} = 1/2 \Rightarrow e^{-k} = (1/2)^{1/3}$ so $T(t) = 20\,(1/2)^{(t/3)} + 70$. Evaluating at $t = 5$, $T(5) \approx 76.3^{\circ}$F

9. $T(t) = -150\left(\dfrac{2}{3}\right)^{t/30} + 300$. $T(-30) = 75^{\circ}$F

11. Using Newton's Law of Cooling, $T(t) = (T_0 - T_s)e^{-kt} + T_s = (200 -$

$68)e^{-kt} + 68$. $T(2) = 132e^{-2k} + 68 = 170 \Rightarrow 132e^{-2k} = 102 \Rightarrow e^{-2k} = 17/22$ $\Rightarrow e^{-k} = (17/22)^{1/2}$. Thus, $T(t) = 132\,(17/22)^{t/2} + 68$. Solving $T(t) = 132\,(17/22)^{t/2} + 68 = 140$ results in $t \approx 4.7$ min.

12. This problem is a great class experiment if time allows. Depending upon your assumptions, the problem can be complicated. Is your model confirmed by real data?

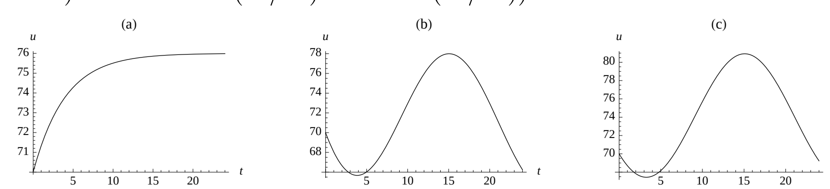
13. $u(t) = -5(9+\pi^2)^{-1}\left[-8\pi^2 - (2\pi^2+27)e^{-t/4} + 3\pi\sin(\pi t/12) + 9\cos(\pi t/12) - 72\right]$

15. $u(t) = -5(9+\pi^2)^{-1}\left(-14\pi^2 + \pi^2 e^{-t/4} + 3\pi\sin(\pi t/12) + 9\cos(\pi t/12) - 126\right)$

17. (a) Solving $y' = 8 - \frac{1}{100}y$, $y(0) = 0$ and using the integrating factor $\mu(t) = e^{\int 1/100\,dt} = e^{t/100}$, we find that $y = \frac{2}{25} + Ce^{-t/100}$. (b) First, $dV/dt = 4 - 2 = 2$ gives us $V = 2t + C$. $V_0 = 500$ so $V(t) = 2t + 500$. Now solve $y' = 4 - \dfrac{1}{t+250}y$, $y(0) = 20$ using the integrating factor $\mu(t) = e^{\int 1/(t+250)\,dt} = e^{\ln(t+250)} = t + 250$ to obtain $y = \dfrac{2t^2 + 1000t + C}{t+250}$. Apply the initial condition to obtain $\dfrac{2t^2 + 1000t + 5000}{t+250}$. (c) $V = -2t + 600$; solve $y' = 4 - \dfrac{3}{300-t}y$, $y(0) = 100$; $y = -2(t-300) + \frac{1}{54000}(t-300)^3$. If $R_1 = R_2$, the volume remains constant, so $V(t) = V_0$. If $R_1 > R_2$, V increases. If $R_1 < R_2$, V decreases.

19. Solve $\dfrac{dy}{dt} + \dfrac{3}{t+200}y = 8$, $y(0) = 10$. The integrating factor is $\mu(t) = e^{\int 3/(t+200)\,dt} = e^{3\ln(t+200)} = (t+200)^3$ and $y = 2t + 400 - 390 \cdot 200^3 (t+200)^{-3}$. $V(t) = t + 200 = 400$ when $t = 200$ and $y(200) = 605$ so the concentration at $t = 200$ is $y(200)/V(200) = 605/400 = 1.5125$.

21. (a) $u(t) = e^{-t/4}(-6 + 76e^{t/4})$; (b) $u(t) = (9+\pi^2)^{-1}(639 + 71\pi^2 + (81 - \pi^2)e^{-t/4} - 90\cos(\pi t/12) - 30\pi\sin(\pi t/12))$; (c) $u(t) = 2(9+\pi^2)^{-1}(333 + 37\pi^2 + (27 - 2\pi^2)e^{-t/4} - 45\cos(\pi t/12) - 15\pi\sin(\pi t/12))$.



23.

$$u(t) = \frac{-4032e^{-2t}u_d - 7\pi^2 e^{-2t}\text{ud} + 287424e^{-2t} + 489\pi^2 e^{-2t} - 240\pi\sin\left(\frac{\pi t}{12}\right) - 5760\cos\left(\frac{\pi t}{12}\right)}{4608 + 8\pi^2}$$

the average temperature is

$$\frac{1}{24}\int_0^{24} u(t)\,dt = \frac{7\left(1 + 47e^{48}\right)\left(576 + \pi^2\right)u_d + 9e^{48}\left(250048 + 433\pi^2\right) - 489\pi^2 - 287424}{384e^{48}\left(576 + \pi^2\right)};$$

Solving $u(t) = 70$ for u_d yields $u_d = \dfrac{3\left(95808 + 4410816e^{48} + 163\pi^2 + 7661e^{48}\pi^2\right)}{7\left(1 + 47e^{48}\right)\left(576 + \pi^2\right)} \approx 69.8273$.

Exercises 3.3

1. $m = 1$ so solve $v' = 32 - v$, $v(0) = 0$: $v(t) = 32 - 32e^{-t}$; $v(2) \approx 27.67$ ft/s
3. The mass is $m = W/g = 1/32$ so we solve $\frac{1}{32}v' = 1 - 2v$, $v(0) = 8$:

$$\begin{aligned}
\frac{dv}{dt} + 64v &= 32 \\
e^{64t}\frac{dv}{dt} + 64e^{64t}v &= 32e^{64t} \\
\frac{d}{dt}\left(e^{64t}v\right) &= 32e^{64t} \\
e^{64t}v &= \frac{1}{2}e^{64t} + C \\
v &= \frac{1}{2} + Ce^{-64t}.
\end{aligned}$$

Apply the initial condition to obtain $v(t) = \frac{1}{2}(1 + 15e^{-64t})$; $v(1) \approx 0.5$ ft/s.
5. The ball's weight is 4 ounces = 1/4 pound so its mass is found by solving $\frac{1}{4} = m \cdot 32$ giving us $m = 1/128$ slug. Now solve $v' = 32 - 8v$, $v(0) = -64$ because the ball is tossed *upward*. The integrating factor is $\mu(t) = e^{\int 8\,dt} = e^{8t}$. The solution to the ODE is $v = 4 + Ce^{-8t}$. Applying the initial condition gives us $v(t) = 4 - 68e^{-8t}$; $v(0) = 0$ when $t = \frac{1}{8}\ln 17 \approx 0.354$ s.
7. $dv/dt = 32 - v$, $v(0) = 0$ has solution $v(t) = 32 - 32e^{-t}$; $ds/dt = v(t)$, $s(0) = 0$ has solution $s(t) = 32t + 32e^{-t} - 32$; $s(4) \approx 96.59$ ft < 300 so about 203.41 ft above the ground.
9. We solve $10v' = 10 \cdot 9.8 - 10v$, $v(0) = 0$ by separating variables:

$$\begin{aligned}
\frac{dv}{dt} &= \frac{49}{5} - v \\
\frac{1}{v - \frac{49}{5}}dv &= -dt \\
\ln\left|v - \frac{49}{5}\right| &= t + C \\
v &= \frac{49}{5} + Ce^{-t}.
\end{aligned}$$

Applying the initial condition gives us $v(t) = \frac{49}{5}(1 - e^{-t}$; $\lim_{t\to\infty} v(t) = 49/5$
11. Because the mass of the object is 100 kg, the weight is $mg = 100 \cdot 9.8$. Therefore, we solve $v' = -9.8 - v/1000$, $v(0) = 100$ to obtain $v = -9800 + Ce^{-t/1000}$ and apply the initial condition resulting in $v(t) = -9800 + 9900e^{-t/1000}$; $v(t) = 0$ when $t \approx 10.152$ s; $s(t) = -9800t + 9900000(1 - e^{-t/1000})$; $s(10.152) \approx$ 506.76 m
13. $dv/dt = -g$, $v(0) = v_0$ has solution $v(t) = -gt + v_0$; $ds/dt = v(t)$, $s(0) = s_0$ has solution $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$
15. Because the object reaches its maximum height when $v = -gt + v_0 = 0$, $t = v_0/g$ and the air resistance is ignored, the object hits the ground when $t = 2v_0/g$. The velocity at this time is $v(2v_0/g) = -g(2v_0/g) + v_0 = -v_0$.

17. The solution of $10v' = 98 - cv$, $v(0) = v_0$ is $v(t) = -\frac{98}{c} + \frac{cv_0 + 98}{c} e^{-ct/10}$ with limit $\lim_{t\to\infty} v(t) = -\frac{98}{c} = -19.6$ so $c = 5$.
19. The parachutist's mass is $m = 192/32 = 6$ slugs so we solve $v' = 32 - \frac{1}{2}v^2$, $v(0) = 60$. Here, we use separation of variables:

$$\begin{aligned} \frac{1}{32 - \frac{1}{2}v^2} dv &= dt \\ \frac{1}{8}\left(\frac{1}{8+v} + \frac{1}{8-v}\right) &= dt \\ \ln|8+v| - \ln|8-v| &= 8t + C \\ \frac{8+v}{8-v} &= Ce^{8t}. \end{aligned}$$

Now we find v:

$$\begin{aligned} \frac{8+v}{8-v} &= Ce^{8t} \\ 8 + v &= 8Ce^{8t} - Cve^{8t} \\ v + Cve^{8t} &= 8Ce^{8t} - 8 \\ v &= \frac{8Ce^{8t} - 8}{1 + Ce^{8t}} \end{aligned}$$

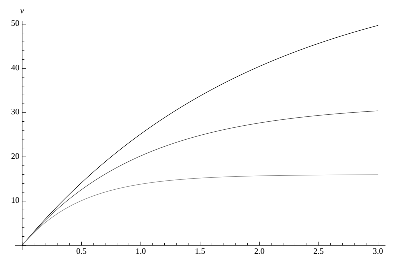
and apply the initial condition

$$v(0) = 60 \Rightarrow \frac{8C - 8}{C + 1} = 60 \Rightarrow C = -\frac{17}{13}$$

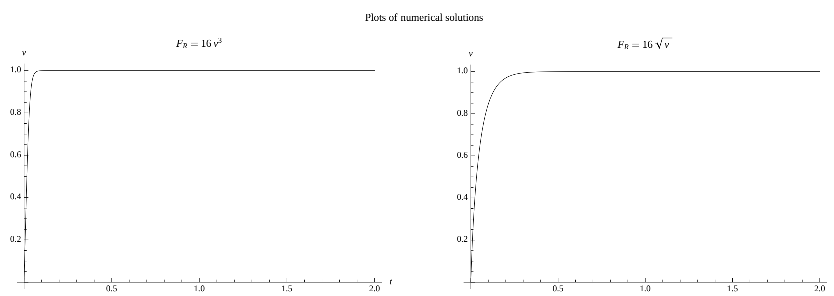
to see that $v(t) = 8\frac{17e^{8t} + 13}{17e^{8t} - 13}$. The limiting velocity is $\lim_{t\to\infty} v(t) = 8$.
21. (b) $\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\,\frac{dv}{dr}$; (c) $\lim_{t\to\infty} v^2 = {v_0}^2 - 2gR$
23. $g \approx 32\,\text{ft/s}^2 \approx 0.006\,\text{mi/s}^2$; $v_0 = \sqrt{2gR} = \sqrt{2 \cdot .165 \cdot 0.006 \cdot 1080} \approx 1.46\text{m/s}$
25. In standard form the equation is $\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{E_0}{R}$. The corresponding homogeneous equation is $\frac{dQ}{dt} + \frac{1}{RC}Q = 0$ with general solution $Q_h = C_1 e^{-t/(RC)}$. Because E_0/R is a constant and *not* a solution of the corresponding homogeneous equation, we assume that a particular solution of the nonhomogeneous equation has the form $Q_p = A$, where A is a constant to be determined. $Q_p' = 0$ and substituting into the nonhomogeneous equation gives us

$$\begin{aligned} Q_p' + \frac{1}{RC}Q_p &= \frac{E_0}{R} \\ 0 + \frac{1}{RC}A &= \frac{E_0}{R} \\ A &= CE_0 \end{aligned}$$

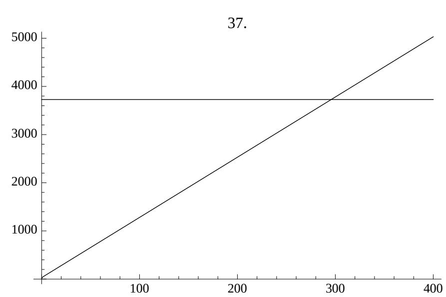
so a particular solution of the nonhomogeneous equation is $Q_p = CE_0$. Therefore, a general solution of the nonhomogeneous equation is $Q(t) = Q_p)t_+Q_h(t) = CE_0 + C_1e^{-t/(RC)}$. Applying the initial condition gives us $Q(t) = E_0C + e^{-t/(RC)}(-E_0C+Q_0)$; differentiating Q yields $I(t) = -\dfrac{1}{RC}e^{-t/(RC)}(-E_0C+Q_0)$.
27. The solution of $v' + \frac{1}{3}v = 16 - 4\sqrt{3}$, $v(0 = 0$ is $v(t) = 12(4-\sqrt{3}) - 12(4-\sqrt{3})e^{-t/3}$; $x(t) = 12(4-\sqrt{3})t + 36(4-\sqrt{3})e^{-t/3} - 36(4-\sqrt{3})$.
29. $v(t) = 32$, $\lim_{t\to\infty}(32 + Ce^{-t}) = 32$
31. $v(t) = -gm/c$, $\lim_{t\to\infty}(-gm/c + Ce^{-ct/m}) = -gm/c$
33. $c = 1/2$: $v(t) = 64e^{-t/2}(-1+e^{t/2})$; $c = 1$: $v(t) = 32e^{-t}(-1+e^{t})$; $c = 2$: $v(t) = 16e^{-2t}(-1+e^{2t})$



35.

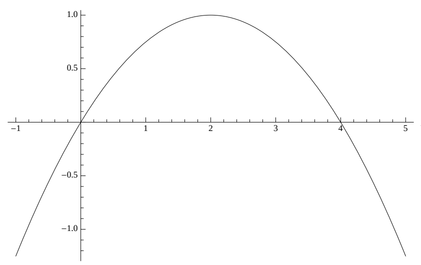
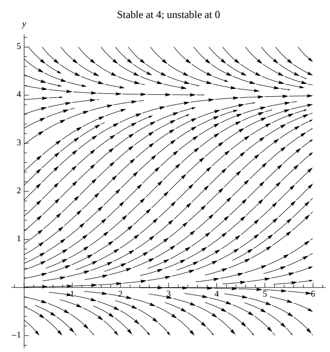


37. The woman's mass is $m = 125/32$ slugs. At $t = 5$, she falls $s_1 = 272.479$ ft. After the parachute opens, she falls approximately $4000 - 272.479300.772 = 3727.52$ ft. in approximately 295.772 seconds. The total time of the fall is $295.772 + 5.000 = 300.772$ seconds.

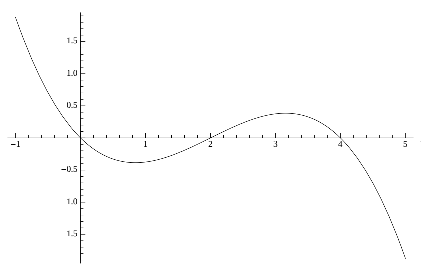
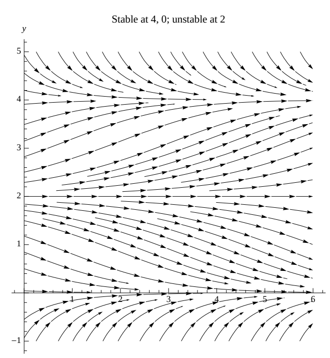


Chapter 3 Review Exercises

1. $y = 0$ is unstable; $y = 1/2$ is stable
3. $y = 0$ is unstable; $y = 4$ is stable

5. $y = 0$: Stable; $y = 2$: Unstable; $y = 4$: Stable

5. $y = y_0 3^{t/4}$; $y(t) = 5y_0 \Rightarrow 5 = 3^{t/4} \Rightarrow \ln 5 = \frac{1}{4} t \ln 3 \Rightarrow t = (4 \ln 5)/\ln 3 \approx 5.86$ days

7. WIth $y(t) = y_0 e^{kt}$, $y(4) = y_0 e^{4k} = 3y_0 \Rightarrow e^k = 3^{1/4}$ so $y(t) = y_0 \cdot 3^{t/4}$. $y_0 \cdot 3^{t/4} = 5y_0 \Rightarrow \frac{1}{4} t \ln 3 = \ln 5$ so $t = (4 \ln 5)/(\ln 3) \approx 5.86$ days.
9. $y(t) = y_0 e^{kt}$; $y(1700) = y_0 e^{1700k} = \frac{1}{2} y_0 \Rightarrow e^k = (1/2)^{1/1700}$. Then $y = y_0 \cdot 2^{-t/1700}$, $y(50) \approx 0.9798 y_0$ (97.98% of y_0)

11. First we solve $y' = ky(1000 - y)$, $y(0) = 250$:

$$\begin{aligned}
\frac{dy}{dt} &= ky(1000-y) \\
\frac{1}{y(1000-y)}dy &= k\,dt \\
\frac{1}{1000}\left(\frac{1}{y} + \frac{1}{1000-y}\right)dy &= k\,dt \\
\ln|y| - \ln|1000-y| &= 1000kt + C \\
\frac{y}{1000-y} &= Ce^{1000kt} \\
y &= \frac{1000Ce^{1000kt}}{1+Ce^{1000kt}}.
\end{aligned}$$

Applying the initial condition gives us

$$y(0) = \frac{1000C}{1+C} = 250 \Rightarrow C = \frac{1}{3}$$

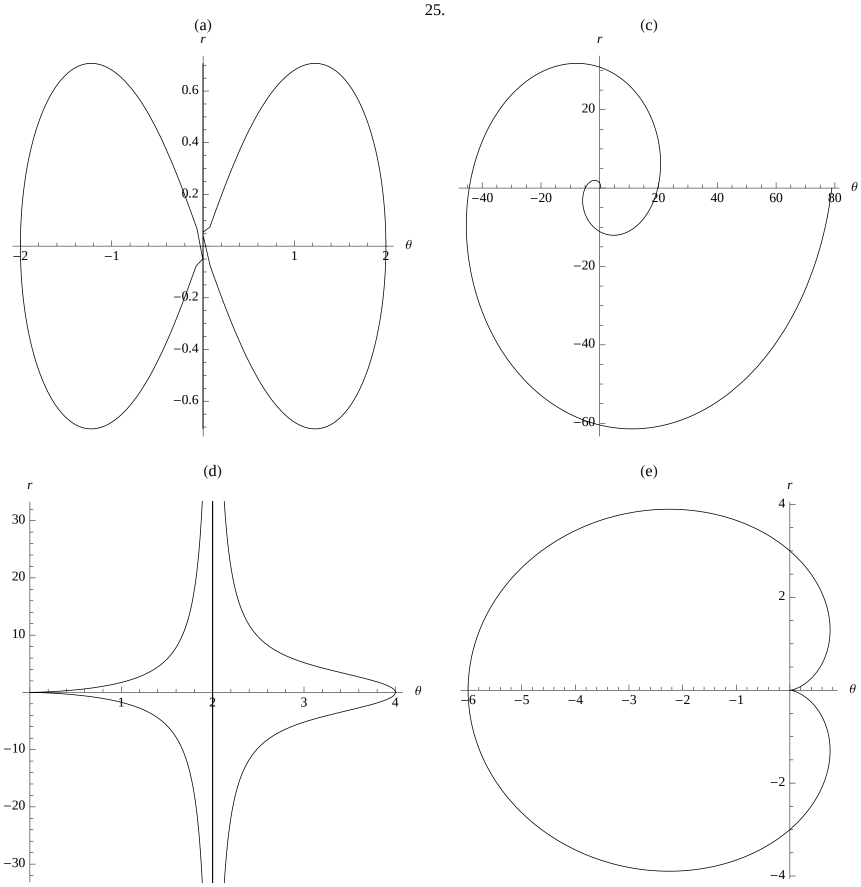
so $y = \dfrac{1000e^{1000kt}}{3+e^{1000kt}}$. Then $y(1) = \dfrac{1000e^{1000k}}{3+e^{1000k}} = 500 \Rightarrow e^{1000k} = 3 \Rightarrow e^k = 3^{1/1000}$
so $y = \dfrac{1000 \cdot 3^t}{3+3^t} = 1000(1+3^{1-t})^{-1}$; $y = 750$ implies $t = (\ln 9)/(\ln 3) \approx 2$ days
13. Using Newton's law of cooling, we have $T_0 = 40$ and $T_s = 90$ so $T(t) = 90 - 50e^{kt}$. Next $T(20) = 90 - 50e^{20k} = 65 \Rightarrow -50e^{20k} = -25 \Rightarrow e^{20k} = 1/2 \Rightarrow e^k = 2^{-1/20}$ so $T(t) = 90 - 50 \cdot 2^{-t/20}$; $T(30) = 90 - 50 \cdot 2^{-3/2} \approx 72.3^{\circ}$F
15. We need to solve $T' = k(T - 325)$, $T(0) = 100$, $T(45) = 150$ and then determine $T(-60)$. $T(t) = Ce^{kt} + 325$; $T(0) = 100 \Rightarrow C = -225 \Rightarrow T(t) = -225e^{kt} + 325$. $T(45) = 150 \Rightarrow -225e^{45k} + 325 = 150 \Rightarrow e^{45k} = \frac{175}{225} = \frac{7}{9} \Rightarrow e^k = \left(\frac{7}{9}\right)^{1/45}$. Therefore, $T(t) = -225 \cdot (7/9)^{t/45} + 325$ so $T(-60) \approx 10.43^{\circ}$F
17. If the object weighs 4 pounds, then its mass is found with $F = mg$ or $4 = m{\cdot}32$ to be $m = 1/8$ slug. Therefore, we solve the initial value problem $\frac{1}{8}v' = 4 - v$, $v(0) = 0$ to find that $v(t) = 4 - 4e^{-8t}$. Then$v(3) = 4 - 4e^{-24} \approx 4$ ft/s. The distance traveled by the rock after t seconds is found by solving $s' = 4 - 4e^{-8t}$, $s(0) = 0$, which has solution $s(t) = 4t + \frac{1}{2}e^{-8t} - \frac{1}{2}$. After 3 seconds, the rock has fallen $s(3) \approx 11.5$ ft
19. $dv/dt = -9.8 - v$, $v(0) = 40$ has solution $v(t) = \frac{1}{5}(-99 + 299e^{-t})$; $v(t) = 0$ when $t = \ln(299/99) \approx 1.11$ s
$ds/dt = v$, $s(0) = 0$ has solution $s(t) = \frac{1}{5}(-99t - 299e^{-t} + 299)$; $s(\ln(299/99)) \approx 18.11$ ft
21. $dv/dt = 32 - \frac{1}{2}v^2$, $v(0) = 30$ has solution $v(t) = 8\dfrac{19e^{8t}+11}{19e^{8t}-11}$; $\lim_{t\to\infty} v(t) = 8$ ft/s
23. With $m = 230$ kg, $dv/dt = 82/115 - 637v/230000$, $v(0) = 0$, which has solution $v(t) = \frac{164000}{637}(1 - e^{-637t/230000})$; $v(t) = 12$ when $t = \dfrac{230000}{637}\ln\dfrac{41000}{39089} \approx 17.23$ s

$dy/dt = v$, $y(0) = 0$ has solution $y = \frac{164000}{637}t + \frac{37720000000}{405769}(e^{-637t/230000} - 1)$

$H = y(17.23) \approx 104.17798$

25. (a) $r^2 = 4\cos^2 2\theta$; (b) $r = 0$; (c) $r = \frac{1}{2}\theta^2$; (d) $r = 2(1 + \sec\theta)$; (e) $r = -3(-1 + \cos\theta)$

25.



27. (a)

$$P = \int_0^\infty e^{-0.05t}\left(p - \frac{c}{q}\right) \cdot \frac{1}{2}\,dt = \lim_{M\to\infty} \frac{1}{2 \cdot -0.05}\left(p - \frac{c}{q}\right)\left[e^{-0.05t}\right]_0^M$$
$$= -10\left(p - \frac{c}{q}\right)\lim_{M\to\infty}\left(e^{-0.05M} - 1\right) = 10\left(p - \frac{c}{q}\right)$$

(b)

$$\frac{1}{[(q-qE)-ax]x}dx = dt$$

$$\frac{1}{r-qE}\left(\frac{1}{x}+\frac{a}{(r-qE)-ax}\right)dx = dt$$

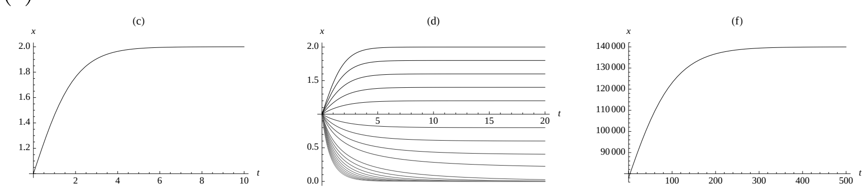
$$\frac{1}{r-qE}\left(\ln|x|-\ln|(r-qE)-ax|\right) = t+C$$

$$\ln\left|\frac{x}{(r-qE)-ax}\right| = (r-qE)(t+C)$$

$$\frac{x}{(r-qE)-ax} = Ke^{(r-qE)t}.$$

$x(0)=x_0$ implies that $K=\dfrac{x_0}{(r-qE)-ax_0}$. Substitution gives us $\dfrac{x}{(r-qE)-ax}=\dfrac{x_0}{(r-qE)-ax_0}e^{(r-qE)t}$. Solving for x we find that $x=\dfrac{x_0(r-qE)e^{(r-qE)t}}{ax_0e^{(r-qE)t}+(r-qE)-ax_0}$. Because $h(t)=qEx(t)$, $h(t)=\dfrac{qEx_0(r-qE)e^{(r-qE)t}}{ax_0e^{(r-qE)t}+(r-qE)-ax_0}$.

(c)



(d) From the graph, we see that if $qE\approx 0.5$, the whale population remains constant. (e) For (i) we obtain approximately 3.819×10^7 while for (ii) we obtain approximately -1.530×10^7. The negative result in (ii) indicates that the whale population becomes extinct. (Then more effort is used to catch whales, more whales are caught and they become extinct; for moderate efforts, less whales are caught, but they do not become extinct.)

(e)

$$\begin{pmatrix} 1000 & 3.28883\times 10^{9} \\ 1500 & -3.84615\times 10^{9} \\ 2000 & -3.84615\times 10^{9} \\ 2500 & 2.95858\times 10^{15} \\ 3000 & 2.95858\times 10^{15} \\ 3500 & 2.95858\times 10^{15} \\ 4000 & 2.95858\times 10^{15} \\ 4500 & 2.95858\times 10^{15} \end{pmatrix}$$

(f) We see that the maximum profit results if $E\approx 2500$. Under the conditions, the whale population increases and approaches a limiting population of about 140000. (g) We see that if there is no harvesting, the limiting population is about 400,000, which is about the optimal stock level of approximately $227,500$. *What advice would you give the whaling industry?*

29. (a) Differentiating $y+2x=c$ with respect to x gives us $y'+2=0 \Rightarrow y'=-2$ so we must solve $y'=1/2$. Integrating yields $y=x/2+k$, k constant. Observe that the original family is the family of lines with slope -2 so it is no surprise that the family of orthogonal trajectories is the family of lines with slope $1/2$. (b) Differentiating the equation yields $dy/dx=ce^{cx}$ and solving the original equation for c gives us $\ln y=cx \Rightarrow c=(\ln y)/x$. Therefore, we must solve

$$\frac{dy}{dx}=-\frac{1}{ce^{cx}}=-\frac{1}{\dfrac{\ln y}{x}y}=-\frac{x}{y\ln y}.$$

Separating variables and integrating with integration by parts gives us

$$\begin{aligned} y\ln y\,dy &= -x\,dx \\ y^2\ln y-\frac{1}{2}y^2+x^2 &= k. \end{aligned}$$

(c) In this case $2yy'=2x$ so $y'=x/y$ and we must solve $dy/dx=-y/x$. Separating variables and integrating gives us

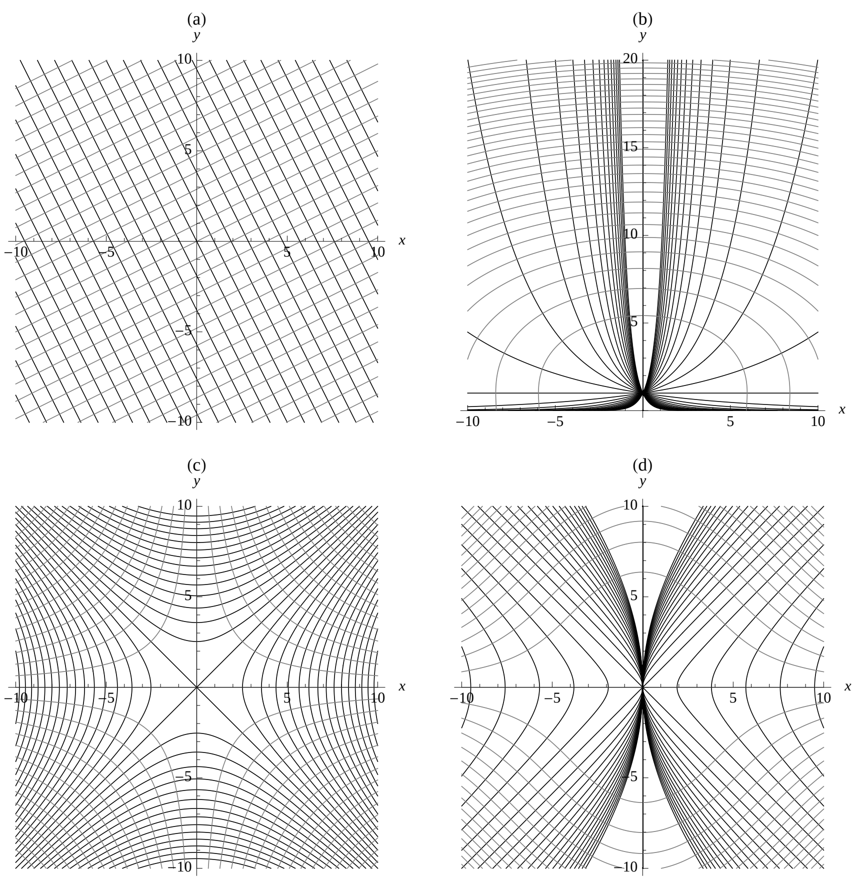
$$\frac{1}{y}dy=-\frac{1}{x}dx \Rightarrow \ln|xy|=k \Rightarrow xy=k.$$

(d) Differentiating the equation gives us $2yy'=2x+c$ and solving for c yields $c=(y^2-x^2)/x$. Then,

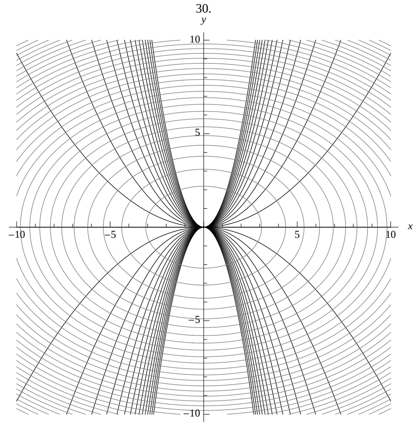
$$\begin{aligned} 2yy' &= 2x+c \\ 2yy' &= 2x+\frac{y^2-x^2}{x} \\ \frac{dy}{dx} &= \frac{x^2+y^2}{2xy}. \end{aligned}$$

so we must solve $dy/dx=-2xy/(x^2+y^2)$, which is homogeneous of degree 2. In this case, we let $x = vy \Rightarrow dx = v\,dy + y\,dv$. Then,

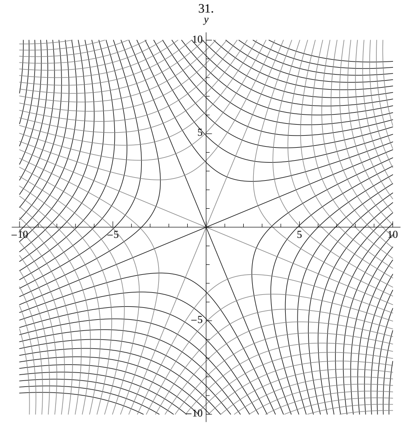
$$\begin{aligned}
2x\,dx + (x^2 + y^2)\,dy &= 0\\
2vy^2(y\,dv + v\,dy) + y^2(v^2+1)dy &= 0\\
2v(y\,dv + v\,dy) + (v^2+1)dy &= 0\\
2vy\,dv + (3v^2+1)dy &= 0\\
\frac{2v}{3v^2+1}\,dv &= -\frac{1}{y}\,dy\\
\frac{1}{3}\ln(3v^2+1) &= -\ln|y| + k\\
\ln\left(\frac{3x^2+y^2}{y^2}\right) &= -3\ln|y| + k\\
x^2y + \frac{1}{3}y^3 &= k.
\end{aligned}$$



30.



31.



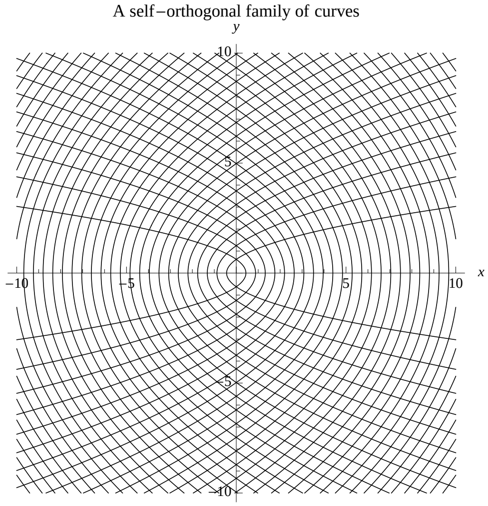
33. $\dfrac{dy}{dx} = 2k_1x,\ k_1 = \dfrac{y-c}{x^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{2(y-c)}{x}$; Solve: $\dfrac{dy}{dx} = -\dfrac{x}{2(y-c)} \Rightarrow \frac{1}{2}y^2 - cy = -\frac{1}{4}x^2 + K$, Multiply by 4: $x^2 + 2y^2 = 4cy + 4K \Rightarrow 4c = 1 \Rightarrow c = 1/4$.
35. $y^2 - x^2 = C$

37. (a) Implicitly differentiating $y^2 = 2cx + 2c^2$, we have $2yy' = 2c$. Using the quadratic formula with $2c^2 + 2xc - y^2 = 0$, we find that $c = \frac{1}{2}\left(-x \pm \sqrt{x^2 + 2y^2}\right)$. If $c = \frac{1}{2}\left(-x - \sqrt{x^2 + 2y^2}\right)$, then the orthogonal trajectories must satisfy $\dfrac{dy}{dx} = \dfrac{2y}{x + \sqrt{x^2 + y^2}}$, which is a first-order homogeneous equation. With the substitution $x = uy \Rightarrow dx = ydu + udy$, we have

$$
\begin{aligned}
2u(u\,dy + y\,du) - \left(uy + \sqrt{u^2y^2 + 2y^2}\right) dy &= 0 \\
2y\,du &= \left(\sqrt{u^2 + 2} - u\right) du \\
\frac{1}{\sqrt{u^2 + 2} - u}\,du &= \frac{1}{2y}\,dy \\
\frac{1}{4}u^2 + \frac{1}{4}u\sqrt{u^2 + 2} + \frac{1}{2}\ln\left|u + \sqrt{u^2 + 2}\right| &= \frac{1}{2}\ln|y| + C \\
\frac{1}{4}\frac{x^2}{y^2} + \frac{1}{4}x\sqrt{\frac{x^2}{y^2} + 2} + \frac{1}{2}\ln\left|\frac{x}{y} + \sqrt{\frac{x^2}{y^2} + 2}\right| &= \frac{1}{2}\ln|y| + C
\end{aligned}
$$

Similarly, if $c = \frac{1}{2}\left(-x + \sqrt{x^2 + 2y^2}\right)$, then the orthogonal trajectories satisfy $\dfrac{dy}{dx} = \dfrac{4y}{x - \sqrt{x^2 + 2y^2}}$. Therefore, $\dfrac{1}{4}\dfrac{x^2}{y^2} - \dfrac{1}{4}x\sqrt{\dfrac{x^2}{y^2} + 2} - \dfrac{1}{2}\ln\left|\dfrac{x}{y} + \sqrt{\dfrac{x^2}{y^2} + 2}\right| = \dfrac{1}{2}\ln|y| + C$. This family of curves is self-orthogonal. (Both sets of graphs are obtained from the same family.)

A self−orthogonal family of curves



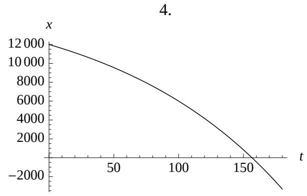
Differential Equations at Work

A. Mathematics of Finance

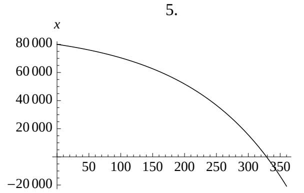
2. We compute $x = 1000e^{.08t}$ for the t-values: \$1491.82, \$2225.54, \$3320.12, \$4953.03.

4. 157 months

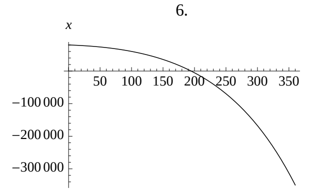
4.



5.



6.



5. $x' - \ln(1 + .08/12)x = -599$, $x(0) = 80000$ has solution $x = 90149.2 - 10149.2e^{0.00664454t}$. $x(t) = 0$ when $t = 329$.

6. $x' - \ln(1 + .08/12)x = -599 \cdot 1.0025^t$, $x(0) = 80000$ has solution $x = 144419e^{0.00249688t} - 64418.7e^{0.00664454t}$ and $x(t) = 0$ when $t = 195$.

7. $x' - \ln(1+.1/12)x = 250$, $x(0) = 0$ has solution $x = 30124.8e^{0.0082988t} - 30124.8$.

Years	Months	Balance
10	120	51424.30
20	240	190632.00
30	360	567473.00

8. $x' - \ln(1+.1/12)x = 250 \cdot 1.005^t$, $x(0) = 0$ has solution $x = 75499.9e^{0.0082988t} - 75499.9e^{0.00498754t}$.

Years	Months	Balance
10	120	67017.1
20	240	303349.
30	360	1.04302×10^6

9.

Years	Months	Balance
10	120	51424.3
20	240	190632.
30	360	567473.

;

Years	Months	Balance
10	35	133381.
20	45	345957.
30	55	897323.

11. $S' - rS = I(t) - E(t)$; $\frac{d}{dt}\left(e^{-rt}S\right) = e^{-rt}(I(t) - E(t))$; $e^{-rt}S = \int_0^t e^{-ru}(I(u) - E(u))\,du + C$; $S(t) = e^{rt}\int_0^t e^{-ru}(I(u) - E(u))\,du + Ce^{rt}$; Applying $S(0) = S_0$ yields $C = S_0$.

12. Here, $E(t) = E(0)e^{it} = 18000e^{0.03t}$ and

$$I(t) = \begin{cases} I(0)e^{jt}, \ 0 \le t \le T \\ F + Ve^{i(t-T)}, \ t > T \end{cases} = \begin{cases} 20000e^{0.05t}, \ 0 \le t \le T \\ 0.2 \cdot 18000e^{0.03T} + 0.3 \cdot 18000e^{0.03T}e^{0.03(t-T)}, \ t > T \end{cases}$$

$$= \begin{cases} 20000e^{0.05t}, \ 0 \le t \le T \\ 18000(0.2e^{0.03T} + 0.3e^{0.03t}), \ t > T \end{cases}.$$

We will assume that $S_0 = 0$ so the balance of the account

$$S(t_0) = e^{rt_0}\left(S_0 + \int_0^{t_0}(I(t) - E(t)e^{-rt}\,dt\right)$$

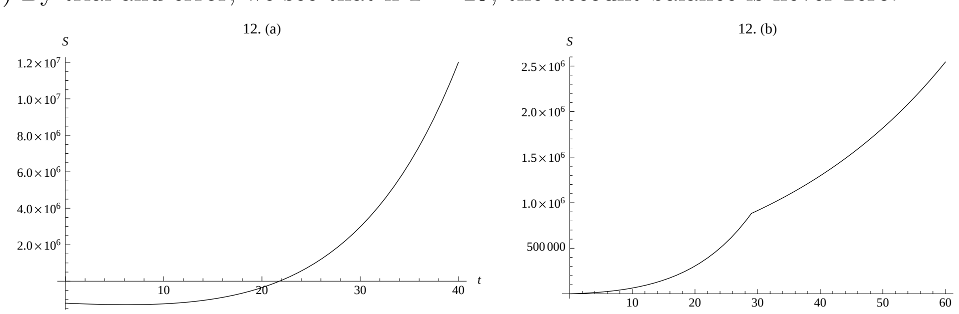
can be rewritten as $S(t_0) = e^{rt_0}\int_0^{t_0}(I(t) - E(t)e^{-rt}\,dt$. Thus, the balance of the account thirty years after retirement is given by

$$S(T+30) = e^{0.06(T+30)}\int_0^{T+30}(I(t) - E(t))e^{-0.06t}\,dt,$$

which is a function of T:

$$S(T+30) = e^{0.06(T+30)}\left\{20000\int_0^T e^{-0.01t}\,dt + 18000\int_T^{T+30}\left(0.2e^{0.03T} + 0.3e^{0.03t}\right)e^{-0.06t}\,dt - 18000\int_0^{T+30} e^{-0.03t}\,dt\right\}.$$

(b) By trial and error, we see that if $T = 29$, the account balance is never zero.



B. Algae Growth

1. Note that the equation is separable as well as first-order linear. We solve it by viewing it as a linear equation and using undetermined coefficients. The corresponding homogeneous equation is $\dfrac{dN}{dt} - \mu N = 0$ with characteristic equation $r - \mu = 0$ to $r = \mu$ and a general solution of the corresponding homogeneous solution is $N_h = Ce^{\mu t}$. A particular solution takes the form $N_p = B$, where B is a constant to be determined. Substituting into the nonhomogeneous equation gives us ${N_p}' - \mu N_p = -\mu B = \mu A$ so $B = -A$ and $N_p = -A$. The general solution of the nonhomogeneous equation is then $N = Ce^{\mu t} - A$. Applying the initial condition gives us $C = A$ so $N(t) = Ae^{\mu t} - A$.
2. $\mu = 0.693$
3. $3e^{0.693 \cdot 24} - 3 \approx 5.033 \times 10^7$ and $3e^{0.693 \cdot 36} - 3 \approx 2.062 \times 10^{11}$

C. Dialysis

1. Use a computer algebra system to perform the algebra and calculus. A general solution of $z' = -\alpha z$ is $z = Ce^{-\alpha x}$. Because $z = u - v$, $u - v = Ce^{-\alpha x}$. Using $Q_B u' = -k(u - v)$ yields $Q_B u' = -kCe^{-\alpha x}$ so dividing by Q_B and integrating results in

$$u = \frac{Ck}{\alpha Q_B e^{\alpha x}} + C_1 = \frac{Ck + \alpha C_1 Q_B e^{\alpha x}}{\alpha Q_B e^{\alpha x}}.$$

Solving for v and substituting results in,

$$v = \frac{1}{k}(Q_B u' + ku) = \frac{1}{\alpha Q_B e^{\alpha x}} \left(Ck - \alpha C Q_B + \alpha C_1 Q_B e^{\alpha x}\right)$$

Applying the initial conditions $u(0) = u_0$ and $v(L) = 0$ leads to the system of equations

$$\begin{aligned} \frac{1}{\alpha Q_B}\left(Ck + C_1 \alpha Q_B\right) = u_0 \\ \frac{1}{\alpha Q_B e^{\alpha L}}\left(Ck - \alpha C Q_B + \alpha C_1 Q_B e^{\alpha L}\right) = 0 \end{aligned},$$

which has solutions $C = \dfrac{k u_0 Q_B e^{\alpha L}}{ke^{\alpha L} + \alpha Q_B - k}$ and $C_1 = \dfrac{u_0(\alpha Q_B - k)}{ke^{\alpha L} + \alpha Q_B - k}$. Substituting these values and simplifying the results into the formulas for u and v yields

$$u = u_0 \frac{Q_B e^{\alpha x} - Q_D e^{\alpha L}}{e^{\alpha x}\left(Q_B - Q_D e^{\alpha L}\right)} \quad \text{and} \quad v = u_0 \frac{e^{\alpha L} - e^{\alpha x}}{e^{\alpha x} \dfrac{Q_D}{Q_B}\left(e^{\alpha L} - 1\right)}.$$

2. In this case, $u(x) = 50.0623e^{-0.5625x} - 14.2623$. After dialysis has been performed once, waste levels are approximately $u(1) \approx 14.2623$, which are within the average range.
3. $\int_0^L k(u(x) - v(x))\, dx = u_0 k Q_B \dfrac{e^{\alpha L} - 1}{\alpha Q_B + ke^{\alpha L} - k}$ and $Q_B(U_0 - u(L)) = u_0 k Q_B \dfrac{e^{\alpha L} - 1}{\alpha Q_B + ke^{\alpha L} - k}$,

which are the same.

4.

$$CL = \frac{Q_B}{u_0}(u_0 - u(L)) = kQ_B \frac{e^{\alpha L} - 1}{\alpha Q_B + ke^{\alpha L} - k} = Q_B \frac{e^{\alpha L} - 1}{\frac{\alpha Q_B - k}{k} + e^{\alpha L}}$$

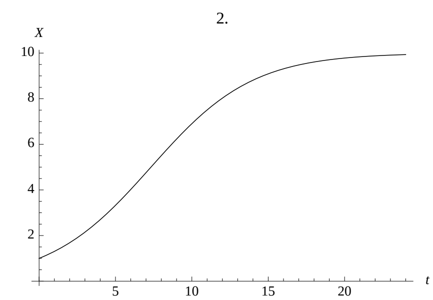
$$= Q_B \frac{e^{\alpha L} - 1}{-\frac{\alpha Q_B}{Q_D} + e^{\alpha L}} = Q_B \frac{1 - e^{\alpha L}}{1 - \frac{\alpha Q_B}{Q_D} e^{-\alpha L}}.$$

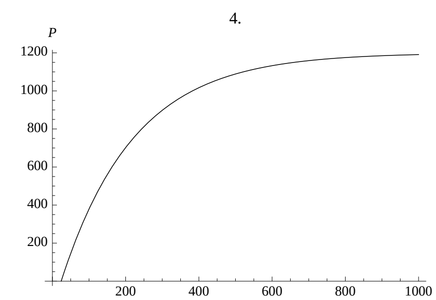
D. Antibiotic Production

1. $X = \dfrac{X_{max} e^{\mu_{max} t}}{X_{max} - 1 + e^{\mu_{max} t}}$

2.

Hours	Mass
4	2.69487
8	5.50521
12	8.02624
16	9.3104
20	9.78178
24	9.93326





3. $P(t) = \dfrac{1}{K_d}\left(10Ee^{K_d t}(e^{K_d t} - e^{24K_d})\right)$