Exercises 4.1

1. $W(S) = \begin{vmatrix} t & 4t-1 \\ 1 & 4 \end{vmatrix} = 1$; linearly independent
3. $W(S) = \begin{vmatrix} e^{-6t} & e^{-4t} \\ -6e^{-6t} & -4e^{-4t} \end{vmatrix} = 2e^{-10t}$; linearly independent
5. Linearly independent; $W(S) = \begin{vmatrix} t^{-1} & t^{-2} \\ -t^{-2} & -2t^{-3} \end{vmatrix} = -2t^{-4} + t^{-}4 = -t^{-4}$.
7. Linearly independent; $W(S) = 0$
9. $W(\{e^t, e^{-t}\}) = -2$
11. $W(\{t^{-1/2}, t^3\}) = \frac{7}{2}t^{3/2}$
13. $y(t) = -2e^{2t} + e^{-t}$
15. $y = -t^{-1/3} + 2t^3$
17. $y = \sin t + \cos t + t\sin t$
19. (a) $\frac{d}{dt}(\cosh t) = \frac{d}{dt}\left(\frac{1}{2}(e^t + e^{-t})\right) = \frac{1}{2}(e^t - e^{-t}) = \sinh t$ (b) $\frac{d}{dt}(\sinh t) = \frac{d}{dt}\left(\frac{1}{2}(e^t - e^{-t})\right) = \frac{1}{2}(e^t + e^{-t}) = \cosh t$ (c) $\cosh^2 t - \sinh^2 t = \frac{1}{4}(e^{2t} + 2 + e^{-2t}) - \frac{1}{4}(e^{2t} - 2 + e^{-2t}) = 1$ (d) $W(\{\cosh t, \sinh t\}) = \cosh^2 t - \sinh^2 t = 1$
21. $W(S) = \begin{vmatrix} \cos 4t & \sin 4t \\ -4\sin 4t & 4\cos 4t \end{vmatrix} = 4\cos^2 4t + 4\sin^2 4t = 4.$
23. $W(S) = \begin{vmatrix} t^{-1} & t \\ -t^{-2} & 1 \end{vmatrix} = 2t^{-1}.$
25.
$$y_2 = e^{3t}\int \frac{e^{\int 5\,dt}}{[e^{3t}]^2}\,dt = e^{3t}\int \frac{e^{5t}}{e^{6t}}\,dt = e^{3t}\int e^{-t}\,dt = -e^{2t}.$$

Because every linear combination (including constant multiples) of solutions to a linear homogeneous equation is a solution, this means that a second linearly independent solution of the equation is $y_2 = e^{2t}$. Therefore, a fundamental set of solutions is $S = \{e^{2t}, e^{3t}\}$ and a general solution of the linear homogeneous equation is $y = c_1e^{2t} + c_2e^{3t}$.

27. Here $p(t) = -4$ and $y_1 = e^{2t}$ so a second linearly independent solution is

$$y_2 = e^{2t}\int \frac{e^{-\int -4\,dt}}{[e^{2t}]^2}\,dt = e^{2t}\int \frac{e^{4t}}{e^{4t}}\,dt = te^{2t}.$$

This means that a fundamental set of solutions is $S = \{e^{2t}, te^{2t}\}$ and a general solution of the homogeneous equation is $y = e^{2t}(c_1 + c_2t)$. Applying the initial conditions yields $y = te^{-2t}$.

29. $S = \{\cos 3t, \sin 3t\}$, $y = c_1\cos 3t + c_2\sin 3t$, $y = \cos 3t - \frac{4}{3}\sin 3t$
31. $S = \{t^{-4}, t\}$, $y = c_1t^{-4} + c_2t$
33. For $t > 0$, $y_2(t) = \dfrac{\sin t}{\sqrt{t}}$ so a general solution is $y = t^{-1/2}(c_1\cos t + c_2\sin t)$.
35. Substituting $y = e^{-t/2}$ and $y = e^{t/3}$ into the equation $y'' + (b/a)\,y' + (c/a)\,y = 0$ results in the system of equations $2\,b/a - 4\,c/a = 1$ and $3\,b/a - 9\,c/a = 1$ so $b/a = 5/6$ and $c/a = 1/6$. Hence, the equation is $6y'' + 5y' + y = 0$.
37. Substitution of either function into the differential equation and equating coefficients gives us $a - b = -3$ and $2a = 4$ so $a = 2$ and $b = 5$.

39. $W\left(\left\{f(t), \int \frac{1}{(f(t))^2} e^{-\int p(t)\,dt}\,dt\right\}\right) = e^{-\int p(t)\,dt}$
41. First, we calculate

$$\begin{aligned}\frac{d}{dt}W(\{y_1, y_2\}) &= \frac{d}{dt}\left(y_1y_2' - y_1'y_2\right)\\ &= y_1y_2'' + y_1'y_2' - y_1'y_2' - y_1''y_2\\ &= y_1y_2'' - y_1''y_2.\end{aligned}$$

Because y_1 and y_2 are both solutions of the equation, $y_1'' = -p(t)y_1' - q(t)y_1$ and $y_2'' = -p(t)y_2' - q(t)y_2$ so

$$\begin{aligned}\frac{d}{dt}W(\{y_1, y_2\}) &= y_1y_2'' - y_1''y_2\\ &= y_1[-p(t)y_2' - q(t)y_2] - y_2[-p(t)y_1' - q(t)y_1]\\ &= -p(t)[y_1y_2' - y_1'y_2] = -p(t)W(\{y_1, y_2\}).\end{aligned}$$

Therefore, $W(\{y_1, y_2\})$ satisfies the first-order linear separable equation $W' + p(t)W = 0 \Rightarrow \frac{dW}{dt} = -p(t)W \Rightarrow \frac{1}{W}dW = -p(t)dt$ with solution $\ln|W| = -\int p(t)\,dt$ or $W(\{y_1, y_2\}) = Ce^{-\int p(t)\,dt}$.
43. $y = c_1 + c_2\tan(c_3 + c_4\ln t)$ is a solution of the equation if (a) $y = c_1$ (note that if $c_2 = 0$ or $c_4 = 0$, y is a constant function); or (b) $y = -\frac{1}{2} + c_2\tan(c_3 + c_2\ln t)$. The Principle of Superposition does not hold if $c_1 \neq 0$ and $y = c_1 + c_2\tan(c_3 + c_2\ln t)$,

$$ty'' - 2yy' = -\frac{2}{t}c_1c_2^2\sec^2(c_3 + c_2\ln t) \neq 0.$$

45. $y = -\pi t^{-1}\cos 4t$
47. $y = t^{-3}(a\cos t + b\sin t)$; $y(\pi = -a\pi^{-3} = 0$ implies $a = 0$; $y(2\pi) = \frac{18}{a}\pi^{-3} = 0$ implies that $a = 0$. $y = t^{-3}\sin t$, C arbitrary
49. We solve $y_1'' + p(t)y_1' + q(t)y_1 = 0$, $y_2'' + p(t)y_2' + q(t)y_2 = 0$ for $p(t)$ and $q(t)$: $p(t) = \frac{y_1''y_2 - y_1y_2''}{W(\{y_1, y_2\})}$ and $q(t) = \frac{y_1'y_2'' - y_1''y_2'}{W(\{y_1, y_2\})}$.

Exercises 4.2

1. $y'' = 0$ has characteristic equation $r^2 = 0$ so $r = 0$ has multiplicity two. Two linearly independent solutions to the equation are $y_1 = 1$ and $y_2 = t$; a fundamental set of solutions is $S = \{1, t\}$; and a general solution is $y = c_1 + c_2t$.
3. $y'' + y' = 0$ has characteristic equation $r^2 + r = 0$, which has solutions $r_1 = 0$ and $r_2 = -1$. Two linearly independent solutions to the equation are $y_1 = 1$ and $y_2 = e^{-t}$; a fundamental set of solutions is $S = \{1, e^{-t}\}$; and a general solution is $y = c_1 + c_2e^{-t}$.
5. $y = c_1e^{-6t} + c_2e^{-2t}$
7. $y = c_1e^{-t/4} + c_2e^{-t/2}$
9. The characteristic equation is $r^2 + 16 = 0$ with roots $r_{1,2} = \pm 4i$ so a fundamental set of solutions is $S = \{\cos 4t, \sin 4t\}$ and a general solution is

$y = c_1 \cos 4t + c_2 \sin 4t$
11. The characteristic equation is $r^2 + 7 = 0$ with roots $r_{1,2} = \pm i\sqrt{7}$ so a fundamental set of solutions is $S = \{\cos(\sqrt{7}t), \sin(\sqrt{7}t)\}$ and a general solution is $y = c_1 \cos(\sqrt{7}t) + c_2 \sin(\sqrt{7}t)$
13. The characteristic equation is $7r^2+4r-3 = (r+1)(7r-3) = 0$ with solutions $r_1 = -1$ and $r_2 = 3/7$. Thus, a fundamental set of solutions is $S = \{e^{-t}, e^{3t/7}\}$ and a general solution is $y = c_1 e^{-t} + c2e^{3t/7}$.
15. The characteristic equation is $r^2 - 6r + 9 = (r-3)^2 = 0$ with solutions $r_{1,2} = 3$. Then, one solution is $y_1 = e^{3t}$ and a second linearly independent solution found through reduction of order is $y_2 = te^{3t}$. Thus, a fundamental set of solutions is $S = \{e^{3t}\, te^{3t}\}$ and a general solution is $y = c_1 e^{3t} + c_2 te^{3t}$.
17. General: $y = c_1 + c_2 e^{t/3}$; IVP: $y = -21(1 - e^{t/3})$
19. General: $y = c_1 e^{3t} + c_2 e^{4t}$; IVP: $y = 14e^{3t} - 11e^{4t}$
21. General: $y = c_1 e^{5t} + c_2 e^{2t}$; IVP: $y = e^{5t}$
23. General: $y = c_1 \cos 10t + c_2 \sin 10t$; IVP: $y = \cos 10t + \sin 10t$
25. General: $y = e^{-2t}(c_1 + c_2 t)$; IVP: $y = e^{-2t}(1 + 5t)$
27. General: $y = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t)$; IVP: $y = e^{-2t}(2\cos 4t + \sin 4t)$
29. The characteristic equation is $r^2 + r + 1 = 0$. Using the quadratic formula,

$$r_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

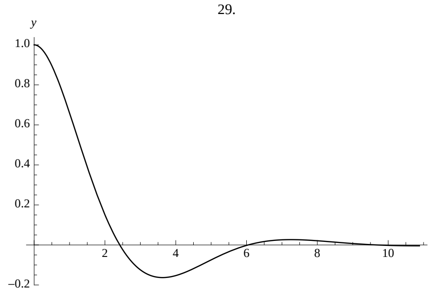
Therefore, a fundamental set of solutions for the equation is $S = \left\{ e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right), e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) \right\}$ and a general solution is $y = c_1 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right)$. Differentiating y,

$$y' = \frac{1}{2}e^{-t/2}\left(\left(-c_1 + \sqrt{3}c_2\right)\cos\left(\frac{\sqrt{3}}{2}t\right) + \left(-\sqrt{3}c_1 - c_2\right)\sin\left(\frac{\sqrt{3}}{2}t\right)\right).$$

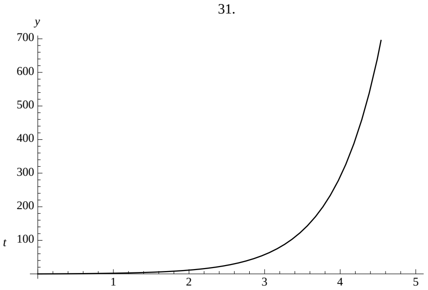
Applying the initial conditions gives us $y(0) = c_1 = 1$ and $y'(0) = -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0$ so $c_1 = 1$ and $c_2 = \frac{1}{\sqrt{3}}$ so the solution to the initial value problem is $y = e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}} e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right)$.

31. $y = -\dfrac{e^{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)t} - e^{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)t}}{\sqrt{5}}$

29.

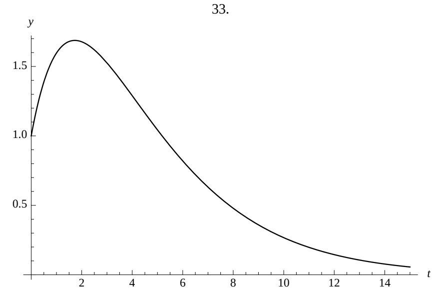


31.

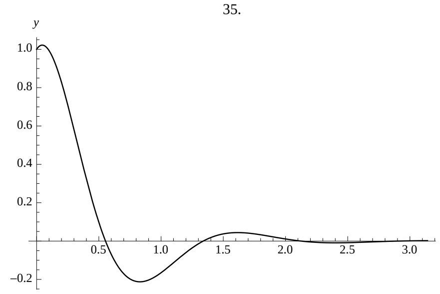


33. A fundamental set of solutions for the equation is $S = \{e^{-t/3}, e^{-t/2}\}$ and a general solution is $y = c_1 e^{-t/3} + c_2 e^{-t/2}$. The solution that satisfies the initial conditions is $y = 9e^{-t/3} - 8e^{-t/2}$.

33.

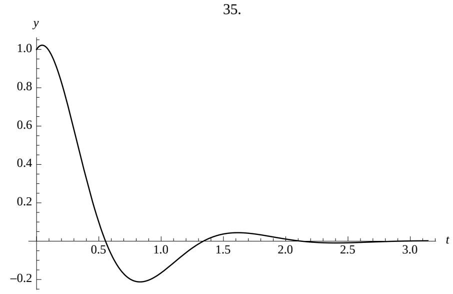


35.



35. A fundamental set of solutions for the equation is $S = \{e^{-2t}\cos 4t, e^{-2t}\sin 4t\}$ and a general solution is $y = c_1 e^{-2t}\cos 4t + c_2 e^{-2t}\sin 4t$. The solution that satisfies the initial conditions is $y = \frac{3}{4}e^{-2t}\sin(4t) + e^{-2t}\cos(4t)$.

35.



37. (a) $y = c_1 t^{2/3} + c_2 t$; (b) $y = t(c_1 + c_2 \ln t)$

39. The characteristic equation of $ay'' + 2by' + cy = 0$ is $ar^2 + 2br + c = 0$, which has solutions $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - ac}}{a}$. With $b^2 - ac > 0$, two linearly independent

solutions are

$$y_1 = \exp\left(\frac{-b+\sqrt{b^2-ac}}{a}t\right) = \exp\left(-\frac{b}{a}t\right)\exp\left(\frac{\sqrt{b^2-ac}}{a}t\right)$$

and

$$y_2 = \exp\left(\frac{-b-\sqrt{b^2-ac}}{a}t\right) = \exp\left(-\frac{b}{a}t\right)\exp\left(-\frac{\sqrt{b^2-ac}}{a}t\right)$$

Two other linearly independent solutions are given by

$$\begin{aligned}\frac{1}{2}(y_1+y_2) &= \exp\left(-\frac{b}{a}t\right)\cdot\frac{1}{2}\left(\exp\left(\frac{\sqrt{b^2-ac}}{a}t\right)+\exp\left(-\frac{\sqrt{b^2-ac}}{a}t\right)\right)\\ &= \exp\left(-\frac{b}{a}t\right)\cosh\left(\frac{\sqrt{b^2-ac}}{a}t\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}(y_1-y_2) &= \exp\left(-\frac{b}{a}t\right)\cdot\frac{1}{2}\left(\exp\left(\frac{\sqrt{b^2-ac}}{a}t\right)-\exp\left(-\frac{\sqrt{b^2-ac}}{a}t\right)\right)\\ &= \exp\left(-\frac{b}{a}t\right)\sinh\left(\frac{\sqrt{b^2-ac}}{a}t\right).\end{aligned}$$

41. $y = Ce^{-t}\sin 2t$, $y = 0$, $y = e^{-t}\cos 2t$
43. $y = \left(\frac{3}{2}a+\frac{1}{2}b\right)e^{-t} + \left(-\frac{1}{2}a-\frac{1}{2}b\right)e^{-3t}$ so $y' = -\left(\frac{3}{2}a+\frac{1}{2}b\right)e^{-t} - 3\left(-\frac{1}{2}a-\frac{1}{2}b\right)e^{-3t}$; $y' = 0$ if $t = -\ln\left(\pm\frac{1}{3}\,\frac{\sqrt{3}\sqrt{(b+a)\,(3\,a+b)}}{b+a}\right)$; For none, $(b+a)(3a+b) \le 0$ while for one, $(b+a)(3a+b) > 0$
45. (a) No (b) To be a general solution, a fundamental set for the equation is $S = \{t\cos t, t\sin t\}$. Now substitute each of these functions into the differential equation and set the result equal to zero. Solve the resulting system for $p(t)$ and $q(t)$ to obtain $p(t) = -2/t$ and $q(t) = (t^2+2)/t^2$.

Exercises 4.3

1. $F = \{t, 1\}$
3. $F_1 = \{e^{2t}\}$, $F_2 = \{1\}$
5. $F_1 = \{e^{-t}\}$, $F_2 = \{t^4, t^3, t^2, t, 1\}$
7. $F_1 = \{\cos 2t, \sin 2t\}$, $F_2 = \{e^{-4t}\}$
9. $F_1 = \{\cos 3t, \sin 3t\}$, $F_2 = \{\cos 2t, \sin 2t\}$
11. $F_1 = \{e^{-t}\cos 2t, e^{-t}\sin 2t\}$, $F_2 = \{1\}$
13. $y_p = Ae^{2t}$ $(y = c_1\cos t + c_2\sin t + \frac{8}{5}e^{2t})$
15. $y_p = Ate^{3t}$ $(y = c_1e^{3t} + c_2e^t + te^{3t})$
17. $y_p = At^2 + Bt + C$ $(y = e^t(c_1 + c_2 t) + t^2 + 4t + 6)$

19. $y_p = A\cos 2t + B\sin 2t$ $(y = c_1\cos t + c_2\sin t - \frac{1}{3}\cos 2t)$
21. $y_p = At\cos 2t + Bt\sin 2t + Ct + D$ $(y = c_1\cos 2t + c_2\sin 2t + \frac{1}{4}t\sin 2t + \frac{1}{4}t)$
23. $y_p = At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2$ $(y = c_1 + c_2 t + \frac{1}{10}t^6 - \frac{1}{3}t^3)$
25. A fundamental set of solutions for the corresponding homogeneous equation is $S\{e^{-2t}, e^t\}$ so a general solution of the corresponding homogeneous equation is $y_h = c_1e^{-2t} + c_2e^t$. The associated set of functions is $F = \{1\}$. Because no element of F is a solution of the corresponding homogeneous equation, we assume a particular solution takes the form $y_p = A \cdot 1 = A$ with derivatives $y_p'' = y_p' = 0$. Substituting y_p into the *nonhomogeneous* equation gives us $y_p'' + y_p' - 2y_p = -2A = -1$ so $A = 1/2$ and a particular solution oft the nonhomogeneous equation is $y_p = 1/2$. Thus, $y = y_h + y_p = c_1e^t + c_2e^{-2\,t} + 1/2$
27. $y = c_1e^{4\,t} + c_2e^{-2\,t} + 1 - 4\,t$
29. The characteristic equation of the corresponding homogeneous equation is $r^2 + 2r + 26 = 0$ with solutions $r_{1,2} = -1 \pm 5i$. A fundamental set of solutions for the corresponding homogeneous equation is $S = \{e^{-t}\cos 5t, e^{-t}\sin 5t\}$ and a general solution is $y_h = e^{-t}(c_1\cos 5t + c_2\sin 5t)$. The associated set of functions is $F = \{t, 1\}$. Because no element of F is a solution to the corresponding homogeneous equation, we assume that a particular solution has the form $y_p = At + B$ with derivatives $y_p' = A$ and $y_p'' = 0$. Substituting y_p into the *nonhomogeneous* equation gives us $y_p'' + 2y_p' + 26y_p = 26At + (2A + 26B) = -338t$ so $26A = -338 \Rightarrow A = -13$ and $2A + 26B = 0 \Rightarrow B = 1$ so a particular solution of the nonhomogeneous equation is $y_p = -13t + 1$. Thus, $y = y_h + y_p = c_1e^{-t}\sin(5\,t) + c_2e^{-t}\cos(5\,t) + 1 - 13\,t$
31. $y = 280 - 60\,t + 5\,t^2 + c_1e^{-1/2\,t} + c_2e^{-1/4\,t}$
33. $S = \{1, e^{2t}\}$, $y_h = c_1 + c_2e^{2t}$, $F = \{\cos 3t, \sin 3t\}$, $y_p = A\cos 3t + B\sin 3t$, $y = -4\sin(3\,t) + 8/3\cos(3\,t) + c_1e^{2\,t} + c_2$
35. $y = c_1e^{-3\,t} + c_2e^{3\,t} - \frac{216}{169}\cos(2\,t) - \frac{54}{13}\sin(2\,t)\,t$
37. $S = \{e^{-2t}, te^{-2t}\}$, $y_h = e^{-2t}(c_1 + c_2t)$, $F = \{t^2\cos 2t, t\cos 2t, \cos 2t, t^2\sin 2t, t\sin 2t, \sin 2t\}$ $y_p = (At^2 + Bt + C)\cos 2t + (Et^2 + Ft + G)\sin 2t$, $y = c_1e^{-2\,t} + c_2e^{-2\,t}t + (-4\,t + 3)\cos(2\,t) - 4\,t\sin(2\,t)(t - 1)$
39. $y = c_1e^{-t} + c_2e^{5\,t} + \left(-36\,t^3 + 18\,t^2 - 6\,t\right)e^{5\,t}$
41. The corresponding homogeneous equation is $y'' + 4y' = 0$ with characteristic equation $k^2 + 4k = k(k+4) = 0 \to k = 0$ or $k = -4$ so $S = \left\{1, e^{-4t}\right\}$ is a fundamental set for the corresponding homogeneous equation and $y_h = c_1 + c_2e^{-4t}$. The forcing function is $f(t) = 8e^{4t} - 4e^{-4t}$ so $F_1 = \left\{e^{4t}\right\}$ and $F_2 = \left\{e^{-4t}\right\}$. An element of F_2 is a solution to the corresponding homogeneous equation so we multiply F_2 by t giving us $tF_2 = \left\{te^{-4t}\right\}$. No element of F_1 or tF_2 is a solution to the corresponding homogeneous equation so we assume that a particular solution has the form $y_p = Ae^{4t} + Bte^{-4t}$. Differentiating twice gives us $y_p' = 4Ae^{4t} - 4Bte^{-4t} + Be^{-4t}$ and $y_p'' = 16Ae^{4t} + 16Bte^{-4t} - 8Be^{-4t}$ and substituting into the nonhomogeneous equation gives us

$$\begin{aligned} y_p'' + 4y_p' &= 32Ae^{4t} - 4Be^{-4t} \\ &= 8e^{4t} - 4e^{-4t} \end{aligned}$$

and equating coefficients we see that $32A = 8$ so $A = 1/4$ and $-4B = -4$ so $B = 1$. Thus, $y_p = \frac{1}{4}e^{4t} + te^{-4t}$ and $y = y_h + y_p = c_1 + c_2e^{-4t} + \frac{1}{4}e^{4t} + te^{-4t}$.
43. The corresponding homogeneous equation is $y'' + 4y' = 0$ with characteristic equation $k^2 + 4k = k(k+4) = 0 \to k = 0$ or $k = -4$ so $S = \left\{1, e^{-4t}\right\}$ is a fundamental set for the corresponding homogeneous equation and $y_h = c_1 + c_2e^{-4t}$. The forcing function is $f(t) = -24t - 6 - 4te^{-4t} + e^{-4t}$ so $F_1 = \{t, 1\}$ and $F_2 = \left\{te^{-4t}, e^{-4t}\right\}$. We multiply F_1 by t and F_2 by t so that no element of $tF_1 = \left\{t^2, t\right\}$ is a solution of the corresponding homogeneous equation and so that no element of $tF_2 = \left\{t^2e^{-4t}, te^{-4t}\right\}$ is a solution of the corresponding homogeneous equation. We now assume that a particular solution has the form $y_p = At^2 + Bt + Ct^2e^{-4t} + Dte^{-4t}$. Then, $y_p' = 2At + B - 4Ct^2e^{-4t} + (2C - 4D)te^{-4t} + De^{-4t}$ and $y_p'' = 2A + 16Ct^2e^{-4t} + (-16C + 16D)te^{-4t} + (2C - 8D)e^{-4t}$. Substituting into the nonhomogeneous equation

$$\begin{aligned} y_p'' + 4y_p' &= 8At + (2A + 4B) - 8Cte^{-4t} + (2C - 4D)e^{-4t} \\ &= -24t - 6 - 4te^{-4t} + e^{-4t} \end{aligned}$$

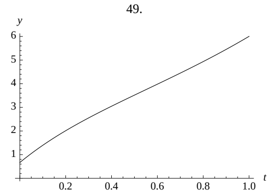
and equating coefficients gives us

$$\begin{array}{cccccc} 8A & & & & = & -24 \\ 2A & +4B & & & = & -6 \\ & & -8C & & = & -4 \\ & & 2C & -4D & = & 1 \end{array}$$

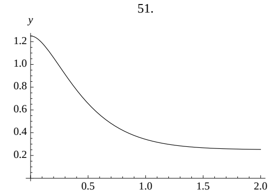
Thus, $A = -3$, $B = 0$, $C = 1/2$ and $D = 0$ so $y_p = -3t^2 + \frac{1}{2}t^2e^{-4t}$. A general solution is then $y = y_h + y_p = c_1 + c_2e^{-4t} - 3t^2 + \frac{1}{2}t^2e^{-4t}$ with derivative $y' = -6t - 2t^2e^{-4t} + te^{-4t} - 4c_2e^{-4t}$.
45. $S = \{1, t\}$, $y_h = c_1 + c_2t$, $F_1 = \{t^2, t, 1\}$, $t^2F_1 = \{t^4, t^3, t^2\}$, $F_2 = \{e^t\}$, $F_3 = \{\cos t, \sin t\}$, $y_p = At^4 + Bt^3 + Ct^2 + Ee^t + F\cos t + G\sin t$, $y = c_1t + c_2 + \frac{1}{12}t^4 + e^t - \sin t$
47. $S = \{e^{2t}, e^{-2t}\}$, $y_h = c_1e^{2t} + c_2e^{-2t}$, $F = \{1\}$, $y_p = A$, $y = 2\,e^{-t} + 2\,e^t - 4$
49. The corresponding homogeneous equation is $y'' + 2y' - 3y = 0$, which has characteristic equation $r^2 + 2r - 3 = (r+3)(r-1) \Rightarrow r_1 = -3$, $r_2 = 1$. Then, a fundamental set of solutions for the corresponding homogeneous equation is $S = \{e^{-3t}, e^t\}$ and a general solution of the corresponding homogeneous equation is $y_h = c_1e^{-3t} + c_2e^t$. The associated set of functions for the forcing function, $f(t) = -2$ is $F = \{1\}$. Because $y = 1$ is not a solution to the corresponding homogeneous equation, we assume that a particular solution of the nonhomogeneous equation has the form $y_p = A \cdot 1 = A$. Differentiating, $y_p'' = y_p' = 0$ and substituting into the *nonhomogenous* equation gives us $-3A = -2 \Rightarrow A = 2/3$ so $y_p = 2/3$ and a general solution of the nonhomogeneous equation is $y = y_h + y_p = c_1e^{-3t} + c_2e^t + 2/3$. Application of the initial conditions yields $y = 2\,e^t - 2\,e^{-3\,t} + 2/3$.
51. $y = e^{-4\,t} + 4\,e^{-4\,t}t + 1/4$
53. The corresponding homogeneous equation is $y'' + 6y'25y = 0$, which has characteristic equation $r^2 + 6r + 25 = 0 \Rightarrow r_{1,2} = -3 \pm 4i$. Then, a fundamental set of solutions for the corresponding homogeneous equation is $S =$

$\{e^{-3t}\cos 4t, e^{-3t}\sin 4t\}$ and a general solution of the corresponding homogeneous equation is $y_h = e^{-3t}(c_1\cos 4t + c_2\sin 4t)$. The associated set of functions for the forcing function, $f(t) = -1$ is $F = \{1\}$. Because $y = 1$ is not a solution to the corresponding homogeneous equation, we assume that a particular solution of the nonhomogeneous equation has the form $y_p = A \cdot 1 = A$. Differentiating, $y_p'' = y_p' = 0$ and substituting into the *nonhomogenous* equation gives us $25A = -1 \Rightarrow A = -1/25$ so $y_p = -1/25$ and $y = y_h + y_p = e^{-3t}(c_1\cos 4t + c_2\sin 4t) - 1/25$. Application of the initial conditions yields $y = 7/4\, e^{-3\,t}\sin(4\,t) - 1/25$.

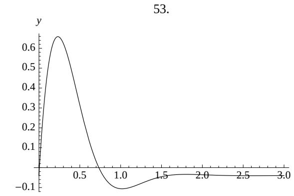
49.



51.



53.



55. $S = \{1, e^t\}$, $F_1 = \{t, 1\}$, $tF_1 = \{t^2, t\}$, $F_2 = \{t^2e^{2t}, te^{2t}, e^{2t}\}$, $y_p = At^2 + Bt + Ct^2e^{2t} + Ete^{2t} + Fe^{2t}$, $y = -2\,t^2e^{2\,t} + 6\,e^{2\,t}t - 7\,e^{2\,t} + 3/2\,t^2 + 5\,e^t + 3\,t - 3/2$

57. The corresponding homogeneous equation is $y'' + 4y' = 0$ with characteristic equation $k^2 + 4k = k(k+4) = 0 \to k = 0$ or $k = -4$ so $S = \left\{1, e^{-4t}\right\}$ is a fundamental set for the corresponding homogeneous equation and $y_h = c_1 + c_2e^{-4t}$. The forcing function is $f(t) = -24t - 6 - 4te^{-4t} + e^{-4t}$ so $F_1 = \{t,\, 1\}$ and $F_2 = \left\{te^{-4t},\, e^{-4t}\right\}$. We multiply F_1 by t and F_2 by t so that no element of $tF_1 = \left\{t^2,\, t\right\}$ is a solution of the corresponding homogeneous equation and so that no element of $tF_2 = \left\{t^2e^{-4t},\, te^{-4t}\right\}$ is a solution of the corresponding homogeneous equation. We now assume that a particular solution has the form $y_p = At^2 + Bt + Ct^2e^{-4t} + Dte^{-4t}$. Then, $y_p' = 2At + B - 4Ct^2e^{-4t} + (2C - 4D)te^{-4t} + De^{-4t}$ and $y_p'' = 2A + 16Ct^2e^{-4t} + (-16C + 16D)te^{-4t} + (2C - 8D)e^{-4t}$. Substituting into the nonhomogeneous equation

$$\begin{aligned} y_p'' + 4y_p' &= 8At + (2A + 4B) - 8Cte^{-4t} + (2C - 4D)e^{-4t} \\ &= -24t - 6 - 4te^{-4t} + e^{-4t} \end{aligned}$$

and equating coefficients gives us

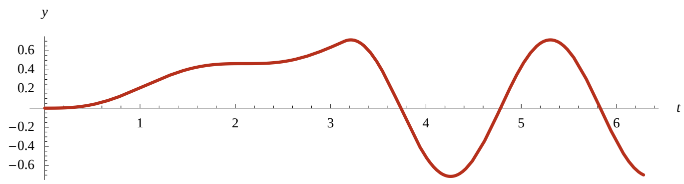
$$\begin{array}{lllll} 8A & & & & = -24 \\ 2A & +4B & & & = -6 \\ & & -8C & & = -4 \\ & & 2C & -4D & = 1 \end{array}$$

Thus, $A = -3$, $B = 0$, $C = 1/2$ and $D = 0$ so $y_p = -3t^2 + \frac{1}{2}t^2e^{-4t}$. A general solution is then $y = y_h + y_p = c_1 + c_2e^{-4t} - 3t^2 + \frac{1}{2}t^2e^{-4t}$ with derivative $y' = -6t - 2t^2e^{-4t} + te^{-4t} - 4c_2e^{-4t}$. Applying the initial conditions gives us

$$\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y'(0) &= -4c_2 = 0 \end{aligned}$$

so $c_1 = c_2 = 0$ and the solution to the initial value problem is $y = -3t^2 + \frac{1}{2}t^2 e^{-4t}$.
59. The corresponding homogeneous equation is $y'' + 9y = 0$ with general solution $y_h = c_1 \cos 3t + c_2 \sin 3t$. For $0 \le t < \pi$, a particular solution has the form $y_p = At + B$ and we find that $y_p = \frac{2}{9}t$ and for $0 \le t < \pi$, the solution to the initial value problem is $y = \frac{2}{9}t - \frac{2}{27}\sin 3t$. Evaluating $y = \frac{2}{9}t - \frac{2}{27}\sin 3t$ and y' at $t = \pi$ shows us that $y(\pi) = 2\pi/9$ and $y'(\pi) = 4/9$ so for $t \ge \pi$, we solve $y'' + 9y = 0$, $y(\pi) = 2\pi/9$, $y'(\pi) = 4/9$, which has solution $y = -\frac{2}{9}\cos 3t - \frac{4}{27}\sin 3t$. Thus,

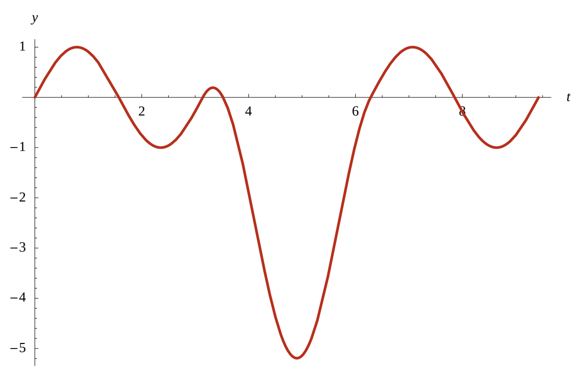
$$y = \begin{cases} \frac{2}{9}t - \frac{2}{27}\sin 3t, \ 0 \le t < \pi \\ -\frac{2}{9}\cos 3t - \frac{4}{27}\sin 3t, \ t \ge \pi \end{cases}.$$



Note: *A CAS was used to construct the solution.*

61.

$$y = \begin{cases} \sin 2t, \ 0 \le t < \pi \\ \frac{1}{2}\left(5 - 5\cos 2t + 2\sin 2t\right), \ \pi \le t < 2\pi \\ \sin 2t, \ t \ge 2\pi \end{cases}.$$



Note: *A CAS was used to construct the solution.*
63. (a) Let $y_p(t)$ be a particular solution to the nonhomogeneous equation. The characteristic equation of the corresponding homogeneous equation is $ar^2 + br + c$ with solutions $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where our convention is that r_1 correspond to the "+" solution and r_2 corresponds to the "−" solution. Note that because b is assumed to be positive, $-b$ is negative. If $b^2 - 4ac > 0$, $r_{1,2}$ are both negative (See Exercise 36 in Section 4.2) and there are constants c_1 and c_2 so that $y_1 = c_1 e^{r_1 t} + c_2 e^{r_2 t} + y_p(t)$ and there are constants b_1 and b_2 so that $y_2 = b_1 e^{r_1 t} + b_2 e^{r_2 t} + y_p(t)$. Then,

$$\lim_{t\to\infty}(y_2 - y_2) = \lim_{t\to\infty}\left((b_1 - c_1)e^{r_1 t} + (b_2 - c_2)e^{r_2 t}\right) = 0 - 0 = 0.$$

If $b^2 - 4ac = 0$, $r = r_{1,2}$ is negative and there are constants c_1 and c_2 so that $y_1 = e^{r_1 t}(c_1 + c_2 t) + y_p(t)$ and there are constants b_1 and b_2 so that $y_2 = e^{r_1 t}(b_1 + b_2 t) + y_p(t)$. Subtracting and taking the limit as $t \to \infty$ yields 0. Finally, if $b^2 - 4ac < 0$, $-b/(2a) < 0$ and there are constants c_1 and c_2 so that

$$y_1 = e^{-bt/(2a)}\left(c_1 \cos\left(\frac{\sqrt{4ac - b^2}}{2a}t\right) + c_2 \sin\left(\frac{\sqrt{4ac - b^2}}{2a}t\right)\right) + y_p(t)$$

and there are constants b_1 and b_2 so that

$$y_2 = e^{-bt/(2a)}\left(b_1 \cos\left(\frac{\sqrt{4ac - b^2}}{2a}t\right) + b_2 \sin\left(\frac{\sqrt{4ac - b^2}}{2a}t\right)\right) + y_p(t)$$

Subtracting and taking the limit as $t \to \infty$ yields 0.
(b) No. For example both $y_1(t) = \frac{1}{2}t \sin t$ and $y_2(t) = \sin t + \frac{1}{2}t \sin t$ are solutions of $y'' + y = \cos t$ and $\lim_{t\to\infty}(y_2(t) - y_1(t)) = \lim_{t\to\infty} \sin t$ does not exist.
(c) Refer to Exercise in Section 4.2. Every solution, $y_h(t)$, of $ay'' + by' + cy = 0$ has the property that $\lim_{t\to\infty} y_h(t) = 0$. A particular solution of $ay'' + by' + cy = 0$ is $y_p = k/c$ so a general solution is $y = y_h(t) + k/c$. Thus, $\lim_{t\to\infty}(y_h(t) + k/c) = 0 + k/c = k/c$.
(d) $y = c_1 + c_2 e^{-bt/a} + kt/b$ and $\lim_{t\to\infty} y(t) = \infty$ if $k > 0$ and $\lim_{t\to\infty} y(t) = -\infty$ if $k < 0$.
(e) $y = c_1 + c_2 t + \frac{1}{2}kt^2$; $\lim_{t\to\infty} y(t) = \infty$ if $k > 0$ and $\lim_{t\to\infty} y(t) = -\infty$ if $k < 0$.
67. $\omega = 2$ because a general solution of $y'' + 4y = 0$ is $y_h = c_1 \cos 2t + c_2 \sin 2t$
69. Resonance occurs when $c = 3$. For $c \neq 3$, $x(t) = c_1 \cos 3t + c_2 \sin 3t - \frac{1}{c^2 - 9} \sin ct$. If $c = 3$, $x(t) = c_1 \cos 3t + c_2 \sin 3t - \frac{1}{6}t \cos 3t$.

Exercises 4.4

1. A fundamental set of solutions for the corresponding homogeneous equation is $S = \{\cos 2t, \sin 2t\}$ and a general solution is $y_h = c_1 \cos 2t + c_2 \sin 2t$. Assuming that a particular solution has the form $y_p = u_1 y_1 + u_2 y_2$, we first compute $W(S) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$. Next,

$$u_1 = \int \frac{1}{W(S)} \begin{vmatrix} 0 & \sin 2t \\ 1 & 2\cos 2t \end{vmatrix} dt = \frac{1}{2}\int -\sin 2t \, dt = \frac{1}{4}\cos 2t$$

and

$$u_2 = \int \frac{1}{W(S)} \begin{vmatrix} \cos 2t & 0 \\ -2\sin 2t & 1 \end{vmatrix} dt = \frac{1}{2}\int \cos 2t \, dt = \frac{1}{4}\sin 2t.$$

Therefore, a particular solution is $y_p = u_1 y_1 + u_2 y_2 = \frac{1}{4}\cos^2 2t + \frac{1}{4}\sin^2 2t = \frac{1}{4}$ and a general solution is $y = y_h + y_p = c_1 \sin(2t) + c_2 \cos(2t) + 1/4$ (Undetermined coefficients is easier.)
3. $y = c_1 e^{5t} + c_2 e^{2t} - \frac{1}{2} e^{3t}$

5. $y = c_1 e^{-2\,t}\sin(4\,t) + c_2 e^{-2\,t}\cos(4\,t) + \frac{1}{8}\,te^{-2\,t}$
7. $S = \{\cos(4t), \sin(4t)\}$, $y_h = c_1\cos(4t) + c_2\sin(4t)$, $W(S) = 4$. $y_p = u_1 y_1 + u_2 y_2 \Rightarrow$

$$u_1 = \int \frac{1}{W(S)} \begin{vmatrix} 0 & \sin 4t \\ \csc 4t & 4\cos 4t \end{vmatrix} dt = \frac{1}{4}\int -1\,dt = -\frac{1}{4}t.$$

and

$$u_2 = \int \frac{1}{W(S)} \begin{vmatrix} \cos 4t & 0 \\ -4\sin 4t & \csc 4t \end{vmatrix} dt = \frac{1}{4}\int \cot 4t\,dt = -\frac{1}{16}\ln\csc 4t = \frac{1}{16}\ln\sin 4t.$$

Thus, $y = y_h + y_p = c_1\sin(4\,t) + c_2\cos(4\,t) + 1/16\,\ln(\sin(4\,t))\sin(4\,t) - 1/4\,t\cos(4\,t)$
9. $y(t) = c_1 e^{-t}\sin(7t) + c_2 e^{-t}\cos(7t) - \frac{1}{7}e^{-t}t\cos(7t) + \frac{1}{49}e^{-t}\sin(7t)\log(\sin(7t))$
11. $S = \{e^t\cos 5t, e^t\sin 5t\}$, $y_h = e^t(c_1\cos 5t + c_2\sin 5t)$, $W(S) = 5e^{2t}$. $y_p = u_1 y_1 + u_2 y_2 \Rightarrow$

$$u_1 = \int \frac{1}{W(S)} \begin{vmatrix} 0 & e^t\sin 5t \\ e^t(\sec 5t + \csc 5t) & e^t(5\cos 5t + \sin 5t) \end{vmatrix} dt = -\frac{1}{5}\int \sin 5t\,(\sec 5t + \csc 5t)\,dt$$
$$= -\frac{1}{5}\int(\tan 5t + 1)\,dt = -\frac{1}{5}\left(\frac{1}{5}\ln\sec 5t + t\right)$$

and

$$u_2 = \int \frac{1}{W(S)} \begin{vmatrix} e^t\cos 5t & 0 \\ e^t(\cos 5t - 5\sin 5t) & e^t(\sec 5t + \csc 5t) \end{vmatrix} dt$$
$$= \frac{1}{5}\int(1 + \cot 5t)\,dt = \frac{1}{5}\left(t - \frac{1}{5}\ln\csc 5t\right) = \frac{1}{5}\left(t + \frac{1}{5}\ln\sin 5t\right).$$

Therefore,

$$y = y_h + y_p = c_1 e^t\sin(5\,t) + c_2 e^t\cos(5\,t) - 1/5\,\left(-1/5\,\cos(5\,t)\ln(\cos(5\,t)) - 1/5\,\sin(5\,t)\ln(\sin(5\,t)) + t\,(-\sin(5\,t) + \cos(5\,t))\right)$$

13. $S = \{e^{3t}\cos 5t, e^{3t}\sin 5t\}$, $y_h = e^{3t}(c_1\cos 5t + c_2\sin 5t)$, $W(S) = 5e^{6t}$. $y_p = u_1 y_1 + u_2 y_2 \Rightarrow$

$$u_1 = \int \frac{1}{W(S)} \begin{vmatrix} 0 & e^{3t}\sin 5t \\ e^{3t}\tan 5t & e^{3t}(5\cos 5t + 3\sin 5t) \end{vmatrix} dt$$
$$= -\frac{1}{5}\int \sin 5t\tan 5t\,dt = -\frac{1}{5}\int \frac{\sin^2 5t}{\cos 5t}\,dt = -\frac{1}{5}\int \frac{1-\cos^2 5t}{\cos 5t}\,dt$$
$$= -\frac{1}{5}\int(\sec 5t - \cos 5t)\,dt = -\frac{1}{25}\left(\ln(\sec 5t + \tan 5t) - \sin 5t\right)$$
$$= -\frac{1}{25}\left(\ln\left(\frac{1 + \sin 5t}{\cos 5t}\right) - \sin 5t\right)$$

and

$$u_2 = \int \frac{1}{W(S)} \begin{vmatrix} e^{3t}\cos 5t & 0 \\ e^{3t}(3\cos 5t - 5\sin 5t) & e^{3t}\tan 5t \end{vmatrix} dt$$
$$= \frac{1}{5}\int \sin 5t\,dt = -\frac{1}{25}\cos 5t.$$

Therefore, $y = c_1 e^{3\,t}\sin(5\,t) + c_2 e^{3\,t}\cos(5\,t) - 1/25\,e^{3\,t}\cos(5\,t)\ln\left(\dfrac{1+\sin(5\,t)}{\cos(5\,t)}\right)$

15. $y = c_1 e^{6\,t}\sin(t) + c_2 e^{6\,t}\cos(t) + e^{6\,t}\left(t\sin(t) + \ln(\cos(t))\cos(t)\right)$

17. $y = c_1 e^{3t} + c_2 e^{-3t} + \frac{1}{18}e^{-3t}\left(-e^{3t} + e^{6t}\ln\left(e^{-3t}+1\right) - \ln\left(e^{3t}+1\right)\right)$

19. $y = c_1 e^{t} + c_2 e^{-t} + 1/2\,e^{-t}\left((t + 1/2\,\cosh(2\,t) - 1/2\,\sinh(2\,t))\,e^{2\,t} - 1/2\,\cosh(2\,t) - 1/2 +\right.$

21. $y = c_1 e^{2\,t} + c_2 e^{2\,t} - e^{2\,t}(\ln(t) + 1)$

23. $S = \{e^{-3t}, te^{-3t}\}$, $W(S) = e^{-6t}$, $y_h = e^{-3t}(c_1 + c_2 t)$. $y_p = u_1 y_1 + u_2 y_2 \Rightarrow$

$$u_1 = \int \frac{1}{W(S)}\begin{vmatrix} 0 & te^{-3t} \\ t^{-1}e^{-3t} & e^{-3t}(1-t)\end{vmatrix}dt = \int -1\,dt = -t$$

and

$$u_2 = \int \frac{1}{W(S)}\begin{vmatrix} e^{-3t} & 0 \\ -3e^{-3t} & t^{-1}e^{-3t}\end{vmatrix}dt = \int t^{-1}\,dt = \ln t.$$

Therefore, $y = c_1 e^{-3\,t} + c_2 e^{-3\,t}t + t\left(-1 + \ln(t)\right)e^{-3\,t}$

25. $y = \left(2\,\dfrac{(\tan(1/2\,e^{t}))^2}{\left(1 + (\tan(1/2\,e^{t}))^2\right)e^{t}} - c_1 e^{-t} + c_2\right)e^{-t}$

27. $S = \{e^t, te^t\}$, $y_h = e^t(c_1 + c_2 t)$, $W(S) = e^{2t}$. $y_p = u_1 y_1 + u_2 y_2 \Rightarrow$

$$u_1 = \int \frac{1}{W(S)}\begin{vmatrix} 0 & te^{t} \\ e^{t}\sqrt{1-t^2} & e^{t}(1+t)\end{vmatrix}dt = -\int t\sqrt{1-t^2}dt = -\frac{1}{3}\sqrt[3]{1-t^2}$$

and

$$u_2 = \int \frac{1}{W(S)}\begin{vmatrix} e^{t} & 0 \\ e^{t} & e^{t}\sqrt{1-t^2}\end{vmatrix}dt = \int \sqrt{1-t^2}\,dt = \frac{1}{2}\left(t\sqrt{1-t^2} + \sin^{-1} t\right).$$

Therefore, $y = y_h + y_p = c_1 e^t + c_2 e^t t + \frac{1}{6}\,e^{t}\left(2\sqrt{1-t^2} + \sqrt{1-t^2}t^2 + 3\,t\arcsin(t)\right)$

29. $y = c_1 e^{2\,t} + c_2 e^{2\,t}t + \frac{1}{2}\,e^{2\,t}\left(\arctan(t)\,t^2 + t - \arctan(t) - t\ln\left(t^2+1\right)\right)$

31. $S = \{\cos(t/2), \sin(t/2)\}$,$y_h = c_1\cos(t/2) + c_2\sin(t/2)$, $W(S) = 1/2$. Assuming that $y_p = u_1 y_1 + u_2 y_2 \Rightarrow$

$$u_1 = \int \frac{1}{W(S)}\begin{vmatrix} 0 & \sin\left(\frac{1}{2}t\right) \\ \sec\left(\frac{1}{2}t\right) + \csc\left(\frac{1}{2}t\right) & \frac{1}{2}\cos\left(\frac{1}{2}t\right)\end{vmatrix}dt = -2\int\left(\tan\left(\frac{1}{2}t\right) + 1\right)\,dt$$
$$= -2\left(2\ln\sec\left(\frac{1}{2}t\right) + t\right)$$

and

$$u_2 = \int \frac{1}{W(S)}\begin{vmatrix} \cos\left(\frac{1}{2}t\right) & 0 \\ -\frac{1}{2}\sin\left(\frac{1}{2}t\right) & \sec\left(\frac{1}{2}t\right) + \csc\left(\frac{1}{2}t\right)\end{vmatrix}dt = 2\int\left(1 + \cot\left(\frac{1}{2}t\right)\right)\,dt$$
$$= 2\left(t - \ln\csc\left(\frac{1}{2}t\right)\right) = 2\left(2 + \ln\sin\left(\frac{1}{2}t\right)\right).$$

Therefore, $y = y_h + y_p = c_1\sin(1/2\,t) + c_2\cos(1/2\,t) + \cos(1/2\,t)\ln(\cos(1/2\,t)) + \sin(1/2\,t)\ln(\sin(1/2\,t)) - 1/2\,t\left(-\sin(1/2\,t) + \cos(1/2\,t)\right)$

33. $y = c_1 \cos 3t + c_2 \sin 3t + \frac{1}{9} \cos 3t \ln \cos 3t + \frac{1}{3} t \sin 3t$

35. $y_h = c_1 \cos 2t + c_2 \sin 2t$ and $S = \{\cos 2t, \sin 2t\}$ so $W(S) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$. Then, $u_1' = \frac{1}{2} \begin{vmatrix} 0 & \sin 2t \\ \tan 2t & 2\cos 2t \end{vmatrix} = -\frac{1}{2} \sin 2t \tan 2t$ and $u_2' = \frac{1}{2} \begin{vmatrix} \cos 2t & 0 \\ -2\sin 2t & \tan 2t \end{vmatrix} = \sin t \cos t$ so $u_1 = -\frac{1}{4} (\ln(\sec 2t + \tan 2t) - \sin 2t)$ and $u_2 = \frac{1}{2} \cos^2 2t$ so $y_p = -\frac{1}{4} \cos 2t (\ln(\sec 2t + \tan 2t) - \sin 2t) + \frac{1}{2} \sin 2t \cos^2 2t$ and $y = y_h + y_p = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{4} \cos 2t (\ln(\sec 2t + \tan 2t) - \sin 2t) + \frac{1}{2} \sin 2t \cos^2 2t$.

37. $y_h = c_1 \cos 2t + c_2 \sin 2t$ and $S = \{\cos 2t, \sin 2t\}$ so $W(S) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$. Then, $u_1' = \frac{1}{2} \begin{vmatrix} 0 & \sin 2t \\ \tan t & 2\cos 2t \end{vmatrix} = -\sin^2 t$ and $u_2' = \frac{1}{2} \begin{vmatrix} \cos 2t & 0 \\ -2\sin 2t & \tan t \end{vmatrix} = \sin t \cos t$ so $u_1 = \frac{1}{4} (\sin 2t - 2t)$ and $u_2 = \frac{1}{2} \left(\ln \cos t - \frac{1}{2} \cos 2t\right)$ so $y_p = \frac{1}{4} \cos 2t (\sin 2t - 2t) + \frac{1}{2} \sin 2t \left(\ln \cos t - \frac{1}{2} \cos 2t\right)$ and $y = y_h + y_p = c_2 \cos 2t + c_2 \sin 2t + \frac{1}{4} \cos 2t (\sin 2t - 2t) + \frac{1}{2} \sin 2t \left(\ln \cos t - \frac{1}{2} \cos 2t\right)$.

39. $y_h = c_1 \cos 2t + c_2 \sin 2t$ and $S = \{\cos 2t, \sin 2t\}$ so $W(S) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$. Then, $u_1' = \frac{1}{2} \begin{vmatrix} 0 & \sin 2t \\ \sec 2t \tan 2t & 2\cos 2t \end{vmatrix} = -\frac{1}{2} \tan^2 2t$ and $u_2' = \frac{1}{2} \begin{vmatrix} \cos 2t & 0 \\ -2\sin 2t & \sec 2t \tan 2t \end{vmatrix} = \frac{1}{2} \tan 2t$ so $u_1 = \frac{1}{4} (2t - \tan 2t)$ and $u_2 = \frac{1}{4} \ln \sec 2t$ so

$$\begin{aligned} y_p &= \frac{1}{4} \cos 2t (2t - \tan 2t) + \frac{1}{4} \sin 2t \ln \sec 2t \\ &= \frac{1}{2} t \cos 2t + \frac{1}{4} \sin 2t \ln \sec 2t \end{aligned}$$

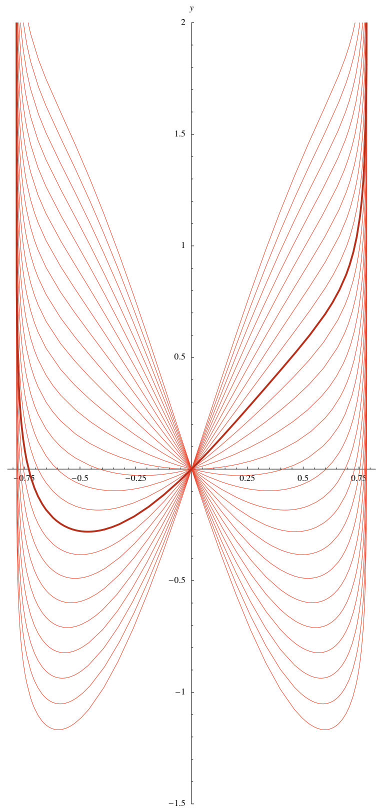
because $-\frac{1}{4} \cos 2t \tan 2t = -\frac{1}{4} \sin 2t$ is a solution of the corresponding homogeneous equation. Hence, a general solution is $y = y_h + y_p = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{2} t \cos 2t + \frac{1}{4} \sin 2t \ln \sec 2t$.

41. $y_h = c_1 \cos 2t + c_2 \sin 2t$ and $S = \{\cos 2t, \sin 2t\}$ so $W(S) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$. Then, $u_1' = \frac{1}{2} \begin{vmatrix} 0 & \sin 2t \\ \sec^2 2t & 2\cos 2t \end{vmatrix} = -\frac{1}{2} \sec 2t \tan 2t$ and $u_2' = \frac{1}{2} \begin{vmatrix} \cos 2t & 0 \\ -2\sin 2t & \sec^2 2t \end{vmatrix} = \frac{1}{2} \sec 2t$ so $u_1 = -\frac{1}{4} \sec 2t$ and $u_2 = \frac{1}{4} \ln(\sec 2t + \tan 2t)$ so $y_p = -\frac{1}{4} + \frac{1}{4} \sin 2t \ln(\sec 2t + \tan 2t)$ and $y = y_h + y_p = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{4} + \frac{1}{4} \sin 2t \ln(\sec 2t + \tan 2t)$ with first derivative $y' = \frac{1}{2} [\cos 2t (4c_2 + \ln(\sec 2t + \tan 2t)) - 4c_1 \sin 2t + \tan 2t]$.

Application of the initial conditions results in

$$\begin{aligned} -\frac{1}{4} + c_1 &= 0 \\ 2c_2 &= 1 \end{aligned}$$

so $c_1 = 1/4$ and $c_2 = 1/2$ and the solution to the initial value problem is $y = \frac{1}{4} \cos 2t + \frac{1}{2} \sin 2t - \frac{1}{4} + \frac{1}{4} \sin 2t \ln(\sec 2t + \tan 2t)$. The following figure shows the graph of $y'' + 4y = \sec^2 2t$, $y(0) = 0$, $y'(0) = c$ for various values of c.



43. $S = \{\cos t, \sin t\}$, $y_h = c_1 \cos t + c_2 \sin t$, $W(S) = 1$. $y_p = u_1 y_1 + u_2 y_2 \Rightarrow$

$$\begin{aligned} u_1 &= \int \frac{1}{W(S)} \begin{vmatrix} 0 & \sin t \\ \tan^2 t & \cos t \end{vmatrix} dt = -\int \sin t \tan^2 t \, dt \\ &= -\int \sin t \left(\sec^2 t - 1\right) dt = -\int \left(\sec t \tan t - \sin t\right) dt \\ &= -\sec t - \cos t \end{aligned}$$

and

$$\begin{aligned} u_2 &= \int \frac{1}{W(S)} \begin{vmatrix} \cos t & 0 \\ -\sin t & \tan^2 t \end{vmatrix} dt = \int \frac{\sin^2 t}{\cos t} dt \\ &= \int \frac{1 - \cos^2 t}{\cos t} dt = \int \left(\sec t - \cos t\right) dt \\ &= \ln(\sec t + \tan t) - \sin t. \end{aligned}$$

Therefore, a general solution is given by $y = y_h + y_p = y_h + u_1 y_1 + u_2 y_2 = c_1 \cos(t) + \sin(t) \left(c_2 - \log\left(\cos\left(\frac{t}{2}\right) - \sin\left(\frac{t}{2}\right)\right) + \log\left(\sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)\right)\right) - 2$. Applying the initial conditions gives us $y = 2\cos t + \sin t - 2 + \sin t - \ln(\sec t +$

$\tan t$).
45. $y = \left(-\frac{1}{6}\sqrt{2} + \frac{1}{36}\pi\right)\cos 3t + \frac{1}{36}\sqrt{2}(6+\sqrt{2}\ln 2)\sin 3t - \frac{1}{3}t\cos 3t + \frac{1}{9}\sin 3t \ln\sin 3t$
51. $y = \frac{1}{k}\left[\left(\int_0^t f(z)\cos kz\,dz\right)\sin kt - \left(\int_0^t f(z)\sin kz\,dz\right)\cos kt\right]$
53. $y = \frac{1}{2k}\left[\left(\int_0^t e^{-kz}f(z)\,dz\right)e^{kt} - \left(\int_0^t e^{kz}f(z)\,dz\right)e^{-kt}\right]$
55. $S = \{t^{-1}, t^{-1}\ln t\}$, $y_h = t^{-1}(c_1 + c_2\ln t)$, $W(S) = t^{-3}$. Assuming that $y_p = u_1y_1 + u_2y_2 \Rightarrow$

$$\begin{aligned} u_1 &= \int \frac{1}{W(S)}\begin{vmatrix} 0 & t^{-1}\ln t \\ \ln t & t^{-2}(1-\ln t)\end{vmatrix} dt = -\int t^2(\ln t)^2\,dt \\ &= -\frac{2}{27}t^3 + \frac{2}{9}t^2\ln t - \frac{1}{3}(\ln t)^2 \end{aligned}$$

and

$$\begin{aligned} u_2 &= \int \frac{1}{W(S)}\begin{vmatrix} t^{-1} & 0 \\ -t^{-2} & \ln t\end{vmatrix} dt = \int t^2\ln t\,dt \\ &= -\frac{1}{9}t^3 + \frac{1}{3}t^3\ln t. \end{aligned}$$

Therefore, $y = y_h + y_p = c_1t^{-1} + c_2t^{-1}\ln t + \ln t - 2$
57. $y = c_1t^6 + c_2t^{-1} - \frac{1}{3}\ln t + \frac{5}{18}$
59. If $y = e^{u(t)} = e^u$, then $y' = e^u u'$ and $y'' = e^u(u')^2 + e^u u''$. Substitution into the equation and dividing bye^{2u} gives us

$$\begin{aligned} e^{-2t}\left[e^u\cdot(e^u(u')^2 + e^u u'') - e^{2u}(u')^2\right] - 2t(1+t)e^{2u} = 0 \\ e^{-2t}\left[((u')^2 + u'') - (u')^2\right] - 2t(1+t) = 0 \\ u'' = 2t(1+t)e^{2t}. \end{aligned}$$

Integrating twice gives us $u = \frac{1}{2}t^2e^{2t} - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + c_1t + c_2$ so $y = \exp\left(\frac{1}{2}t^2e^{2t} - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + c_1t + c_2\right)$.
61. (a) $y = t^2(c_1\cos t + c_2\sin t) + t$ (b) $y = t^2(c_1\cos t + c_2\sin t) + t$ (infinitely many solutions)
63. (a) $y = t^{-1/2}(c_1\cos 2t + c_2\sin 2t) + t^{-1/2}$ (b) No solution (c) $y = t^{-1/2}(1 - \cos 2t)$ 65. A general solution is $y = c_1t + c_2\sin t + \cos t$ and the solution to the initial value problem is $y = \frac{4\sqrt{2}}{\pi-4}t + \frac{\pi+4}{\pi-4}\sin t + \cos t$.

Exercises 4.5

1. $W(S) = \begin{vmatrix} 3t^2 & t & 2t - 2t^2 \\ 6t & 1 & 2 - 4t \\ 6 & 0 & -4 \end{vmatrix} = 0$, linearly dependent

3. $W(S) = \begin{vmatrix} e^{-t} & e^{3t} & te^{3t} \\ -e^{-t} & 3e^{3t} & e^{3t} + 3te^{3t} \\ e^{-t} & 9e^{3t} & 6e^{3t} + 9te^{3t} \end{vmatrix} = 16e^{5t}$, linearly independent

5. $W(S) = \begin{vmatrix} 1 & t & t^2 & t^3 & t^4 \\ 0 & 1 & 2t & 3t^2 & 4t^3 \\ 0 & 0 & 2 & 6t & 12t^2 \\ 0 & 0 & 0 & 6 & 24t \\ 0 & 0 & 0 & 0 & 24 \end{vmatrix} = 288$, linearly independent

7. 4th order, $y = (c_1 + c_2 t + c_3 t^2)e^{-2t} + c_4 e^{2t}$

9. 4th order, $y = c_1 + c_2 t + c_3 \cos 3t + c_4 \sin 3t$

11. 6th order, $y = e^{-3t}\left[(c_1 + c_2 t)\cos 4t + (c_3 + c_4 t)\sin 4t\right] + c_5 e^{-5t} + c_6 e^{-t/3}$

13. All are solutions of $y''' = 0$ but the Wronskian of the set is 0 so they cannot be a fundamental set.

15. Yes: $y^{(4)} - \frac{1}{4}y''' - 2y'' - \frac{7}{2}y' + y = 0$

17. $k = 0$ with multiplicity 3 is the solution of the characteristic equation so $S = \left\{1,\, t,\, t^2\right\}$ is a fundamental set of solutions and, thus, $y = c_0 + c_1 t + c_2 t^2$ is a general solution.

19. $y = c_1 + c_2\, e^{5\,t} + c_3\, e^{5\,t} t$

21. The characteristic equation is $k^4 + 16k^2 = 0$ and the left-hand side of the equation factors resulting in $k^2(k^2 + 16) = 0$ so $k_{1,2} = 0$ is a 0 of multiplicity 2 and $k^2 + 16 = 0$ has roots $k_{3,4} = \pm 4i$. Thus, a fundamental set of solutions is $S = \{1,\, t,\, \cos 4t,\, \sin 3t\}$ and a general solution of the equation is $y = c_1 + c_2 t + c_3 \cos 4t + c_4 \sin 4t$.

23. The characteristic equation is $3k^3 - 4k^2 - 5k + 2 = 0$. Using synthetic division, we see that $k = -1$ is a zero of $3k^3 - 4k^2 - 5k + 2$:

$$\begin{array}{r|rrrr} -1 & 3 & -4 & -5 & 2 \\ \hline & 3 & -7 & 2 & 0 \end{array}$$

This means that $3k^3 - 4k^2 - 5k + 2 = (k+1)(3k^2 - 7k + 2)$ and the quadratic factors quickly so we see that $3k^3 - 4k^2 - 5k + 2 = (k+1)(3k^2 - 7k + 2) = (k+1)(3k-1)(k-2) = 0$ so $k_1 = -1$ has corresponding solution $y_1 = e^{-t}$, $k_2 = 1/3$ has corresponding solution $y_2 = e^{t/3}$, and $k_3 = 2$ has corresponding solution $y_3 = e^{2t}$. A fundamental set of solutions is then $S = \left\{e^{-t},\, e^{t/3},\, e^{2t}\right\}$ and a general solution is $y = c_1 e^{-t} + c_2 e^{t/3} + c_3 2t$.

25. The characteristic equation is $k^3 - 5k + 2 = 0$. Using synthetic division, we see that $k = 2$ is a zero of $k^3 - 5k + 2$:

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -5 & 2 \\ \hline & 1 & 2 & -1 & 0 \end{array}$$

This means that $k^3 - 5k + 2 = (k-2)(k^2 - 5k + 2)$ and although $k^2 + 2k - 1$ does not "factor easily," using the quadratic formula we see that $k^2 + 2k - 1 = 0$ if $k_{2,3} = -1 \pm \sqrt{2}$. Thus, $k_1 = 2$ has corresponding solution $y_1 = e^{2t}$, $k_2 = -1 - \sqrt{2}$ has corresponding solution $y_2 = e^{(-1-\sqrt{2})t}$, and $k_3 = -1 + \sqrt{2}$ has corresponding solution $y_3 = e^{(-1+\sqrt{2})t}$ so a fundamental set of solutions is $S = \left\{e^{2t},\, e^{(-1-\sqrt{2})t},\, e^{(-1+\sqrt{2})t}\right\}$ and a general solution is $y = c_1 e^{2t} + c_2 e^{(-1-\sqrt{2})t} +$

$c_3e^{(-1+\sqrt{2})t}$.
27. The characteristic equation is $k^4+k^3=k^3(k+1)=0$ so $k_{1,2,3}=0$ is a zero of multiplicity three and $k_4=-1$ is a zero of multiplicity one. A fundamental set is $S=\{1,\, t,\, t^2,\, e^{-t}\}$; a general solution is $y=c_1+c_2t+c_3t^2+c_4e^{-t}$.
29. The characteristic equation is $r^4-16=(r^2+4)(r^2-4)=0$ with solutions $r_1=2$, $r_2=-2$, and $r_{3,4}=\pm 2i$. Therefore, four linearly independent solutions to the equation are $y_1=e^{2t}$, $y_2=e^{-2t}$, $y_3=\cos 2t$, and $y_4=\sin 2t$. A general solution is then $y=c_1\,e^{-2\,t}+c_2\,e^{2\,t}+c_3\,\sin(2\,t)+c_4\,\cos(2\,t)$
31. The characteristic equation is $r^4+7r^3+6r^2-32r-32=0$. Observe that the first three terms have a common factor of r^2 and the last two have a common factor of -32 so the equation can be rewritten as $r^2(r^2+7r+6)-32(r+1)=0$. $r^2+7r+6=(r+6)(r+1)$ so the characteristic equation further factors as $r^2(r+6)(r+1)-32(r+1)=(r+1)[r^2(r+6)-32]=(r+1)(r-2)(r+4)^2=0$. The solutions of the characteristic equation are $r_1=-1$ with corresponding solution $y_1=e^{-t}$, $r_2=2$ with corresponding solution $y_2=e^{2t}$, $r_{3,4}=-4$ with solutions $y_3=e^{-4t}$, and $t_4=te^{-4t}$. Therefore, a general solution is $y=c_1\,e^{2\,t}+c_2\,e^{-t}+c_3\,e^{-4\,t}+c_4\,e^{-4\,t}t$
33. The characteristic equation is $k^5+4k^4=k^4(k+4)=0$ so $k_{1,2,3,4}=0$ is a zero of multiplicity four and $k_5=-4$ is a zero of multiplicity one. Four linearly independent solutions corresponding to $k_{1,2,3,4}=0$ are $y_1=1$, $y_2=t$, $y_3=t^2$, and $y_4=t^3$. One solution corresponding to $k_5=-4$ is $y_5=e^{-4t}$. Therefore, a fundamental set is $S=\{1,\, t,\, t^2, t^3,\, e^{-4t}\}$; a general solution is $y=c_1+c_2t+c_3t^2+c_4t^3+c_5e^{-4t}$.
35. The characteristic equation is $k^5+3k^4+3k^3+k^2=k^2(k^3+3k^2+3k+1)=k^2(k+1)^3=0$ so $k_{1,2}=0$ is a zero of multiplicity two with corresponding solutions $y_1=1$ and $y_2=t$ and $k_{3,4,5}=-1$ is a zero of multiplicity three with corresponding solutions $y_3=e^{-t}$, $y_4=te^{-t}$, and $y_5=t^2e^{-t}$. Thus, a fundamental set is $S=\{1,\, t,\, e^{-t},\, te^{-t},\, t^2e^{-t}\}$ and a general solution is $y=c_1+c_2t+c_3e^{-t}+c_4te^{-t}+c_5t^2e^{-t}$
37. The characteristic equation is $r^4+8r^2+16=(r^2+4)^2=0$, which has solution $r_{1,2,3,4}=-2i$ so four linearly independent solutions are $y_1=\cos 2t$, $y_2=t\cos 2t$, $y_3=\sin 2t$, $y_4=t\sin 2t$. Thus, a general solution is $y=(c_1+c_2t)\cos 2t+(c_3+c_4t)\sin 2t$
39. The solutions of $r^6+12r^4+48r^2+64=0$ are $r_{1,2,3,4,5,6}=\pm 2i$. Thus, six linearly independent solutions are $y_1=\cos 2t$, $y_2=t\cos 2t$, $y_3=t^2\cos 2t$, $y_4=\sin 2t$, $y_5=t\sin 2t$, and $y_6=t^2\sin 2t$. Thus, a general solution is $y=(c_1+c_2t+c_3t^2)\cos 2t+(c_4+c_5t+c_6t^2)\sin 2t$.
41. The characteristic equation of the corresponding homogeneous equation is $r^3-1=(r-1)(r^2+r+1)=0$. One solution of the characteristic equation is $r_1=1$ with corresponding solution of the differential equation $y_1=e^t$. Using the quadratic formula to solve $r^2+r+1=0$, we obtain $r_{1,2}=\frac{1}{2}\left(-1\pm\sqrt{3}i\right)$ so two more linearly independent solutions are $y_2=e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$ and $y_3=e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$. A general solution is then $y=c_1e^t+e^{-t/2}\left(c_2\cos\left(\frac{\sqrt{3}}{2}t\right)+c_3\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$. Applying the initial conditions results in $y=e^t-\sqrt{3}e^{-1/2\,t}\sin\left(1/2\,\sqrt{3}t\right)-$

$e^{-1/2\,t}\cos\left(1/2\sqrt{3}t\right)$
43. A general solution of the equation is $y=(c_1+c_2t)e^{2t}+(c_3+c_4t)e^{-2t}$. Applying the initial conditions yields $y=t(e^{2t}-e^{-2t})$.
45. The characteristic equation is $k^4-5k^2+4=(k^2-4)(k^2-1)=(k+2)(k-2)(k+1)(k-1)=0$. A solution corresponding to $k_1=-2$ is $y_1=e^{-2t}$, a solution corresponding to $k_2=2$ is $y_2=e^{2t}$, a solution corresponding to $k_3=-1$ is $y_3=e^{-t}$ and a solution corresponding to $k_4=1$ is $y_4=e^t$. Thus, a fundamental set is $\left\{e^{-2t},\,e^{2t},\,e^{-t},\,e^t\right\}$ and a general solution is $y=c_1e^{-2t}+c_2e^{2t}+c_3e^{-t}+c_4e^t$ with derivatives $y'=-2c_1e^{-2t}+2c_2e^{2t}-c_3e^{-t}+c_4e^t$, $y'=4c_1e^{-2t}+4c_2e^{2t}+c_3e^{-t}+c_4e^t$, and $y'=-8c_1e^{-2t}+8c_2e^{2t}-c_3e^{-t}+c_4e^t$. Applying the initial conditions gives us

$$\begin{aligned} c_1+c_2+c_3+c_4&=-1\\ -2c_1+2c_2-c_3+c_4&=3\\ 4c_1+4c_2+c_3+c_4&=-7\text{'}\\ -8c_1+8c_2-c_3+c_4&=15 \end{aligned}$$

which has solution $c_1=-2$, $c_2=0$, $c_3=1$, and $c_4=0$ so the solution to the initial value problem is $y=e^{-t}-2e^{-2t}$.
47. The characteristic equation for the corresponding homogeneous equation is $8k^5+4k^4+66k^3-41k^2-37k=k(8k^4+4k^3+66k^2-41k-37)=0$. This shows us that $k_1=0$ is a solution of the characteristic equation with multiplicity one; a solution of the differential equation corresponding to $k_1=0$ is $y_1=1$. Now, by inspection we see that $k_2=1$, with corresponding solution $y_2=e^t$, is a zero of $8k^4+4k^3+66k^2-41k-37$ and using synthetic division,

$$\begin{array}{r|rrrrr} 1 & 8 & 4 & 66 & -41 & -37\\ \hline & 8 & 12 & 78 & 37 & 0 \end{array}\text{'}$$

shows us that $8k^4+4k^3+66k^2-41k-37=(k-1)(8k^3+12k^2+78k+37$. By the Rational Root theorem, the *possible* rational roots of $8k^3+12k^2+78k+37$ are ±1, $\pm1/2$, $\pm1/4$, $\pm1/8$, ±37, $\pm37/2$, $\pm37/4$, and $\pm37/8$. By inspection we note that ±1 are *not* roots of $8k^3+12k^2+78k+37$. Testing $1/2$,

$$\begin{array}{r|rrrr} 1/2 & 8 & 12 & 78 & 37\\ \hline & 8 & 16 & 86 & 80 \end{array}\text{'}$$

shows us that $k=1/2$ is *not* a root of $8k^3+12k^2+78k+37$. On the other hand, testing $-1/2$,

$$\begin{array}{r|rrrr} -1/2 & 8 & 12 & 78 & 37\\ \hline & 8 & 8 & 74 & 0 \end{array}\text{'}$$

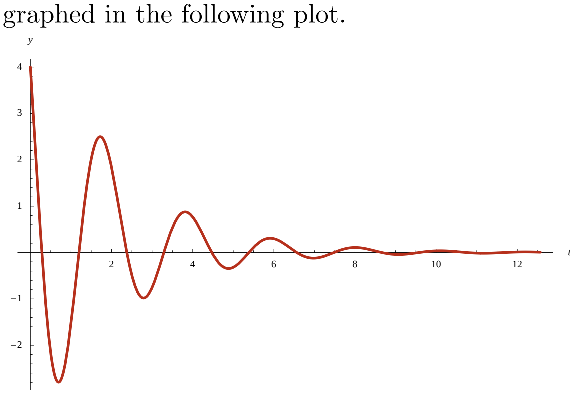
shows us that $k=-1/2$ *is* a root of $8k^3+12k^2+78k+37$ and that $8k^3+12k^2+78k+37=(k-1/2)(8x^2+8x+74)=(2k-1)(4k^2+4k+37)$. A solution corresponding to $k_3=1/2$ is $y_3=e^{t/2}$. We use the quadratic formula to solve $4k^2+4k+37=0$ and obtain $k_{4,5}=-\frac{1}{2}\pm3i$ with corresponding solutions $y_4=e^{-t/2}\cos3t$ and $y_5=e^{-t/2}\sin3t$. Thus, a fundamental set is

$S = \left\{1,\, e^t,\, e^{-t/2},\, e^{-t/2}\cos 3t,\, e^{-t/2}\sin 3t\right\}$ and a general solution is $y = c_1 + c_2e^t + c_3e^{-t/2} + e^{-t/2}(c_4\cos 3t + c_5\sin 3t)$, with derivatives $y' = c_2e^t - \frac{1}{2}c_3e^{-t/2} + e^{-t/2}\left[\left(-\frac{1}{2}c_4 + 3c_5\right)\cos 3t + \left(-3c_4 - \frac{1}{2}c_5\right)\sin 3t\right]$, $y'' = c_2e^t + \frac{1}{4}c_3e^{-t/2} + e^{-t/2}\left[\left(-\frac{35}{4}c_4 - 3c_5\right)\cos 3t + \left(3c_4 - \frac{35}{4}c_5\right)\sin 3t\right]$, $y''' = c_2e^t - \frac{1}{8}c_3e^{-t/2} + e^{-t/2}\left[\left(\frac{107}{8}c_4 - \frac{99}{5}c_5\right)\cos 3t + \left(\frac{99}{4}c_4 + \frac{107}{8}c_5\right)\sin 3t\right]$, and $y^{(4)} = c_2e^t - \frac{1}{16}c_3e^{-t/2} + e^{-t/2}\left[\left(\frac{1081}{16}c_4 + \frac{105}{2}c_5\right)\cos 3t + \left(-\frac{105}{2}c_4 + \frac{1081}{16}c_5\right)\sin 3t\right]$.

Applying the initial conditions results in the system

$$\begin{aligned}
y(0) &= c_1 + c_2 + c_3 + c_4 = 4 \\
y'(0) &= c_2 - \frac{1}{2}c_3 - \frac{1}{2}c_4 + 3c_5 = -14 \\
y''(0) &= c_2 + \frac{1}{4}c_3 - \frac{35}{4} - 3c_5 = -14 \\
y'''(0) &= c_2 - \frac{1}{8}c_3 + \frac{107}{8}c_4 - \frac{99}{4}c_5 = 139 \\
y^{(4)}(0) &= c_2 + \frac{1}{16}c_3 + \frac{1081}{16}c_4 + \frac{105}{2}c_5 = -\frac{29}{4}
\end{aligned},$$

which has solution $c_1 = 0$, $c_2 = 0$, $c_3 = 1$, $c_4 = 3$, and $c_5 = -4$. Thus, the solution to the initial value problem is $y = e^{-t/2}\left(1 + 3\cos 3t - 4\sin 3t\right)$, which is graphed in the following plot.



Note: *A CAS was used to assist in constructing the solution.*

49. The characteristic equation is $r^5 + 8r^4 = r^4(r+8) = 0$ with solutions $r_{1,2,3,4} = 0$ and $r_5 = -8$. For the first four roots, solutions are $y_1 = 1$, $y_2 = t$, $y_3 = t^2$, and $y_4 = t^3$. For the fifth, we obtain $y_5 = e^{-8t}$. Therefore a general solution is $y = c_1 + c_2t + c_3t^2 + c_4t^3 + c_5e^{-8}$. Application of the initial conditions results in $y = 8t^3 + 4t + 9$.

53. (a) 3 (b) $t-1$ (c) 2 (d) 0

59. $W(S) = \begin{vmatrix} e^{-3t} & e^{-t} & e^{-4t}\cos(3t) & e^{-4t}\sin(3t) \\ -3e^{-3t} & -e^{-t} & -4e^{-4t}\cos(3t) - 3e^{-4t}\sin(3t) & -4e^{-4t}\sin(3t) + 3e^{-4t}\cos(3t) \\ 9e^{-3t} & e^{-t} & 7e^{-4t}\cos(3t) + 24e^{-4t}\sin(3t) & 7e^{-4t}\sin(3t) - 24e^{-4t}\cos(3t) \\ -27e^{-3t} & -e^{-t} & 44e^{-4t}\cos(3t) - 117e^{-4t}\sin(3t) & 44e^{-4t}\sin(3t) + 117e^{-4t}\cos(3t) \end{vmatrix} = 1080e^{-12t}$

61. (b) $W(S) = \begin{vmatrix} f(t) & tf(t) & t^2 f(t) \\ \frac{d}{dt} f(t) & f(t) + t \frac{d}{dt} f(t) & 2tf(t) + t^2 \frac{d}{dt} f(t) \\ \frac{d^2}{dt^2} f(t) & 2 \frac{d}{dt} f(t) + t \frac{d^2}{dt^2} f(t) & 2f(t) + 4t \frac{d}{dt} f(t) + t^2 \frac{d^2}{dt^2} f(t) \end{vmatrix} =$ $2(f(t))^3$
63. (a) $y = -0.09090909091\, e^{-3.0\,t} + 0.5142594770 \sin(1.414213562\,t) + 0.09090909091 \cos(1.4$
(b) $y = 2.020725943\, e^{2.0\,t} \sin(1.732050808\,t) + e^{2.0\,t} \cos(1.732050808\,t) - 0.5773502693\, e^{2.0\,t}$ sin
$3.500000000\, e^{2.0\,t} \cos(1.732050808\,t)\, t$ (c) $y = 0.0002889303734\, e^{-36.99601821\,t} -$ $1.061192194\, e^{0.4334929696\,t} \sin(0.5236492683\,t) - 1.000288930\, e^{0.4334929696\,t} \cos(0.5236492683\,t)$
65. $y = 0.05460976246\, y_0 e^{0.4712630674\,t} + 0.8550147286\, y_0 e^{-0.2076539530\,t} \sin(1.126208087\,t) -$ $0.05460976246\, y_0 e^{-0.2076539530\,t} \cos(1.126208087\,t)$
67. $y(t) = c_2 \cos^2\left(\frac{1}{2}(2c_1 - t)\right)$

Exercises 4.6

1. $y = c_1 + c_2 t + c_3 e^{-t} + \frac{1}{2} e^t$
3. The characteristic equation for the corresponding homogeneous equation is $r^5 - r^4 = r^4(r-1) = 0$ with roots $r_{1,2,3,4} = 0$ and $r_5 = 1$. Therefore, a fundamental set of solutions is $S = \{1, t, t^2, t^3, e^t\}$ and a general solution of the corresponding homogeneous equation is $y_h = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t$. For the forcing function, the associated set of functions is $F = \{1\}$. Because the function 1 is a solution of the corresponding homogeneous equation, we multiply F by t^n where n is the lowest positive integral power so that no element of $t^n F$ is a solution of the corresponding homogeneous equation. In this case $t^4 F = \{t^4\}$ and we see that t^4 is not a solution of the corresponding homogeneous equation. Therefore, we assume that a particular solution takes the form $y_p = At^4$ with derivatives $y_p' = 4At^3$, $y_p'' = 12At^2$, and so on. Substituting y_p into the *nonhomogeneous* equation results in $y_p^{(5)} - y_p^{(4)} = -24A = 1 \Rightarrow A = -\frac{1}{24}{}^4$ so a particular solution of the nonhomogeneous equation is $y_p(t) = -1/24$. Therefore, a general solution of the equation is $y = y_h + y_p = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t - \frac{1}{24} t^4$
5. $y = c_1 + c_2 t + c_3 e^{-3t} + c_4 e^{3t} + \frac{1}{6} t e^{3t}$
7. The corresponding homogeneous equation has general solution $y_h = c_1 e^{3t} + c_2 e^{2t} + c_3 e^t$. For the forcing function, we have $F_1 = \{e^{-3t}\}$. Because e^{-3t} is a solution of the corresponding homogeneous equation we multiply F_1 by t^n where n is the smallest positive integer so that no element of $t^n F_1$ is a solution of the corresponding homogeneous equation. In this case, $tf_1 = \{te^{-3t}\}$. For the second term, $F_2 = \{te^{-t}, e^{-t}\}$. No element of F_2 is a solution to the corresponding homogeneous equation. Combining F_1 and F_2, we search for a particular solution of the nonhomogeneous equation of the form $y_p = Ate^{-3t} + Bte^{-t} + Ce^{-t}$. Computing the derivatives and substituting into the *nonhomogeneous* equation, solving for A, B, and C, forming y_h and then $y = y_h + y_p$ gives us the general solution of the equation: $y = e^{-3\,t} t + 3/4\, te^{-t} - \frac{7}{8} e^{-t} + 3/2\, e^{-3\,t} - 1/4\, e^{-t} t^2 + c_1 e^{-3\,t} + c_2 e^{-2\,t} + c_3 e^{-t}$
9. $y = e^t + c_1 e^{4\,t} + c_2 e^{-5\,t} \cos(t) + c_3 e^{-5\,t} \sin(t)$
11. The corresponding homogeneous equation is $y''' + 4y' = 0$ which has general

solution $y_h = c_1 + c_2 \cos 2t + c_3 \sin 2t$. A fundamental set is $S = \{1, \cos 2t, \sin 2t\}$ with Wronskian

$$W(S) = \begin{vmatrix} 1 & \cos 2t & \sin 2t \\ 0 & -2\sin 2t & 2\cos 2t \\ 0 & -4\cos 2t & -4\sin 2t \end{vmatrix} = 8$$

Next,

$$u_1' = \frac{1}{8} \begin{vmatrix} 0 & \cos 2t & \sin 2t \\ 0 & -2\sin 2t & 2\cos 2t \\ \tan 2t & -4\cos 2t & -4\sin 2t \end{vmatrix} = \frac{1}{4}\tan 2t$$

so $u_1 = \frac{1}{8} \ln \sec 2t$;

$$u_2' = \frac{1}{8} \begin{vmatrix} 1 & 0 & \sin 2t \\ 0 & 0 & 2\cos 2t \\ 0 & \tan 2t & -4\sin 2t \end{vmatrix} = \sin^2 2t$$

so $u_2 = \frac{1}{8}(4t - \sin 4t)$; and

$$u_3' = \frac{1}{8} \begin{vmatrix} 1 & \cos 2t & 0 \\ 0 & -2\sin 2t & 0 \\ 0 & -4\cos 2t & \sec 2t \end{vmatrix} = -\frac{1}{4}\tan 2t$$

so $u_3 = -\frac{1}{8} \ln \sec 2t$, Thus,

$$y_p = \frac{1}{8}\ln \sec 2t + \frac{1}{8}\cos 2t(4t - \sin 4t) - \frac{1}{8}\sin 2t \ln \sec 2t$$

and a general solution is $y = y_h + y_p = c_1 + c_2 \cos 2t + c_3 \sin 2t + \frac{1}{8}\ln \sec 2t + \frac{1}{8}\cos 2t(4t - \sin 4t) - \frac{1}{8}\sin 2t \ln \sec 2t$.

13. The corresponding homogeneous equation is $y^{(4)} + 4y'' = 0$ with characteristic equation $k^4 + 4k^2 = k^2(k+4) = 0$ so a fundamental set is $S = \{1,\ t,\ \cos 2t,\ \sin 2t\}$ and a general solution of the corresponding homogeneous equation is $y_h = c_1 + c_2 t + c_3 \cos 2t + c_4 \sin 2t$. Next,

$$W(S) = \begin{vmatrix} 1 & t & \cos 2t & \sin 2t \\ 0 & 1 & -2\sin 2t & 2\cos 2t \\ 0 & 0 & -4\cos 2t & -4\sin 2t \\ 0 & 0 & 8\sin 2t & -8\cos 2t \end{vmatrix} = 32.$$

Next,

$$u_1' = \frac{1}{32} \begin{vmatrix} 0 & t & \cos 2t & \sin 2t \\ 0 & 1 & -2\sin 2t & 2\cos 2t \\ 0 & 0 & -4\cos 2t & -4\sin 2t \\ \sec^2 2t & 0 & 8\sin 2t & -8\cos 2t \end{vmatrix} = -\frac{1}{4} t \sec^2 2t$$

so $u_1 = \frac{1}{16}(\ln \sec 2t - 2t \tan 2t)$;

$$u_2' = \frac{1}{32} \begin{vmatrix} 1 & 0 & \cos 2t & \sin 2t \\ 0 & 0 & -2\sin 2t & 2\cos 2t \\ 0 & 0 & -4\cos 2t & -4\sin 2t \\ 0 & \sec^2 2t & 8\sin 2t & -8\cos 2t \end{vmatrix} = \frac{1}{4}\sec^2 2t$$

so $u_{\frac{1}{8}} \tan 2t$;

$$u_3' = \frac{1}{32}\begin{vmatrix} 1 & t & 0 & \sin 2t \\ 0 & 1 & 0 & 2\cos 2t \\ 0 & 0 & 0 & -4\sin 2t \\ 0 & 0 & \sec^2 2t & -8\cos 2t \end{vmatrix} = \frac{1}{8}\sec 2t \tan 2t$$

so $u_3 = \frac{1}{16}\sec 2t$; and

$$u_4' = \frac{1}{32}\begin{vmatrix} 1 & t & \cos 2t & 0 \\ 0 & 1 & -2\sin 2t & 0 \\ 0 & 0 & -4\cos 2t & 0 \\ 0 & 0 & 8\sin 2t & \sec^2 2t \end{vmatrix} = -\frac{1}{8}\sec 2t$$

so $u_4 = -\frac{1}{16}\ln(\sec 2t + \tan 2t)$. Hence,

$$\begin{aligned} y_p &= \frac{1}{16}(\ln \sec 2t - 2t\tan 2t) + \frac{1}{8}t\tan 2t + \frac{1}{16}\cos 2t \sec 2t - \frac{1}{16}\sin 2t \ln(\sec 2t + \tan 2t) \\ &= \frac{1}{16}\left(\ln \sec 2t - \sin 2t \ln(\sec 2t + \tan 2t)\right) \end{aligned}$$

and a general solution is

$$y = y_h + y_p = c_1 + c_2 t + c_3\cos 2t + c_4 \sin 2t + \frac{1}{16}\left(\ln\sec 2t - \sin 2t\ln(\sec 2t + \tan 2t)\right).$$

15. The corresponding homogeneous equation is $y''' + 9y' = 0$, which has characteristic equation $r^3 + 9r = r(r^2+9) = 0$ with solutions $r_1 = 0$ and $r_{2,3} = \pm 3i$. Therefore, a fundamental set of solutions is $S = \{1, \cos 3t, \sin 3t\}$ and $W(S) = \begin{vmatrix} 1 & \cos(3t) & \sin(3t) \\ 0 & -3\sin(3t) & 3\cos(3t) \\ 0 & -9\cos(3t) & -9\sin(3t) \end{vmatrix} = 17$. A general solution of the corresponding homogeneous equation is $y_h = c_1 + c_2\cos 3t + c_2 \sin 3t$. Next,

$$u_1 = \frac{1}{27}\int \begin{vmatrix} 0 & \cos(3t) & \sin(3t) \\ 0 & -3\sin(3t) & 3\cos(3t) \\ \sec^2(3t) & -9\cos(3t) & -9\sin(3t) \end{vmatrix} dt = \frac{1}{9}\int \sec^2 3t\, dt = \frac{1}{27}\tan 3t,$$

$$u_2 = \frac{1}{27}\int \begin{vmatrix} 1 & 0 & \sin(3t) \\ 0 & 0 & 3\cos(3t) \\ 0 & \sec^2(3t) & -9\sin(3t) \end{vmatrix} dt = -\frac{1}{9}\int \sec(3t)\, dt = -\frac{1}{27}\ln(\sec(3t)+\tan(3t)),$$

and

$$u_3 = \frac{1}{27}\int \begin{vmatrix} 1 & \cos(3t) & 0 \\ 0 & -3\sin(3t) & 0 \\ 0 & -9\cos(3t) & \sec^2(3t) \end{vmatrix} dt = -\frac{1}{9}\int \sec(3t)\ \tan(3t)\, dt = -\frac{1}{27}\sec(3t).$$

Therefore,

$$y = \frac{1}{27}\left[(c_1 + \ln(\cos(3t/2) - \sin(3t/2)) - \ln(\cos(3t/2) + \sin(3t/2)))\cos 3t + c_2 \sin 3t + c_3\right].$$

17. A general solution of the corresponding homogeneous equation, $y''' + 4y' = 0$ is $y_h = c_1 + c_2 \cos 2t + c_3 \sin 2t$. Using variation of parameters to find a particular solution of the nonhomogeneous equation yields $y_p = \frac{1}{8}(-2t\cos(2t) - \log(\cos(t) - \sin(t)) + \log(\sin(t) + \cos(t)) + \sin(2t)\log(\cos(2t)))$. Therefore, $y = y_h + y_p = \frac{1}{8}\left(c_1 + c_2 \cos^2 t - 2t\cos 2t - \ln(\cos t - \sin t) + \ln(\cos t + \sin t) + c_3 \sin 2t + \ln(\cos 2t)\sin 2t\right)$

19. The characteristic equation for the corresponding homogeneous equation is $r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0$ with solutions $r_{1,2,3} = 1$. Therefore, a fundamental set of solutions for the corresponding homogeneous equation is $S = \{e^t, te^t, t^2e^t\}$ and a general solution of the corresponding homogeneous equation is $y_h = e^t(c_1 + c_2t + c_3t^2)$. With variation of parameters, a particular solution of the nonhomogeneous equation is found to be $y_p = \frac{1}{4}e^t t^2(2\log(t) - 3)$. Therefore, a general solution of the nonhomogeneous equation is $y = y_h + y_p = -3/4\, t^2e^t + 1/2\, \ln(t)\, t^2e^t + c_1\, e^t + c_2\, te^t + c_3\, t^2e^t$

21. Use undetermined coefficients. $y = 1/6\, e^{-3\,t} + c_1\, e^{-4\,t} + c_2\, e^{-2\,t} + c_3\, e^{3\,t}$

23. The characteristic equation for the corresponding homogeneous equation is $r^3 + 3r^2 + 2r = r(r^2 + 3r + 2) = r(r+1)(r+2) = 0$ so a fundamental set of solutions for the corresponding homogeneous equation is $S = \{1, e^{-t}, e^{-2t}\}$ and a general solution of the corresponding homogeneous equation is $y_h = c_1 + c_2e^{-t} + c_3e^{-2t}$. Using undetermined coefficients, assume that a particular solution has the form $y_p = A\cos t + B\sin t$. Substitute y_p into the nonhomogeneous equation to determine y_p. Then a general solution of the nonhomogeneous equation is give by $y = y_h + y_p = 1/10\, \sin(t) - 3/10\, \cos(t) + c_1e^{-2\,t} + c_2e^{-t} + c_3$

25. $y = -\frac{1}{2}t^2 + c_1 + c_2\cos t - \ln(-\cos(t/2) - \sin(t/2)) - \ln(-\cos(t/2) + \sin(t/2)) - (c_3 - \ln(\cos(t/2) - \sin(t/2)) + \ln(\cos(t/2) + \sin(t/2)))\sin t$

27. $y = 1 + t - \cos t - \ln(\cos t) - 2\tanh^{-1}(\tan(t/2))\, \sin t$

29. A general solution of the equation is $y = c_1e^{-t} + c_3t + c_2 - \dfrac{\sin(t)}{2} - \dfrac{\cos(t)}{2}$. Apply the initial conditions to solve the initial value problem: $y = 2 - 2\cos t - \frac{1}{2}t\sin t$

31. First write the equation in standard form: $y''' - \dfrac{1}{t\ln t}y'' + \dfrac{1}{t^2\ln t}y' = \dfrac{1}{t^2\ln t}$ to see that the forcing function is $f(t) = \dfrac{1}{t^2\ln t}$. Next, $W(\{1, t^2, t\ln t\|) = \begin{vmatrix} 1 & t^2 & t\ln(t) \\ 0 & 2t & \ln(t) + 1 \\ 0 & 2 & \dfrac{1}{t} \end{vmatrix} = -2\ln(t)$. Now,

$$u_1 = -\int \frac{1}{2\ln t}\begin{vmatrix} 0 & t^2 & t\ln(t) \\ 0 & 2t & \ln(t) + 1 \\ \dfrac{1}{t^2\ln t} & 2 & \dfrac{1}{t} \end{vmatrix} dt = \frac{t}{2\ln t},$$

$$u_2 = -\int \frac{1}{2\ln t} \begin{vmatrix} 1 & 0 & t\ln(t) \\ 0 & 0 & \ln(t)+1 \\ 0 & \dfrac{1}{t^2\ln t} & \dfrac{1}{t} \end{vmatrix} dt = -\frac{1}{2t\ln t},$$

and

$$u_3 = -\int \frac{1}{2\ln t} \begin{vmatrix} 1 & t^2 & 0 \\ 0 & 2t & 0 \\ 0 & 2 & \dfrac{1}{t^2\ln t} \end{vmatrix} dt = \frac{1}{\ln t}.$$

Therefore, $y = y_h + y_p = c_1 + c_2t^2 + c_3t\ln t + t$

33. A general solution is $y = c_1t + c_2t\ln t + c_3\sqrt{t} - t^2$; the solution of the initial value problem is $y = t - t^2 + 2t\ln t$.

35. A general solution is $y = c_1 + c_2t + c_3t\ln t + c_4t^2 + 2t^{-1/2}$; the solution of the initial value problem is $y = \frac{1}{8}\left(-41 + 16t^{-1/2} + 12t + 13t^2 - 30t\ln t\right)$

Exercises 4.7

1. Substituting $y = x^r$ into the equation and simplifying gives us $4r(r-1) - 8r + 5 = (2r-5)(2r-1) = 0$ so $r_1 = 5/2$ and $r_2 = 1/2$. A fundamental set of solutions is $S = \{x^{1/2}, x^{5/2}$; a general solution is $y = c_1x^{1/2} + c_2x^{5/2}$.

3. $y = c_1x + c_2x^4$

5. Substituting $y = x^r$ into the equation and simplifying gives us $4r(r-1)+17 = 4r^2 - 4r + 17 = 0$. Using the quadratic formula to solve for r gives us $r_{1,2} = \frac{1}{2} \pm 2i$. Therefore, a fundamental set of solutions is $S = \{\sqrt{x}\cos(2\ln x), \sqrt{x}\sin(2\ln x)\}$ and a general solution is $y = \sqrt{x}(c_1\cos(2\ln x) + c_2\sin(2\ln x))$.

7. $y = x(c_1\cos(3\ln x) + c_2\sin(3\ln x))$

9. Substituting $y = x^r$ into the equation and simplifying gives us $4r(r-1) + 8r + 1 = 4r^2 + 4r + 1 = 0$ with solutions $r_{1,2} = -1/2$. Therefore, a fundamental set of solutions is $S = \{1/\sqrt{x}, \ln x/\sqrt{x}\}$ and a general solution is $y = x^{-1/2}(c_1 + c_2\ln x)$. (We are assuming that $x > 0$.)

11. $y = x^3(c_1 + c_2\ln x)$

13. Substituting $y = x^r$ into the equation and simplifying gives us $r(r-1)(r-2) + 22r(r-1) + 124r + 140 = r^3 + 19r^2 + 104r + 140 = (r+2)(r+7)(r+10) = 0$ so $r_1 = -10$ $r_2 = -7$, and $r_3 = -2$. A fundamental set of solutions is then $S = \{x^{-10}, x^{-7}, x^{-2}\}$ and a general solution is $y = c_1x^{-10} + c_2x^{-7} + c_3x^{-2}$.

14. $y = c_1x^{-5} + c_2x^2 + c_3x^{10}$

16. $y = c_1x^{-1} + c_2\cos(2\ln x) + c_3\sin(2\ln x)$

17. Substituting $y = x^r$ into the equation $x^3y''' + 2xy' - 2y = 0$ and simplifying gives us $r(r-1)(r-2) + 2r - 2 = (r-1)(r^2 - 2r + 2) = 0$ with solutions $r_1 = 1$ and $r_{2,3} = 1 \pm i$. Therefore, a fundamental set of solutions is $S = \{x, x\cos(\ln x), x\sin(\ln x)\}$ and a general solution is $y = c_1x + x(c_2\cos(\ln x) + c_3\sin(\ln x))$.

19. $y = x^{-1}(c_1 + c_2\ln x + c_3(\ln x)^2)$

21. A fundamental set of solutions for the corresponding homogeneous equation is $S = \{x^{-2}, x^{-2}\ln x\}$ with Wronskian $W(S) = x^{-5}$. In standard form, the equation is $y'' + \dfrac{5}{x}y' + \dfrac{4}{x^2}y = x^{-7}$, which shows us that the forcing function is

$f(x) = x^{-7}$. Now, we use variation of parameters to form a particular solution of the form $y_p = u_1 y_1 + u_2 y_2$:

$$u_1 = \int x^5 \begin{vmatrix} 0 & \frac{\log(x)}{x^2} \\ \frac{1}{x^7} & \frac{1}{x^3} - \frac{2\log(x)}{x^3} \end{vmatrix} dx = -\int \frac{\ln x}{x^4} dx = \frac{1}{9x^3} + \frac{\ln x}{3x^3}$$

and

$$u_2 = \int x^5 \begin{vmatrix} \frac{1}{x^2} & 0 \\ -\frac{2}{x^3} & \frac{1}{x^7} \end{vmatrix} dx = \int x^{-4} dx = -\frac{1}{3x^3}.$$

After forming y_p, $y = y_h + y_p = \dfrac{c_1}{x^2} + \dfrac{c_2 \ln(x)}{x^2} + 1/9\, x^{-5}$

23. $y = c_1 \sin(\ln(x)) + c_2 \cos(\ln(x)) + 1/5\, x^{-2}$

25. With the substitution $x = e^t$, the equation becomes $d^2y/dt^2 + dy/dt - 6y = 2e^t$. The corresponding homogeneous equation has solution $y_h = c_1 e^{2t} + c_2 e^{-3t} = c_1 x^2 + c_2 x^{-3}$. Using undetermined coefficients, we assume that a particular solution of the transformed equation takes the form $y_p = Ae^t$. Substituting y_p into the transformed nonhomogeneous equation gives us $Ae^t + Ae^t - 6Ae^t = -4Ae^t = 2e^t$ so $A = -1/2$ and $y_p = -2e^t = -1/2\, x$. Therefore, a general solution of the nonhomogeneous equation is $y = y_h + y_p = \dfrac{c_1}{x^{-3}} + c_2 x^2 - \frac{1}{2}\, x$.

27. With the substitution $x = e^t$, the equation becomes $d^2y/dt^2 + 4y = 8$. The corresponding homogeneous equation has general solution $y_h = c_1 \cos 2t + c_2 \sin 2t = c_1 \sin(2\ln(x)) + c_2 \cos(2\ln(x))$. We now assume that a particular solution of the transformed equation has the form $y_p = A$. Substituting into the nonhomogeneous equation gives us $y_p'' + 4y_p = 4A = 8$ so $A = 2$ an $y_p = 2$. Therefore, a general solution of the nonhomogeneous equation is $y = y_h + y_p = c_1 \sin(2\ln(x)) + c_2 \cos(2\ln(x)) + 2$

29. $y = 1/25\, x^{-3} + c_1 x^2 + \dfrac{c_2}{x^4} + c_3\, x^2 \ln(x)$

31. $y = 9/5\, \sqrt[3]{x} + 1/5\, x^2$

33. A general solution of the equation is $y = c_1 \cos(2\ln x) + c_2 \sin(2\ln x)$ an the solution to the initial value problem is $y = \cos(2\ln(x))$.

35. A general solution of the equation is $y = c_1 x^{-10} + c_2 x + c_3 x^2$ and the solution to the initial value problem is $y = -3/4\, x^2 + 1/44\, x^{-10} + \frac{8}{11}\, x$

37. $y = 3/2\, x + 1/2\, x^2 \sin(\ln(x)) - 3/2\, x^2 \cos(\ln(x))$

39. A general solution of the equation is $y = c_1\sqrt{x} + c_2 x^{-1} + \frac{1}{5} x^{-2}$ and the solution to the initial value problem is $y = \frac{22}{15}\sqrt{x} + 1/5\, x^{-2} - 5/3\, x^{-1}$

41. A general solution of the equation is $y = c_1\sqrt{x} + c_2\sqrt{x}\ln x + \frac{1}{25} x^3$ and the solution to the initial value problem is $y = \frac{24}{25}\sqrt{x} - 8/5\sqrt{x}\ln(x) + 1/25\, x^3$.

43. $W(S) = \begin{vmatrix} x^{r_1} & x^{r_2} \\ x^{r_1-1} r_1 & x^{r_2-1} r_2 \end{vmatrix} = -x^{r_1+r_2-1}(-r_2 + r_1)$

49. $y = c_1 + c_2 \cos(6\ln x) + c_3 \sin(6\ln x)$

51. $y = x(c_1 + c_2 \ln x + c_3 (\ln x)^2)$

53. (c) $y = c_1 \sin(2\arctan(x)) + c_2 \cos(2\arctan(x))$ (d) $y = c_1 \sin(2\arctan(x)) + c_2 \cos(2\arctan(x)) + 1/4 \arctan(x)$ (e) $y = 1/2 \sin(2\arctan(x))$ (f) $y = 3/8 \sin(2\arctan(x)$ $1/4 \arctan(x)$

65. $y = \frac{106}{25} x^{-2} + \frac{14}{5} \frac{\sin(5\ln(x))}{x^2} - \frac{56}{25} \frac{\cos(5\ln(x))}{x^2}$

67. $y = \pm\sqrt{2c_1 \ln(x) + 2c_2}$

Exercises 4.8

1. $x = 0$, $R \geq 1$

3. $x = \pm 2$, $R \geq 1$

5. $y = a_0 + a_1 x + (11/2\, a_1 - 15\, a_0)\, x^2 + \left(\frac{91}{6}\, a_1 - 55\, a_0\right) x^3 + \left(\frac{671}{24}\, a_1 - \frac{455}{4}\, a_0\right) x^4 + \left(\frac{4651}{120}\, a_1 - \frac{671}{4}\, a_0\right) x^5 + \cdots$

7. $y = a_0 + a_1 x + \left(1/2\, a_1 + a_0 + \frac{1}{2}\right) x^2 + \left(1/2\, a_1 + 1/3\, a_0 + \frac{1}{3}\right) x^3 + \left(\frac{5}{24}\, a_1 + 1/4\, a_0 + \frac{5}{24}\right) x^4 + \left(\frac{11}{120}\, a_1 + 1/12\, a_0 + \frac{1}{12}\right) x^5 + \cdots$

9. $y = a_0 + a_1 x - 1/4\, a_1 x^3 + 3/16\, a_1 x^4 - \frac{9}{80}\, a_1 x^5 + \cdots$

11. $y = a_0 + a_1 x + (1/2\, a_1 - a_0)\, x^2 + (-1/3\, a_1 - 1/3\, a_0)\, x^3 + \left(-\frac{5}{24}\, a_1 + 1/6\, a_0\right) x^4 + \left(-\frac{1}{120}\, a_1 + 1/15\, a_0\right) x^5 + \cdots$

13. $y = a_0 + a_1 x + 3/4\, a_0 x^2 + 1/12\, a_1 x^3 - \frac{5}{32}\, a_0 x^4 - 1/32\, a_1 x^5 + \cdots$

15. $y = 1 + \frac{1}{3} x^4 + \frac{1}{42} x^8 + \frac{1}{1386} x^{12} + \cdots$

17. $y = a_1 + a_2\, (x-1) + 1/8\, a_2\, (x-1)^2 + \frac{5}{96}\, a_2\, (x-1)^3 + \frac{15}{512}\, a_2\, (x-1)^4 + \frac{39}{2048}\, a_2\, (x-1)^5 + O\left((x-1)^6\right)$

19. $y = 1 + \frac{1}{6} x^3 - \frac{1}{30} x^5 + \frac{1}{240} x^7 - \frac{37}{90720} x^9 + \cdots$

21. $y = x + \frac{1}{2} x^2 - \frac{1}{8} x^4 - \frac{1}{8} x^5 - \frac{7}{120} x^6 + \frac{11}{1680} x^7 + \frac{89}{2688} x^8 + \frac{79}{2880} x^9 + \cdots$

23. (a) $y = a_1 + a_2\, x - 1/2\, a_1\, x^2 + 1/6\, a_2\, x^3 - 1/8\, a_1\, x^4 + 1/24\, a_2\, x^5 - \frac{7}{240}\, a_1\, x^6 + \frac{1}{112}\, a_2\, x^7 - \frac{11}{1920}\, a_1\, x^8 + \frac{13}{8064}\, a_2\, x^9 + \cdots$ (b) $y = a_1 + a_2\, x - 3/2\, a_1\, x^2 - 1/6\, a_2\, x^3 - 1/8\, a_1\, x^4 - 1/40\, a_2\, x^5 - 1/48\, a_1\, x^6 - \frac{1}{240}\, a_2\, x^7 - \frac{3}{896}\, a_1\, x^8 - \frac{11}{17280}\, a_2\, x^9 + \cdots$; $y = a_1 + a_2\, x - 1/2\, k a_1\, x^2 + (1/3\, a_2 - 1/6\, k a_2)\, x^3 + \left(-1/6\, k a_1 + 1/24\, k^2 a_1\right) x^4 + \left(-1/15\, k a_2 + \frac{1}{120}\, k^2 a_2 + 1/10\, a_2\right) x^5 + \left(\frac{1}{60}\, k^2 a_1 - \frac{1}{720}\, k^3 a_1 - \frac{2}{45}\, k a_1\right) x^6 + (-\frac{23}{1260}\, k a_2 + \frac{1}{280}\, k$ $(-\frac{1}{1680}\, k^3 a_1 + \frac{1}{40320}\, k^4 a_1 + \frac{11}{2520}\, k^2 a_1 - \frac{1}{105}\, k a_1)\, x^8 + (\frac{43}{45360}\, k^2 a_2 - \frac{1}{11340}\, k^3 a_2 - \frac{11}{2835}\, k a_2 +$ $\cdots$

25. $2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \frac{2}{11}$

27. $y = 1 - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{24} x^4 - \frac{1}{24} x^5 + \frac{1}{40} x^6 - \frac{17}{5040} x^7 - \frac{53}{20160} x^8 + \frac{11}{7560} x^9 - \frac{41}{302400} x^{10} - \frac{3097}{19958400} x^{11} + \frac{8219}{119750400} x^{12} - \frac{3431}{2075673600} x^{13} + \cdots$

Exercises 4.9

1. $x = 0$, regular

3. $x = 4$, regular

5. $y = c_1\, x^{7/2} \left(1 + \frac{1}{3} x + \frac{1}{22} x^2 + \frac{1}{286} x^3 + \frac{1}{5720} x^4 + \frac{3}{486200} x^5 + \cdots\right) + c_2\, (1 - \frac{3}{5} x + \frac{3}{10} x^2 - \frac{3}{10} x^3$

7. $y = c_1 \left(1 + \frac{1}{4} x - \frac{3}{104} x^2 - \frac{29}{6864} x^3 + \frac{13}{65472} x^4 + \frac{251}{11348480} x^5 + \cdots\right) x^{-5/9} + c_2 \left(1 + \frac{1}{14} x - \frac{13}{644} x^2 - \frac{59}{61824} x^3 + \frac{29}{247296} x^4 + \frac{53}{12364800} x^5 + \cdots\right)$

9. $y = c_1 x \left(1 - \frac{1}{3} x + \frac{1}{24} x^2 - \frac{1}{360} x^3 + \frac{1}{8640} x^4 - \frac{1}{302400} x^5 + \cdots\right) + c_2\, (\ln(x) \left(x^2 - \frac{1}{3} x^3 + \frac{1}{24} x^4 - \frac{1}{360} x^5 + \cdots\right) x^{-1} + \left(-2 - 2\, x + \frac{4}{9} x^3 - \frac{25}{288} x^4 + \frac{157}{21600} x^5 + \cdots\right) x$

11. $y = c_1\left(1+\frac{3}{2}x^2-\frac{9}{40}x^4+\cdots\right)x^{-2}+c_2\sqrt[3]{x}\left(1-\frac{3}{26}x^2+\frac{9}{1976}x^4+\cdots\right)$
13. $y = c_1x^{7/4}\left(1+\frac{7}{8}x+\frac{77}{160}x^2+\frac{77}{384}x^3+\frac{209}{3072}x^4+\frac{4807}{245760}x^5+\cdots\right)+$
$c_2\left(\ln(x)\left(\frac{15}{8}x^3+\frac{105}{64}x^4+\frac{231}{256}x^5+\cdots\right)x^{-5/4}+(12+15\,x+\frac{15}{4}x^2-\frac{13}{2}x^3-\frac{1741}{256}x^4-\frac{4141}{1024}x^5+\cdots x^{-5/4}\right)$
15. $y = c_1x^{-7}+c_2x$
17. $y = c_1x^2+c_2x^2\ln x$
21. $y = \left(c_1\left(-2\,\frac{-1+x}{\sqrt{x\,(-1+x)}}+\ln\left(-1/2+x+\sqrt{-x+x^2}\right)\right)+c_2\right)\sqrt{x}\,(-1+x)^{-3/2}$
23. $y = c_1\ (1-2\,x)+c_2\ ((-1+2\,x)\ln(x)-2+(1-2\,x)\ln(-1+x))$
27. (a) $= x^{-1/2}(c_1\cos x+c_2\sin x)$ (b) $y = c_1J_5(4x)+Y_5(4x)$
33. $y = 1+(1-x)+\frac{3}{8}(-1+x)^2-\frac{1}{3}(-1+x)^3+\frac{101}{384}(-1+x)^4-\frac{29}{128}(-1+x)^5+\ldots$

Chapter 4 Review Exercises

1. Both are solutions of $y'''-25y'=0$ and $W(\{e^{5t},1\}) = \begin{vmatrix} e^{5\,t} & 1 \\ 5\,e^{5\,t} & 0 \end{vmatrix} = -5e^{5t} \neq 0$; linearly independent
3. $W(S) = \begin{vmatrix} t & t\ln(t) \\ 1 & \ln(t)+1 \end{vmatrix} = t$; linearly independent
5. All three are solutions of $y''''+y''=0$ and $W(S) = \begin{vmatrix} t & \cos(t) & \sin(t) \\ 1 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{vmatrix} = t$; linearly independent
11. First, write the equation in standard form: $y''+(2/t)y'+y=0$ to see that $p(t)=2/t$. Using the reduction of order formula (4.4), $y_2(t)=v(t)f(t)$ where

$$v(t) = \int \frac{1}{(f(t))^2}e^{-\int p(t)\,dt}\,dt = \int \frac{1}{(t^{-1}\sin t)^2}e^{-\int 2/t\,dt}dt = \int \csc^2 t\,dt = -\cot t.$$

Then, a second linearly independent solution of the equation is $y_2(t) = t^{-1}\sin t\cot t = t^{-1}\cos t$ and a general solution is $y = t^{-1}(c_1\cos t+c_2\sin t)$.
13. The characteristic equation is $6r^2+5r-4=(2r-1)(3r+4)=0$ with solutions $r_1=1/2$ and $r_2=-4/3$. Therefore, a fundamental set of solutions is $S=\{e^{t/2},e^{-4t/3}\}$ and a general solution is $y = c_1\,e^{1/2\,t}+c_2\,e^{-4/3\,t}$.
15. $y = c_1\,e^{-2\,t}+c_2\,e^{-t}$
17. $y = c_1\,e^{1/2\,t}+c_2\,e^{2\,t}$
19. $y = c_1\,e^{-1/4\,t}+c_2\,e^{1/5\,t}$
21. The characteristic equation is $2r^3+3r^2+r=r(r+1)(2r+1)=0$ with solutions $r_1=0$, $r_2=-1$, and $r_3=-1/2$. Therefore, a fundamental set of solutions is $S=\{1,e^{-t},e^{-t/2}\}$ and a general solution is $y\,(t) = c_1+c_2\,e^{-1/2\,t}+c_3\,e^{-t}$.
23. The characteristic equation is $9r^3+12r^2+13r = r(9r^2+12r+13)=0$ with solutions $r_1=0$ and $r_{2,3}=-2/3\pm i$. Therefore, a fundamental set of solutions is $S=\{1,e^{-2t/3}\cos t,e^{-2t/3}\sin t\}$ and a general solution is $y = c_1+c_2\,e^{-2/3\,t}\sin(t)+c_3\,e^{-2/3\,t}\cos(t)$.

25. A fundamental set of solutions for the corresponding homogeneous equation is $S = \{1, e^{-5t}\}$ and a general solution of the corresponding homogeneous equation is $y_h = c_1 + c_2 e^{-5t}$. To find a particular solution, we use undetermined coefficients. The associated set of functions is $F = \{t^2, t, 1\}$. Because 1 is a solution of the corresponding homogeneous equation we multiply F by t giving us $tF = \{t^3, t^2, t\}$. No element of tF is a solution of the corresponding homogeneous equation so we assume that a particular solution has the form $y_p = At^3 + Bt^2 + Ct$. Substituting y_p into the *nonhomogeneous* equation and solving for A, B, and C gives us $y_p = \frac{1}{3}t^3 - \frac{1}{5}t^2 + \frac{2}{25}t$. A general solution of the nonhomogeneous equation is then $y = y_h + y_p = -1/5\,t^2 + 1/3\,t^3 + c_1\,e^{-5\,t} + \frac{2}{25}\,t + c_2$.
27. $y = c_1 e^{-t}\sin(2\,t) + c_2 e^{-t}\cos(2\,t) + \frac{3}{17}\sin(2\,t) - \frac{12}{17}\cos(2\,t)$
29. A fundamental set of solutions for the corresponding homogeneous equation is $S = \{1, e^{2t}\}$ and a general solution of the corresponding homogeneous equation is $y_h = c_1 + c_2 e^{2t}$. To find a particular solution, we use variation of parameters. First, we compute $W(S) = 2e^{2t}$. Assuming that $y_p = u_1 y_1 + u_2 y_2$,

$$\begin{aligned} u_1 &= \int \frac{1}{W(S)} \begin{vmatrix} 0 & e^{2t} \\ 1/(1+e^{2t}) & 2e^{2t} \end{vmatrix} dt = -\frac{1}{2}\int e^{-2t}\cdot\frac{e^{2t}}{1+e^{2t}}\,dt \\ &= -\frac{1}{2}\left(t - \frac{1}{2}\ln\left(1+e^{2t}\right)\right) \end{aligned}$$

and

$$\begin{aligned} u_2 &= \int \frac{1}{W(S)} \begin{vmatrix} 1 & 0 \\ 0 & 1/(1+e^{2t}) \end{vmatrix} dt = \frac{1}{2}\int e^{-2t}\frac{1}{1+e^{2t}}\,dt \\ &= \frac{1}{2}\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\ln\left(1+e^{2t}\right)\right). \end{aligned}$$

Therefore, $y = y_h + y_p = 1/4\,\ln\left(1+e^{2\,t}\right) + 1/4\,e^{2\,t}\ln\left(1+e^{2\,t}\right) - 1/4 - 1/2\,t - 1/2\,t\,e^{2\,t} + c_1\,e^{2\,t} + c_2$
31. $y = e^{3t}(c_1\cos 2t + c_2\sin 2t) + \frac{3}{29}e^{-2t}$
33. Use a CAS to help you with the algebra. A fundamental set of solutions for the corresponding homogeneous equation is $S = \{e^{-4t}, e^{-3t}\}$ and a general solution is $y_h = c_1 e^{-3t} + c_2 e^{-4t}$. To find a particular solution, we use undetermined coefficients. The associated set of functions is $F = \{t^2 e^{-4t}, te^{-4t}, e^{-4t}\}$. Multiplying F by t gives us $tF = \{t^3 e^{-4t}, t^2 e^{-4t}, te^{-4t}\}$ and we assume that a particular solution takes the form $y_p = At^3 e^{-4t} + Bt^2 e^{-4t} + Cte^{-4t}$. Substituting y_p into the *nonhomogeneous* equation and solving for A, B, and C gives us $y_p = -e^{-4t}\left(t^3 + 3t^2 + 6t + 6\right)$. Therefore a general solution of the nonhomogeneou equation is $y = y_h + y_p = c_1 e^{-3\,t} + c_2\,e^{-4\,t} - t\left(6 + 3\,t + t^2\right)e^{-4\,t}$
35. Use a CAS to help you with the algebra. A fundamental set of solutions for the corresponding homogeneous equation is $S = \{e^{-2t}, te^{-2t}, e^{4t}\}$ and a general solution of the corresponding homogenous equation is $y_h = (c_1 + c_2 t)e^{-2t} + c_3 e^{4t}$. We use undetermined coefficients to find a particular solution of the nonhomogeneous equation. For the first term in the forcing function, the associated set of functions is $F_1 = \{e^{4t}\}$. Multiplying F_1 by t gives us

$tF_1 = \{te^{4t}\}$. For the second term, the associated set of functions is $F_2 = \{e^{-2t}\}$. Multiplying F_2 by t^2 gives us $t^2F_2 = \{t^2e^{-2t}\}$. Therefore, we assume that a particular solution of the nonhomogeneous equation has the form $y_p = Ate^{4t} + Bt^2e^{-2t}$. Substituting y_p into the *nonhomogeneous* equation and solving for A and B results in $y_p = \frac{1}{216}e^{-2t}\left(18t^2 + 6e^{6t}t + 6t - 2e^{6t} + 1\right)$. Therefore, a general solution of the nonhomogeneous equation is $y = y_h + y_p = \left(1/36\,te^{10\,t} - \frac{1}{108}\,e^{10\,t} + 1/12\,t^2e^{4\,t} + 1/36\,te^{4\,t} + \frac{1}{216}\,e^{4\,t}\right)e^{-6\,t} + c_1\,e^{-2\,t} + c_2e^{4\,t} + c_3\,e^{-2\,t}t$

37. $y = \frac{4}{625} - \frac{8}{125}\,t + 1/25\,t^2 + c_1e^{-t}\cos(2\,t) + c_2e^{-t}\sin(2\,t) + c_3\,e^{-t}\cos(2\,t)\,t + c_4\,e^{-t}\sin(2\,t)\,t$

39. The characteristic equation is $r^2+10r+16 = (r+8)(r+2) = 0$ with solutions $r_1 = -8$ and $r_2 = -2$ so a fundamental set of solutions is $S = \{e^{-8t}, e^{-2t}\}$ and a general solution is $y = c_1e^{-8t} + c_2e^{-2t}$. Applying the initial conditions results in $y = 2/3\,e^{-2\,t} - 2/3\,e^{-8\,t}$.

41. The characteristic equation is $r^2 + 25 = 0$ with solutions $r_{1,2} = \pm 5i$. Therefore, a fundamental set of solutions is $S = \{\cos 5t, \sin 5t\}$ and a general solution is $y = c_1\cos 5t + c_2\sin 5t$. Applying the initial conditions results in $y = \cos(5\,t)$.

43. $y = \frac{19}{25}\,e^t - \frac{19}{25}\,e^{-4\,t} + 1/5\,te^t$

45. $y = 1/2\,\sin(t)\,t$

47. The characteristic equation for the corresponding homogeneous equation is $r^2 + 1 = 0$ with solutions $r_{1,2} = \pm i$ so a fundamental set of solutions for the corresponding homogeneous equation is $S = \{\cos t, \sin t\}$ with $W(S) = 1$ and a general solution is $y_h = c_1\cos t + c_2\sin t$. Using variation of parameters, we assume that a particular solution takes the form $y_p = u_1y_1 + u_2y_2$. Using the variation of parameters formulas,

$$u_1 = \int \frac{1}{W(S)}\begin{vmatrix} 0 & \sin t \\ \csc t & \cos t \end{vmatrix}\,dt = -\int dt = -t$$

and

$$u_2 = \int \frac{1}{W(S)}\begin{vmatrix} \cos t & 0 \\ -\sin t & \csc t \end{vmatrix}\,dt = \int \cot t\,dt = -\ln\csc t = \ln\sin t.$$

Therefore, $y = y_h + y_p = c_1\sin(t) + c_2\cos(t) + \ln(\sin(t))\sin(t) - t\cos(t)$

49. $y = \frac{1}{20}e^{4t}\left(t^5 + 20t\right)$

51. $y = \frac{1}{4}e^{t-1}\left(8 - e - 4t + 4et - 3et^2 + 2et^2\ln t\right)$

55. (a) $y = -e^{-t-1} + \frac{1}{2}e^{t-1}$

57. Assuming that $y = x^r$ and substituting into the equation yields $r(r-1) - 4r + 6 = (r-2)(r-3) = 0$ with solutions $r_1 = 2$ and $r_2 = 3$. Therefore, a fundamental set of solutions if $S = \{x^2, x^3\}$ and a general solution of the equation is $y = c_1x^2 + c_2x^3$.

59. $y = -x^{-1} + 2x^{-1/2}$

61. Assuming that $y = x^r$ and substituting into the equation yields $r(r-1) - 7r + 25 = 0$ with solutions $r_{1,2} = 4 \pm 3i$. Therefore, a fundamental set of solutions is $S = \{x^4\cos(3\ln x), x^4\sin(3\ln x)\}$ and a general solution is $y = x^4(c_1\cos(3\ln x) + c_2\sin(3\ln x))$.

63. $y = a_1 + a_2x + (2\,a_2 - 2\,a_1)\,x^2 + (2\,a_2 - 8/3\,a_1)\,x^3 + (4/3\,a_2 - 2\,a_1)\,x^4 + \left(2/3\,a_2 - \frac{16}{15}\,a_1\right)x^5 + \cdots$
65. $y = a_1 + a_2x - 3/2\,a_1x^2 - 1/6\,a_2x^3 - 5/8\,a_1x^4 - 1/8\,a_2x^5 + \cdots$
67. $y = c_1x^{-2} + c_2x^{1/2}$
69. $y = c_1\,(-1+x)^4\,(x-1/6) + c_2\,\left(-6\,(-1+x)^4\,(x-1/6)\ln(-1+x) + 6\,(-1+x)^4\,(x-1\right.$
73. $y = \sqrt{2t - 4\ln t - 1}$, $t \geq 1$

Differential Equations at Work

B. Modeling the Motion of a Skier

1. With $v = s' = ds/dt$, $s'' = k^2 - h^2(s')^2$ becomes $v' = k^2 - h^2v^2$. Then,

$$\begin{aligned}
\frac{1}{k^2 - h^2v^2}dv &= dt \\
\frac{1}{2k}\left(\frac{1}{k+hv} + \frac{1}{k-hv}\right) &= dt \\
\frac{1}{2hk}\left(\ln(k+hv) - \ln(k-hv)\right) &= t + C \\
\ln\left(\frac{k+hv}{k-hv}\right) &= 2hkt + C \\
\frac{k+hv}{k-hv} &= Ce^{2hkt} \\
v &= \frac{k}{h}\frac{Ce^{2hkt}-1}{Ce^{2hkt}+1} \\
v &= \frac{k}{h}\tanh\left(hkt + \tanh^{-1}\left(\frac{hv_0}{k}\right)\right)
\end{aligned}$$

Applying the initial condition gives us $v(0) = k(C-1)/(Ch+h) = v_0$ so $C = (k+hv_0)/(k-hv_0)$ and then

$$v = \frac{k}{h}\frac{hv_0 - k + (hv_0+k)e^{2hkt}}{k - hv_0 + (hv_0+k)e^{2hkt}}.$$

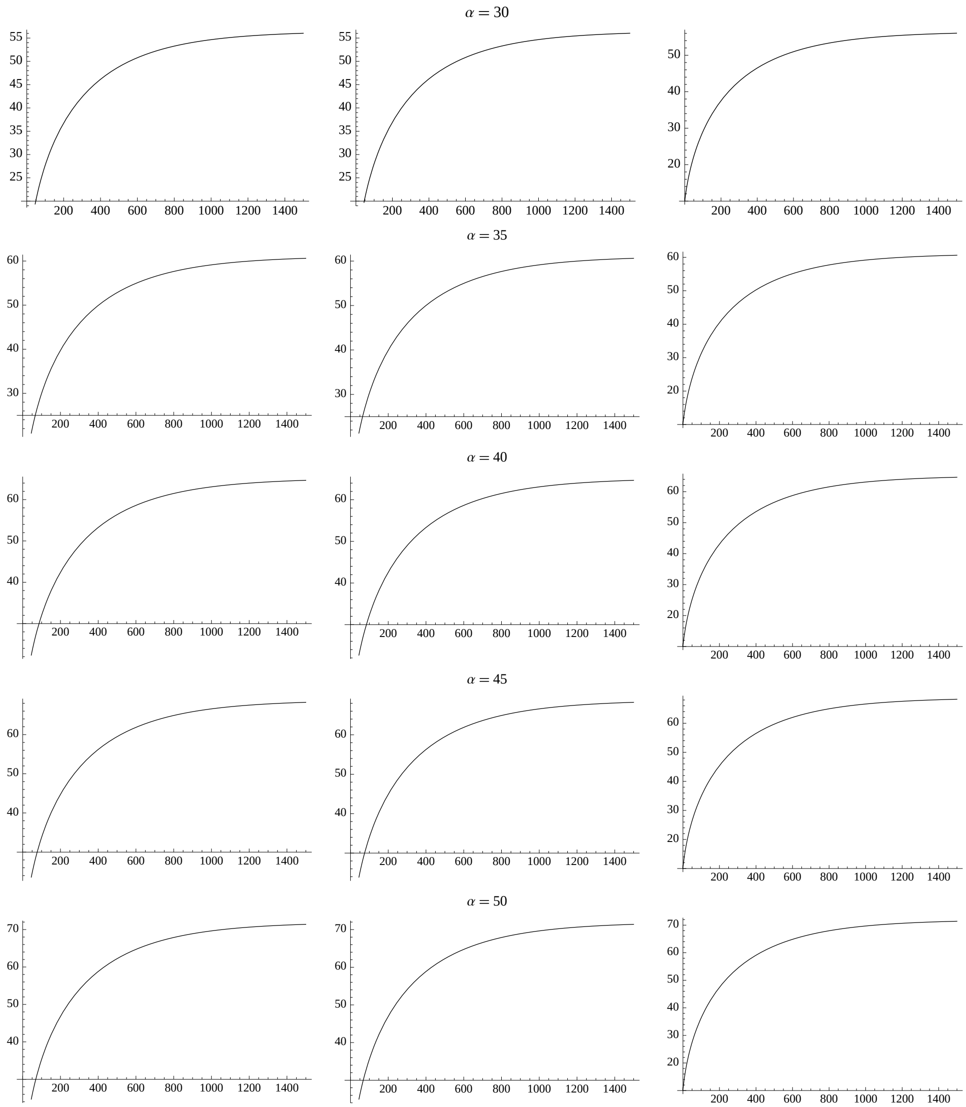
2. Integrating gives us

$$s = C + \frac{\ln\left(-2\left(hv_0\left(e^{2hkt}-1\right) + k\left(e^{2hkt}+1\right)\right)\right)}{h^2} - \frac{kt}{h}.$$

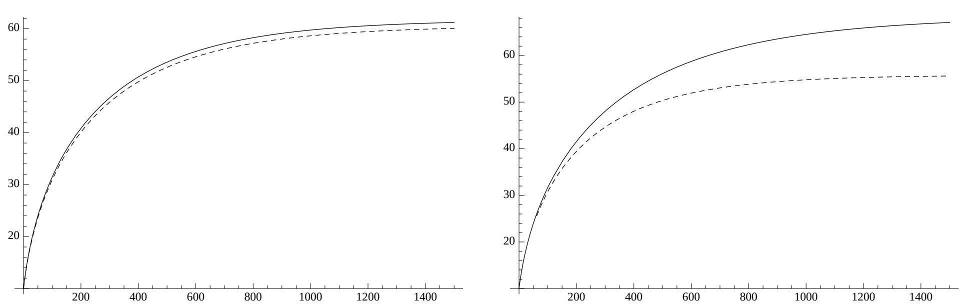
Then, $s(0) = C + \ln(-4k)/h^2 = 0$ so $C = -\ln(-4k)/h^2$.

3. $\begin{pmatrix} 30. & 4.39078 & 0.001376 \\ 35. & 5.13939 & 0.001376 \\ 40. & 5.84888 & 0.001376 \\ 45. & 6.51387 & 0.001376 \\ 50. & 7.12928 & 0.001376 \end{pmatrix}$

7.



8.



10. Students should notice that for longer races, drag affects the skier the most. Minimizing body area by wearing tight clothing can help skiers increase their speed substantially.

C. The Schrödinger Equation

1. In this problem, remember that e represents the **elementary charge** rather than the unique number whose natural logarithm has value one. For the given values, $a_0 = \dfrac{4\pi\epsilon_0 h^2}{\mu e^2} = 5.29465 \times 10^{11}$, $2a_0 = 1.05893 \times 10^{10}$. $E = -\dfrac{h^2}{2\mu {a_0}^2} = 2.17869 \times 10^{18}$.

5.

$$u(r) = c_1\, r^{\ell}((1 - 2\,\frac{(\ell+1)p}{-2\,\ell-2} r + 2\,\frac{4\,p^2 + 6\,\ell p^2 + 2\,\ell^2 p^2 - 2\,Z\ell - 2\,Z}{(-2\,\ell-2)(-6-4\,\ell)} r^2 - 8\,\frac{p(6\,p^2 + 11\,\ell p^2 + 6\,\ell^2 p^2 - 3\,Z\ell^2 - 9\,Z\ell - 6\,Z + \ell^3 p^2)}{(-2\,\ell-2)(-6-4\,\ell)(-12-6\,\ell)} r^3 + \cdots)) + c_2\, r^{-\ell-1}((1 + 2\,\frac{\cdots}{\ell^2 + 4\cdots} \cdots 2\,\frac{4\,p^2 + 6\,(-\ell-1)p^2 + 2\,(-\ell-1)^2 p^2 + Z\ell^2 + Z\ell - 3\,Z(-\ell-1) - 2\,Z - Z(-\ell-1)^2}{(\ell^2 + 4\,\ell + 1 - (-\ell-1)^2)(-1 + \ell^2 + 6\,\ell - (-\ell-1)^2)} r^2 - \cdots$$

6.

$$R(r) = c_1\, r^{\ell}((1 + 200\,\frac{Z}{-2\,\ell-2} r + 40000\,\frac{Z^2}{(-2\,\ell-2)(-6-4\,\ell)} r^2 + \cdots)) + c_2\, r^{-\ell-1}((1 + 200\,\frac{Z}{\ell^2 + 4\,\ell + 1 - (-\ell-1)^2} r + 40000\,\frac{Z^2}{(\ell^2 + 4\,\ell + 1 - (-\ell-1)^2)(-1 + \ell^2 \cdots}$$