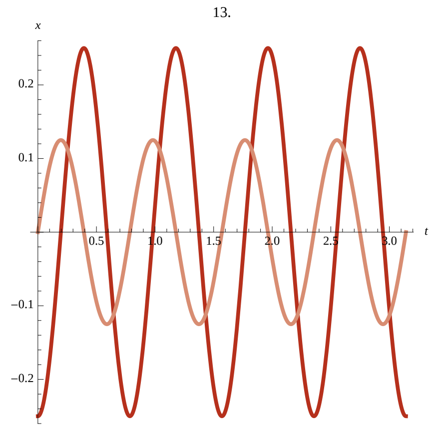
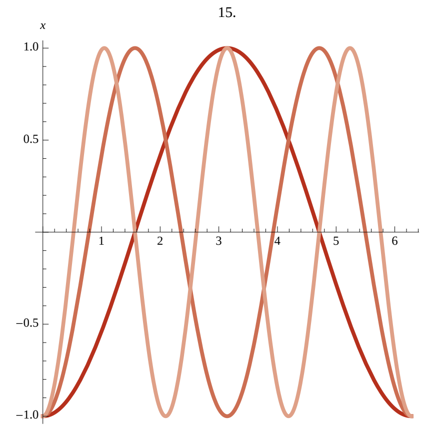
Exercises 5.1

1. $m = 4$ slugs, $k = 9$ lb/ft; released 1 ft above equilibrium with zero initial velocity
3. $m = 1/4$ slugs, $k = 16$ lb/ft; released 3/4 ft (8 in) below equilibrium with an upward initial velocity of 2 ft/s
5. A general solution of $x'' + x = 0$ is $x(t) = c_1 \cos t + c_2 \sin t$. Application of the initial condition yields $x(t) = 3\cos t - 4\sin t = 5\cos(t - \phi)$, $\phi = -\cos^{-1}(3/5) \approx 0.93$ rads; period $= 2\pi$, amplitude $= 5$
7. A general solution of $x'' + 16x = 0$ is $c_1 \cos 4t + c_2 \sin 4t$. Then, $x'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t$; $x(0) = c_1 = -2$, $x'(0) = 4c_2 = 1 \Rightarrow c_2 = 1/4$ so $x(t) = -2\cos 4t + \frac{1}{4}\sin 4t$, which we can rewrite as

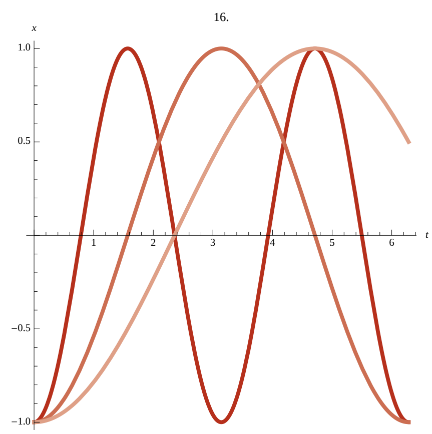
$$x(t) = \sqrt{(-2)^2 + \left(\frac{1}{4}\right)^2}\cos(4t - \phi) = \frac{\sqrt{65}}{4}\cos(4t - \phi),$$

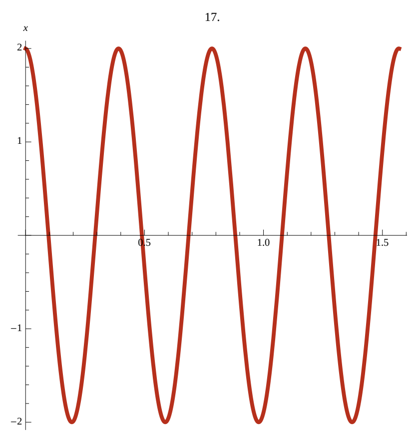
where $\phi = \cos^{-1}(-8/\sqrt{65}) \approx 3.02$ rads; period $= \pi/2$; amplitude $= \sqrt{65}/4$
9. A general solution of $x'' + 9x = 0$ is $x(t) = c_1 \cos 3t + c_2 \sin 3t$. Application of the initial conditions yields $x(t) = \frac{1}{3}\cos 3t - \frac{1}{3}\sin 3t$. Thus, $x(t) = \frac{\sqrt{2}}{3}\cos(3t - \phi)$, $\phi = -\cos^{-1}(1/\sqrt{2}) = -\pi/4$ rads; period $= 2\pi/3$; amplitude $= \sqrt{2}/3$
11. Here, $16 = k \cdot 1/2 \Rightarrow k = 32$ and $16 = m \cdot 32 \Rightarrow m = 1/2$ so we solve $\frac{1}{2}x'' + 32x = 0$, $x(0) = 1$, $x'(0) = 0$. A general solution of $x'' + 64x = 0$ is $x(t) = c_1 \cos 8t + c_2 \sin 8t$ and application of the initial conditions yields $x(t) = \cos 8t$, maximum displacement $= 1$ ft when $-8\sin 8t = 0$ or $8t = n\pi$ so that $t = n\pi/8$, $n = 0, 1, 2, \ldots$.
13. We first determine the spring constant k and the mass m: $F = ks \Rightarrow 6 = \frac{1}{2} \cdot k \Rightarrow k = 12$. Next, $F = mg \Rightarrow 6 = m \cdot 32 \Rightarrow m = 3/16$. Thus, we solve $\frac{3}{16}x'' + 12x = 0$ or, equivalently, $x'' + 64x = 0$. A general solution of this equation is $x(t) = c_1 \cos 8t + c_2 \sin 8t$. If the object is lifted 3 in. = 1/4 ft. above the equilibrium position and released, we find the solution that satisfies the initial conditions $x(0) = -1/4$ and $x'(0) = 0$: $x(0) = c_1 = -1/4$, $x'(0) = 8c_2 = 0 \Rightarrow x(t) = -\frac{1}{4}\cos 8t$; $t = \pi/16$ sec; $x(5) \approx 0.167$ ft.; $x(t) = \frac{1}{8}\sin 8t$; $t = \pi/8$ sec





15. General solutions are $x = c_1 \cos t + c_2 \sin t$, $x = c_1 \cos 2t + c_2 \sin 2t$, and $x = c_1 \cos 3t + c_2 \sin 3t$. Applying the initial conditions yields $x = -\cos t$, $x = -\cos 2t$, and $x = -\cos 3t$, respectively. As k increases, the frequency at which the spring-mass system passes through equilibrium increases.





17. Here, $1 = k \cdot 1/8 \Rightarrow k = 8$ and $1 = m \cdot 32 \Rightarrow m = 1/32$ so we solve $\frac{1}{32}x'' + 8x = 0$, $x(0) = b$, $x'(0) = -1$. A general solution of $x'' + 256x = 0$ is $x(t) = c_1 \cos 16t + c_2 \sin 16t$ and application of the initial conditions yields $x(t) = b\cos 16t - \frac{1}{16}\sin 16t$. The amplitude of the solution is $A = \sqrt{b^2 + \dfrac{1}{k/m}} = \sqrt{b^2 + \dfrac{1}{256}}$ and $\sqrt{b^2 + \dfrac{1}{256}} = 2 \Rightarrow b = \sqrt{1023}/16$

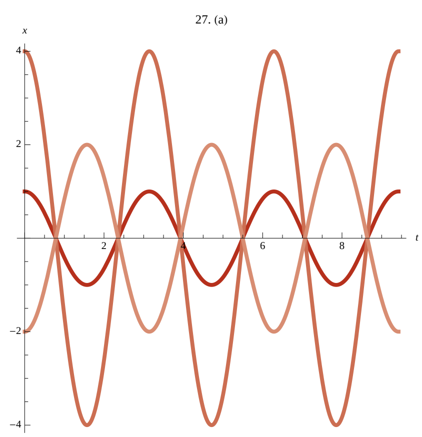
19. $T = 2\pi\sqrt{m/k} \Rightarrow 2\pi\sqrt{m/32} = \pi/2 \Rightarrow m = 2$ slugs

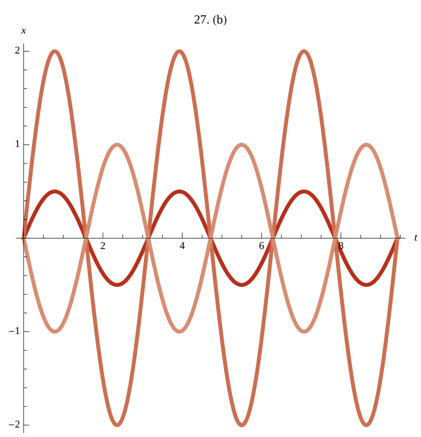
21. First, $x(t) = \sqrt{\alpha^2 + \dfrac{\beta^2}{\omega^2}}\cos(\omega t - \phi) \Rightarrow v(t) = x'(t) = -\omega\sqrt{\alpha^2 + \dfrac{\beta^2}{\omega^2}}$. The maximum value of $-\omega\sqrt{\alpha^2 + \dfrac{\beta^2}{\omega^2}}\sin(\omega t - \phi)$ is $\omega\sqrt{\alpha^2 + \dfrac{\beta^2}{\omega^2}}$.

23. $y(t) = 1.61\,\cos 3.5t - 0.856379\,\sin 3.5t$; maximum displacement $= \sqrt{(1.61)^2 + (0.856379)^2} =$ 1.8236 ft. For (c), the result is $\dfrac{d^2y}{dt^2} + \dfrac{\pi r^2 \rho}{m} y = 0$

25. First, we find the mass of the cylinder: $512 = m \cdot 32 \Rightarrow m = 16$ slugs. Also, $h = mg/(\pi R^2 \rho) = 512/(62.5\pi) \approx 2.61$ ft $\Rightarrow$ Equilibrium position is $4 - 2.61 \approx 1.39$ feet and $y(0) = 3 - 1.39 \approx 1.61$ feet so we solve the initial value problem $y'' + \dfrac{\pi \cdot 62.5}{16} y = 0$, $y(0) = 1.61$, $y'(0) = -3$. A general solution of $y'' + \dfrac{\pi \cdot 62.5}{16} y = 0$ is $y(t) = c_1 \cos 3.5t + c_2 \sin 3.5t$ and applying the initial conditions results in $y(t) = 1.61 \cos 3.5t - 0.856379 \sin 3.5t$. Maximum displacement is $1.61^2 + 0.8564444^2 \approx 1.8236$ ft.

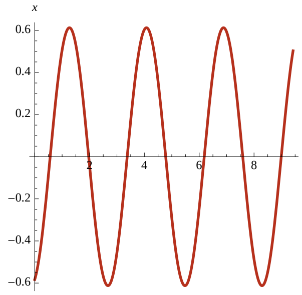
27. For (a), we solve $x'' + 4x = 0$, $x(0) = \alpha$, $x'(0) = 0$ for $\alpha = 1, 4, -2$. $\alpha = 1$: $x = \cos 2t$; $\alpha = 4$: $x = 4\cos 2t$; $\alpha = -2$: $x = -2\cos 2t$. From the graph we see that varying α affects the amplitude of the displacement but not the times at which the mass passes through the equilibrium position.



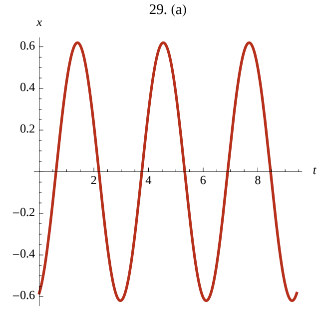


For (b), we solve $x'' + 4x = 0$, $x(0) = 0$, $x'(0) = \beta$ for $\beta = 1, 4, -2$. $\beta = 1$: $x = \frac{1}{2}\sin 2t$; $\beta = 4$: $x = 2\sin 2t$; $\beta = -2$: $x = -\sin 2t$. From the graph, we see that varying β not only affects the initial velocity but also the amplitude of the displacement. The time at which the object passes through the equilibrium position is not affected.

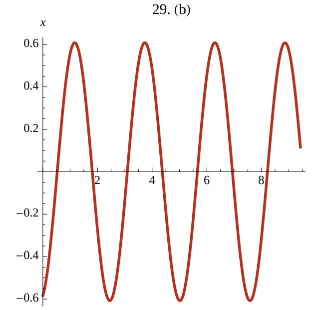
28.



29. (a)

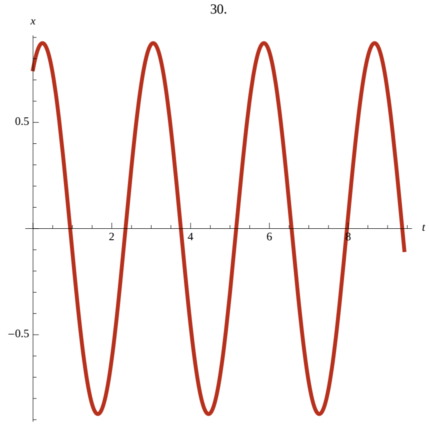


29. (b)

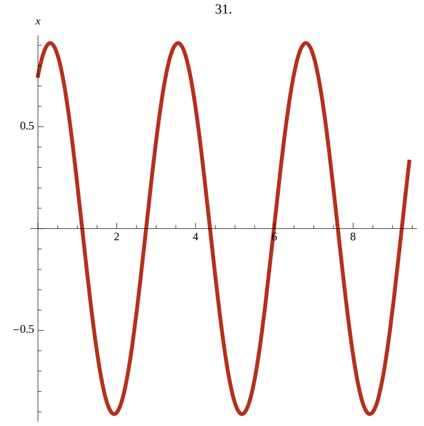


29. For (a) the solution is $x = \frac{1}{24}\left(-14\cos 2t + 5\sin 2t\right)$ and for (b) the solution is $x = \frac{1}{72}\left(-42\cos\sqrt{6}t + 5\sqrt{6}\sin\sqrt{6}t\right)$. For (a), the period is π and the amplitude is $\frac{1}{6}\sqrt{137/2} \approx 1.37941$. For (b) the period is $\sqrt{2/3}\pi$ and the amplitude is $\frac{1}{12}\sqrt{199} \approx 1.17556$.

30.



31.



31. The solution is

$$x = \frac{1}{60}\left(45\cos\left(\frac{1}{2}\sqrt{15}t\right) + 8\sqrt{15}\sin\left(\frac{1}{2}\sqrt{15}t\right)\right).$$

The period is $4\pi/\sqrt{15}$ and the amplitude is $\frac{1}{4}\sqrt{199/15} \approx 0.910586$

Exercises 5.2

1. $m = 1$, $c = 4$, $k = 3$; released from equilibrium with an upward initial velocity of 4 ft/s.
3. $m = 1/4$, $c = 2$, $k = 1$; released 6 inches above equilibrium with a downward initial velocity of 1 ft/s.
5. A general solution of $x'' + 4x' + 13x = 0$ is $x = e^{-2t}(c_1\cos 3t + c_2\sin 3t)$. Application of the initial conditions yields $c_1 = 1$, $-2c_1 + 3c_2 = -1 \Rightarrow c_1 = 1$, $c_2 = 1/3$ and $x(t) = e^{-2t}\left(\cos 3t + \frac{1}{3}\sin 3t\right) = \frac{\sqrt{10}}{3}e^{-2t}\cos(3t - \phi)$, $\phi = \cos^{-1}(3/\sqrt{10}) \approx$

0.32 rads, Q.P.: $2\pi/3$, $t \approx 0.63$
7. The characteristic equation of $x'' + 2x' + 26x = 0$ is $r^2 + 2r + 26 = 0$ with solutions $r_{1,2} = -1 \pm 5i$ so a general solution is $x = e^{-t}(c_1 \cos 5t + c_2 \sin 5t)$. Application of the initial conditions yields $c_1 = 1$, $-c_1 + 5c_1 = 1 \Rightarrow c_1 = 1$, $c_2 = 2/$ so $x = e^{-t}\left(\cos 5t + \frac{2}{5}\sin 5t\right)$. Thus, $x(t) = \frac{\sqrt{29}}{5}e^{-t}\cos(5t - \phi)$, $\phi = \cos^{-1}(5/\sqrt{29}) \approx 0.38$ rads, Q.P.: $2\pi/5$, $t \approx 0.39$
9. A general solution of $x'' + 8x' + 15x = 0$ os $x = c_1 e^{-5t} + c_2 e^{-3t}$ and application of the initial conditions yields $c_1 + c_2 = 0$, $-5c_1 - 3c_2 = 1 \Rightarrow c_1 = -1/2$, $c_2 = 1/2 \Rightarrow x(t) = -\frac{1}{2}e^{-5t} + \frac{1}{2}e^{-3t}$; overdamped; does not pass through equilibrium; $x\left(\frac{1}{2}\ln(5/3)\right) \approx 0.093$
11. The characteristic equation is $r^2 + \frac{3}{2}r + \frac{1}{2} = \left(r + \frac{1}{2}\right)(r+1) = 0$ so a general solution is given by $x = c_1 e^{-t} + c_2 e^{-t/2}$. Application of the initial conditions yields the system of equations $c_1 + c_2 = -1$, $-c_1 - \frac{1}{2}c_2 = 2 \Rightarrow c_1 = -3$, $c_2 = 2$. Thus, $x(t) = -3e^{-t} + 2e^{-t/2}$; overdamped; $x(t) = 0$ implied $t = 2\ln(3/2) \approx 0.811$; maximum displacement $= 1$ at $t = 0$.
13. A general solution of $x'' + 8x' + 16x = 0$ is $x = c_1 e^{-4t} + c_2 t e^{-4t}$ and application of the initial conditions yields $c_1 = 4$, $-4c_1 + c_2 = -2$ so $c_1 = 4$ and $c_2 = 14$ so $x(t) = 4e^{-4t} + 14te^{-4t}$; critically damped; does not pass through equilibrium; maximum displacement $= 4$ at $t = 0$
15. The characteristic equation of $x'' + 10x' + 25x = 0$ is $r^2 + 10r + 25 = (r+5)^2 = 0$ with solutions $r_{1,2} = -5$ so a general solution of the ODE is $x = e^{-5t}(c_1 + c_2 t)$. Application of the initial conditions yields $c_1 = -5$ and $-5c_1 + c_2 = 1$ so $c_1 = -5$ and $c_2 = -24$ and $x(t) = -5e^{-5t} - 24te^{-5t}$; critically damped; does not pass through equilibrium; maximum displacement $= 5$ at $t = 0$
17. Because the spring-mass system is critically damped, we have $c^2 - 4mk = 0 \Rightarrow c^2 - 4 \cdot 1 \cdot 1 = 0 \Rightarrow c^2 = 4 \Rightarrow c = 2$.
19. The object's mass is given by $F = mg \Rightarrow 32 = m \cdot 32 \Rightarrow m = 1$. We solve the initial-value problem $x'' + 10x' + 24x = 0$, $x(0) = -1/2$, $x'(0) = 0$. The characteristic equation is $r^2 + 10r + 24 = 0 \Rightarrow r_1 = -4$, $r_2 = -6$ so a general solution of the differential equation is $x = c_1 e^{-4t} + c_2 e^{-6t}$. Application of the initial conditions yields the system of equations $c_1 + c_2 = -1/2$, $-4c_1 - 6c_2 = 0$, which has solutions $c_1 = -3/2$ and $c_2 = 1$ so $x(t) = -\frac{3}{2}e^{-4t} + e^{-6t}$; does not pass through equilibrium; maximum displacement $= 1/2$ at $t = 0$
21. First we find the spring constant: $m = 70 \Rightarrow F = mg = 70 \cdot 9.8 = 686 \Rightarrow k = F/s = 686/-.25 = 2744$. We solve the initial-value problem $70x'' + 280x' + 2744x = 0$, $x(0) = -3$, $x'(0) = 0$. A general solution of the equation is

$$x = e^{-2t}\left(c_1 \cos\left(4\sqrt{\frac{11}{5}}t\right) + c_2 \sin\left(4\sqrt{\frac{11}{5}}t\right)\right).$$

Application of the initial conditions yields the system of equations $c_1 = -3$, $-2c_1 + 4\sqrt{\frac{11}{5}}c_2 = 0$, which has solution $c_1 = -3$ and $c_2 = -\frac{3}{2}\sqrt{5/11}$ so $x(t) = e^{-2t}\left(-3\cos(4\sqrt{11/5}t) - \frac{3}{2}\sqrt{5/11}\sin(4\sqrt{11/5}t)\right)$ or $x(t) = \frac{21}{2\sqrt{11}}\cos(4\sqrt{11/5}t -$

ϕ), where $\phi = \cos^{-1}(-2\sqrt{11}/7) \approx 2.81646$; $x(t) = 0$ when $t = \frac{1}{4}\sqrt{5/11}\left(\frac{1}{2}(2n+1)\pi + \phi\right)$, n an integer or $t \approx 0.739471, 1.26899, 1.7985, 2.32802, 2.85753, \ldots$

23. The motion is critically damped if $c^2 - 4mk = 0 \Rightarrow c^2 - 4 \cdot 4 \cdot 64 = 0 \Rightarrow c = 32$. The motion is underdamped if $0 < c < 32$.

25. The quasi-period is

$$\frac{4\pi m}{\sqrt{4km - c^2}} = \frac{4\pi \frac{1}{13}}{\sqrt{4 \cdot 13 \cdot \frac{1}{13} - c^2}} = \frac{\pi}{6}$$

$\Rightarrow c = 10/13$.

27. The solution of $mx'' + cx' + kx = 0$, $x(0) = \alpha$, $x'(0) = 0$ is

$$x = \frac{\alpha e^{-\frac{t\left(\sqrt{c^2-4km}+c\right)}{2m}}\left(c\left(e^{\frac{t\sqrt{c^2-4km}}{m}} - 1\right) + \sqrt{c^2-4km}\left(e^{\frac{t\sqrt{c^2-4km}}{m}} + 1\right)\right)}{2\sqrt{c^2-4km}}$$

while the solution of $mx'' + cx' + kx = 0$, $x(0) = 0$, $x'(0) = \beta$ is

$$x = \frac{m\beta e^{-\frac{t\left(\sqrt{c^2-4km}+c\right)}{2m}}\left(e^{\frac{t\sqrt{c^2-4km}}{m}} - 1\right)}{\sqrt{c^2-4km}}$$

Use the Principle of Superposition to form the result.

29. $x(t) = c_1 e^{-pt} + c_2 t e^{-pt}$, $\rho = c/(2m)$; $x(0) = \alpha$, $x'(0) = \beta$ implies that $x(t) = \alpha e^{-\rho t}(1 + \rho t)$; $x(t) = 0$ implies $t = -1/\rho < 0$

31. The mass passes through the equilibrium when $\cos(\mu t - \phi) = 0$ which occurs when $\mu t - \phi = \frac{1}{2}\pi + n\pi$, $n = 0, \pm 1, \pm 2, dots$ or $t = \frac{1}{\mu}\left(\frac{1}{2}\pi + n\pi + \phi\right)$. The difference between two consecutive tiems is

$$\frac{1}{\mu}\left(\frac{1}{2}\pi + (n+1)\pi + \phi\right) - \frac{1}{\mu}\left(\frac{1}{2}\pi + n\pi + \phi\right) = \frac{\pi}{\mu},$$

which is exactly one-half of the quasiperiod.

33. Suppose that two successive minima or maximum occur when

$$t_n = \frac{1}{\mu}\left(\tan^{-1}(-\rho/\mu) + \phi + n\pi\right)$$

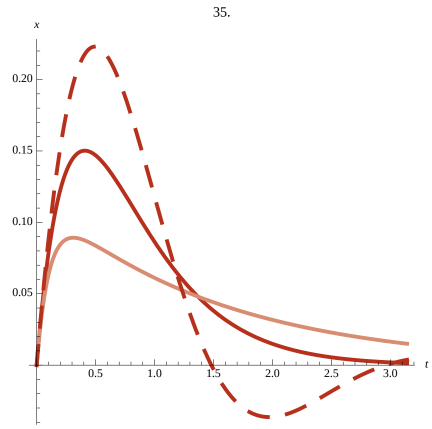
and

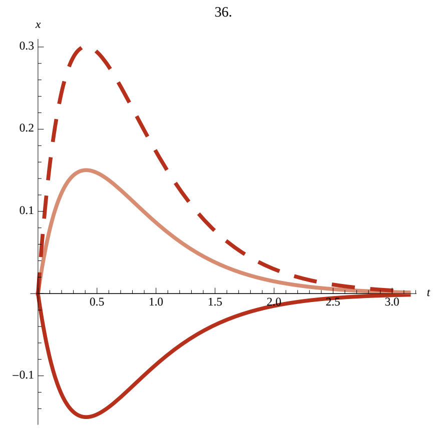
$$t_{n+2} = \frac{1}{\mu}\left(\tan^{-1}(-\rho/\mu) + \phi + (n+2)\pi\right)$$

Then, $\cos(\mu t_n - \phi) = \cos(\mu t_{n+2} - \phi)$ and

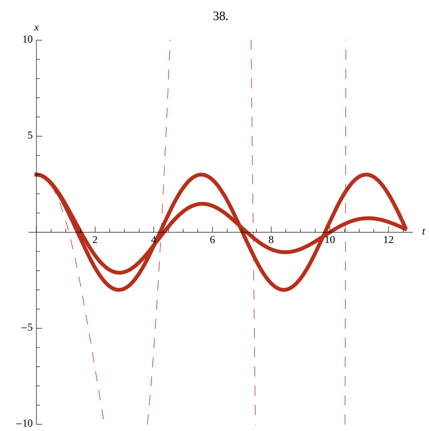
$$\frac{x(t_n)}{x(t_{n+2})} = \exp\left(2c\pi/\sqrt{4mk - c^2}\right).$$

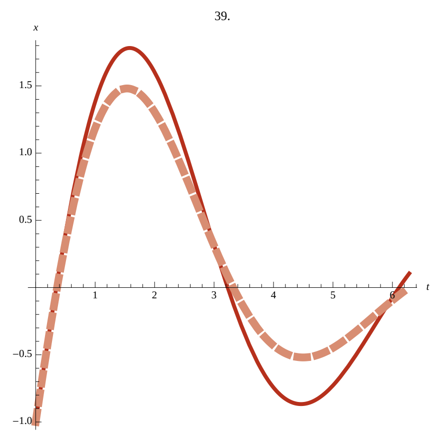
35. Note that $c = 2\sqrt{6}$ results in critical damping, $c = 4\sqrt{6}$ in overdamping and $c = -\sqrt{6}$ in underdamping. $c = 2\sqrt{6}$: $x = te^{-t\sqrt{6}}$; $c = 4\sqrt{6}$: $x = \frac{1}{6\sqrt{2}}e^{-3\sqrt{2}(1+\sqrt{3})t}(e^{6t\sqrt{2}} - 1)$; $c = \sqrt{6}$: $x = \frac{1}{3}\sqrt{2}e^{-t\sqrt{3/2}}\sin\left(\frac{3}{\sqrt{2}}t\right)$





39. $c = 2$: $x = e^{-t}(2t-1)$; $c = \sqrt{8}$: $x = (1/\sqrt{2}-2)e^{-(1+\sqrt{2})t} + (1-1/\sqrt{2})e^{(1-\sqrt{2})t}$





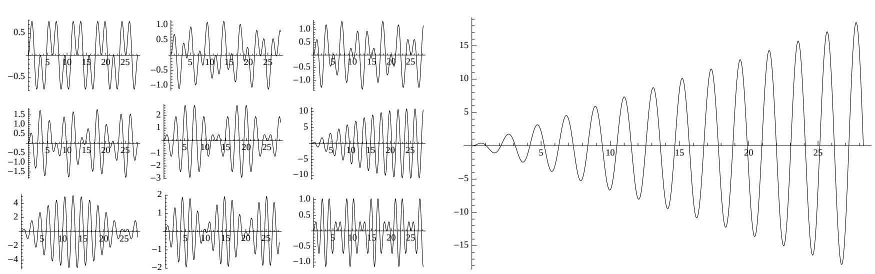
Exercises 5.3

1. First, we determine the mass and the value of the spring constant. $F = ks \Rightarrow 8 = 1 \cdot k \Rightarrow k = 8$ and $F = mg \Rightarrow m = F/g = 16/32 = 1/2$. Then we solve the initial value problems $x'' + 16x = 2\cos 3t$, $x(0) = 0$, $x'(0) = 2$. A general solution of the corresponding homogeneous equation is $x_h = c_1 \cos 4t + c_2 \sin 4t$. Using the method of undetermined coefficients we assume that a particular solution of the nonhomogenous equation has the form $x_p = A\cos 3t + B\sin 3t$. Substitution of x_p into the *nonhomogeneous* equation yields $7A\cos 3t + 7B\sin 3t = 2\cos 3t \Rightarrow A = 2/7$, $B = 0$ so a general solution of the nonhomogeneous equation is $x(t) = c_1 \cos 4t + c_2 \sin 4t + \frac{2}{7}\cos 3t$. Application of the initial con-

ditions yields $2/7+c_1=0$, $4c_2=2 \Rightarrow c_1=-2/7$, $c_2=1/2$ so $x(t) = -\frac{2}{7}\cos 4t+\frac{1}{2}\sin 4t+\frac{2}{7}\cos 3t$. The natural frequency of the system is $\omega=4$.
3. First, we determine the mass and the value of the spring constant. $F=16 \Rightarrow m=F/g=16/32=1/2$ and $F=16 \Rightarrow k=F/s=16/(2/3)=24$. ($s=8$ inches $=2/3$ foot.) Then, we solve the initial-value problem $x''+48x=4\cos t$, $x(0)=1/3$, $x'(0)=0$. The characteristic equation of the corresponding homogeneous equation $x''+48x=0$ is $r^2+48=0 \Rightarrow r_{1,2}=\pm 4\sqrt{3}i$ so a general solution of the corresponding homogeneous equation is $x_h=c_1\cos\left(4\sqrt{3}t\right)+c_2\sin\left(4\sqrt{3}t\right)$. Using the method of undetermined coefficients, we assume that there is a particular solution of the nonhomogeneous equation of the form $x_p=A\cos t+B\sin t$. SUbstitution of this function and its derivatives into the nonhomogeneous equation yields $47A\cos t+47B\sin t=4\cos t$ so $A=4/47$ and $B=0$. Thus, a general solution of the nonhomogeneous euqation is $x=c_1\cos\left(4\sqrt{3}t\right)+c_2\sin\left(4\sqrt{3}t\right)+\frac{4}{47}\cos t$. Applying the initial conditions results in the system of equations $c_1+4/47=1/3$, $4\sqrt{3}c_2=0$, which has solution $c_1=35/141$ and $c_2=0$. Thus, $x(t)=\frac{35}{141}\cos(4\sqrt{3}t)+\frac{4}{47}\cos t$.
5. We solve $x''+9x=4\cos\omega t$, $x(0)=0$, $x'(0)=0$. A general solution of the corresponding homogeneous equation is $x_h=c_1\cos 3t+c_2\sin 3t$. If $\omega\neq 3$, we assume that a particular solution of the nonhomogeneous equation is $s_p=A\cos\omega t+B\sin\omega t$. Substitution of this function into the nonhomogenous equation yields

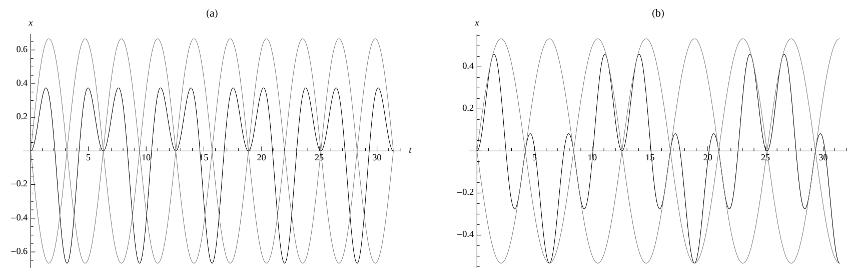
$$A(9-\omega^2)\cos\omega t+B(9-\omega^2)\sin\omega t=4\cos\omega t \Rightarrow A=4/(9-\omega^2),\ B=0$$

so a general solution of the nonhomogeneous equation is $x=c_1\cos 3t+c_2\sin 3t+\dfrac{4}{9-\omega^2}\cos\omega t$. $x(t)=\begin{cases}\frac{4}{\omega^2-9}(\cos 3t-\cos\omega t),\ \omega\neq 3\\ \frac{2}{3}t\sin 3t,\ \omega=3\end{cases}$; resonance occurs if $\omega=3$



7. We solve $x''+8x'+25x=\cos t-\sin t$, $x(0)=x'(0)=0$. A general solution of the corresponding homogeneous equation is $x_h=e^{-4t}(c_1\cos 3t+c_2\sin 3t)$. Using the method of undetermined coefficients, there is a particular solution of the form $x_p=A\cos t+B\sin t$. Substitution of x_p into the nonhomogeneous equation yields $x_p=\frac{1}{20}\cos t-\frac{1}{40}\sin t$ so a general solution of the nonhomogeneous equation is $x=e^{-4t}(c_1\cos 3t+c_2\sin 3t)+\frac{1}{20}\cos t-\frac{1}{40}\sin t$. Application of the initial conditions yields $x(t)=-e^{-4t}\left(\frac{1}{20}\cos 3t+\frac{7}{120}\sin 3t\right)\frac{1}{40}(2\cos t-\sin t)$.

9. We solve $x'' + 4x = \cos t$, $x(0) = x'(0) = 0$; $x_h = c_1 \cos 2t + c_2 \sin 2t$; $x_p = A\cos t + B \sin t = \frac{1}{3}\cos t \Rightarrow x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3}\cos t$. Application of the initial conditions yields $x(t) = \frac{1}{3}(\cos t - \cos 2t)$; the envelope functions are $\pm\frac{2}{3}\sin\left(\frac{1}{2}t\right)$. If the external force is changed to $f(t) = \cos(t/2)$, we solve $x'' + 4x = \cos(t/2)$, $x(0) = x'(0) = 0$. $x_p = A\cos(t/2) + B\sin(t/2) = \frac{4}{15}\cos(t/2)$. Application of the initial conditions yields $x(t) = \frac{4}{15}\cos(t/2) - \frac{4}{15}\cos 2t$; the envelope functions are $\pm\frac{8}{15}\sin\frac{3}{4}t$. The maximum displacement decreases



11. Solve: $4x''+4x'+26x = 250\sin t$, $x(0) = x'(0) = 0$; $x(t) = e^{-t/2}\left(c_1\cos(\frac{5}{2}t) + c_2\sin(\frac{5}{2}t)\right) - 2\cos t + 11\sin t$; $x(t) = e^{-t/2}\left(2\cos(\frac{5}{2}t) - 4\sin(\frac{5}{2}t)\right) - 2\cos t + 11\sin t$; transient: $e^{-t/2}\left(2\cos(\frac{5}{2}t) - 4\sin(\frac{5}{2}t)\right)$; steady-state: $-2\cos t + 11\sin t$

13. First, write the equation as $x'' + (k/m)x = (F/m)\sin\omega t$. Both k and m are positive so $x_h = c_1\cos(\sqrt{k/m}t) + c_2\sin(\sqrt{k/m}t)$. $x_p = A\cos\omega t + B\sin\omega t = \frac{F}{k-m\omega^2}\sin\omega t$. (a) $x(t) = \alpha\cos(\sqrt{k/m}t) - \frac{F\omega}{k-m\omega^2}\sqrt{m/k}\sin(\sqrt{k/m}t) + \frac{F}{k-m\omega^2}\sqrt{m/k}\sin\omega t$; (b) $x(t) = \left(\beta - \frac{F\omega}{k-m\omega^2}\right)\sqrt{m/k}\sin(\sqrt{k/m}t) + \frac{F}{k-m\omega^2}\sqrt{m/k}\sin\omega t$; (c) $x(t) = \alpha\cos(\sqrt{k/m}t) + \left(\beta - \frac{F\omega}{k-m\omega^2}\right)\sqrt{m/k}\sin(\sqrt{k/m}t) + \frac{F}{k-m\omega^2}\sqrt{m/k}\sin\omega t$

15. First solve $x''+x = 1$, $x(0) = x'(0) = 0 \Rightarrow x(t) = 1 - \cos t$. Because $x(\pi) = 2$ and $x'(\pi) = 0$ next solve $x'' + x = 0$, $x(\pi) = 2$, $x'(\pi) = 0 \Rightarrow x(t) = -2\cos t$. Thus, $x(t) = \begin{cases} 1 - \cos t,\ 0 \le t \le \pi \\ -2\cos t,\ t > \pi \end{cases}$

17. First solve $x'' + x = t$, $x(0) = x'(0) = 0 \Rightarrow x(t) = t - \sin t$. $x(1) = 1 - \sin 1$ and $x'(1) = 1 - \cos 1$ so next solve $x'' + x = 2 - t$, $x(1) = 1 - \sin 1$, $x'(1) = 1 - \cos 1 \Rightarrow x(t) = (\sin 2 - 2\sin 1)\cos t + 4\cos 1\sin^2\frac{1}{2}\ \sin t$. Thus,

$$x(t) = \begin{cases} t - \sin t,\ 0 \le t \le 1 \\ -2\sin 1\cos t + (2\cos 1 - 1)\sin t + 2 - t,\ 1 < t \le 2 \\ (\sin 2 - 2\sin 1)\cos t + 4\cos 1\sin^2\frac{1}{2}\ \sin t,\ t > 2 \end{cases}$$

19. The roots of the characteristic equation $r^2 + 2\lambda r + \omega^2 = 0$ are $r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$. Therefore, $x_h(t) = e^{-\lambda t}(c_1\cos\sqrt{\omega^2-\lambda^2}t + c_2\sin\sqrt{\omega^2-\lambda^2}t)$, which can be written as

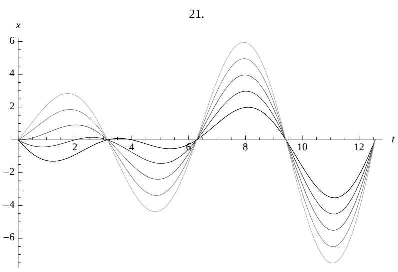
$$x_h(t) = Ae^{-\lambda t}\sin\left(\sqrt{\omega^2-\lambda^2}t - \phi\right) = Ae^{-\lambda t}\left[\sin\left(\sqrt{\omega^2-\lambda^2}t\right)\cos\phi - \cos\left(\sqrt{\omega^2-\lambda^2}t\right)\sin\phi\right],$$

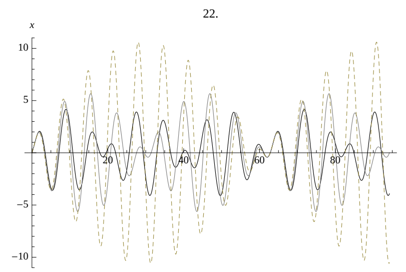
where $A = \sqrt{c_1^2 + c_2^2}$, $\sin\phi = c_1/A$, and $\cos\phi = c_2/A$. A particular solution of

$x'' + 2\lambda x' + \omega^2 x = F \sin \gamma t$ is

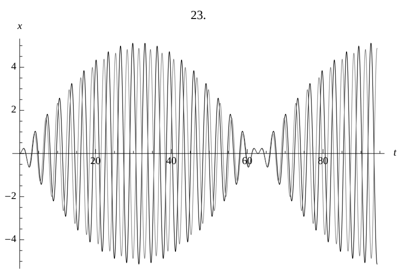
$$x_p(t) = \frac{F\left(\omega^2 - \gamma^2\right)\sin(\gamma t) - 2F\gamma\lambda\cos(\gamma t)}{\gamma^4 + \gamma^2\left(4\lambda^2 - 2\omega^2\right) + \omega^4}.$$

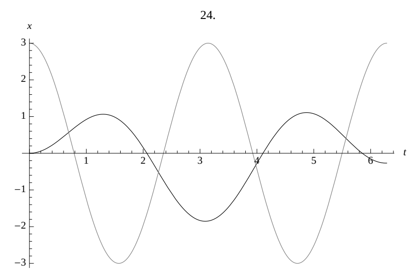
With $\sin(A+B) = \sin A \cos B + \cos A \sin B$, we have $x_p(t) = \dfrac{F}{(\omega^2-\gamma^2)^2 + 4\lambda^2\gamma^2}\sin(\gamma t + \theta)$,si
$\dfrac{-2\lambda\gamma}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}$ and $\cos\phi = \dfrac{\omega^2-\gamma^2}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}$.
21. $b = 0$: $x = \frac{1}{2} t \sin t$; $b = 1$: $x = \frac{1}{2}(2\sin t + t\sin t)$. As $\lambda \to 0$, the motion approaches resonance.





23. (a) $x(t) = \frac{100}{39}(\cos(19t/10) - \cos 2t)$; (b) $x(t) = \frac{100}{41}(\cos 2t - \cos(21t/10))$



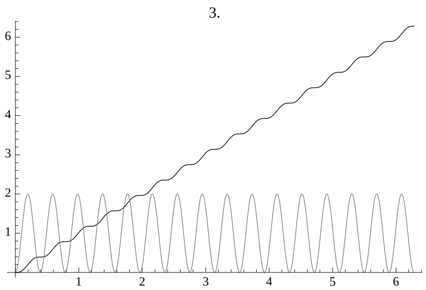


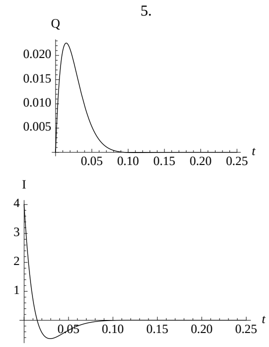
25. $x(t) = \begin{cases} t - \sin t + a\cos t + b\sin t, \ 0 \le t \le 1 \\ -t - \sin t + 2\sin(t-1) + 2 + a\cos t + b\sin t, \ 1 < t \le 2 \\ -\sin t + 2\sin(t-1) - \sin(t-2) + a\cos t + b\sin t, \ t > 1 \end{cases}$
27. $x(t) = -\frac{5000}{89733775561}(\cos(299581t/50) - \cos(20\sqrt{15}t))$

Exercises 5.4

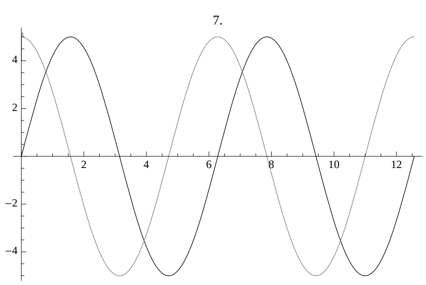
1. Solve $2Q'' + 32Q = 220$ or $Q'' + 16Q = 110$, $Q(0) = Q'(0) = 0$. A general solution is $Q = c_1 \cos 4t + c_2 \sin 4t + 55/8$. Application of the initial condition yields $Q(t) = \frac{55}{8}(1 - \cos 4t)$. $I(t) = Q'(t) = \frac{55}{2}\sin 4t$.
3. Solve $\frac{1}{4}Q'' + 64Q = 16t$ or $Q'' + 256Q = 64t$, $Q(0) = Q'(0) = 0$. A general solution of the equation is $Q = c_1\cos 16t + c_2 \sin 16t + t/4$ and application of the initial conditions yields $Q(t) = \frac{1}{64}(16t - \sin 16t)$. $I(t) = Q'(t) = \frac{1}{4}(1 - \cos 16t)$.
5. Solve $Q'' + 125Q' + 5000Q = 0$, $Q(0) = 0$, $Q'(0) = 4$. A general solution

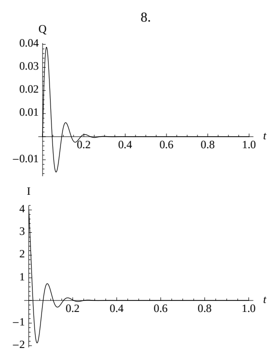
is $Q(t) = e^{-125t/2}\left(c_1 \cos\left(\frac{25}{2}\sqrt{7}t\right) + c_2 \sin\left(\frac{25}{2}\sqrt{7}t\right)\right)$. Application of the initial conditions yields $Q(t) = \frac{8\sqrt{7}}{175}e^{-125t/2}\sin\left(\frac{25\sqrt{7}}{2}t\right)$; max.: 0.0225 at $t \approx 0.0147$





7. Solve $Q'' + 125Q' + 5000Q = 630\cos t_2 5000\sin t$. A general solution is $Q(t) = e^{-125t/2}\left(c_1 \cos\left(\frac{25}{2}\sqrt{7}t\right) + c_2 \sin\left(\frac{25}{2}\sqrt{7}t\right)\right) + \frac{12185}{12502813}\cos t + \frac{62526875}{12502813}\sin t$. Application of the initial conditions yields $Q(t) = e^{-125t/2}\left(-\frac{12185}{12502813}\cos\left(\frac{25\sqrt{7}}{2}t\right) + \frac{26554371}{312570325\sqrt{7}}\sin\left(\frac{25\sqrt{7}}{2}t\right)\right) + \frac{12185}{12502813}\cos t + \frac{62526875}{12502813}\sin t$; steady-state charge: $\lim_{t\to\infty} Q(t) = \frac{12185}{12502813}\cos t + \frac{62526875}{12502813}\sin t$; steady-state current: $-\frac{12185}{12502813}\sin t + \frac{62526875}{12502813}\cos t$





9. (a) $s(x) = \frac{1}{300}x^4 + \frac{1}{3}x^2 - \frac{1}{15}x^3$; (b) $s(x) = \frac{1}{30}x^4 + \frac{10}{3}x^2 - \frac{2}{3}x^3$; (c) $s(x) = \frac{1}{3}x^4 + \frac{100}{3}x^2 - \frac{20}{3}x^3$

11. In each case, we consider the boundary value problem $EI\frac{d^4s}{dx^4} = 8$, $s(0) = s''(0) = 0$, $s(10) = s''(10) = 0$. (a) $s(x) = \frac{1}{300}x^4 + \frac{10}{3}x - \frac{1}{15}x^3$; (b) $s(x) = \frac{1}{30}x^4 + \frac{100}{3}x - \frac{2}{3}x^3$; (c) $s(x) = \frac{1}{3}x^4 + \frac{1000}{3}x - \frac{20}{3}x^3$; simple support leads to larger maximum displacement

13 and 22. (a) $s(x) = \frac{1}{360}x^6 + \frac{10000}{9}x - \frac{125}{9}x^3$; (d) $s(x) = \frac{1}{360}x^6 + \frac{1250}{3}x - \frac{125}{18}x^3$

14 and 23. (a) $s(x) = 480000\pi^{-4}\sin(\pi x/10)$; (b) no solution; (c) $s(x) = -80\pi^{-4}\left(-600\pi x - 300\pi^3 x - +\pi^3 x^3 - 6000\sin(\pi x/10)\right)$ (d) $s(x) = 240\pi^{-4}\left(-100\pi x + \pi x^3 + 2000\sin(\pi x/10)\right)$

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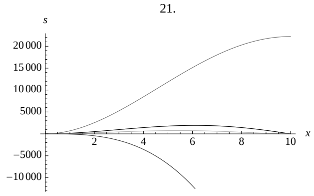

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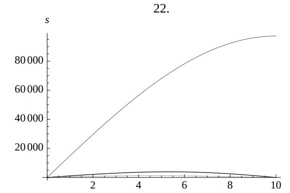

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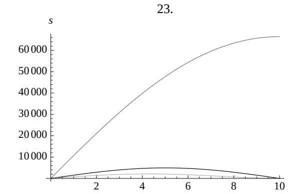


15. A general solution of $LQ''+\frac{1}{C}Q=0$ is $Q=c_1\cos\left(\frac{1}{\sqrt{LC}}t\right)+c_2\sin\left(\frac{1}{\sqrt{LC}}t\right)$. Application of the initial conditions yields $Q(t)=Q_0\cos(t/\sqrt{LC})$; differentiation yields $I(t)=-Q_0/\sqrt{LC}\,\sin(t/\sqrt{LC})$; maximum $Q=Q_0$; maximum $I=Q_0/\sqrt{LC}$

17. If $CR^2-4L<0$, general solution of the corresponding homogeneous equation is

$$Q_h=-\exp\left(-\frac{R}{2L}t\right)\left(c_1\cos\left(\frac{\sqrt{4L-R^2C}}{2L\sqrt{C}}t\right)+c_2\sin\left(\frac{\sqrt{4L-R^2C}}{2L\sqrt{C}}t\right)\right)$$

A particular solution of the nonhomogeneous equation is

$$Q_p=-\frac{CE_0}{1-2CL\omega^2+C^2\omega^2(L^2\omega^2+R^2)}\left(C\omega R\cos\omega t+(CL\omega^2-1)\sin\omega t\right)$$

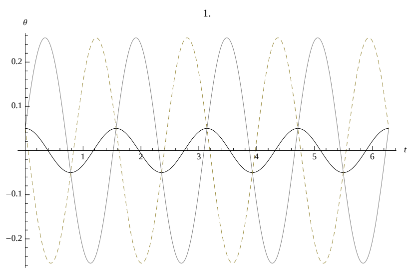
and $I_p=E_0((L\omega-1/(C\omega)^2+R^2))^{-1}\left(R\sin\omega t-(L\omega-1/(C\omega)\cos\omega t\right)$

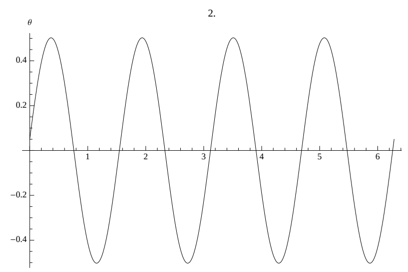
19. $I_p=\frac{E_0}{(L\omega-1/(C\omega))^2+R^2}\sqrt{(L\omega-1/(C\omega))^2+R^2}\cos(\omega t-\phi)$, $\cos\phi=-(L\omega-1/(C\omega))/\sqrt{(L\omega-1/(C\omega))^2+R^2}$. Let $g(\omega)=\sqrt{(L\omega-1/(C\omega))^2+R^2}$; $g'(\omega)=0\Rightarrow\omega=1/\sqrt{LC}$.

The motion stops in (a) because $\lim_{t\to\infty}\theta(t)=0$; it eventually oscillates in (b).
25. (a) $s(x)=\frac{1}{36000}x^6-\frac{5}{36}x^3+\frac{100}{9}x$; (b) no solution; (c) $s(x)=\frac{1}{36000}x^6-\frac{5}{9}x^3+150x$; (d) $s(x)=\frac{1}{36000}x^6-\frac{5}{72}x^3+\frac{25}{6}x$

Exercises 5.5

1. $g/L=32/2=16\Rightarrow$ we solve $\theta''+16\theta=0$ subject to the indicated conditions. The characteristic equation is $r^2+16=0\Rightarrow r_{1,2}=\pm4i$ so a general solution is $\theta(t)=c_1\cos4t+c_2\sin4t$. (a) $\theta(t)=\frac{1}{20}\cos4t$; (b) $\theta(t)=\frac{1}{20}\cos4t+\frac{1}{4}\sin4t$; (c) $\theta(t)=\frac{1}{20}\cos4t-\frac{1}{4}\sin4t$; maximum displacement: (a) $1/20$; (b) and (c): $\sqrt{26}/20\approx0.255$





3. The characteristic equation of $\theta'' + 2\sqrt{7}\theta' + 16\theta = 0$ has roots $r_{1,2} = \frac{1}{2}\left(-2\sqrt{7} \pm 6i\right) = -\sqrt{7} \pm 3i$. Therefore, a general solution of the equation is $\theta(t) = e^{-t\sqrt{7}}\left(c_1 \cos 3t + c_2 \sin 3t\right)$. (a) $\theta(t) = \frac{1}{60}e^{-t\sqrt{7}}(\sqrt{7}\sin 3t + 3\cos 3t)$; (b) $\theta(t) = \frac{1}{60}e^{-t\sqrt{7}}\left((\sqrt{7}+20)\sin 3t + 3\cos 3t\right)$; (c) $\theta(t) = \frac{1}{60}e^{-t\sqrt{7}}\left((\sqrt{7}-20)\sin 3t + 3\cos 3t\right)$
5. $\theta'(t) = -\theta_0\omega\sin\omega t + v_0\cos\omega t$; $\theta''(t) = -\theta_0\omega^2\cos\omega t - v_0\omega\sin\omega t$; $\theta'' + \omega^2\theta = -\theta_0\omega^2\cos\omega t - v_0\omega\sin\omega t + \theta_0\omega^2\cos\omega t + v_0\omega\sin\omega t = 0$; $\theta(0) = \theta_0$, $\theta'(0) = v_0$
7. Because $\theta = \theta_0\cos\omega t + (v_0/\omega)\sin\omega t$ and $\theta = A\cos(\omega t - \phi) = A\cos\omega t\cos\phi + A\sin\omega t\sin\phi$, we have $A\cos\phi = \theta_0$ and $A\sin\phi = v_0/\omega$. Therefore, $\cos\phi = \theta_0/A$ and $\sin\phi = v_0/(\omega A)$, so $(\theta_0/A)^2 + (v_0/(\omega A))^2 = 1$l Solving for A, we find that $A = \sqrt{\theta_0^2 + v_0^2/\omega^2}$. The phase angle is determined by finding the angle ϕ that satisfies $\cos\phi = \theta_0/A$ and $\sin\phi = v_0/(\omega A)$.
9. $T = 2\pi\sqrt{2/9.8} \approx 2.83$ s
11. $T = 2\pi\sqrt{8/32} \approx 3.14$ s
13. $2\pi\sqrt{L/9.8} = 1$ if $L = 0.248$ m
15. Maximum displacement: $\sqrt{\theta_0^2 + v_0^2/\omega^2}$ when $\cos(\omega t - \phi) = \pm 1 \Rightarrow \omega t - \phi = n\pi$, $(n = 0, \pm 1, \pm 2, \ldots) \Rightarrow t = (1/\omega)(n\pi + \phi)$; $\omega = \sqrt{g/L}$, $\phi = \cos^{-1}\left(\theta_0/\sqrt{\theta_0^2 + v_0^2/\omega^2}\right)$
17. The characteristic equation is $Lr^2 + br + g = 0$ so that that roots are $r_{1,2} = \frac{1}{2L}\left(-b \pm \sqrt{b^2 - 4Lg}\right)$. Thus, $b^2 - 4Lg > 0$, overdamped with solution

$$\theta = \frac{1}{2\sqrt{b^2 - 4Lg}}\left[(b\theta_0 + \sqrt{b^2 - 4Lg}\theta_0 + 2Lv_0)\exp\left(\frac{-b + \sqrt{b^2 - 4Lg}}{2L}t\right) + (-b\theta_0 + \sqrt{b^2 - 4Lg}\theta_0 - 2Lv_0)\exp\left(\frac{-b - \sqrt{b^2 - 4Lg}}{2L}t\right)\right];$$

If $b^2 - 4Lg = 0$, critically damped with solution $\theta = \theta_0 e^{-bt/(2L)} + \left(\frac{b\theta_0}{2L} + v_0\right)te^{-bt/(2L)}$; if $b^2 - 4Lg < 0$, underdamped with solution

$$\theta = Ce^{-bt/(2L)}\left[F\cos Wt + \frac{\sqrt{b^2 - 4Lg}}{4Lg - b^2}\sin Wt\right] + C\left[-F\cos\gamma t + \frac{b\gamma}{L\gamma^2 - g}\sin\gamma t\right],$$

where $C = \frac{L\gamma^2 - g}{b^2\gamma^2 + L^2\gamma^4 - 2L\gamma^2 g + g^2}$ and $S = \frac{1}{2L}\sqrt{b^2 - 4gL}$. The pendulum eventually oscillates.

19. $x(t) = A\cos\omega t \Rightarrow x'(t) = -A\omega\sin\omega t$. Substitution into the nonlinear term yields

$$\epsilon\left(x^2-1\right)\frac{dx}{dt} = -\epsilon A^3\omega\cos^2\omega t\,\sin\omega t + \epsilon A\omega\sin\omega t.$$

However, $\cos^2\omega t = 1-\sin^2\omega t$, so we have

$$\epsilon\left(x^2-1\right)\frac{dx}{dt} = -\epsilon A^3\omega\sin\omega t\,\sin\omega t + \epsilon A^3\omega\sin^3\omega t + \epsilon A\omega\sin\omega t.$$

With the identity $\sin^3\omega t = -\frac{1}{4}\sin 3\omega t + \frac{3}{4}\sin\omega t$, it follows that

$$\epsilon\left(x^2-1\right)\frac{dx}{dt} = \left(-\frac{1}{4}\epsilon A^3\omega + \epsilon A\omega\right)\sin\omega t - \frac{1}{4}\epsilon A^3\omega\sin 3\omega t$$

21. (a) $\theta(t) = 2\sin t$; (b) $\theta(t) = 2\cos t$; (c) $\theta(t) = -2\cos t$; (d) $\theta(t) = -\sin t$; (e) $\theta(t) = -2\sin t$; (f) $\theta(t) = \cos t - \sin t$; (g) $\theta(t) = -\cos t + \sin t$

21.

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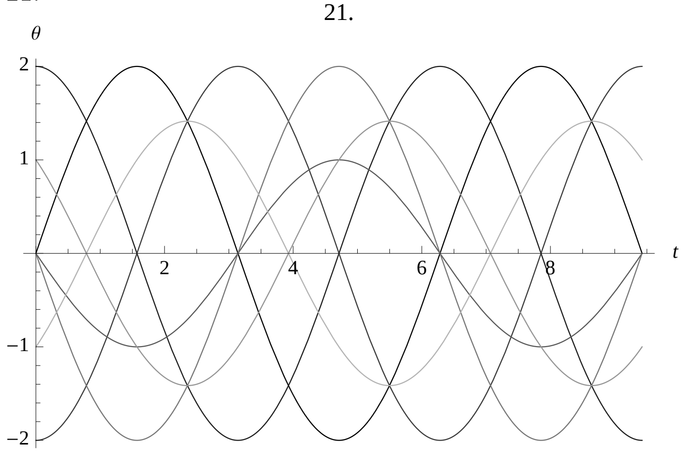


22 and 23. $\theta(t) = \frac{1}{15}e^{-t/4}\left(\sqrt{15}\sin\left(\frac{\sqrt{15}t}{4}\right) + 15\cos\left(\frac{\sqrt{15}t}{4}\right)\right), -\frac{1}{15}e^{-t/4}\left(\sqrt{15}\sin\left(\frac{\sqrt{15}}{4}\right.\right.$

$$\frac{4e^{-t/4}\sin\left(\frac{\sqrt{15}t}{4}\right)}{\sqrt{15}}, -\frac{4e^{-t/4}\sin\left(\frac{\sqrt{15}t}{4}\right)}{\sqrt{15}}, \frac{1}{3}e^{-t/4}\left(\sqrt{15}\sin\left(\frac{\sqrt{15}t}{4}\right) + 3\cos\left(\frac{\sqrt{15}t}{4}\right)\right),$$

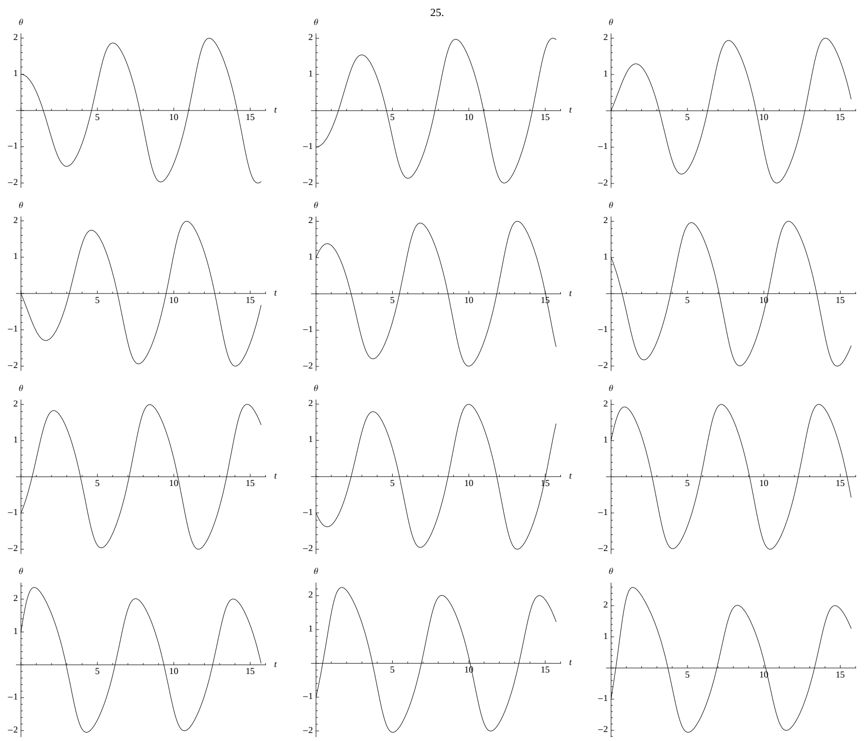
$$-\frac{1}{5}e^{-t/4}\left(\sqrt{15}\sin\left(\frac{\sqrt{15}t}{4}\right) - 5\cos\left(\frac{\sqrt{15}t}{4}\right)\right), \frac{1}{5}e^{-t/4}\left(\sqrt{15}\sin\left(\frac{\sqrt{15}t}{4}\right) - 5\cos\left(\frac{\sqrt{15}t}{4}\right)\right)$$

$$-\frac{1}{3}e^{-t/4}\left(\sqrt{15}\sin\left(\frac{\sqrt{15}t}{4}\right) + 3\cos\left(\frac{\sqrt{15}t}{4}\right)\right), \frac{1}{5}e^{-t/4}\left(3\sqrt{15}\sin\left(\frac{\sqrt{15}t}{4}\right) + 5\cos\left(\frac{\sqrt{15}t}{4}\right)\right.$$

$$\frac{1}{15}e^{-t/4}\left(13\sqrt{15}\sin\left(\frac{\sqrt{15}t}{4}\right) + 15\cos\left(\frac{\sqrt{15}t}{4}\right)\right), \frac{1}{15}e^{-t/4}\left(7\sqrt{15}\sin\left(\frac{\sqrt{15}t}{4}\right) - 15\cos\left(\sqrt{}\right.\right.$$

$$\frac{1}{15}e^{-t/4}\left(11\sqrt{15}\sin\left(\frac{\sqrt{15}t}{4}\right) - 15\cos\left(\frac{\sqrt{15}t}{4}\right)\right)$$

25.



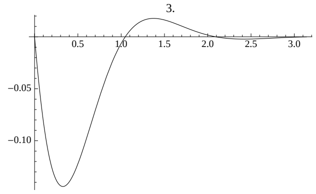
Chapter 5 Review Exercises

1. First, we find $m = W/g = 32/32 = 1$. Next, we find the spring constant with $F = k \cdot s \Rightarrow 32 = 1/2 \cdot k \Rightarrow k = 64$. Now, we solve the initial-value problem $x'' + 64x = 0$, $x(0) = 1/3$, $x'(0) = 0$. A general solution is $x = c_1 \cos 8t + c_2 \sin 8t$ and a solution of the initial-value problem is $x(t) = \frac{1}{3}\cos 8t$; maximum displacement $= 1/3$; $t = \pi/16, \pi/8$

3. Solve $5x'' + 20x' + 65x = 0$, $x(0) = 0$, $x'(0) = -1$. A general solution is $x = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t)$. Application of the initial conditions yields $x(t) = -\frac{1}{3}e^{-2t}\sin 3t$, $\lim_{t\to\infty} x(t) = 0$; quasiperiod $= 2\pi/3$; maximum displacement $= \left|x(\frac{1}{3}\tan^{-1}(3/2))\right| \approx 0.144$; $t = \pi/3$

5. Solve $4x'' + 16x = 4$ or $x'' + 4x = 1$, $x(0) = x'(0) = 0$. The differential equation has general solution $x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4}$ and application of the initial conditions results in $x(t) = \frac{1}{4}(1 - \cos 2t)$; maximum displacement is $1/2$ and occurs when $t = \pi/2$

3.



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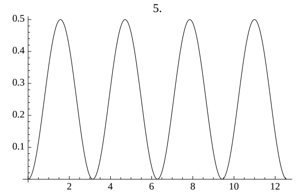

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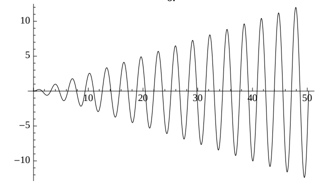
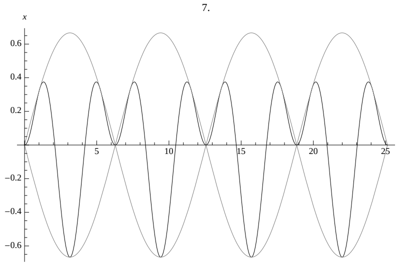
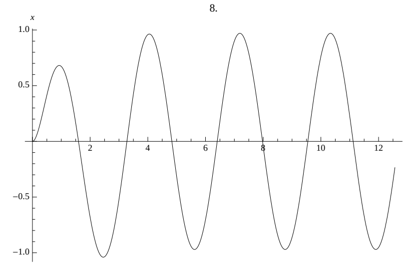


7. Solve $4x'' + 16x = 4\cos t$ or $x'' + 4x = \cos t$, $x(0) = x'(0) = 0$. A general solution is $x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3}\cos 2t$ and application of the initial conditions yields $x(t) = \frac{1}{3}\cos t - \frac{1}{3}\cos 2t$; beats; The envelope functions are $y = \pm\frac{2}{3}\sin(t/2)$.

7.



8.



9. Solve $4Q'' + 80Q' + 436Q = 100$, $Q(0) = Q'(0) = 0$. A general solution is $Q(t) = e^{-10t}(c_1 \cos 3t + c_2 \sin 3t + 25/109$ an application of the initial conditions yields $Q(t) = \frac{25}{327}[3 - 3e^{-10t}(\cos 3t - 10\sin 3t)]$; $I(t) = \frac{25}{3}e^{-10t}\sin 3t$; $\lim_{t\to\infty} Q(t) = 25/109$; $\lim_{t\to\infty} I(t) = 0$
11. Solve $Q'' + 10^4 Q = 220$, $Q(0) = Q'(0) = 0$. A general solution is $Q = c_1 \cos 100t + c_2 \sin 100t + 11/500$ and application of the initial conditions yields $Q(t) = \frac{11}{500}(1 - \cos 100t)$; $I(t) = \frac{11}{5}\sin 100t$; $\lim_{t\to\infty} Q(t)$ and $\lim_{t\to\infty} I(t)$ do not exist.
12. Solve $Q'' + 10^4 Q = 100\sin 10t$, $Q(0) = Q'(0) = 0$. A general solution of the equation is $Q = c_1 \cos 100t + c_2 \sin 100t + \frac{24}{2499}\sin 2t$ and application of the initial conditions results in $Q = \frac{1}{2499}\left(-\frac{1}{2}\sin 100t + 25\sin 2t\right)$; $I = \frac{50}{2499}(-\cos 100t + \cos 2t)$; the limits do not exist.
13. $s^{(4)} = 0$, $s(0) = s'(0) = s(10) = s'(10) = 0 \Rightarrow s(x) = \frac{250}{3}x^2 - \frac{125}{9}x^3 + \frac{1}{12}x^5 - \frac{1}{360}x^6$
15. We solve $s^{(4)} = x(10 - x)$, $s(0) = s'(0) = s''(10) = s'''(10) = 0$ with solution $s(x) = \frac{1250}{3}x^2 - \frac{250}{9}x^3 + \frac{1}{12}x^5 - \frac{1}{360}x^6$
17. Solve $\frac{1}{2}\theta'' + 32\theta = 0$, $\theta(0) = 1$, $\theta'(0) = 0 \Rightarrow \theta(t) = \cos 8t$; maximum displacement $= 1$; $t = \pi/16$
19. Solve $\frac{1}{2}\theta'' + 8\theta' + 32\theta = 0$, $\theta(0) = 1$, $\theta'(0) = 0 \Rightarrow \theta(t) = c_1 e^{-8t} + c_2 t e^{-8t}$. Application of the initial conditions yields $\theta(t) = e^{-8t} + 8te^{-8t}$; motion is not periodic
21. $\theta(t) = 0.2168\,\cos 3.7t + 0.0471622\sin 3.7t$
23. The solution to $y'' + y = 1 + t^2$, $y(0) = y'(0) = 0$ is $y = 3 - t^2 - 3\cos t$. Then, $y(1) = 2 - 3\cos 1$ and $y'(1) = -2 + 3\sin 1$ and the solution to $y'' + y = 0$, $y(1) = 2 - 3\cos 1$, $y'(0) = -2 + 3\sin 1$ is $y = (-3 + 2\cos 1 + 2\sin 1)\cos t - 2(\cos 1 -$

$\sin 1)\sin t$. Thus, the solution to $y'' + y = \begin{cases} 1 - t^2, \ 0 \le t < 1 \\ 0, \ t \ge 1 \end{cases}$, $y(0) = y'(0) = 0$

is $y(t) = \begin{cases} 3 - t^2 - 3\cos t, \ 0 \le t \le 1 \\ (-3 + 2\cos 1 + 2\sin 1)\cos t - (2\cos 1 - 2\sin 1 + \sin 2)\sin t, \ t > 1 \end{cases}$

25. $x(t) = 25\cos\left(\frac{7\sqrt{5}}{25}t\right)$

27. $x(t) = A\sin\omega_n t + B\cos\omega_n t + F/k$, $x(t) = (x_0 - F/k)\cos\omega_n t + F/k$

Differential Equations at Work

A. Rack-and-Gear Systems

1. Natural frequency: ${\omega_n}^2 = \dfrac{kr^2}{2\bar{I} + (W/g)r^2 + m_3 r^2/3}$ so $\omega_n = \sqrt{\dfrac{kr^2}{2\bar{I} + (W/g)r^2 + m_3 r^2/3}}$

2. $\theta(t) = \theta_0 \sin\omega_n t$

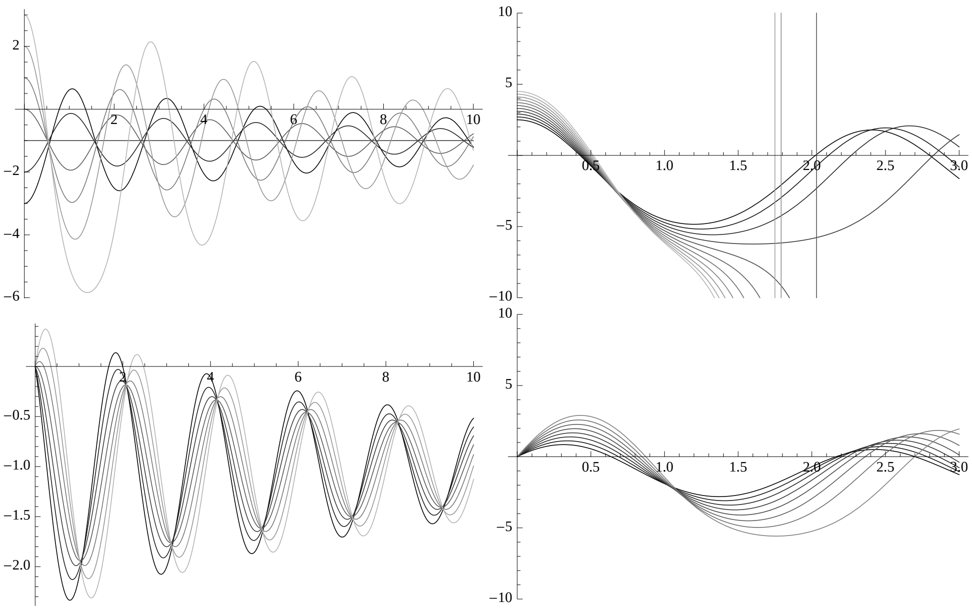
3. $\theta(t) = \theta_0 \sin\omega_n t$ so $\dot{\theta}(t) = \theta_0\omega_n \cos\omega_n t$. The maximum value of $\dot{\theta}$ is $\theta_0\omega_n$, so the maximum value of T is $T_{max} = \left(\bar{I} + \dfrac{1}{2}\dfrac{W}{g}r^2 + \dfrac{m_3 r^2}{2\cdot 3}\right)(\theta_0\omega_n)^2$; the maximum value of $\theta(t)$ is θ_0, so the maximum value of U is $U_{max} = \frac{1}{2}kr^2\theta_0^2$. The value for ω_n that was found in (1) is found with $T_{max} = U_{max}$ also.

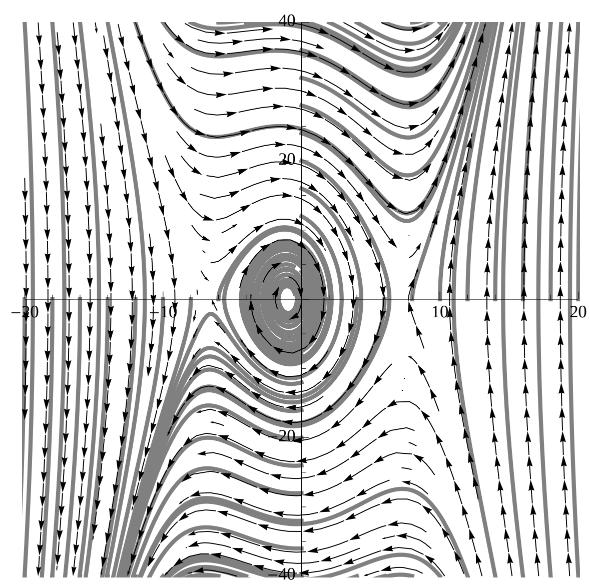
4. The natural frequency approaches a finite value found with L'Hopital's Rule to be

$$\lim_{r\to\infty}\sqrt{\frac{kr^2}{2\bar{I} + (W/g)r^2 + m_3 r^2/3}} = \sqrt{\frac{k}{W/g + m_3/3}}.$$

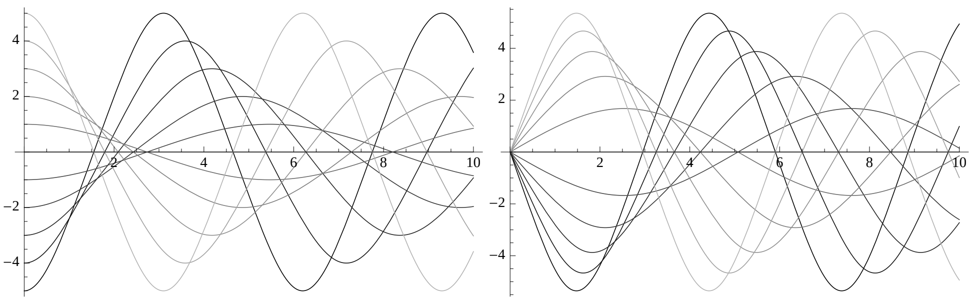
The method for finding the natural frequency of a mechanical system by solving $T_{max} = U_{max}$ for ω_n is called **Rayleigh's Energy Method**. Rayleigh discovered he did not need the exact shape of the displacement function to obtain a good approximation of the natural frequency. In fact, we can test this with the spring-mass system without damping modeled by $mx'' + kx = 0$. Assume that the displacement is $x(t) = x_0 \sin\omega_n t$ and use Rayleigh's Energy Method to show that the natural frequency is $\omega_n = \sqrt{k/m}$ as expected. Notice that the term x_0^2 appears in both T_{max} and U_{max} so they cancel in the calculation of ω_n. Therefore, $omega_n$ does not depend on the amplitude of $x(t)$ as is the case with any linear ordinary differential equation. This means that Rayleigh's Energy Method can be used to approximate the natural frequency of the system without actually solving a differential equation.

B. Soft Springs

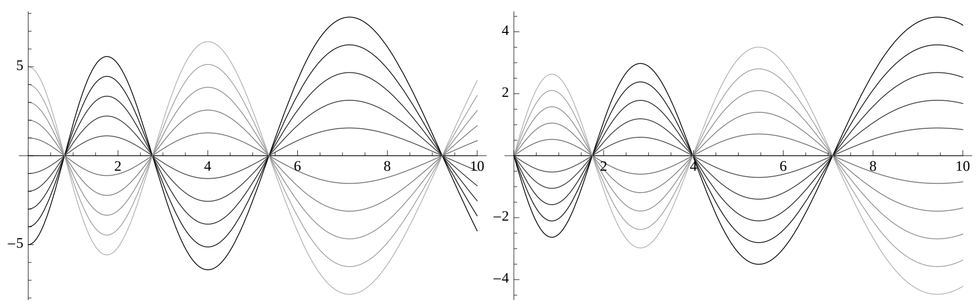




C. Hard Springs

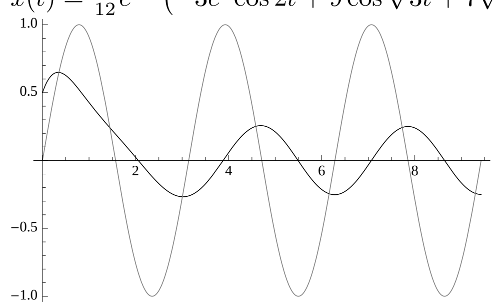


D. Aging Springs

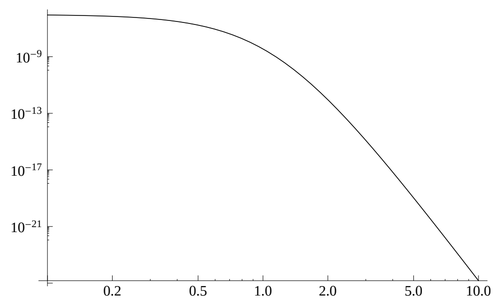


E. Bodé Plots

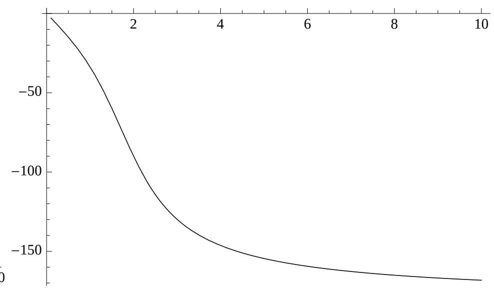
1. $x(t) = \frac{1}{12} e^{-t} \left(-3e^{t} \cos 2t + 9 \cos \sqrt{3}t + 7\sqrt{3} \sin \sqrt{3}t \right)$



2.
3. $M(2) \approx .25$; $\phi(2) \approx 90$
4.



10^−9
10^−13
10^−17
10^−21
0.2
0.5
1.0
2.0
5.0
10.0



2
4
6
8
10
−50
−100
−150