Exercises 6.1

1. Remember that x and y are functions of t: $x = x(t)$ and $y = y(t)$. The solution of $x' = 6$ or $dx/dt = 6$ is $x = x(t) = 6t + c_1$ because $\int 6\,dt = 6t + c_1$. For the second equation, $y' = \cos t$ or $dy/dt = \cos t$, integrating gives us $\int \cos t\,dt = \sin t + c_2$. Thus, $y = \sin t + c_2$.

3. $x' = 0 \Rightarrow x = c_1$. $y' = -2y \Rightarrow \dfrac{1}{y}dy = -2\,dt \Rightarrow y = c_2 e^{-2t}$. If you had solved the problem in a different order than is shown here, you might have arrived at the solution $\left\{x(t) = c_2,\, y(t) = c_1\, e^{-2\,t}\right\}$.

5. Refer back to Chapter 2 for help with solving first-order equations. For the first equation, $x_1' = -3x_1$, $x_1(0) = -1$ has solution $x_1 = -e^{-3t}$. Next, $x_2' = 1$, $x_2(0) = 1$ has solution $x_2 = t + 1$.

7. Differentiating the x equation gives us $x'' = -3x' + 6y'$ while solving the x' equation for y gives us $y = \frac{1}{6}(x' + 3x)$. Now, substitute the y and y' equations into the x'' equation and set the equation equal to 0:

$$\begin{aligned} x'' &= -3x' + 6y' \\ x'' &= -3x' + 6\,(4x - y) \\ x'' &= -4x' + 21x \\ x'' + 4x' - 21x &= 0. \end{aligned}$$

This second order equation with constant coefficients has characteristic equation $r^2 + 4r - 21 = 0 \Rightarrow r_1 = -7, r_2 = 3$ so $\left\{x(t) = -\frac{3}{2}c_1\, e^{-7\,t} + c_2\, e^{3\,t},\, y(t) = c_1\, e^{-7\,t} + c_2\, e^{3\,t}\right\}$.

9. $\{x(t) = c_1 \sin(t) + c_2 \cos(t)\,,\, y(t) = -1/2\, c_1 \cos(t) + 1/2\, c_2 \sin(t) - 1/2\, c_1 \sin(t) - 1/2\, c_2 \cos(t)\}$

11. Differentiating the x' equation and substituting the y' equation into the result gives us

$$\begin{aligned} x'' &= y' \\ x'' &= -x + 1 \\ x'' + x &= 1. \end{aligned}$$

For $x'' + x = 1$, the corresponding homogeneous equation is $x'' + x = 0$, which has general solution $x_h = c_1 \cos t + c_2 \sin t$. Use undetermined coefficients to find a particular solution of the nonhomogeneous equation. Practice a lot and a thoughtful guess gives us $x_p = 1$. Therefore, $x = c_1 \cos t + c_2 \sin t + 1$. Choosing a different linear combination gives us $\{x(t) = -c_1 \cos(t) + c_2 \sin(t) + 1,\, y(t) = c_1 \sin(t) + c_2 \cos(t)\}$.

13. $\{x' = y,\, y' = -4x + 3y\}$

15. $\{x' = y,\, y' = -16x + t \sin t$

17. $\{y' = x,\, x' = z,\, z' = -6x - 3y - 3z + t\}$

21. Using $D = d/dt$, in operator notation the system becomes $Dx = -2x - 2y + 4$, $Dy = -5x + y$ or $\begin{cases} (D+2)x + 2y = 4 \\ 5x + (D-1)y = 0 \end{cases}$. Applying $D-1$ to the first equation and -2 to the second equation gives us $\begin{cases} (D-1)(D+2)x + 2(D-1)y = (D-1)(4) \\ -10x + -2(D-1)y = 0 \end{cases}$.

Adding these equations gives us $[(D-1)(D+2)-10](x)=-4$ or $x''+x'-12x=-4$, which has solution $x=c_1e^{4t}+c_2e^{3t}+\frac{1}{3}$. From the x' equation, $x'=-2x-2y+4$, $y=\frac{1}{2}(4-2x-x')=c_1e^{4t}-\frac{5}{2}c_2e^{3t}+\frac{5}{3}$.

23. $x(t)=e^{-3t}\left(2c_2t+c_1\right)-\dfrac{e^t}{4}+\frac{2}{9}$, $y(t)=c_2e^{-3t}+\frac{1}{3}$

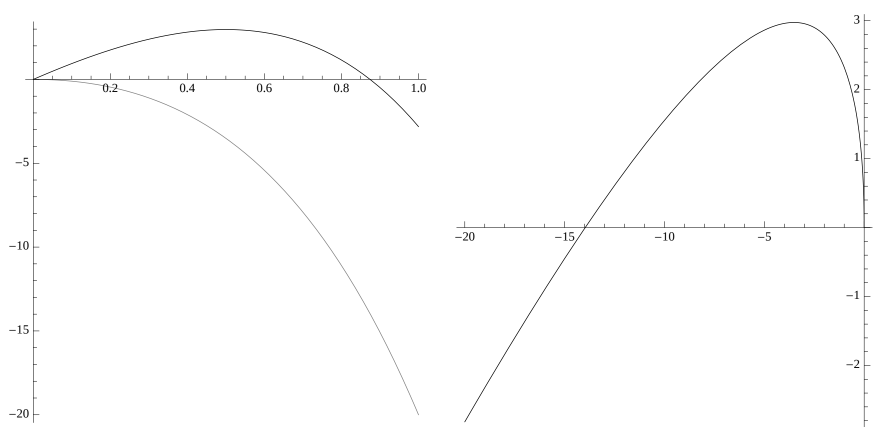
25. $x(t)=e^t\left(c_1(1-2t)-\left(c_3+c_4\right)t+c_2+c_4\right)+\left(c_1+c_3\right)\sin(t)-\left(c_2+c_4\right)\cos(t)$, $y(t)=\frac{1}{2}e^{-it}\left(2e^{(1+i)t}\left(c_1(2t-1)+c_3t+c_4(t-1)-c_2\right)+(1+i)\left(\left(c_1-i\left(c_2+ic_3+c_4\right)\right)e^{2it}\right.\right.$ or $x=(c_2+c_1)\sin t+c_1\cos t-c_3e^t-c_4te^t$, $y=c_2\cos t+(-c_2-2c_1)\sin t+c_3e^t+c_4te^t$

27. $x=-\frac{3}{10}\cos t-\frac{1}{10}\sin t-2c_1e^{2t}+(3c_3-2c_2)e^{-t}-2c_3te^{-t}$, $y=\frac{1}{2}\sin t+(3c_2-4c_3)e^{-t}+3c_3te^{-t}$, $z=\frac{2}{5}\cos t+\frac{3}{10}\sin t+c_1e^{2t}+c_2e^{-t}+c_3te^{-t}$

29. In operator notation, the system is

$$\begin{cases}(D-3)x+2y=0\\ -2x+(D+1)y=10\end{cases}\quad\Rightarrow\begin{cases}(D-3)(D+1)x+2(D+1)y=0\\ 4x-2(D+1)y=-20\end{cases}$$

Adding, we have $(D^2-2D+1(x=-20$ with general solution $x=c_1e^t+c_2e^t-20$. Solving the first differential equation for y, we have $y=-\frac{1}{2}(x'-3x)=c_1e^t-\frac{1}{2}c_2te^t-30$. Application of the initial conditions yields $x(0)=c_1-20=0$, $y(0)=c_1-\frac{1}{2}c_3-30=0$ with solution $c_1=20$, $c_2=-20$. Therefore $x=20e^t-20te^t-20$, $y=30e^t-20te^t-30$.



31. $\begin{vmatrix}D+1 & e^t\\ D+1 & 2e^t\end{vmatrix}=(D+1)(2e^t)-(D+1)(e^t)=e^t\neq 0$

33. $\begin{cases}D^2x+D^2y=t^2\\ 4D^2x+4D^2y=4t^2\end{cases}$, $\begin{vmatrix}D^2 & D^2\\ 4D^2 & 4D^2\end{vmatrix}=0$, $\begin{vmatrix}t^2 & D^2\\ 4t^2 & 4D^2\end{vmatrix}=0$, $\begin{vmatrix}D^2 & t^2\\ 4D^2 & 4t^2\end{vmatrix}=0$

35. $k=-4$; $\begin{vmatrix}D+1 & -D\\ -4(D+1) & 4D\end{vmatrix}=0$; $\begin{vmatrix}t & -D\\ kt & 4D\end{vmatrix}=0\Leftrightarrow k=-4$; $\begin{vmatrix}D+1 & t\\ -4(D+1) & kt\end{vmatrix}=0\Leftrightarrow k=-4$

37. $c\neq 4$: $\begin{vmatrix}D+1 & -c\\ 2(D+1) & -8\end{vmatrix}=-8+2c=0\Rightarrow c=4$. If $c\neq 4$, $\begin{vmatrix}D+1 & -c\\ 2(D+1) & -8\end{vmatrix}\neq 0$.

However, $\begin{vmatrix} D+1 & e^{-t} \\ 2(D+1) & 2e^{-t} \end{vmatrix} = 0$.

39. (a) $x_1' = x_2$, $x_2' = -3x + y = -3x_1 + y_1$, $y_1' = y_2$, $y_2' = -x_1 - y_1$; (b) $x_1' = x_2$, $x_2' = -x + 2y + \cos 2t = -x_1 + 2y_1 + \cos 2t$, $y_1' = y_2$, $y_2' = -2x_1 + y_1$

Exercises 6.2

1. $\begin{pmatrix} -3 & 5 \\ 3 & 2 \end{pmatrix}$

3. $\begin{pmatrix} -5 & 21 \\ 1 & -3 \end{pmatrix}$

5. $\mathbf{AB} = \begin{pmatrix} -1 & -11 \\ -2 & -28 \end{pmatrix}$; $\mathbf{BA} = \begin{pmatrix} -2 & -3 \\ -16 & -27 \end{pmatrix}$

7. $\mathbf{AB} = \begin{pmatrix} 22 & -5 & -13 \\ 22 & -24 & -17 \\ -31 & -4 & 37 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} 1 & 9 & 16 & -35 \\ -25 & 20 & 20 & 5 \\ 0 & -9 & -8 & -28 \\ -27 & -11 & -22 & 22 \end{pmatrix}$

9. 17

11. -3

13. 1

15. $\begin{pmatrix} 1 & \frac{1}{2} \\ -1 & 0 \end{pmatrix}$

17. $\begin{pmatrix} -1 & -3 & 5 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 1 & 2 & -4 & 2 \end{pmatrix}$

19. $\lambda_1 = -5$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -4$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

21. $\lambda_{1,2} = -2 \pm 2i$, $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \pm 2i \\ 5 \end{pmatrix}$

23. The eigenvalues are found by solving

$$\begin{vmatrix} -1-\lambda & -1 & -2 \\ 2 & 2-\lambda & 2 \\ -2 & -1 & -1-\lambda \end{vmatrix} = -2 + 3\lambda - \lambda^3 = -(\lambda+2)(\lambda-1)^2 = 0,$$

which shows us that $\lambda_1 = -2$ has multiplicity one and $\lambda_{2,3} = 1$ has multiplicity two. Let $\mathbf{v}_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$ denote an eigenvector corresponding to λ_i. For $\lambda_1 = -2$, $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_1 = \mathbf{0}$ has augmented matrix $\begin{pmatrix} 1 & -1 & -2 \\ 2 & 4 & 2 \\ -2 & -1 & 1 \end{pmatrix}$, which reduces to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, $x_1 = z_1$ and $y_1 = -z_1$. Choosing $z_1 = 1$ gives us $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. For $\lambda_{2,3} = 1$, $(\mathbf{A} - \lambda_{2,3}\mathbf{I})\mathbf{v}_{2,3} = \mathbf{0}$ has augmented matrix $\begin{pmatrix} -2 & -1 & -2 \\ 2 & 1 & 2 \\ -2 & -1 & -2 \end{pmatrix}$, which reduces to $\begin{pmatrix} 1 & 1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, $y_{2,3}$ and $z_{2,3}$ are free and $x_{2,3} = -\frac{1}{2}y_{2,3} - z_{2,3}$. More specifically, for s, $t \in R$, $\mathbf{v}_{2,3} = \begin{pmatrix} -1/2s - t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t$ are eigenvectors corresponding to $\mathbf{v}_{2,3}$. In this case, we see that choosing $s = 2$ and $t = 0$ and then $s = 0$ and $t = 1$ gives us two linearly independent eigenvectors corresponding to $\lambda_{2,3}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

25. The eigenvalues are found by solving

$$\begin{vmatrix} -3-\lambda & 0 & -1 \\ -1 & -1-\lambda & -3 \\ 1 & 0 & -3-\lambda \end{vmatrix} = -10 - 16\lambda - 7\lambda^2 - \lambda^3 = -(\lambda+1)(\lambda^2+6\lambda+10) = 0$$

resulting in $\lambda_1 = -1$ and $\lambda_{2,3} = -3 \pm i$.

Let $\mathbf{v}_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$ denote an eigenvector corresponding to λ_i. For $\lambda_1 = -1$, $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}$ has augmented matrix $\begin{pmatrix} -2 & 0 & -1 \\ -1 & 0 & -3 \\ 1 & 0 & -2 \end{pmatrix}$, which reduces to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ so $x_1 = y_1 = 0$ and z_1 is free. Choosing $z_1 = 1$ gives us $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $\lambda_2 = -3 + i$, $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v}_2 = \mathbf{0}$ has augmented matrix $\begin{pmatrix} -2-3i & 0 & -1 \\ -1 & -3i & -3 \\ 1 & 0 & -2-3i \end{pmatrix}$, which reduces to $\begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -1-i \\ 0 & 0 & 0 \end{pmatrix}$ so $x_2 = iz_2$ and $y_2 = (1+i)z_2$. Choosing $z_2 = 1$ gives us $\mathbf{v}_2 = \begin{pmatrix} -i \\ 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} i$. Since eigenvectors of complex conjugate eigenvalues are also complex conjugates, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} i$.

27. $\lambda_1 = -2$ and $\lambda_{2,3,4} = -1$; $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{v}_{3,4} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

29. $\begin{pmatrix} -2e^{-2t} \\ -5e^{-5t} \end{pmatrix}$; $\begin{pmatrix} -\frac{1}{2}e^{-2t} \\ -\frac{1}{5}e^{-5t} \end{pmatrix}$

31. $\begin{pmatrix} -\sin t & \cos t - t\sin t \\ \cos t & t\cos t + \sin t \end{pmatrix}$; $\begin{pmatrix} \sin t & \cos t + t\sin t \\ -\cos t & \sin t - t\cos t \end{pmatrix}$

33. $\begin{pmatrix} 4e^{4t} \\ -3\sin 3t \\ 3\cos 3t \end{pmatrix}$; $\begin{pmatrix} \frac{1}{4}e^{4t} \\ \frac{1}{3}\sin 3t \\ -\frac{1}{3}\cos 3t \end{pmatrix}$

37. For $\mathbf{A}$, the eigenvalues are $\lambda_{1,2} = -1$ and $\lambda_3 = 0$ with corresponding eigenvectors $\mathbf{v}_{1,2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix}$. For $\mathbf{B}$, The eigenvalues are $\lambda_{1,2} = 3 \pm 2\sqrt{11}$ and $\lambda_3 = 3$ with corresponding eigenvectors $\mathbf{v}_{1,2} = \begin{pmatrix} 1 - \frac{1}{5}(3 \pm 2\sqrt{11}) \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} -10 \\ 25 \\ 19 \end{pmatrix}$.

39. $\lambda_{1,2} = 1 \pm ik$; $\lambda_{1,2} = k \pm i$

Exercises 6.3

1. $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

3. $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}$

5. $\begin{pmatrix} t(t^3-1)x' \\ t(t^3-1)y' \end{pmatrix} = \begin{pmatrix} 1 & -t \\ -2t^2 & t^3+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

7. $\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix}$

9. -5, linearly independent

11. $2 - 4\cos^2 2t$, linearly independent

13. $2e^{2t}$, linearly independent

15-22. $\mathbf{\Phi}(t)\mathbf{C}$ where $\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ for 2×2 or $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ for 3×3

23. $\left\{x(t) = e^{-t} + 2e^{4t},\ y(t) = -e^{-t} + 3e^{4t}\right\}$

25. $\{x(t) = \sin(t), y(t) = -\sin(t) + \cos(t)\}$

27. $x = 4t^{-1} - 1$, $y = 4 - t^2$

29. $\{x(t) = t,\ y(t) = \sin(3t) + \cos(3t),\ z(t) = -\cos(3t) + \sin(3t)\}$

33. $\mathbf{\Phi}(t) = \begin{pmatrix} e^t & e^{-t} \\ 2\,e^t & e^{-t} \end{pmatrix}$

Exercises 6.4

Remember that $\mathbf{C}$ denotes a *constant vector*. For $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{C}$ has the form $\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ while for $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{C}$ has the form $\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

1. $\mathbf{X} = \begin{pmatrix} e^{4\,t} & e^{2\,t} \\ e^{4\,t} & 3\,e^{2\,t} \end{pmatrix} \mathbf{C}$

3. $\mathbf{X} = \begin{pmatrix} 6\,e^{-4\,t} & 0 \\ e^{-4\,t} & e^{2\,t} \end{pmatrix} \mathbf{C}$

5. $\mathbf{X} = \begin{pmatrix} e^{2\,t} & e^{2\,t}t \\ e^{2\,t} & e^{2\,t}t + e^{2\,t} \end{pmatrix} \mathbf{C}$

7. $\mathbf{X} = \begin{pmatrix} e^t \sin(2\,t) & e^t \cos(2\,t) \\ -1/2\,e^t \sin(2\,t) + 1/2\,e^t \cos(2\,t) & -1/2\,e^t \cos(2\,t) - 1/2\,e^t \sin(2\,t) \end{pmatrix} \mathbf{C}$

9. $\mathbf{X} = \begin{pmatrix} e^{3\,t} & 0 & 0 \\ 0 & e^{2\,t} & e^{2\,t}t \\ 0 & 0 & e^{2\,t} \end{pmatrix} \mathbf{C}$

11. $\mathbf{X} = \begin{pmatrix} \sin(t) & \cos(t) & 0 \\ -\cos(t) & \sin(t) & 0 \\ 0 & 0 & e^{2\,t} \end{pmatrix} \mathbf{C}$

13. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 15$ and $\lambda_2 = -4$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ so $\mathbf{X} = \begin{pmatrix} e^{-4\,t} & e^{15\,t} \\ 1/2\,e^{-4\,t} & -7/5\,e^{15\,t} \end{pmatrix} \mathbf{C}$

15. The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 6 & -1 \\ 5 & 0 \end{pmatrix}$ are $\lambda_1 = 5$ and $\lambda_2 = 1$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. Thus, a general solution is $\mathbf{X} = \begin{pmatrix} e^t & e^{5\,t} \\ 5\,e^t & e^{5\,t} \end{pmatrix} \mathbf{C}$.

17. The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} 7 & 0 \\ 5 & -8 \end{pmatrix}$ are $\lambda_1 = -8$ and $\lambda_2 = 7$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Thus, a general solution is $\mathbf{X} = \begin{pmatrix} 0 & 3\,e^{7\,t} \\ e^{-8\,t} & e^{7\,t} \end{pmatrix} \mathbf{C}$

19. The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix}$ are $\lambda_1 = -11$ and $\lambda_2 = -4$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Thus, a general solution is $\mathbf{X} = \begin{pmatrix} e^{-4\,t} & e^{-11\,t} \\ 1/3\,e^{-4\,t} & -2\,e^{-11\,t} \end{pmatrix} \mathbf{C}$.

21. The eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} -6 & -4 \\ -3 & -10 \end{pmatrix}$ are $\lambda_1 = -12$ and $\lambda_2 = -4$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Thus, a general solution is $\mathbf{X} = \begin{pmatrix} e^{-4\,t} & e^{-12\,t} \\ -1/2\,e^{-4\,t} & 3/2\,e^{-12\,t} \end{pmatrix} \mathbf{C}$.

23. $\lambda_{1,2} = -8$ and there is only one eigenvector $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. A general solution is then $\mathbf{X} = \begin{pmatrix} e^{-8\,t} & e^{-8\,t}t \\ -e^{-8\,t} & -e^{-8\,t}t + 1/2\,e^{-8\,t} \end{pmatrix} \mathbf{C}$

25. The eigenvalues of the coefficient matrix are $\lambda_{1,2} = \pm 4i$ and the corresponding eigenvectors are $\mathbf{v}_{1,2} = \begin{pmatrix} \pm 2i \\ -1 \end{pmatrix}$. Thus, $\mathbf{X} = \begin{pmatrix} \sin(4\,t) & \cos(4\,t) \\ 1/2\,\cos(4\,t) & -1/2\,\sin(4\,t) \end{pmatrix} \mathbf{C}$.

27. The eigenvalues are 4, -2, and -1 with corresponding eigenvalues of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ so a general solution is $\mathbf{X} = \begin{pmatrix} e^{4\,t} & e^{-t} & 0 \\ 0 & 0 & e^{-2\,t} \\ 0 & -5\,e^{-t} & 0 \end{pmatrix} \mathbf{C}$.

29. The eigenvalues are $\lambda_1 = 3$ and $\lambda_{2,3} = -1$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -7 \\ 4 \\ 6 \end{pmatrix}$ and $\mathbf{v}_{1,2} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$. A general solution is $\mathbf{X} = \begin{pmatrix} e^{-t} & e^{3\,t} & e^{-t}t \\ 0 & -4/7\,e^{3\,t} & -e^{-t} \\ -2\,e^{-t} & -6/7\,e^{3\,t} & -2\,e^{-t}t \end{pmatrix} \mathbf{C}$

31. The eigenvalues are $\lambda_1 = 1$ and $\lambda_{2,3} = 1 \pm 2i$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v}_{2,3} = \begin{pmatrix} -1 \mp \frac{3}{2}i \\ \mp i \\ 1 \end{pmatrix}$; a general solution is

$$\mathbf{X} = \begin{pmatrix} e^{t} & e^{t}\cos(2\,t) & e^{t}\sin(2\,t) \\ 0 & \frac{6}{13}\,e^{t}\cos(2\,t) - \frac{4}{13}\,e^{t}\sin(2\,t) & \frac{6}{13}\,e^{t}\sin(2\,t) + \frac{4}{13}\,e^{t}\cos(2\,t) \\ 0 & -\frac{4}{13}\,e^{t}\cos(2\,t) - \frac{6}{13}\,e^{t}\sin(2\,t) & \frac{6}{13}\,e^{t}\cos(2\,t) - \frac{4}{13}\,e^{t}\sin(2\,t) \end{pmatrix} \mathbf{C}.$$

33. The eigenvalues are $\lambda_{1,2} = -1$ and $\lambda_{3,4} = 0$ with corresponding eigenvectors

$\mathbf{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. A general solution is
$\mathbf{X} = \begin{pmatrix} -2e^t & -2e^t & 1 & 4 \\ 0 & -3e^t & 1 & 1 \\ 0 & 3e^t & 0 & 2 \\ 3e^t & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$

35. A general solution is $x(t) = c_1 e^t + c_2 e^t \left(e^t - 1\right)$, $y(t) = c_2 e^{2t}$ and the solution of the IVP is $x = 4e^t \left(-1 + e^t\right)$, $y = 4e^{2t}$

37. A general solution is $x(t) = c_1 e^{4t}$, $y(t) = 2c_1 e^{4t} t + c_2 e^{4t}$ and the solution of the IVP is $x = 8e^{4t}$, $y = 16e^{4t} t$

39. $x(t) = c_1 \cos(4t) - c_2 \sin(4t)$, $y(t) = c_1 \sin(4t) + c_2 \cos(4t)$; $x = -8 \sin 4t$, $y = 8 \cos 4t$

41. $x(t) = c_1 e^{2t} \left(2e^t - 1\right) + c_3 e^{2t} \left(e^t - 1\right)$, $y(t) = c_3 \left(-e^t\right) \left(e^t - 1\right)^2 - 2c_1 e^{2t} \left(e^t - 1\right) + c_2 e^t$, $z(t) = -2c_1 e^{2t} \left(e^t - 1\right) - c_3 e^{2t} \left(e^t - 2\right)$; $x = -e^{2t} \left(-1 + 2e^t\right)$, $y = 2e^t \left(1 - e^t + e^{2t}\right)$, $z = 2e^{2t} \left(-1 + e^t\right)$

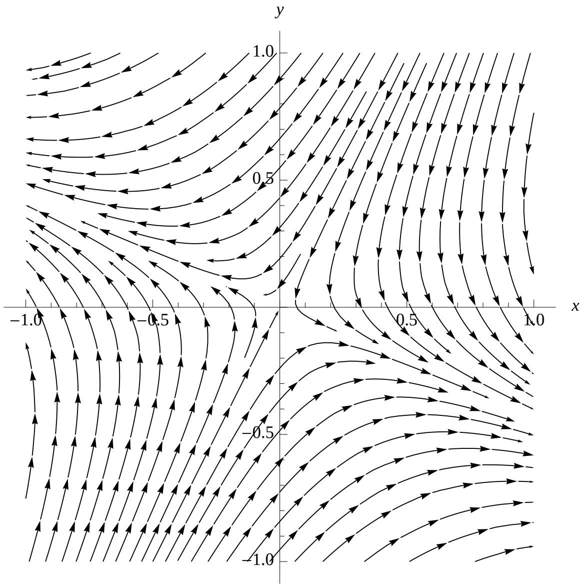
43. $x(t) = c_1 \left(-e^{-t}\right) \left(3e^t - 4\right) + c_2 e^{-t} \left(e^t - 1\right) + 8c_3 e^{-t} \left(e^t - 1\right) - 3c_4 e^{-t} \left(e^t - 1\right)$,
$y(t) = -3c_4 e^{-t} \left(e^t - 1\right)^2 - 2c_1 e^{-t} \left(-3e^t + e^{2t} + 2\right) + c_2 e^{-t} \left(-e^t + e^{2t} + 1\right) + 2c_3 e^{-t} \left(-7e^t + 3e^{\dots}\right.$
$z(t) = -\frac{1}{2} c_2 e^{-t} \left(e^t - 1\right)^2 + \frac{3}{2} c_4 e^{-t} \left(e^t - 1\right)^2 + c_1 e^{-t} \left(-3e^t + e^{2t} + 2\right) - c_3 e^{-t} \left(-8e^t + 3e^{2t} + 4\right)$,
$w(t) = 2c_1 \left(e^t - 1\right) + c_2 \left(1 - e^t\right) - 6c_3 \left(e^t - 1\right) + c_4 \left(3e^t - 2\right)$; $x = 9e^{-t} \left(-1 + e^t\right)$,
$y = e^{-t} \left(9 - 15e^t + 7e^{2t}\right)$, $z = -\frac{1}{2} e^{-t} \left(9 - 18e^t + 7e^{2t}\right)$, $w = -7 \left(-1 + e^t\right)$

48. (a) $x(t) = \frac{1}{10} e^{-\frac{1}{2}\left(\sqrt{5} - 5\right)t} \left(\left(5 - 3\sqrt{5}\right) e^{\sqrt{5}t} + 5 + 3\sqrt{5}\right)$,
$y(t) = \frac{1}{10} e^{-\frac{1}{2}\left(\sqrt{5} - 5\right)t} \left(-\left(\sqrt{5} - 5\right) e^{\sqrt{5}t} + 5 + \sqrt{5}\right)$; (b) $x(t) = e^{2t}$, $y(t) = e^{2t}(3t + 1)$; (c) $x(t) = \dfrac{5 \sin\left(\sqrt{7}t\right)}{\sqrt{7}} + \cos\left(\sqrt{7}t\right)$, $y(t) = \cos\left(\sqrt{7}t\right) - \dfrac{3 \sin\left(\sqrt{7}t\right)}{\sqrt{7}}$

49. $x(t) = \frac{1}{13} e^{-3t} \left(6 \left(c_1 - 3c_2 + 4c_3\right) e^{3t} \sin(2t) + 3 \left(3c_1 + 4c_2 - c_3\right) e^{3t} \cos(2t) + 4c_1 - 12c_2 + \dots\right.$
$y(t) = \frac{1}{26} e^{-3t} \left(\left(c_1 - 16c_2 + 17c_3\right) e^{3t} \sin(2t) + 2 \left(4c_1 + c_2 + 3c_3\right) e^{3t} \cos(2t) - 8c_1 + 24c_2 - 6c_{\dots}\right.$
$z(t) = \frac{1}{13} e^{-3t} \left(-2 \left(3c_1 + 4c_2 - c_3\right) e^{3t} \sin(2t) + 4 \left(c_1 - 3c_2 + 4c_3\right) e^{3t} \cos(2t) - 4c_1 + 12c_2 - 3\dots\right.$

51. $x(t) = \frac{1}{17} e^{-11t/3} \left(14e^{17t/3} x_0 - 7e^{17t/3} y_0 + 3x_0 + 7y_0\right)$, $y(t) = -\frac{1}{17} e^{-11t/3} \left(6e^{17t/3} x_0 - 3e^{\dots}\right.$

57.



Exercises 6.5

1. $\mathbf{X} = \begin{pmatrix} 2e^{-t} & 1 \\ e^{-t} & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -e^t \\ e^t \end{pmatrix}$

3. $x = -c_1 e^{-t} - 2c_2 e^t - t + 1$, $y = c_1 e^{-t} + 3c_2 e^t - 2t$

5. $\mathbf{X} = \begin{pmatrix} e^{-2t} & 3e^{-t} \\ e^{-2t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\cos t \\ \sin t - \cos t \end{pmatrix}$

7. $\mathbf{X} = \begin{pmatrix} -e^{-2t} & -2e^{-t} \\ 2e^{-2t} & 3e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -te^{-2t} \\ 2te^{-2t} - e^{-2t} \end{pmatrix}$

9. $\mathbf{X} = \begin{pmatrix} 3e^{-2t} & e^{-t} \\ 2e^{-2t} & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -2e^{-t} \end{pmatrix}$

11. $\mathbf{X} = \begin{pmatrix} 2te^{-t} + e^{-t} & -2te^{-t} \\ 2te^{-2t} & -2te^{-t} + e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$

13. In matrix form, the system is $\mathbf{X}' = \begin{pmatrix} 1 & -6 \\ 1 & 4 \end{pmatrix} \mathbf{X} + \mathbf{F}(t)$, where $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$.

For (a), $\begin{pmatrix} 1 & -6 \\ 1 & 4 \end{pmatrix}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -1$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Hence, a general solution is

$$\begin{aligned} \mathbf{X} &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} 2e^{-2t} & 3e^{-t} \\ e^{-2t} & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned}$$

This shows us that a fundamental matrix for the system is

$$\mathbf{\Phi} = \begin{pmatrix} 2e^{-2t} & 3e^{-t} \\ e^{-2t} & e^{-t} \end{pmatrix}.$$

Because we started with x, y notation we end with x, y notation:

$$\begin{aligned} x &= 2c_1 e^{-2t} + 3c_2 e^{-t} \\ y &= -c_1 e^{-2t} + c_2 e^{-t} \end{aligned}.$$

For (b), we know that from (a) a general solution of the corresponding homogeneous system is $\mathbf{X}_h = \begin{pmatrix} 2e^{-2t} & 3e^{-t} \\ e^{-2t} & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. Because $\mathbf{a}e^{2t}$ is not a solution for *any* vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, we assume that a particular solution has the form $\mathbf{X}_p = \mathbf{a}e^{2t} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{2t}$, where $\mathbf{A}$ is to be determined. Substituting $\mathbf{X}_p$ into the nonhomogeneous equation gives us

$$\begin{aligned} x' - (x - 6y) &= 6a_2 e^t = 0 \\ y' - (x - 4y) &= -a_1 e^t + 5a_2 e^t = -2e^{-t}. \end{aligned}$$

From the first equation, we see that $a_2 = 0$ and from the second it follows that $a_1 = 2$ so $\mathbf{X}_p == \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{2t}$. As in (a), because we started with x, y notation we end with x, y notation:

$$\begin{aligned} x &= 2c_1 e^{-2t} + 3c_2 e^{-t} + 2e^t \\ y &= -c_1 e^{-2t} + c_2 e^{-t} \end{aligned}.$$

For (c), we know that from (a) a general solution of the corresponding homogeneous system is $\mathbf{X}_h = \begin{pmatrix} 2e^{-2t} & 3e^{-t} \\ e^{-2t} & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. Because $\mathbf{a}e^{-t}$ is a solution for *some* vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, we assume that a particular solution has the form $\mathbf{X}_p = \mathbf{a}e^{-t} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-t}$, where $\mathbf{a}$ is linearly independent of of $\mathbf{v}_2$. Substituting $\mathbf{X}_p$ into the nonhomogeneous equation gives us

$$\begin{aligned} x' - (x - 6y) &= -2(a_1 - 3a_2)e^{-t} = -2e^{-t} \\ y' - (x - 4y) &= -(a_1 - 3a_2)e^{-t} = -e^{-t} \end{aligned}$$

so $a_1 = 1 + 3a_2$. Choosing $a_2 = 0$ gives us $\mathbf{X}_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$ and as in (a) and (b) because we started with x, y notation we end with x, y notation:

$$\begin{aligned} x &= 2c_1 e^{-2t} + 3c_2 e^{-t} + e^{-t} \\ y &= -c_1 e^{-2t} + c_2 e^{-t}. \end{aligned}$$

15. The eigenvalues of $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -3 & -4 \end{pmatrix}$ are $\lambda_{1,2} = -1$ and there is only one linearly independent eigenvector corresponding to this eigenvalue, $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Thus, one solution of the system is $\mathbf{X}_1 = \mathbf{v}e^{-t}$. We now look for a second linearly independent solution of the form $\mathbf{X}_2 = \mathbf{w}_1 te^{-t} + \mathbf{w}_2 e^{-t}$. Differentiating, $\mathbf{X}_2' = -\mathbf{w}_1 te^{-t} + (\mathbf{w}_1 - \mathbf{w}_2)\, e^{-t}$, and substituting into the system gives us

$$\begin{aligned} \mathbf{X}_2' &= -\mathbf{w}_1 te^{-t} + (\mathbf{w}_1 - \mathbf{w}_2)\, e^{-t} \\ &= \mathbf{A}\mathbf{X}_2 \\ &= \mathbf{A}\left(\mathbf{w}_1 te^{-t} + \mathbf{w}_2 e^{-t}\right) \\ &= \mathbf{A}\mathbf{w}_1 te^{-t} + \mathbf{A}\mathbf{w}_2 e^{-t} \end{aligned}$$

Equating coefficients gives us $\mathbf{A}\mathbf{w}_1 = -\mathbf{w}_1$. This means that $\mathbf{w}_1$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue $\lambda = -1$. We choose $\mathbf{w}_1 = \mathbf{v}$. Thus, $\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{v} - \mathbf{w}_2 = \mathbf{A}\mathbf{w}_2$, which shows us that $\mathbf{w}_2$ satisfies $(\mathbf{A} + \mathbf{I})\, \mathbf{w}_2 = \mathbf{v}$:

Remember that when the equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ is satisfied, λ is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v}$.

$$\begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} w_{2,1} \\ w_{2,2} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{2,1} \\ w_{2,2} \end{pmatrix} = \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}$$

Then $w_{2,1} + w_{2,2} = -1/3$. We choose $w_{2,1} = -1/3$, which gives us

$$\mathbf{X}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} e^{-t}.$$

Combining $\mathbf{X}_1$ and $\mathbf{X}_2$, a fundamental matrix for the system is

$$\mathbf{\Phi} = \begin{pmatrix} -e^{-t} & -te^{-t} - \frac{1}{3}e^{-t} \\ e^{-t} & te^{-t} \end{pmatrix}$$

and a general solution of the system is $\mathbf{X} = \begin{pmatrix} -e^{-t} & -te^{-t} - \frac{1}{3}e^{-t} \\ e^{-t} & te^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. When dealing with repeated eigenvalues for which there is only one linearly independent eigenvector, one must be especially carefully in choosing the form of $\mathbf{X}_p$. For (b), we assume that

$$\mathbf{X}_p = \mathbf{a}t^2 e^{-t} + \mathbf{b}te^{-t} + \mathbf{c}e^{-t},$$

where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ are constant vectors to be determined. Substituting $\mathbf{X}_p$ into the nonhomogeneous equation gives us

$$\begin{aligned} \mathbf{X}_p' - \mathbf{A}\mathbf{X}_p &= \begin{pmatrix} -3(a_1 + b_1)t^2 + (2a_1 - 3b_1 - 3b_2)t + (b_1 - 3c_1 - 3c_2) \\ 3(a_1 + b_1)t^2 + (5b_1 + 3b_2) + (b_2 + 3c_1 + 3c_2 \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \end{aligned}$$

and equating coefficients we find that

$$\begin{aligned} a_1 + a_2 &= 0 \\ 2a_1 - 3b_1 - 3b_2 &= 0 \\ b_1 - 3c_1 - 3c_2 &= 1 \\ a_1 + a_2 &= 0 \\ 2a_2 + 3b_1 + 3b_2 &= 0 \\ b_2 + 3c_1 + 3c_2 &= -1, \end{aligned}$$

which has solution $a_1 = 0$, $a_2 = 0$, $b_1 = 1+3c_1+3c_2$, $b_2 = -1-3c_1-3c_2$. Choosing $c_1 = c_2 = 0$, $\mathbf{X}_p = \begin{pmatrix} te^{-t} \\ -te^{-t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} -e^{-t} & -te^{-t} - \frac{1}{3}e^{-t} \\ e^{-t} & te^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} te^{-t} \\ -te^{-t} \end{pmatrix}$.
For (c), we proceed in much the same way as in (b). We assume that

$$\mathbf{X}_p = \mathbf{a}t^2 e^{-t} + \mathbf{b}te^{-t} + \mathbf{c}e^{-t},$$

where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ are constant vectors to be determined. Substituting $\mathbf{X}_p$ into the nonhomogeneous equation gives us

$$\begin{aligned} \mathbf{X}_p' - \mathbf{A}\mathbf{X}_p &= \begin{pmatrix} -3(a_1+b_1)t^2 + (2a_1 - 3b_1 - 3b_2)t + (b_1 - 3c_1 - 3c_2) \\ 3(a_1+b_1)t^2 + (5b_1 + 3b_2) + (b_2 + 3c_1 + 3c_2 \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} \end{aligned}$$

and equating coefficients we find that

$$\begin{aligned} a_1 + a_2 &= 0 \\ 2a_1 - 3b_1 - 3b_2 &= 0 \\ b_1 - 3c_1 - 3c_2 &= 2 \\ a_1 + a_2 &= 0 \\ 2a_2 + 3b_1 + 3b_2 &= 0 \\ b_2 + 3c_1 + 3c_2 &= -1, \end{aligned}$$

which has solution $a_1 = 3/2$, $a_2 = -3/2$, $b_1 = 2 + 3c_1 + 3c_2$, $b_2 = -1 - 3c_1 - 3c_2$. Choosing $c_1 = c_2 = 0$ gives us $\mathbf{X}_p = \begin{pmatrix} \frac{3}{2}t^2e^{-t} + 2te^{-t} \\ -\frac{3}{2}t^2e^{-t} - te^{-t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} -e^{-t} & -te^{-t} - \frac{1}{3}e^{-t} \\ e^{-t} & te^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \frac{3}{2}t^2e^{-t} + 2te^{-t} \\ -\frac{3}{2}t^2e^{-t} - te^{-t} \end{pmatrix}$.
17. $x(t) = (c_1 + t)\cos(t) - \sin(t)\left(c_2 + \log(\cos(t))\right)$, $y(t) = (c_1 + t)\sin(t) + \cos(t)\left(c_2 + \log(\cos(\right.$
19. $x(t) = c_1\cos(t) + \sin(t)\left(c_2 - 2\log\left(\sin\left(\frac{t}{2}\right)\right) + 2\log\left(\cos\left(\frac{t}{2}\right)\right)\right)$, $y(t) = -c_1\sin(t) + \cos(t)\left(c_2 - 2\log\left(\sin\left(\frac{t}{2}\right)\right) + 2\log\left(\cos\left(\frac{t}{2}\right)\right)\right) - 2$
21. $x(t) = \frac{1}{9}e^{-4t}\left(c_1\left(20e^{3t} - 4e^{6t} - 7\right) + 4\left(e^{3t} - 1\right)\left(2\left(c_2 + c_3\right)e^{3t} - 2c_2 + c_3\right)\right) - t^2 + \frac{11t}{4} - \frac{83}{16}$, $y(t) = \frac{1}{96}e^{-4t}\left(32\left(5c_1 - 4c_2 - c_3\right)e^{3t} - 48\left(c_1 - 2\left(c_2 + c_3\right)\right)e^{6t} - 16\left(7c_1 - 8c_2 + \right.\right.$
$z(t) = \frac{1}{144}e^{-4t}\left(16\left(-5c_1 + 4c_2 + c_3\right)e^{3t} - 32\left(c_1 - 2\left(c_2 + c_3\right)\right)e^{6t} + 16\left(7c_1 - 8c_2 + 4c_3\right) + e^{4\right.$

23. In matrix form, we are solving $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$, where $\mathbf{X} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} -2 & 1 & -1 \\ 5 & -2 & 5 \\ 2 & -1 & 1 \end{pmatrix}$, and $\mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix}$. $\mathbf{A}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -1 \pm 2i$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{v}_{2,3} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} i$. One solution of the homogeneous equation is $\mathbf{X}_1 = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix}$.
Two more linearly independent solutions are

$$\mathbf{X}_1 = e^{-t} \left[\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \sin 2t \right] = \begin{pmatrix} -e^{-t} \cos 2t \\ e^{-t} (\cos 2t + 2 \sin 2t) \\ 2e^{-t} \cos 2t \end{pmatrix}$$

and

$$\mathbf{X}_2 = e^{-t} \left[\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \cos 2t \right] = \begin{pmatrix} -e^{-t} \sin 2t \\ e^{-t} (-2\cos 2t + \sin 2t) \\ 2e^{-t} \sin 2t \end{pmatrix}.$$

Thus, a general solution of the system is

$$\mathbf{X} = \begin{pmatrix} -e^{-t} & -e^{-t} \cos 2t & -e^{-t} \sin 2t \\ 0 & e^{-t} (\cos 2t + 2 \sin 2t) & e^{-t} (-2\cos 2t + \sin 2t) \\ e^{-t} & 2e^{-t} \cos 2t & 2e^{-t} \sin 2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

For (b), we assume that a particular solution takes the form

$$\mathbf{X}_p = \mathbf{a}e^t + \mathbf{b}\cos 2t + \mathbf{c}\sin 2t,$$

where the vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ are to be determined.

Substituting into the nonhomogeneous system and simplifying the results gives us

$$\begin{aligned} \mathbf{X}_p' - \mathbf{A}\mathbf{X}_p &= \begin{pmatrix} (3a_1 - a_2 + a_3)e^t + (2b_1 - b_2 + b_3 + 2c_1)\cos 2t + (-2b_1 + 2c_1 - c_2 + c_3)\sin 2t \\ (-5a_1 + 3a_2 - 5a_3)e^t + (-5b_1 + 2b_2 - 5b_3 + 2c_2)\cos 2t + (-2b_2 - 5c_1 + 2c_2 - 5c_3)\sin 2t \\ (-2a_1 + 2a_2)e^t + (-2b_1 + 2b_2 - b_3 + 2c_3)\cos 2t + (-2b_3 - 2c_1 + 2c_2 - c_2)\sin 2t \end{pmatrix} \\ &= \begin{pmatrix} 3e^t - \cos 2t + \sin 2t \\ -5e^t + 2\cos 2t - 7\sin 2t \\ -2e^t + 4\cos 2t - \sin 2t \end{pmatrix} \end{aligned}$$

and equating coefficients results in the system

$$
\begin{aligned}
a_1 - a_2 + a_3 &= 3 \\
2b_1 - b_2 + b_3 + 2c_1 &= -1 \\
-2b_1 + 2c_1 - c_2 + c_3 &= 1 \\
-5a_1 + 3a_2 - 5a_3 &= -5 \\
-5b_1 + 2b_2 - 5b_3 + 2c_2 &= 2 \\
-2b_2 - 5c_1 + 2c_2 - 5c_3 &= -7 \\
-2a_1 + 2a_2 &= -2 \\
-2b_1 + 2b_2 - b_3 + 2c_3 &= 4 \\
-2b_3 - 2c_1 + 2c_2 - c_2 &= -1,
\end{aligned}
$$

which has solution $a_1 = 1$, $a_2 = 0$, $a_3 = 0$, $b_1 = 0$, $b_2 = 1$, $b_3 = 0$, $c_1 = 0$, $c_2 = 0$, and $c_3 = 1$.

Thus, a particular solution of the nonhomogeneous system is

$$\mathbf{X}_p = \begin{pmatrix} e^t \\ \cos 2t \\ \sin 2t \end{pmatrix}$$

and a general solution of the nonhomogeneous system is

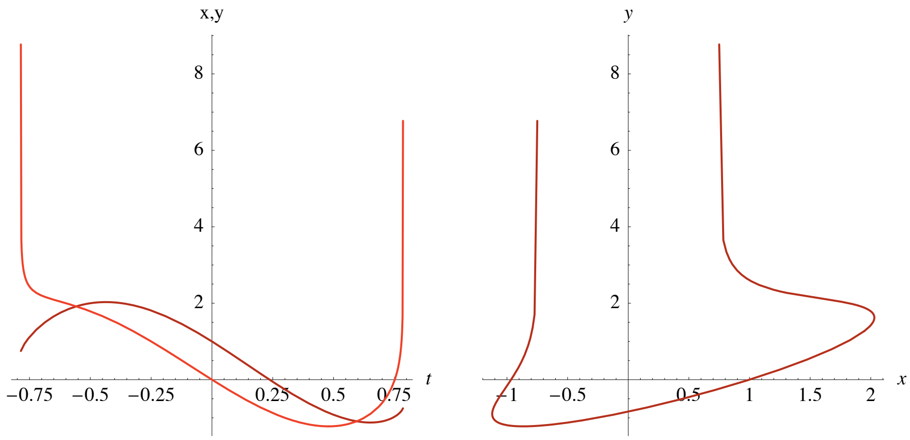
$$
\begin{aligned}
\mathbf{X} &= \mathbf{X}_h + \mathbf{X}_p \\
&= \begin{pmatrix} -e^{-t} & -e^{-t}\cos 2t & -e^{-t}\sin 2t \\ 0 & e^{-t}\left(\cos 2t + 2\sin 2t\right) & e^{-t}\left(-2\cos 2t + \sin 2t\right) \\ e^{-t} & 2e^{-t}\cos 2t & 2e^{-t}\sin 2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} e^t \\ \cos 2t \\ \sin 2t \end{pmatrix}.
\end{aligned}
$$

25. $x(t) = -1/2\,\cos(2\,t)\,c_2 + 1/2\,\sin(2\,t)\,c_1 + 1/8\,\cos(2\,t) + 1/4\,\sin(2\,t)\,t + c_3$, $y(t) = 1/2 c_2 \cos(2\,t) - 1/2 c_2\,\sin(2\,t) - 1/8\,\cos(2\,t) - 1/4\,\sin(2\,t)\,t - 2\,c_3$, $z(t) = \sin(2\,t)\,c_2 + \cos(2\,t)\,c_1 + 1/2\,t\cos(2\,t) + 3\,c_3$
27. $x(t) = -1 + t + e^{-t}$, $y(t) = 0$
29. $x(t) = -2/3\,e^{-t} + 1/6\,e^{5\,t} - 1/2\,e^{t}$, $y(t) = 2/3\,e^{-t} + 1/12\,e^{5\,t} + 1/4\,e^{t}$
31. $x(t) = \frac{20}{13} + e^{2\,t}\left(1/39\,\sin(3\,t) - \frac{7}{13}\,\cos(3\,t)\right)$,
$y(t) = -\frac{30}{13} - 1/2\,e^{2\,t}\left(-\frac{62}{39}\,\sin(3\,t) - \frac{8}{13}\,\cos(3\,t)\right)$
33. $x(t) = -1/4\,\sin(2\,t) + 1/2\,\cos(2\,t) - 1/2\,\sin(2\,t)\ln\left(\frac{1+\sin(2\,t)}{\cos(2\,t)}\right) - 1/4\,\cos(2\,t)\ln\left(\frac{1+\mathrm{s}}{\mathrm{cos}}\right.$
$5/4\,\cos(2\,t)\ln(\cos(2\,t)) + 5/2\,\sin(2\,t)\,t + 1/2$, $y(t) = -1/4\,\sin(2\,t) - 1/4\,\cos(2\,t)\ln\left(\frac{1+\sin(2}{\cos(2\,t}\right.$
$1/2\,\ln(\cos(2\,t))\sin(2\,t) + 1/4\,\cos(2\,t)\ln(\cos(2\,t)) + \cos(2\,t)\,t + 1/2\,\sin(2\,t)\,t$
35.

$$x = \frac{1}{4}\left(5\ln(\cos(t) - \sin(t))\cos(2t) - 5\ln(\cos(t) + \sin(t))\cos(2t) + 4\cos(2t) - 3\sin(2t)\right)$$

and

$$
\begin{aligned}
y = \frac{1}{2}\big(&-\cos^2(2t) + 2\ln(\cos(t) - \sin(t))\cos(2t) - 2\ln(\cos(t) + \sin(t))\cos(2t) + \cos(2t) - \mathrm{s} \\
&\ln(\cos(t) - \sin(t))\sin(2t) + \ln(\cos(t) + \sin(t))\sin(2t) - 2\sin(2t)\big)
\end{aligned}
$$

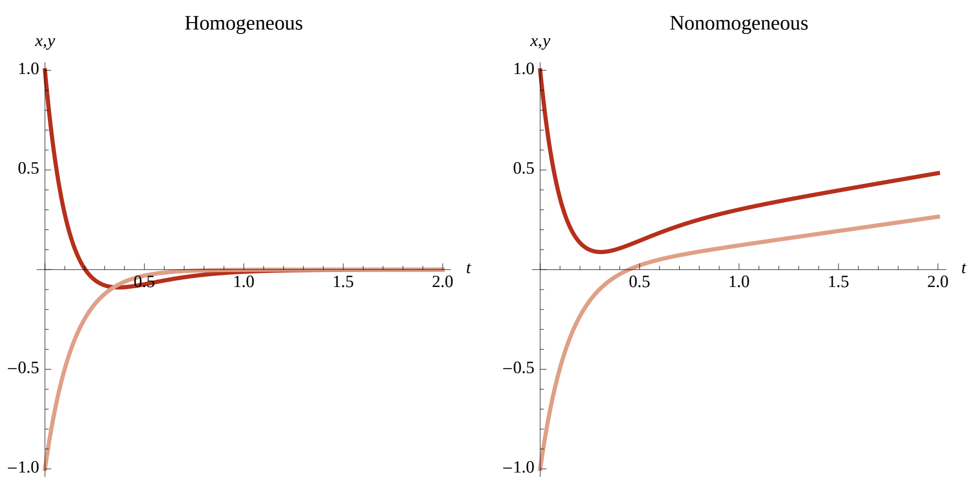


37. $\mathbf{X} = \begin{pmatrix} 1 & \sqrt{t} \\ t & t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix}$

39. $\mathbf{X} = \begin{pmatrix} 1 & f(t) \\ f(t) & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} f(t) \\ f(t) \end{pmatrix}$

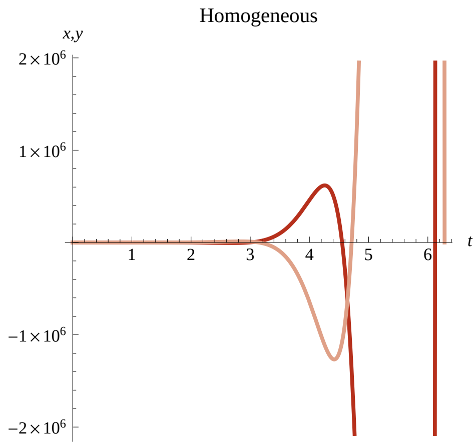
41. (a) $x(t) = -\frac{199}{343} e^{-t} + \frac{453}{1372} e^{-8t} + 1/4 - 3/14\, e^{-t} t^2 + \frac{3}{49} te^{-t}$, $y(t) = \frac{391}{343} e^{-t} + \frac{151}{1372} e^{-8t} - 1/4 + 1/49\, te^{-t} + 3/7\, e^{-t} t^2$ (b) $x(t) = \frac{260}{81} e^{t} - \frac{179}{81} e^{-2t} + \frac{10}{9} te^{t} - 5/3\, t^2 e^{t} - \frac{20}{27} e^{-2t} t - \frac{10}{9} t^2 e^{-2t} - \frac{7}{9} e^{-2t} t^3$, $y(t) = -\frac{173}{81} e^{t} + \frac{173}{81} e^{-2t} - \frac{10}{9} te^{t} + 7/6\, t^2 e^{t} + \frac{7}{9} t^2 e^{-2t} + \frac{14}{27} e^{-2t} t + \frac{7}{9} e^{-2t} t^3$; (c) $x(t) = \frac{279}{289} \cos(4t) - \frac{3479}{4624} \sin(4t) + \frac{10}{289} e^{-t} + 1/2 \sin(4t)\, t - 1/4 \cos(4t)\, t + \frac{5}{17} te^{-t}$, $y(t) = \frac{1321}{2312} \sin(4t) + \frac{300}{289} \cos(4t) - \frac{11}{289} e^{-t} + \frac{3}{17} te^{-t} - 1/2 \cos(4t)\, t$

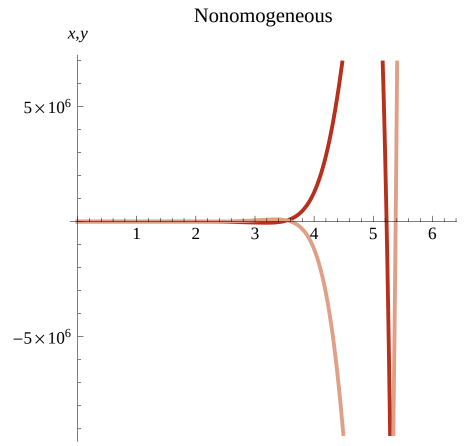
43. Homogeneous: $x(t) = e^{-7t}\left(3 - 2e^{2t}\right)$, $y(t) = -e^{-7t}$; Nonhomogeneous:
$$x(t) = \frac{210t + 3600e^{-7t} - 2548e^{-5t} + 173}{1225}, \quad y(t) = \frac{1}{49}\left(7t - 48e^{-7t} - 1\right)$$



45. Homogeneous: $x(t) = e^{3t}(3\sin(2t) + \cos(2t))$, $y(t) = -4e^{3t}\sin(2t)$; Nonhomo-

geneous: $x(t) = \frac{1}{4}e^{3t}((2t+15)\sin(2t) - (6t+1)\cos(2t) + 5)$, $y(t) = \frac{1}{2}e^{3t}(-9\sin(2t) + (4t+3)\cos(2t) - 3)$

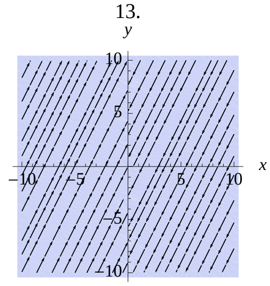




Exercises 6.6

1. $\lambda_{1,2} = 1, -11$; Saddle point, unstable
3. $\lambda_{1,2} = 2$; Deficient node, unstable
5. $\lambda_{1,2} = -8, -12$; Improper node, asymptotically stable
7. $\lambda_{1,2} = -1$; Star node, asymptotically stable
9. $\lambda_{1,2} = 2 \pm 5i\sqrt{2}$; Spiral point, unstable
11. $\lambda_{1,2} = \pm i\sqrt{41}$; Center, stable

13.



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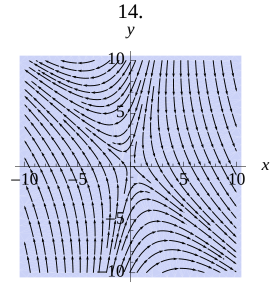

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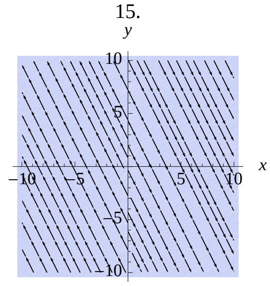

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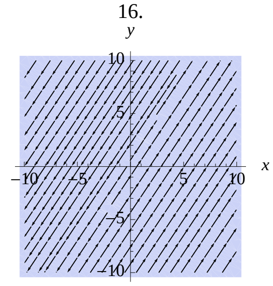
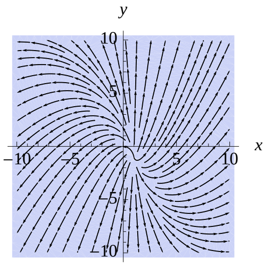


17. $x'' - x' - 2x = 0$, $\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \mathbf{X}$, Saddle point, unstable

19. $x'' - 6x' + 9x = 0$, $\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix} \mathbf{X}$, Deficient node, unstable

21.



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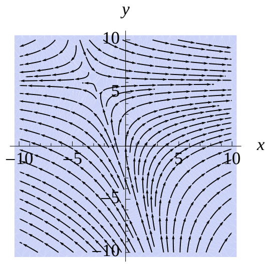

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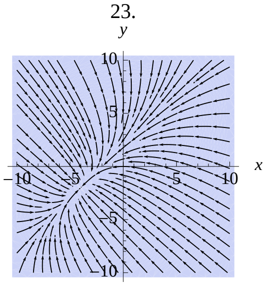
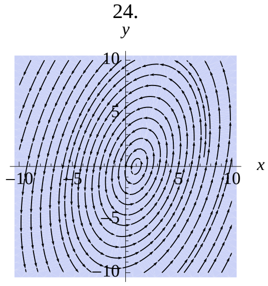


24.

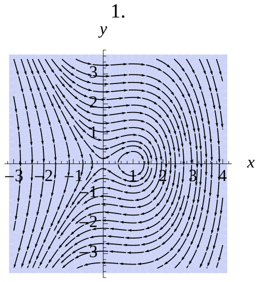


27. $A_1 = K_3C_1/K_1$, $B_1 = K_3C_1/K_2$, C_1 arbitrary.

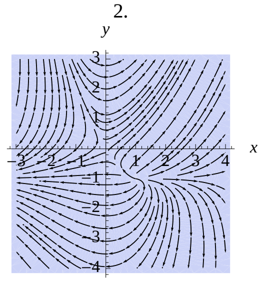
Exercises 6.7

1. $(0,0)$, saddle point, unstable; $(1,0)$ inconclusive–center or spiral point
3. $(0,0)$, saddle point, unstable; $(1,-1)$, spiral point, unstable
5. $(0,0)$, inconclusive–center or spiral point

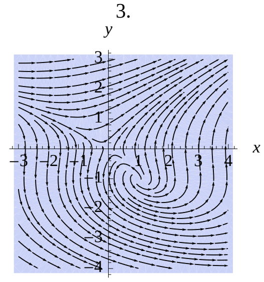
1.



2.

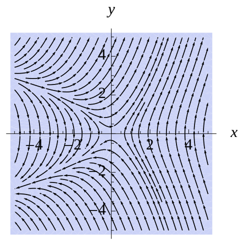


3.

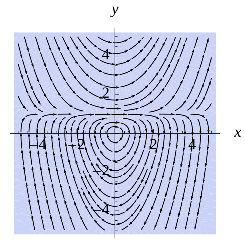


Out[26]=

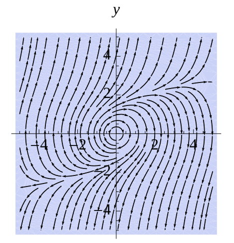
4.



5.



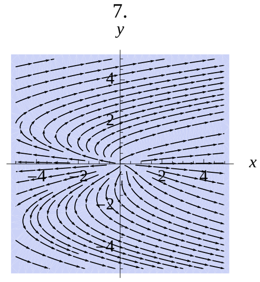
6.



7. $(0,0)$, node or spiral point, unstable

9. $(0,0)$, improper node, asymptotically stable; $(0,4)$, saddle point, unstable; $(2,0)$, saddle point, unstable; $(3,1)$, spiral point, unstable

7.



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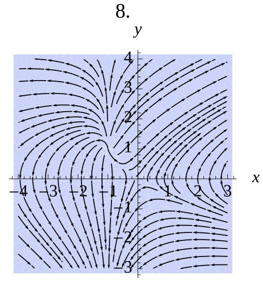
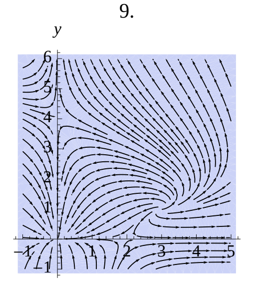


9.

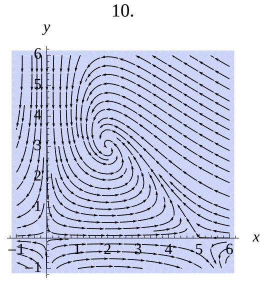


11. $(-4,4)$, saddle point, unstable; $(1,-1)$, saddle point unstable

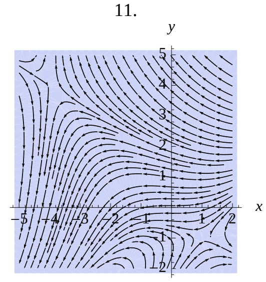
13. $(2,-1)$: $\lambda_{1,2}=-6,\ 1$, saddle; $(8,-1)$: $\lambda_{1,2}=6,\ 1$, unstable node

15. $(0,0)$, $\lambda_{1,2}=\pm\sqrt{2}$, saddle; $(1,1)$, $\lambda_{1,2}=-1\pm i$, stable spiral

10.



11.



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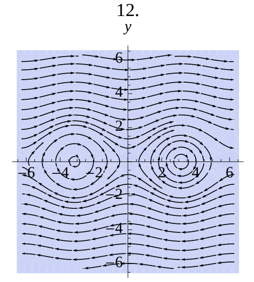

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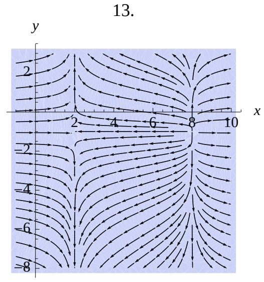

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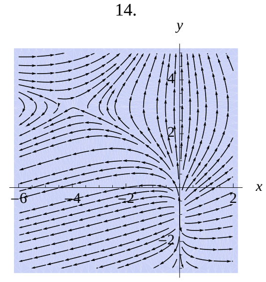

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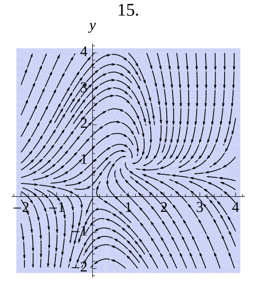
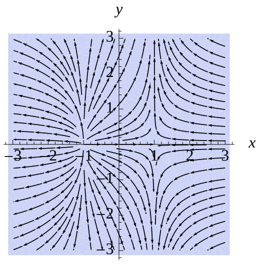


17. $(0,2)$, $\lambda_{1,2}=-6,\ 1$, unstable node; $(0,-2)$, $\lambda_{1,2}=-2,\ -4$, stable node;

$(2,0)$, $\lambda_{1,2} = \pm 2\sqrt{2}$, saddle; $(-2,0)$, $\lambda_{1,2} = \pm 2\sqrt{2}$, saddle
19. $(0,0)$, $\lambda_{1,2} = \pm 2$, saddle; $(-1,-1)$, $\lambda_{1,2} = 1 \pm i\sqrt{3}$, unstable spiral

16.



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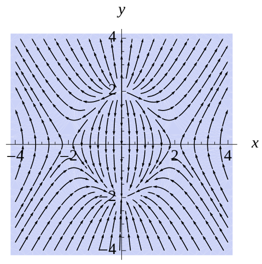
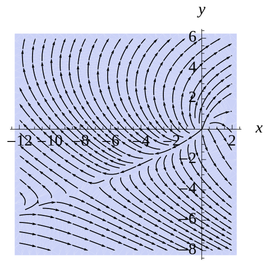
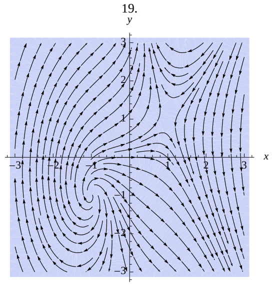


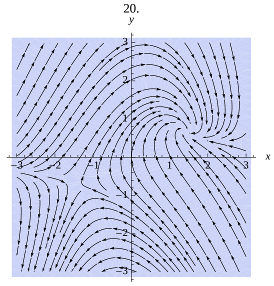
18.



19.

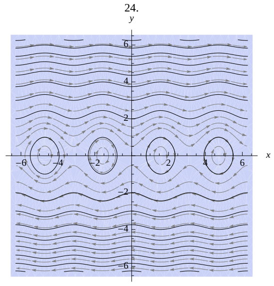


20.

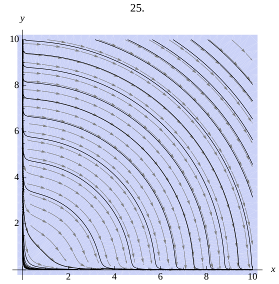


21. $(0,0)$, $\lambda_{1,2} = a_1,\ -a_2$, saddle; $(a_1/b_1, 0)$, $\lambda_{1,2} = -a_1,\ a_1c_2/b_1 - a_2$, stable node; $(-a_2/c_2, (a_1c_2+a_2b_1)/(c_1c_2)$, $\lambda_{1,2} = \dfrac{a_2b_1}{c_2} \pm \sqrt{\left(\dfrac{a_2b_1}{c_2}\right)^2 - \dfrac{4a_2(a_1c_2+a_2b_1)}{b_1}}$, unstable node if $\left(\dfrac{a_2b_1}{c_2}\right)^2 - \dfrac{4a_2(a_1c_2+a_2b_1)}{b_1} \geq 0$; unstable spiral if $\left(\dfrac{a_2b_1}{c_2}\right)^2 - \dfrac{4a_2(a_1c_2+a_2b_1)}{b_1} < 0$
25. $H(x,y) = \int (2y + x^{-1}y^{-2})\,dy + g(x)$; $g(x) = -\int -2x\,x$ so $H(x,y) = x^2 - (xy)^{-1} + y^2$.

24.

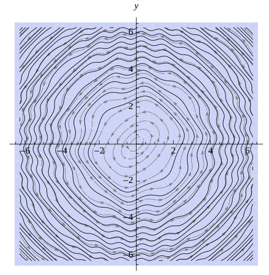


25.

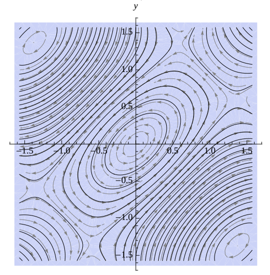


27. $H(x,y) = \int (x\cos xy - \sin 2y)\,dy + g(x)$; $g(x) = -int \sin 2x\,dx = \frac{1}{2}\cos 2x$ so $H(x,y) = \frac{1}{2}\cos 2y + \frac{1}{2}\cos 2x + \sin xy$.

26.

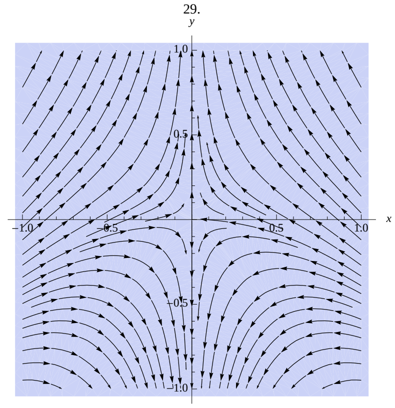


27.



29. $x(t) = c_1 e^t$, $y(t) = -\frac{1}{3} {c_1}^2 e^{2t} + c_2 e^t$

29.



31.

1. Take the appropriate partial derivatives.

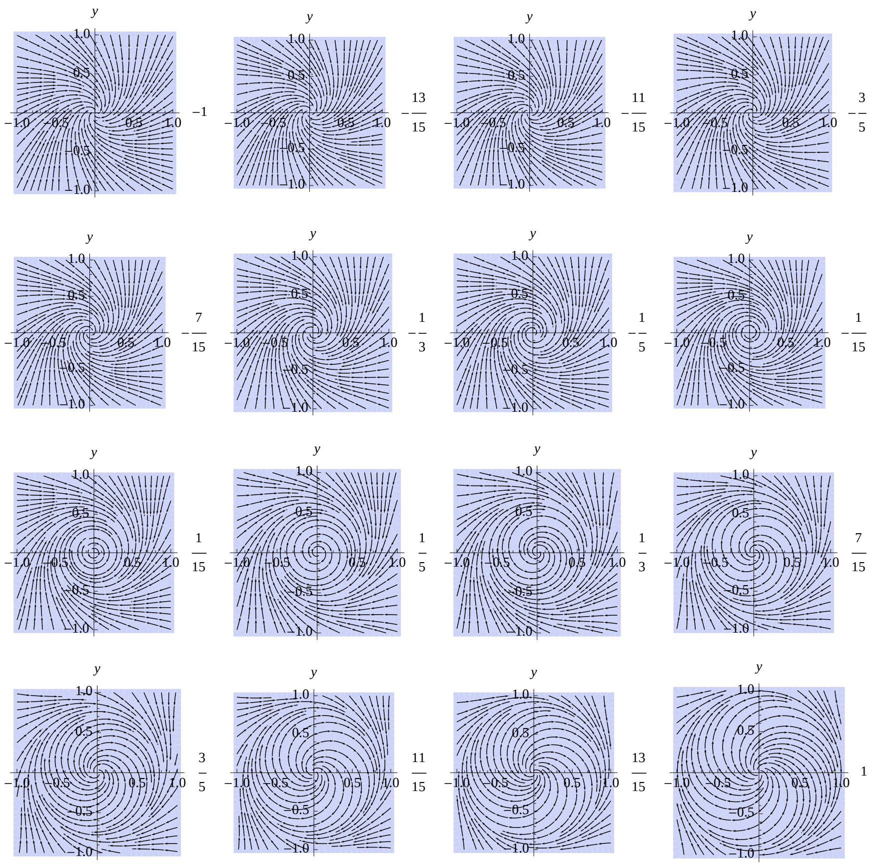
2. Substitute $x = y$ and $z = \dfrac{y^2}{b}$ into the middle equation.

3. The characteristic equation is $(-b - \lambda)\left(\lambda^2 + (a - k_2 + 1)\lambda - a(k_2 - 1 + c + k_1)\right) = 0$, which does not depend on k_3.

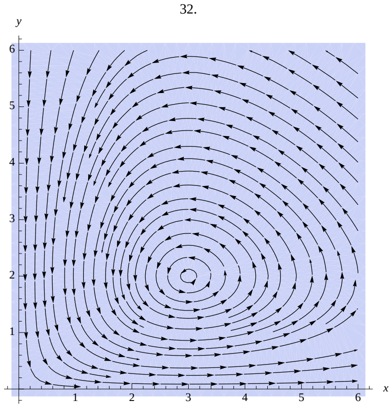
4. Since the eigenvalues do not depend on k_3, we can simplify the problem with $k_3 = 0$. Substitute the values of k_1, k_2 and $k_3 = 0$ into the controlled system.

5. The equilibrium points are $\left(3\sqrt{7}, 3\sqrt{7}, 21\right)$ and $\left(-3\sqrt{7}, -3\sqrt{7}, 21\right)$ with corresponding eigenvalues $\lambda_1 \approx -18.43$, $\lambda_{2,3} \approx 4.2 \pm 14.88i$. The positive real components of $\lambda_{2,3}$ cause the equilibria to be unstable.

33.



32.

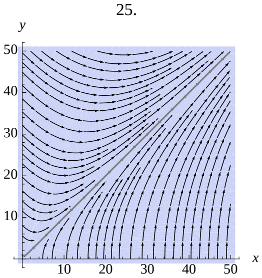


Exercises 6.8

1. $x(1) \approx 8.64479$, $y(1) \approx 8.29263$
3. $x(1) \approx -7.2362$, $y(1) \approx -1.5998$
5. $x(1) \approx -0.113115$, $y(1) \approx -1.06576$

7. $x(1) \approx -326.204$, $y(1) \approx 608$
9. $x(1) \approx 0.164251$, $y(1) \approx -0.504587$
11. $x(1) \approx 1.52319$, $y(1) \approx 0.419975$
13. $x' = y$, $y' = -2x - 3y$, $x(0) = 0$, $y(0) = -3$; Euler's method yields $x(1) \approx -0.723913$, $y(1) \approx 0.40179$; Exact solution is $x(t) = 3e^{-2t} - 3e^{-t}$ so $x(1) = 3e^{-2}(1 - e) \approx -0.697632$.
15. $x' = y$, $y' = -9x$, $x(0) = 0$, $y(0) = 3$; Euler's method yields $x(1) \approx 0.346313$, $y(1) \approx -4.49743$; Exact solution is $x(t) = \sin 3t$ so $x(1) = \sin 3 \approx 0.14112$.
17, $x' = y$, $y' = -t^{-2}(16x + ty)$, $x(1) = 0$, $y(1) = 4$; Euler's method yields $x(2) \approx 0.354942$, $y(2) \approx -2.90834$/ Exact solution is $x(t) = \sin(4 \ln t)$ so $x(2) = \sin(4 \ln 2) \approx 0.360687$
19. (See 13) Runge-Kutta yields $x(1) \approx -0.697621$, $y(1) \approx 0.291602$
21. (See 15) Runge-Kutta yields $x(1) \approx 0.141307$, $y(1) \approx -2.96975$
23. (See 17) Runge-Kutta yields $x(1) \approx 0.360845$, $y(1) \approx -1.86541$
25. $x(t) = \frac{1}{3}e^{-2t}(e^{3t} - 1)$, $y(t) = \frac{1}{3}e^{-2t}(e^{3t} + 2)$
26. $x(t) = -\frac{1}{3}e^{-2t}(3\cos 3t + \sin 3t)$, $y(t) = \frac{1}{3}e^{-2t}(3\cos 3t + 11\sin 3t)$
27. $x(t) = 2e^{t}(t + 1)$, $y(t) = -2te^{t}$

25.



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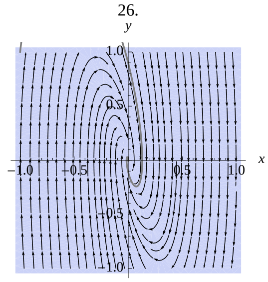

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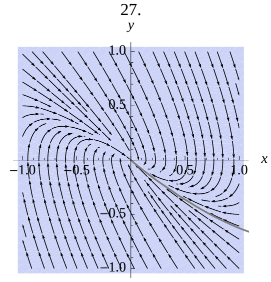
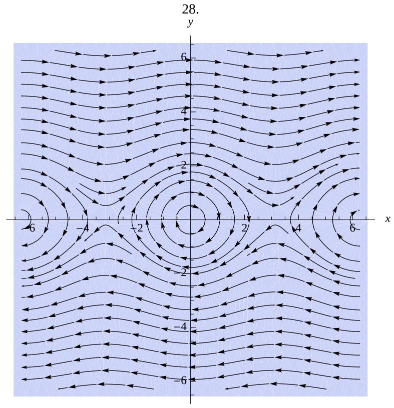


28.

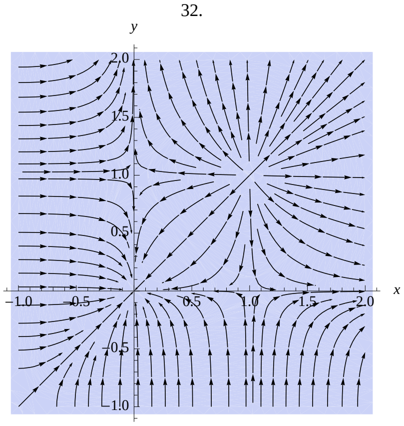
28.



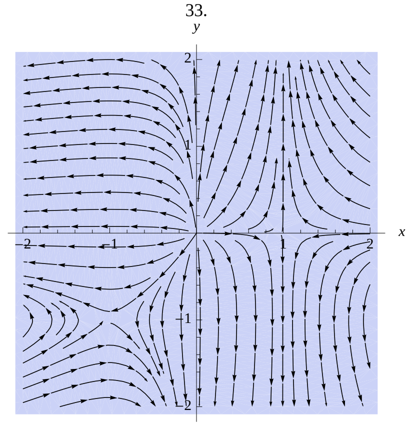
Chapter 6 Review Exercises

1. $\lambda_1 = 11,\ \lambda_2 = -4,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
3. $\lambda_{1,2} = -4,\ \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
5. $\lambda_{1,2} = \pm 3i,\ \lambda_3 = -1,\ \mathbf{v}_{1,2} = \begin{pmatrix} -1 \mp \frac{1}{2}i \\ -1/2 \\ 1 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$
7. $\lambda_{1,2,3} = 0,\ \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
9. $x = \frac{1}{5}e^t\left(\left(4+e^{5t}\right)c_1 - 4\left(-1+e^{5t}\right)c_2\right),\ y = \frac{1}{5}e^t\left(-e^{5t}c_1 + c_1 + 4e^{5t}c_2 + c_2\right)$
11. $x = -e^{-2t} + 2e^t,\ y = 3e^{-2t} - 3e^t,\ x(0) = 1,\ y(0) = 0$
13. $x = 2\sin 4t,\ y = \cos 4t + \sin 4t$
15. $x = \frac{17}{4}e^{-2t}\sin 4t,\ y = \frac{1}{4}e^{-2t}(4\cos 4t - \sin 4t)$
17. $x = e^{-t}\left(2tc_1 + c_1 + 2tc_2\right),\ y = e^{-t}\left(c_2 - 2t\left(c_1 + c_2\right)\right)$
19. $x = e^{-t}\left(\left(5-4e^t\right)c_1 + 2\left(-1+e^t\right)\left(3c_2 - 2c_3\right)\right),\ y = \left(-2+2e^{-2t}\right)c_1 + \left(3-2e^{-2t}\right)c_2 +$
$2\left(-1+e^{-2t}\right)c_3,\ z = e^{-2t}\left(-3c_2\left(-1+e^t\right)^2 + \left(3-5e^t+2e^{2t}\right)c_1 + \left(3-4e^t+2e^{2t}\right)c_3\right)$
21. $x = \cos 3t + \frac{11}{3}\sin 3t,\ y = \frac{1}{3}(-5 + 5\cos 3t + \sin 3t),\ z = -\cos 3t + 5\sin 3t$
23. $x(t) = -e^{7t} + c_2e^{-t},\ y(t) = \frac{7}{9}e^{7t} + c_2e^{-t} + c_1e^{-2t}$
25. $x(t) = c_2e^{-t} + c_1e^{5t} + 1,\ y(t) = -c_2e^{-t} + 1/2\,c_1e^{5t} - 1$
27. $x(t) = c_2\sin(t) + c_1\cos(t) - \cos(t) - t\sin(t),\ y(t) = -c_2\cos(t) + c_1\sin(t) +$
$t\cos(t) - \sin(t)$
29. $x(t) = 15/2\,t^2\ln(t) - \frac{45}{4}t^2 - t\ln(t) + t + c_1 t + c_2,\ y(t) = t\ln(t) - t + 1/9\,c_1 +$
$5t^2\ln(t) - 15/2\,t^2 + 2/3\,c_1 t + 2/3\,c_2$
31. $x = \frac{1}{210}\left(140t^{3/2} + 16t^{7/2}\right),\ y = \frac{1}{15}\left(-10t^{3/2} - 12t^{5/2}\right),\ z = \frac{1}{210}\left(140t^{3/2} + 56t^{5/2} + 16t^{3/2}\right)$

32.



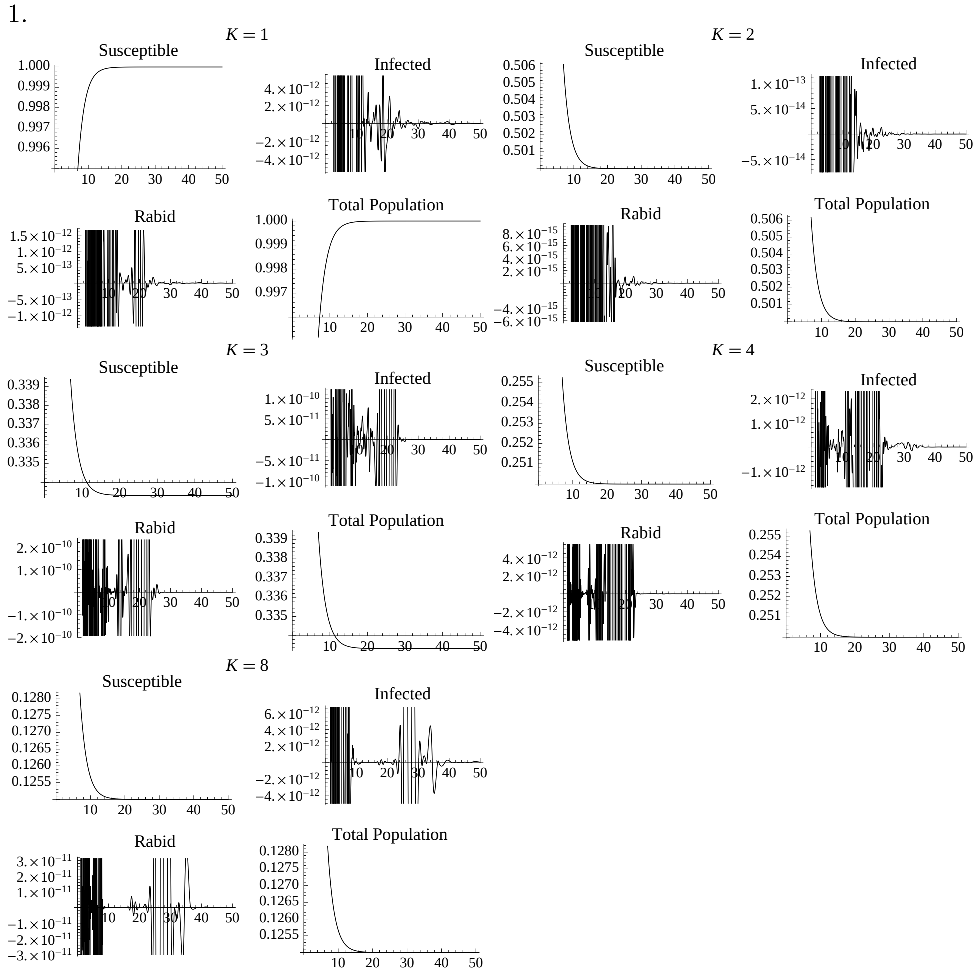
33.



Differential Equations at Work

A. Modeling a Fox Population in which Rabies is Present

1.



6. $[(\sigma + \alpha)(\alpha + \beta)]/(\sigma\beta) = 1.00103$

B. Controlling the Spread of a Disease

1. If $S_0 < \gamma/\lambda$, $I'(0) = \lambda S_0 I_0 - \gamma I_0 < \lambda \cdot \dfrac{\gamma}{\lambda} I_0 - \gamma I_0 = 0$. Thus, the rate of infection immediately begins to decrease. If $I'(0) = \lambda S_0 I_0 - \gamma I_0 > \lambda \cdot \dfrac{\gamma}{\lambda} I_0 - \gamma I_0 = 0$, the rate of infection first increases.
2. The equation is separable. $dI/dS = -1 + \rho/S$ becomes $dI = (-1 + \rho/S)dS$, which has solution $I = -S + \rho \ln S + C$. Applying the initial condition results in $I_0 = -S_0 + \rho \ln S_0 + C$ so $C = I_0 + S_0 - \rho \ln S_0$ and thus $I + S - \rho \ln S = I_0 + S_0 - \rho \ln S_0$.
3. The maximum value of I occurs when $dI/dS = -1 + \rho/S = (\rho - S)/S = 0 \Rightarrow$

$S = \rho$ so the maximum is

$$I = -\rho + \rho \ln \rho + I_0 + S_0 - \rho \ln S_0 = I_0 + S_0 + \rho \cdot \left(\ln \frac{\rho}{S_0} - 1 \right).$$

4. We solve the system $\lambda SI + \mu - \mu S = 0$, $\lambda SI - \gamma I - \mu I = 0$ for S and I. $\lambda SI - \gamma I - \mu I = 0 \Rightarrow I(\lambda S - \gamma - \mu) = 0 \Rightarrow I = 0$ or $\lambda S - \gamma - \mu = 0$ with solution $S = (\gamma + \mu)/\lambda$. If $I = 0$, $S = 1$; if $S = (\gamma + \mu)/\lambda$,

$$\begin{aligned} -\lambda \cdot \frac{\gamma+\mu}{\lambda} I + \mu - \mu \cdot \frac{\gamma+\mu}{\lambda} &= 0 \\ -(\gamma+\mu)I &= \mu \cdot \frac{\gamma+\mu}{\lambda} - \mu \\ I &= -\frac{\mu \cdot \frac{\gamma+\mu}{\lambda} - \mu}{\gamma+\mu} = \frac{\mu}{\lambda} \frac{\lambda - (\gamma+\mu)}{\gamma+\mu}. \end{aligned}$$

5 and 6. To have a "meaningful" equilibrium point, we must have

$$\begin{aligned} S + I &\le 1 \\ \frac{\gamma+\mu}{\lambda} + \frac{\mu}{\lambda}\frac{\lambda-(\gamma+\mu)}{\gamma+\mu} &\le 1 \\ \frac{\gamma^2+\gamma\mu+\lambda\mu}{\lambda(\gamma+\mu)} &\le 1 \\ \gamma^2 = \gamma\mu + \lambda\mu &\le \lambda(\gamma+\mu) \\ \gamma[(\gamma+\mu)-\lambda] &\le 0 \Rightarrow (\gamma+\mu) - \lambda \le 0. \end{aligned}$$

Otherwise, there is no "meaningful" equilibrium point. If there is an equilibrium point, it must be a stable spiral.

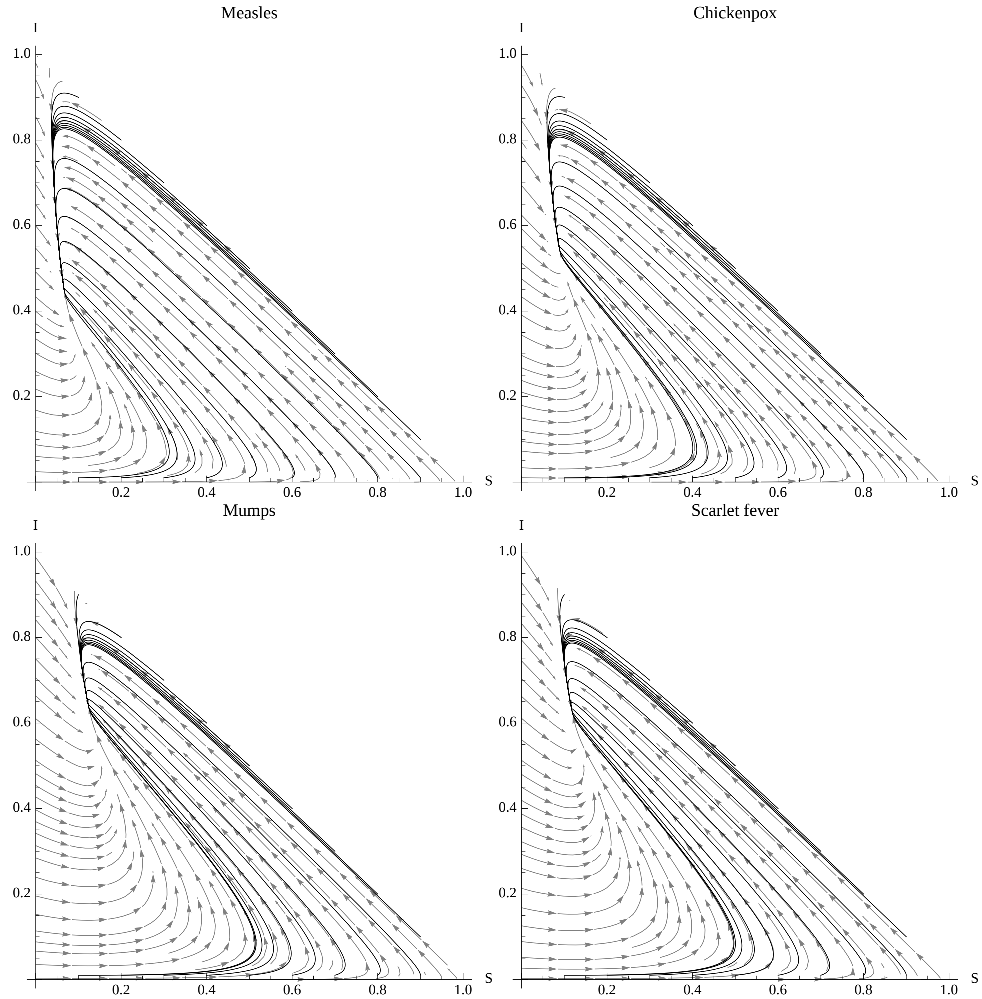
7.

Disease	λ
Measles	4.44
Chickenpox	2.69
Mumps	1.58
Scarlet Fever	1.7

8.

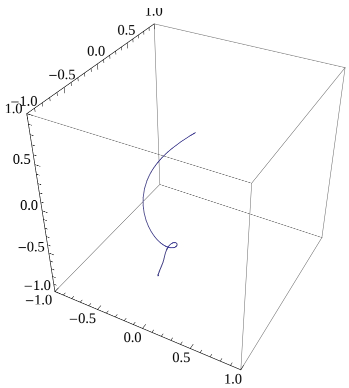
Disease	Minimum R-value
Measles	0.933186
Chickenpox	0.911505
Mumps	0.876544
Scarlet Fever	0.882354

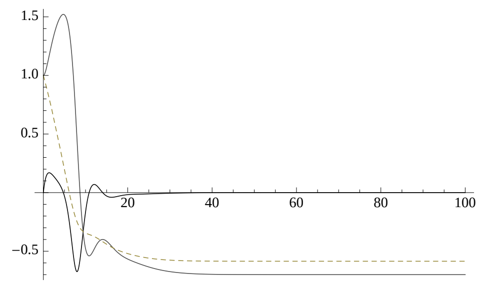
9.

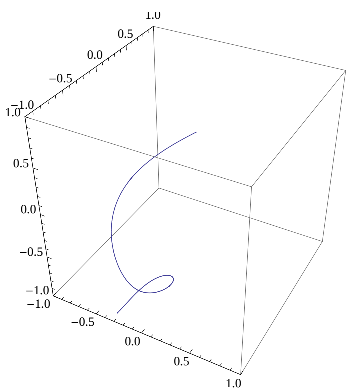


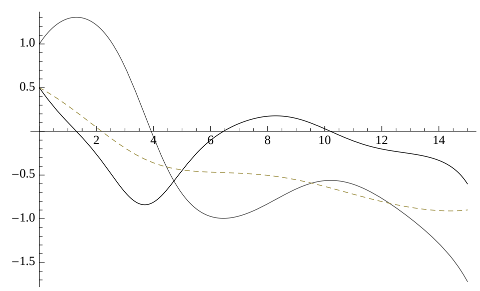
10. A variety of responses are possible here. Some responses might include discussions of immunization programs, isolation programs, or simply letting the epidemic run its course.

C. Fitzhugh-Nagumo Model









D. An Agricultural Model

1. Note that $\dfrac{dp}{dT} = \dfrac{dp}{dP}\dfrac{dP}{dt}\dfrac{dt}{dT} = rK\dfrac{dP}{dt}$ so that substitution yields $rK\dfrac{dP}{dt} = rKP\left(1 - \dfrac{KP}{K}\right) - aKP\dfrac{rS_\omega}{a} - bKP\dfrac{rS_c}{b}$. Similarly, we obtain the other ODEs.

2. If $P = 0$, $S_\omega = 0$ and $S_c = 0$, then all three equations are satisfied. If $P = 0$, we have the eq. point $(0, 0, 0)$ or we obtain $S_\omega = -\delta_\omega W_1 < 0$ and $S_c = -\delta_c V_1 < 0$. The latter case is not feasible because the spider populations are negative.

3. Substitution of $S_\omega = 0$ yields $1 - P - S_c = 0$ so that $\beta P - \delta_c - \dfrac{S_c}{V_1} = \beta P - \delta_c - \dfrac{1}{V_1}(1 - P) = 0$. Solving for P yields the desired result. Then use $S_c = 1 - P$ to solve for S_c. Parameters: $\beta > \delta_c$

4. Substitute $S_c = 0$ to obtain the desired results. Parameters: $\alpha > \delta_\omega$

5. Parameters: $\alpha\,(1 + \delta_c V_1) > \delta_\omega\,(1 + \beta V_1)$ and $\beta\,(1 + \delta_\omega W_1) > \delta_c\,(\alpha W_1 + 1)$.

6. The solutions asymptotically approach the equilibrium solution $(P, S_\omega, S_c) \approx (0.1, 0.35, 0.5)$ so that all three populations persist.

E. Modeling the Spread of Dengue in Indonesia

1. Equate all equations to zero and solve with a computer algebra system.

2. $J = \begin{pmatrix} -\delta_S M_{DS} - \mu_H & 0 & 0 & 0 & -\delta_S H_S \\ -\delta_S M_{DS} & -\gamma - \mu_H & 0 & 0 & \delta_S H_S \\ 0 & \gamma & -\mu_H & 0 & 0 \\ 0 & -\beta_S M_S & 0 & -\beta_S H_D - v_S & 0 \\ 0 & \beta_S M_S & 0 & \beta_S H_D & -v_S \end{pmatrix}$

3. The eigenvalues in this case are $\lambda \approx -0.68, -0.25, -0.1, -0.1, -0.018$, which are all negative. Therefore, E_0^* is stable. In this case, the eigenvalues are $\lambda \approx -0.62, -0.25, -0.1, -0.1, 0.02$. Since not all are negative, E_0^* is unstable. For the set of parameters when $\gamma = 0.35$, $R_0 \approx 0.89 < 1$. Stable as shown above. When $\gamma = 0.25$, $R_0 \approx 1.14 > 1$. Unstable as shown above.

4. The dengue-infected populations approach zero over time. Therefore, this preventive approach reduces the presence of the virus. In the simple model (without wolbachia), the infected populations approach a non-zero limit, so the virus persists.