Exercises 7.1

1. (a) $Q(t) = \frac{3\sqrt{5}}{5}e^{-5t/3}\sin\left(\frac{\sqrt{5}}{3}t\right)$, $I(t) = e^{-5t/3}\left(\cos\left(\frac{\sqrt{5}}{3}t\right) - \sqrt{5}\sin\left(\frac{\sqrt{5}}{3}t\right)\right)$;
(b) $Q(t) = \frac{1}{3}e^{-t} + \frac{1}{15}e^{-5t/3}\left(-5\cos\left(\frac{\sqrt{5}}{3}t\right) + 7\sqrt{5}\sin\left(\frac{\sqrt{5}}{3}t\right)\right)$, $I(t) = -\frac{1}{3}e^{-t} - \frac{2}{3}e^{-5t/3}\left(2\cos\left(\frac{\sqrt{5}}{3}t\right) - \sqrt{5}\sin\left(\frac{\sqrt{5}}{3}t\right)\right)$
3. (a) $Q(t) = 10^{-6}e^{-t}\cos(\sqrt{2}t)$
(b) $Q(t) = \frac{2}{9}e^{-t/2} + \frac{4\sqrt{2}}{9}e^{-t}\sin(\sqrt{2}t) - \frac{1999991}{9000000}e^{-t}\cos(\sqrt{2}t)$, $I_2(t) = \frac{8}{9}e^{-t/2} - \frac{1999991}{4500000\sqrt{2}}e^{-t}\sin(\sqrt{2}t) - \frac{8}{9}e^{-t}\cos(\sqrt{2}t)$
5. $Q(t) = 10^{-6}e^{-2t}(1+t)$, $I_2(t) = 10^{-6}te^{-2t}$ so $I(t) = -10^{-6}e^{-2t}(1+2t)$ and then $I_1(t) = -10^{-6}e^{-2t}(1+t)$
7. (a) $Q(t) = 10^{-6}e^{-t} - \frac{1}{2}\cdot 10^{-6}t^2e^{-t}$, $I_2(t) = 10^{-6}te^{-t}$, $I_3(t) = \frac{1}{2}\cdot 10^{-6}t^2e^{-t}$;
(b) $Q(t) = 10^{-6}e^{-t} + te^{-t} - \frac{1}{2}\cdot 10^{-6}t^2e^{-t}$, $I_2(t) = 10^{-6}te^{-t}$, $I_3(t) = -te^{-t} + \frac{1}{2}\cdot 10^{-6}t^2e^{-t}$
9. (a) $Q(t) = 90 - te^{-t} - 90e^{-t}$, $I_2(t) = e^{-t}$, $I_3(t) = -90 + te^{-t} + 90e^{-t}$; (b) $Q(t) = -te^{-t} + 45e^{-t} + 45\sin t - 45\cos t$, $I_2(t) = e^{-t}$, $I_3(t) = te^{-t} - 45e^{-t} - 45\sin t + 45\cos t$
11. (a) $Q(t) = 90 - \frac{4}{3}e^{-t/4} - \frac{266}{3}e^{-t}$, $I_2(t) = e^{-t/4}$, $I_3(t) = -90 + \frac{4}{3}e^{-t/4} + \frac{266}{3}e^{-t}$;
(b) $Q(t) = -45\cos t + 45\sin t - \frac{4}{3}e^{-t/4} + \frac{139}{3}e^{-t}$, $I_2(t) = e^{-t/4}$, $I_3(t) = 45\cos t - 45\sin t + \frac{4}{3}e^{-t/4} - \frac{139}{3}e^{-t}$
13. $x' = y$, $y' = -9x$, $\lambda = \pm 3i$; undamped
15. $x' = y$, $y' = -9x - 10y$, $\lambda_1 = -1$, $\lambda_2 = -9$; overdamped
17. $x' = y$, $y' = -50x - 10y$, $\lambda_{1,2} = -5 \pm 5i$; underdamped
19. $x' = y$, $y' = -25x - 10y$, $\lambda_{1,2} = -5$; critically damped
21. (13) $x(t) = \cos 3t$, $y(t) = -3\sin 3t$; (15) $x(t) = \frac{9}{8}e^{-t} - \frac{1}{8}e^{-9t}$, $y(t) = -\frac{9}{8}e^{-t} + \frac{9}{8}e^{-9t}$
23. $\lambda = (2m)^{-1}\left(-c \pm \sqrt{c^2 - 4km}\right)$, overdamped if $c^2 > 4km$; critically damped if $c^2 = 4km$; underdamped if $c^2 < 4km$
25. $\lim_{t\to\infty} Q(t)$ does not exist; $\lim_{t\to\infty} I(t)$ does not exist
27. $\lim_{t\to\infty} Q(t)$ does not exist; $\lim_{t\to\infty} I_2(t)$ does not exist
29. $\lim_{t\to\infty} Q(t) \approx 91$; $\lim_{t\to\infty} I_2(t) = 0$; $\lim_{t\to\infty} I_3(t) \approx -43.5$
31. $x(t) = \frac{1}{2}e^{-3t/2}((3e^t - 1)x_0 + 2(e^t - 1)y_0)$, $y(t) = -\frac{1}{4}e^{-3t/2}(3(e^t - 1)x_0 + 2(e^t - 3)y_0)$; stable node

Exercises 7.2

1. $x(t) = \frac{3}{2}(1 - 3e^{-t})$, $y(t) = \frac{3}{2}(1 + 3e^{-t})$, $\lim_{t\to\infty}(x(t), y(t)) = (3/2, 3/2)$
3. $x(t) = \frac{8}{3}(1 - e^{-3t/2})$, $y(t) = \frac{4}{3}(1 + 2e^{-3t/2})$, $\lim_{t\to\infty}(x(t), y(t)) = (8/3, 4/3)$
5. $x(t) = 1 + 3e^{-15t/4}$, $y(t) = 4 - 3e^{-15t/4}$, $\lim_{t\to\infty}(x(t), y(t)) = (1, 4)$
7. $x(t) = 10(1 - e^{-t/5})$, $y(t) = 10(1 - e^{-t/5}) - 2te^{-t/5}$, $\lim_{t\to\infty}(x(t), y(t)) = (10, 10)$, $x(t)$
9. $x(t) = 4e^{-t/5}$, $y(t) = 4e^{-t/5} + \frac{4}{5}te^{-t/5}$, $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$, $x(t)$
11. (a) $x(t) = 300(1 - e^{-t/20})$, $y(t) = 150(1 + e^{-t/10} - 2e^{-t/20})$, (b) $\lim_{t\to\infty}(x(t), y(t)) = (300, 150)$

13. $x(t) = \frac{325}{6} - \frac{125}{12}e^{-3t/25} - \frac{174}{5}e/^{-t/5}$, $y(t) = \frac{200}{3} + \frac{125}{6}e^{-3t/25} - \frac{175}{2}e^{-t/25}$; $\lim_{t\to\infty}(x(t), y(t)) = (325/6, 200/3)$; $x(t) = y(t)$ at $t \approx 29.1$ min.
15. System: $dx/dt = 2y - 2x + 1$, $dy/dt = x - 3y + 1$; $x(t) = \frac{5}{4} - \frac{23}{12}e^{-4t} + \frac{8}{3}e^{-t}$, $y(t) = \frac{3}{4} + \frac{23}{12}e^{-4t} + \frac{4}{3}e^{-t}$; $\lim_{t\to\infty}(x(t), y(t)) = (4/3, 3/4)$; maximum is $x \approx 4$ and $y \approx 1$
17. System: $dx/dt = -\frac{1}{10}x + 10$, $dy/dt = \frac{1}{10}x - \frac{1}{10}y$, $dz/dt = \frac{1}{10}y - \frac{1}{10}z$; $x(t) = 100(1-e^{-t/10})$, $y(t) = 10(10-10e^{-t/10}-te^{-t/10})$, $z(t) = 100-100e^{-t/10} - 10te^{-t/10} - \frac{1}{2}t^2e^{-t/10}$, $\lim_{t\to\infty}(x(t), y(t), z(t)) = (100, 100, 100)$
19. $x(t) = \frac{25}{3}e^{4t} + \frac{5}{3}e^{-2t}$, $y(t) = 25e^{4t} - 5e^{-2t}$
21. $x(t) = 15e^{2t} - 10e^{-t}$, $y(t) = 10e^{-t}$
23. $x(t) = -\frac{67}{7}e^{-10t} + \frac{171}{10}e^{7t} + \frac{663}{70}e^{-3t}$, $y(t) = 5e^{-10t} + \frac{171}{10}e^{7t} - \frac{221}{10}e^{-3t}$, $z(t) = \frac{52}{7}e^{-10t} + \frac{171}{10}e^{7t} + \frac{663}{70}e^{-3t}$; $x(1) \approx z(1) \approx 18752.89$
25. $x(t) = \frac{43}{7} + \frac{46}{21}e^{7t} - \frac{1}{3}e^{-8t}$, $y(z) = \frac{4}{3}e^{7t} + \frac{2}{3}e^{-8t}$, $z(t) = -\frac{43}{7} + \frac{136}{21}e^{7t} - \frac{1}{3}e^{-8t}$; $z(1) \approx 7095.86$
29. $x(t) = x_0e^{-at}$, $y(t) = y_0e^{-at} + ax_0te^{-at}$, $z(t) = x_0 + y_0 + z_0 - (x_0 + y_0)e^{-at} - ax_0te^{-at}$; $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$, $\lim_{t\to\infty} z(t) = x_0 + y_0 + z_0$
33. $x(t) = 7e^{-6t}$, $y(t) = \frac{47}{5}e^{-t} - \frac{42}{5}e^{-6t}$, $z(t) = 16 - \frac{47}{5}e^{-t} + \frac{7}{5}e^{-6t}$; $\lim_{t\to\infty} z(t) = 16$
35. $x(t) = e^{-4t}$, $y(t) = 3e^{-2t} - 2e^{-4t}$, $z(t) = 3 - 3e^{-2t} + e^{-4t}$; $\lim_{t\to\infty} z(t) = 3$
37. $x(t) = 2$, $y(t) = 2 - e^{-t}$, $z(t) = t + 1 + e^{-t}$; $\lim_{t\to\infty} x * t) = \lim_{t\to\infty} y(t) = 2$, $\lim_{t\to\infty} z(t) = \infty$
39. $x(t) = 10 - 2e^{-t}$, $y(t) = 10 - 8e^{-t} - 2te^{-t}$, $z(t) = 10t - 8 + 10e^{-t} + 2te^{-t}$; $\lim_{t\to\infty} x * t) = \lim_{t\to\infty} y(t) = 10$, $\lim_{t\to\infty} z(t) = \infty$
41. $x(t) = (v_1+v_2)^{-1}\left[bV_1\left(1 - \exp(-Pt(v_1 + v_2)/(V_2V_2))\right) + a\left(V_1 + V_2\exp(-Pt(v_1 + v_2)/(V\right.\right.$
$y(t) = (v_1+v_2)^{-1}\left[aV_2\left(1 - \exp(-Pt(v_1 + v_2)/(V_2V_2))\right) + b\left(V_2 + V_1\exp(-Pt(v_1 + v_2)/(V_2V_2\right.\right.$
(a) $\lim_{t\to\infty} x(t) = (a + b)V_1/(V_1 + V_2)$, $\lim_{t\to\infty} y(t) = (a + b)V_2/(V_1 + V_2)$ (b) $V_1 > V_2$, $V_1 = V_2$ (c) $bV_1 - aV_2 > 0$, $aV_2 - bV_1 > 0$, no
43. The problem can be solved as a system or the system can be written as a second-order ODE, where $x'' = (a_1 - a_2)x' + b_1y'$ (derivative of first equation) or $y' = 1/b_1\,(x'' - (a_1 - a_2)x')$. Substitution of y' and $y = 1/b_1\,(x' - (a_1 - a_2)x)$ into the second equation and simplification yields $x'' - [(a_1 - a_2) + (b_1 - b_2)]x' + [(b_1 - b_2)(a_1 - a_2) - a_2b_1]x = 0$.

We consider the characteristics of this equation (same as the eigenvalues of the system).

Periodic if $a_1 + b_1 = a_2 + b_2$ and $a_1b_1 - 2a_2b_1 - a_1b_2 + a_2b_2 > 0$. For example, if $a_1 = a_2 = -1$, $b_1 = b_2 = 1$, $x_0 = 1$, and $y_0 = 0$, then $x(t) = \cos t$, $y(t) = -\sin t$.

Exponential decay if $b_2 - b_1 > a_1 - a_2$ and $(b_1 - b_2)(a_1 - a_2) - a_2b_1 \geq 0$. For example, if $a_1 = -1$, $a_2 = 0$, $b_1 = b_2 = 1$, $x_0 = 1$, $y_0 = 1$, then $x(t) = e^{-t}$, $y(t) = 0$.

Exponential growth if $a_1 - a_2 > b_2 - b_1$ (will have at least one positive eigenvalue). For example, if $a_1 = 2$, $a_2 = 0$, $b_1 = b_1 = 1$, $x_0 = 1$, $y_0 = 0$, then $x(t) = e^{2t}$, $y(t) = 0$.

Exercises 7.3

1. $-a\ln|y| + by - c\ln|x| + dx = C$
3. (a) $(2k, k)$ stable spiral; (b) $(2k, k)$ center
5. $dx/dt = y$, $dy/dt = -\sin x$, $(k\pi, 0)$, $k = 0, \pm 1, \pm 2, \ldots$
7. $dx/dt = y$, $dy/dt = -k^2 x$, $dy/dx = (dy/dt)/(dx/dt) = -k^2 x/y$; $\frac{1}{2}y^2 + \frac{1}{2}k^2 x^2 = C$; $(0, 0)$ center
9. $dx/dt = y$, $dy/dt = -gM_1 x^{-1} + gM_2(R - x)^{-2}$;
$(x_0, y_0) = \left(R(M_1 + \sqrt{M_1 M_2})/(M_1 - M_2), 0\right) = \left(R\sqrt{M_1}/(\sqrt{M_1} - \sqrt{M_2}), 0\right)$; saddle
11. If $dx/dt = y - F(x) = y - \int_0^x f(u)\,du$ and $dy/dt = -g(x)$, then $d^2x/dt^2 = dy/dt - f(x)\,dx/dt = -g(x) - f(x)\,dx/dt$, which is equivalent to $d^2x/dt^2 + f(x)\,dx/dt + g(x) = 0$.
13. In each case, no limit cycle in the xy-plane: (a) $f_x(x, y) + g_y(x, y) = 3x^2 + 1 + x^2 = 4x^2 + 1 > 0$; (b) $f_x(x, y) + g_y(x, y) = -1 - 1 = -2 < 0$; (c) $f_x(x, y) + g_y(x, y) = y^2 + 8 > 0$.
15. $du/d\theta = y$, $dy/d\theta = \alpha - u = ku^2$; $y = 0$ and $\alpha - u - ku^2 = 0$ so $u = \frac{1}{2k}(1 \pm \sqrt{1 - 4k\alpha})$. We choose $u = \frac{1}{2k}(1 - \sqrt{1 - 4k\alpha})$ because it is closer to $u = 0$ than $u = \frac{1}{2k}(1 + \sqrt{1 - 4k\alpha})$, and we are considering a small change in the orbit. The eigenvalues of $J\left(\frac{1}{2k}(1 - \sqrt{1 - 4k\alpha}), 0\right) = \begin{pmatrix} 0 & 1 \\ -\sqrt{1 - 4k\alpha} & 0 \end{pmatrix}$ satisfy $\lambda^2 + \sqrt{1 - 4k\alpha} = 0$; center.

1. Differentiating $\frac{1}{2}m(x)(dx/dt)^2 + V(x) = E$ with respect to t yields $\frac{1}{2}m'(x)(dx/dt)^3 + m(x)\,dx/dt\,d^2x/dt^2 + V'(x)\,dx/dt = dx/dt\,\left[\frac{1}{2}m'(x)(dx/dt)^2 + m(x)\,d^2x/dt^2 + V'(x)\right] = 0$, so

$$\begin{aligned} \frac{1}{2}m'(x)\left(\frac{dx}{dt}\right)^2 + m(x)\frac{d^2x}{dt^2} + V'(x) &= 0 \\ \frac{1}{\sqrt{m(x)}}\left[\frac{1}{2}m'(x)\left(\frac{dx}{dt}\right)^2 + m(x)\frac{d^2x}{dt^2} + V'(x)\right] &= 0 \\ \frac{1}{2}\frac{m'(x)}{\sqrt{m(x)}}\left(\frac{dx}{dt}\right)^2 + \sqrt{m(x)}\frac{d^2x}{dt^2} + \frac{V'(x)}{\sqrt{m(x)}} &= 0 \\ \frac{d}{dt}\left[\sqrt{m(x)}\frac{dx}{dt}\right] + \frac{V'(x)}{\sqrt{m(x)}} &= 0. \end{aligned}$$

Notice that $du/dx = \sqrt{m(x)}$, so the equation is

$$\frac{d}{dt}\left[\frac{du}{dx}\frac{dx}{dt}\right] + \frac{V'(x)}{\sqrt{m(x)}} = 0.$$

Notice also that $du/dt = (du/dx)\,(dx/dt)$, so we have

$$\frac{d}{dt}\left(\frac{du}{dt}\right) + \frac{V'(x)}{\sqrt{m(x)}} = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} + + \frac{V'(x)}{\sqrt{m(x)}} = 0.$$

19. paths; circles; equilibrium point; center; agree
21. (a) center if $x = 1$; saddle if $x = -1$; (b) saddle if $x = 0$; center if $x = 1$, $x = -1$
23. $d\theta/dt = y$, $dy/dt = -FR\,\text{sgn}(y)/I$; equilibrium points: $(\theta, 0)$; if $d\theta/dt > 0$, then we integrate $I\dfrac{d\theta}{dt}\dfrac{d}{d\theta}\left(\dfrac{d\theta}{dt}\right) = -FR$ with respect to θ to obtain the parabolas $\dfrac{1}{2}I\left(\dfrac{d\theta}{dt}\right)^2 = -FR\theta + C$, $d\theta/dt > 0$. Similar calculations follow for $d\theta/dt < 0$.
25. (a) With $' = d/dt$, $(x^2 + y^2 + x^2)' = 2xx' + 2yy' + 2zz'$ and substitute for x', y', and z', where $x' = y$, $y' = -x$ on $x^2 + y^2 + z^2 = 1$. (b) $z(t) = C_1$, so $z(t) = z_0$. (c) Use the change of variables in Exercise 10.
26. (a) $(-1.44225, -0.44225)$; $J(x, y) = \begin{pmatrix} 1 - x^2 & -1 \\ c & -bc \end{pmatrix}$; eigenvalues of $J(-1.44225, -0.44225$ are $\lambda_{1,2} = -0.0977925 \pm 0.431302i$; spiral point, asymptotically stable
27. Use a numerical solver with initial guess $x_0 = 1$ and graph the solution. Observe the graph to see that $x_0 = 2$.

Chapter 7 Review Exercises

1. (a) $Q(t) = \frac{1}{500000}\left(\frac{3}{2}e^{-2t}3 - e^{-t}\right)$, $I(t) = \frac{1}{500000}\left(-e^{-2t}3 + e^{-t}\right)$; (b) $Q(t) = -\frac{4500001}{500000}e^{-t} - 3te^{-t} + \frac{9000003}{1000000}e^{-2t/3}$, $I(t) = \frac{3000001}{500000}e^{-t} + 3te^{-t} - \frac{3000001}{1000000}e^{-2t/3}$
3. (a) $Q(t) = \frac{1}{1000000}e^{-t}\cos t$, $I_2(t) = \frac{1}{1000000}e^{-t}\sin t$; (b) $Q(t) = 60 + 60e^{-t} - \frac{59999999}{1000000}e^{-t}\cos t$, $I_2(t) = 60 - \frac{59999999}{1000000}e^{-t}\sin t - 60e^{-t}\cos t$
5. (a) $x(t) = e^{-t} + te^{-t}$, $y(t) = -te^{-t}$ (critically damped); (b) $x(t) = \cos 2t$, $y(t) = -2\sin 2t$ (undamped)
7. $x(t) = \frac{5}{2}(3 - e^{-t})$, $y(t) = \frac{5}{2}(3 + e^{-t})$
9. $x(t) = 15e^{3t} + 5e^{t}$, $y(t) = 15e^{3t} - 5e^{t}$
11. $x(t) = -\frac{5}{3}e^{-5t} - \frac{128}{51}e^{-14t} + \frac{139}{17}e^{3t}$, $y(t) = \frac{7}{6}e^{-5t} - \frac{64}{51}e^{-14t} + \frac{139}{34}e^{3t}$, $z(t) = -\frac{1}{6}e^{-5t} + \frac{208}{51}e^{-14t} + \frac{139}{34}e^{3t}$
12. $J(x, y) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos x & -b \end{pmatrix}$; $J(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}n\pi & -b \end{pmatrix}$; if n is odd: $\lambda_{1,2} = -\frac{1}{2}(-b \pm \sqrt{b^2 + 4g/L})$, $\lambda_2 < 0 < \lambda_1$ real; if n is even $\lambda_{1,2} = -\frac{1}{2}(-b \pm \sqrt{b^2 - 4g/L})$, complex conjugate pair or two negative real
13. $(0, 0)$; $\lambda_1 - a > 0$, $\lambda_2 = -c < 0$, saddle point, unstable; $(a/k, 0)$, $\lambda_1 = -a < 0$, $\lambda_2 = -c + ad/k = d(a/k - c/d) > 0$, saddle point, unstable; $(c/d, (ad - ck/(bd))$, $\lambda_{1,2} = \frac{1}{2d}(-ck \pm \sqrt{-4acd^2 + 4c^2dk + c^2k^2})$, where $4dc(cd - ad) < 0$, so $-ck - \sqrt{-4acd^2 + 4c^2dk + c^2k^2} < 0$ if $-4acd^2 + 4c^2dk + c^2k^2 > 0$ (improper node) and $\lambda_{1,2}$ has negative real part if $-4acd^2 + 4c^2dk + c^2k^2 < 0$ (spiral point); asymptotically stable.
w 18. Solve $\frac{dr}{dt} = -kr$, $r(0) = r_0$ for r to yield $r(t) = r_0e^{-kt}$. Substitute and then solve for V using $r(0) = r_0$ and $V_0 = \ln e^{V_0}$.

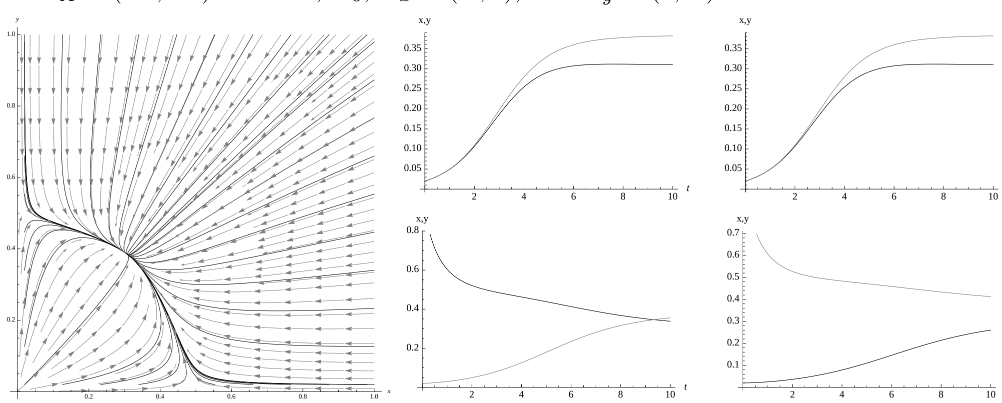
Differential Equations at Work

A. Competing Species

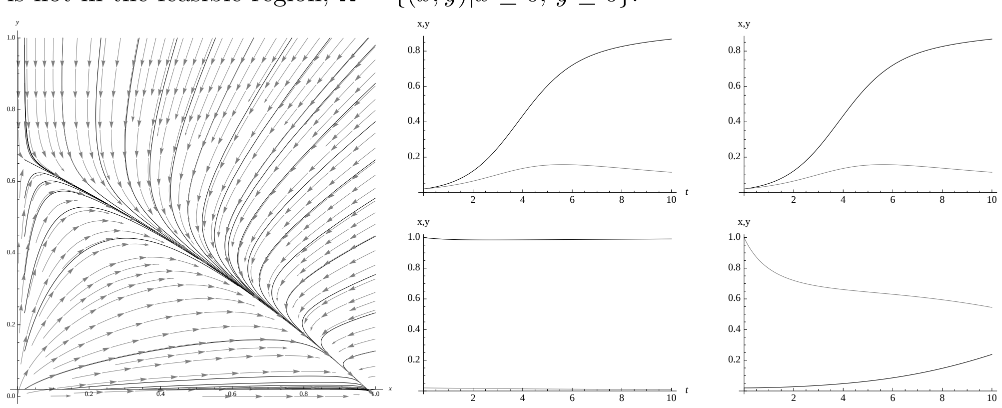
1. The equilibrium points are found by solving $x(a - b_1x - b_2y) = 0$, $y(c - d_1x - d_2y) = 0$ for x and y: $E_0 = (0,0)$, $E_x = (a/b_1, 0)$, $E_y = (0, c/d_2)$, and $E_A = ((b_2c - ad_2)/(b_2d_1 - b_1d_2), (b_1c - ad_1)/(b_1d_2 - b_2d_1))$. The Jacobian is $\mathbf{J} = \begin{pmatrix} a - 2b_1x - b_2y & -b_2x \\ -d_1y & -d_1x - 2d_2y \end{pmatrix}$. The eigenvalues of $\mathbf{J}(E_0)$ are a and c so E_0 is always unstable. At E_x the Jacobian has eigenvalues $-a$ and $c - ad_1/b_1$; at E_y the Jacobian has eigenvalues $-c$ and $a - b_2c/d_2$; and at E_A the Jacobian has eigenvalues

$$\frac{1}{2b_2d_1 - 2b_1d_2}\Bigg(- b_1b_2c + ab_2d_2 + b_1cd_2 - ad_1d_2 \pm \sqrt{4(b_1c - ad_1)(b_2c - ad_2)(b_1d_2 - b_2d_1) + (ad_1d_2 + b_1(b_2c - (a+c)d_2))^2}\Bigg)$$

1. $E_A = (.31, .38)$ is stable, E_0, $E_x = (.5, 0)$, and $E_y = (0, .5)$ are unstable



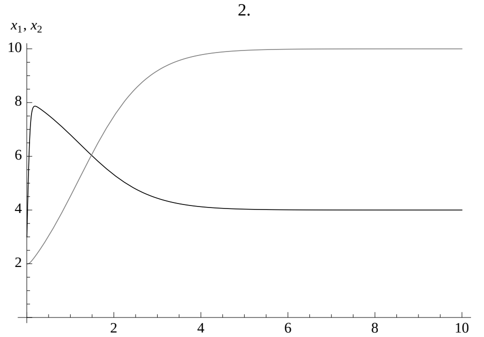
2. E_0 and $E_y = (0, 0.67)$ are unstable. $E_x = (1, 0)$ is stable. $E_A = (1.32, -.32)$ is not in the feasible region, $\Omega = \{(x,y) | x \geq 0,\ y \geq 0\}$.



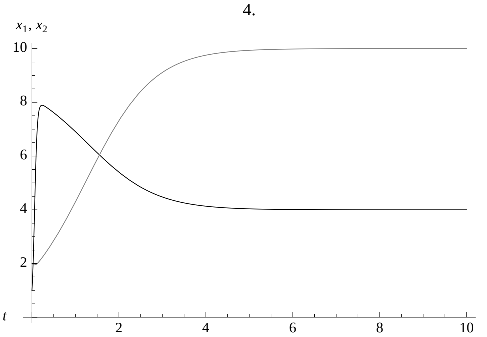
B. Food Chains

1. To solve $ax_1 - bx_1^2 - cx_1x_2 = 0$, $x_2(-d + ex_1) = 0$ for x_1 and x_2 observe that from the second equation $x_2 = 0$ or $x_1 = d/e$. If $x_2 = 0$, then $x_1(a - bx_1) = 0$ resulting in rest points $(0, 0)$ and $(a/b, 0)$. If $x_1 = d/e$ then, $x_1(a - bx_1 - c \cdot \frac{d}{e}) = 0$ so $x_1 = 0$ or $bx_1 = (ae - cd)/e$ resulting in $x_1 = (ae - cd)/(ce)$.
2. $(0, 0)$, $(4, 10)$, $(9, 0)$
4. $(0, 0)$, $(4, 10)$, $(9, 0)$
5. $(0, .29)$, $(8.71, 0$, $(4, 8.75)$
7. $(.59, 0)$, $(8.41, 0)$, $(4, 7.5)$
8. $(.76, 0)$, $(8.24, 0)$, $(4, 6.88)$

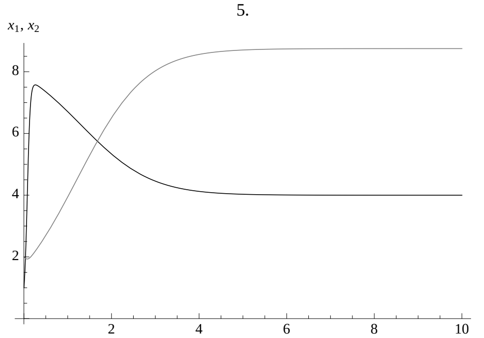
2.



4.



5.



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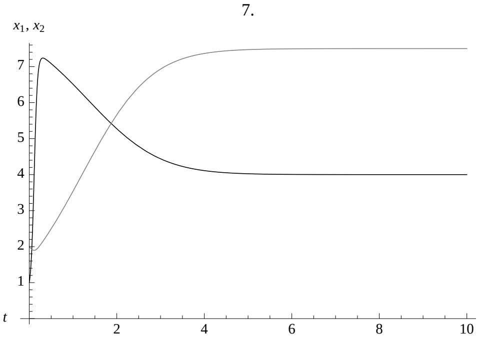

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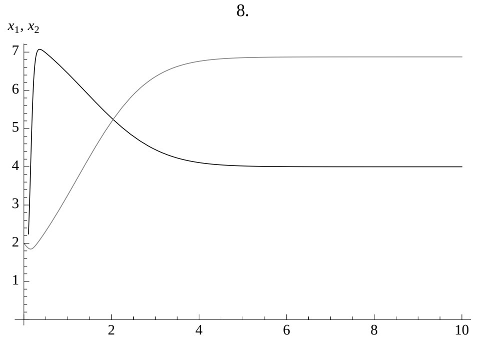

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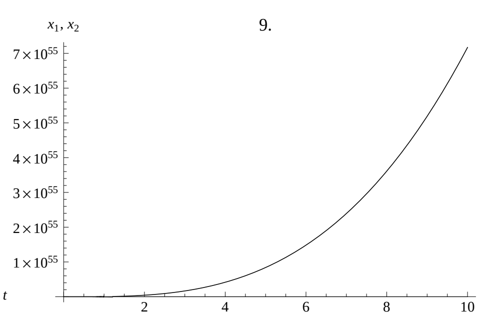

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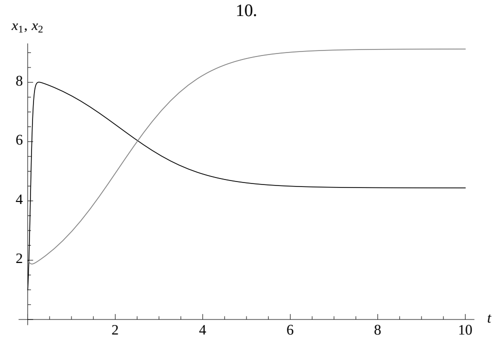

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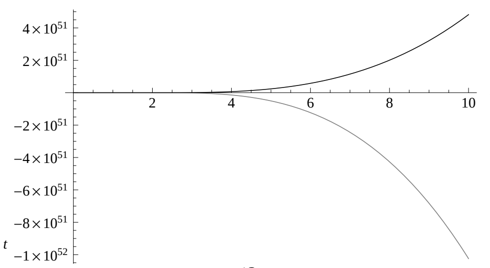

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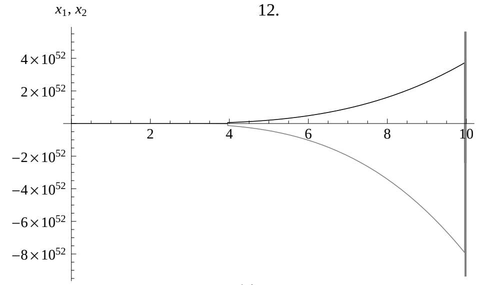

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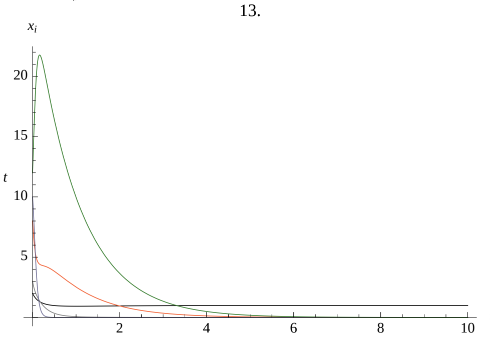

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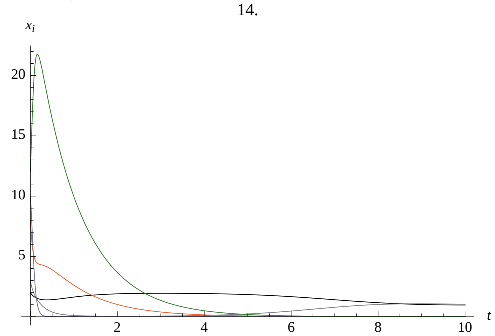

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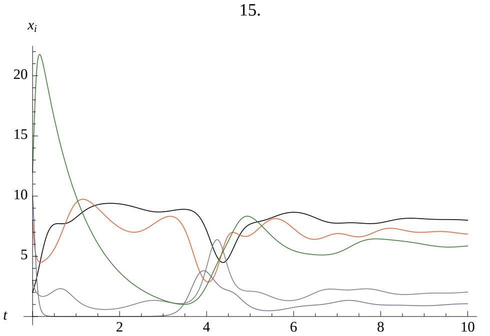

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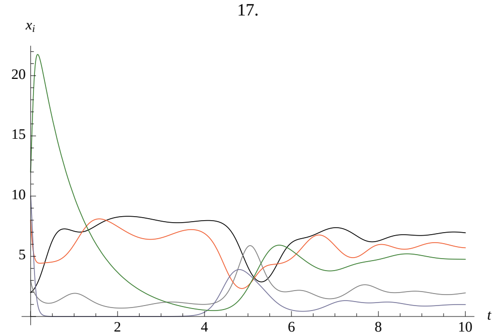
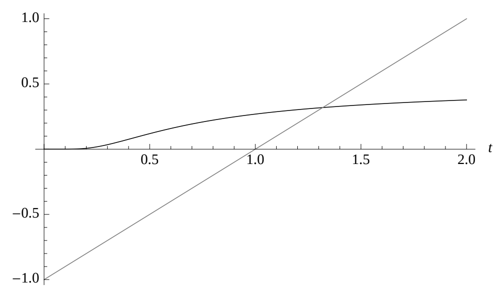
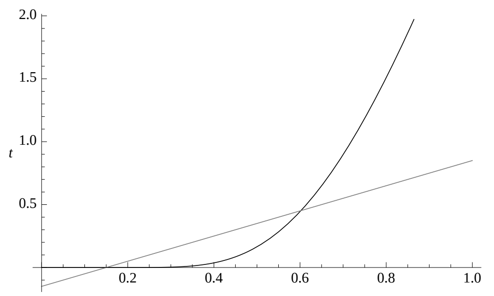


C. Chemical Reactor

1. $(Vc)' = Vc'$ because V is assumed to be constant. Divide both sides of the equation by V.
2. Divide both sides by VCp.
4. $t \approx 1.31905$
5. $t \approx 0.15$, $t \approx 0.602161$





E. The Rössler System and Attractor

1. $0 < a < 2$; since the eigenvalues have a positive real component, the origin is unstable with an outward spiral.

2. $\left(\frac{1}{2}\left(c+\sqrt{c^2-4ab}\right), \frac{1}{2a}\left(-c-\sqrt{c^2-4ab}\right), \frac{1}{2a}\left(c+\sqrt{c^2-4ab}\right)\right)$ and $\left(\frac{1}{2}\left(c-\sqrt{c^2-4a\right.\right.$

   The Jacobian is $\begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{pmatrix}$. For the given parameter values, the equilibrium solutions are $(5.69297, -28.4649, 28.4649)$ and $(0.0070262, -0.035131, 0.035$ The eigenvalues for the first eq. solution are $\lambda_{1,2} = -4.59607 \times 10^{-6} \pm 5.42803i$ and $\lambda_3 = 0.192983$. Those for the second are $\lambda_{1,2} = 0.0970009 \pm 0.995193i$ and $\lambda_3 = -5.68698$. The second equilibrium solution is more central to the attractor. It has a strong level of attraction due to the negative eigenvalue and a low level of repulsion because of the small magnitude of the real component. The first equilibrium has little influence. It primarily pushes the system towards the central equilibrium solution. NOTE: This may be one that we would like to make a MANIPULATE command to show a nice 3D view of the attractor.

F. Cell Dynamics in Colon Cancer

1. Solution to Malthus (exponential growth/decay) model.

2. $\frac{dN_1}{dt} - \beta N_1 = N_{00}e^{\alpha t}$. Solve as a first-order linear IVP to obtain $N_1(t) = \frac{N_{00}}{\alpha-\beta}e^{\alpha t} + Ce^{\beta t}$. Applying the IC yields $N_{10} = \frac{N_{00}}{\alpha-\beta} + C$ so that $N_1(t) = \frac{N_{00}}{\alpha-\beta}e^{\alpha t} + \left(N_{10} - \frac{N_{00}}{\alpha-\beta}\right)e^{\beta t}$.

3. Solve $\frac{dN_2}{dt} + \gamma N_2 = \beta_2 N_1$ as a first-order linear IVP with $N_2(0) = N_{20}$ to yield $N_2(t) = \frac{\beta_2 N_{00}}{(\alpha-\beta)(\alpha+\gamma)} e^{\alpha t} + \frac{\beta_2}{\beta+\gamma}\left(N_{10} - \frac{N_{00}}{\alpha-\beta}\right) e^{\beta t} + Ce^{-\gamma t}$ where $C = N_{20} - \frac{\beta_2 N_{00}}{\alpha-\beta} \frac{\beta-\alpha}{(\alpha+\gamma)(\beta+\gamma)} - \frac{\beta_2 N_{10}}{\beta+\gamma}$.
4. Substitute $\alpha_1 + \alpha_2 = 1 - \alpha_3$ into $\alpha = 0$.
5. Take the limit as $t \to \infty$ of the expressions for $N_1(t)$ and $N_2(t)$ assuming that $\alpha = 0$, $\beta < 0$ and $\gamma > 0$.