Exercises 8.1

1. We use integration by parts once ($\int ue^u\,du = e^u(u-1)+C$) to evaluate the improper integral: $\mathcal{L}\{21t\} = \int_0^\infty e^{-st}\cdot 21t\,dt = \lim_{M\to\infty}\left(-\frac{21}{s^2}e^{-st}(st+1)\right)\Big|_0^M = 21\lim_{M\to\infty}\frac{1}{s^2}(1-e^{-Ms}(Ms+1)) = 21s^{-2}$

3. $\mathcal{L}\{2e^t\} = \int_0^\infty e^{-st}\cdot 2e^t\,dt = 2\int_0^\infty e^{t-st}\,dt = \lim_{M\to\infty}\frac{2}{1-s}e^{(1-s)t}\Big|_0^M = \frac{2}{s-1}$

5. Use integration by parts twice or use the formula $\int e^u\sin u\,du = \frac{1}{2}e^u(-\cos u + \sin u)$. Then, $\mathcal{L}\{2\sin 2t\} = \int_0^\infty e^{-st}\cdot 2\sin 2t\,dt = \lim_{M\to\infty} -\frac{2}{s^2+4}e^{-st}(2\cos 2t + s\sin 2t)\Big|_0^M = \frac{4}{s^2+4}$

7. $\mathcal{L}\{f(t)\} = \int_1^\infty e^{-st}\cdot 1\,dt = \lim_{M\to\infty} -\frac{1}{s}e^{-st}\Big|_1^M = e^{-s}s^{-1}$

9. $\mathcal{L}\{f(t)\} = \int_0^{\pi/2} e^{-st}\cdot\cos t\,dt = \frac{1}{s^2+1}e^{-st}(-s\cos t + \sin t)\Big|_0^{\pi/2} = \frac{s+e^{-\pi s/2}}{s^2+1}$

11. $\mathcal{L}\{f(t)\} = \int_0^3 e^{-st}\cdot(3-t)\,dt = \frac{1}{s^2}e^{-st}(1+s(t-3))\Big|_0^3 = (3s-1+e^{-3s})s^{-2}$

13. $\mathcal{L}\{f(t)\} = \int_0^{10} e^{-st}\cdot 1\,dt + \int_{10}^\infty e^{-st}\cdot -1\,dt = -\frac{1}{s}e^{-st}\Big|_0^{10} + \lim_{M\to\infty}\frac{1}{s}e^{-st}\Big|_{10}^M = (1-2e^{-10s})s^{-1}$

15. $\mathcal{L}\{\sin kt\} = \int_0^\infty e^{-st}\sin kt\,dt = \lim_{M\to\infty} -\frac{1}{s^2+k^2}e^{-st}(k\cos kt + s\sin kt)\Big|_0^M = k/(s^2+k^2)$

21. $(s-1)/[(s-1)^2+9]$

23. $5/[(s+1)^2+25]$

25. $16(2s+1)^{-3}$

27. $-18(s-3)^{-1}$

29. $\frac{s^2-25}{(s^2+25)^2}$

31. $\frac{1}{s}+\frac{5}{s^2+25}$

33. $\frac{s^2+2s-3}{(s^2+2s+5)^2}$

35. $(s+1)^{-8}$

37. $s^{-6}-s^{-4}+s^{-2}$

39. $\frac{24(5s^4-10s^2+1)}{(s^2+1)^5}$

41. $\frac{6s}{(s^2-9)^2}$

43. $\frac{14s}{(s^2+49)^2}$

45. $\dfrac{2(s^3+27s)}{(s^2-9)^3}$

47. $\dfrac{2}{s^2+2s-3}$

49. $\dfrac{s+2}{(s+2)^2+16}$

51. $\dfrac{4}{(s+5)^2+16}$

55. (a) $\dfrac{s^2+2k^2}{s^3+4k^2s}$ (b) $\dfrac{2k^2}{s^3+4k^2s}$

61. No to all.

Exercises 8.2

1. $\frac{1}{5040}t^7$
3. e^{-5t}
5. $\frac{1}{2}t^2e^{-6t}$
7. $\frac{1}{3}\sin 3t$
9. $8e^{-8t}-7e^{-7t}$
11. $\frac{2}{9}e^{-2t}+\frac{7}{9}e^{7t}$
13. $\frac{1}{2}+\frac{1}{2}e^{4t}$
15. $-\frac{1}{8}e^{-2t}+\frac{1}{8}e^{6t}$
17. $2-\sin t$
19. $2+e^{2t}$
21. $t+\cos 2t$
23. $1+t+t^2$
25. $t-\frac{1}{2}t^2+\frac{1}{3}t^3-\frac{1}{4}t^4$

27. (a) $\mathcal{L}\{a\sin bt - b\sin at\} = \dfrac{a^3b-ab^3}{(s^2+a^2)(s^2+b^2)}$ and $\mathcal{L}\{\cos bt - \cos at\} = \dfrac{a^2s-b^2s}{(s^2+a^2)(s^2+b^2)}$; (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \dfrac{a\sin bt - b\sin at}{ab(a^2-b^2)}$ and $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)(s^2}\right.$ $\dfrac{\cos bt-\cos at}{a^2-b^2}$.

29. (b) $\mathcal{L}^{-1}\left\{\dfrac{k}{s(s^2+k^2)}\right\} = \int_0^t \sin k\tau\, d\tau = \dfrac{1}{k}(1-\cos kt)$ and $\mathcal{L}^{-1}\left\{\dfrac{k}{s^2(s^2+k^2)}\right\} = \int_0^t\int_0^\tau \sin k\lambda\, d\lambda\, d\tau = \dfrac{1}{k^2}(kt-\sin kt)$.

Exercises 8.3

For each problem, let $\mathcal{L}\{y(t)\} = Y(s)$. Apply the Laplace transform to both sides of the equation, apply the initial conditions and then solve for $Y(S)$. Find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ by taking the inverse Laplace transform.

1. $24Y(s)+s^2Y(s)+11(sY(s)-y(0))-sy(0)-y'(0)$. Application of the initial conditions yields $(s^2+11s+24)Y(s)+s+11=0$ so $Y(s)=\dfrac{-s-11}{s^2+11s+24}$. Taking the inverse Laplace transform results in $y=\frac{1}{5}e^{-8t}(3-8e^{5t})$

3. $(s^2+3s-10)Y(s)-(s+3)y(0)-y'(0)=0$; $(s^2+3s-10)Y(s)+s+2=0$;
$Y(s)=\dfrac{-s-2}{s^2+3s-10}$; $y=-\frac{1}{7}e^{-5t}(3+4e^{7t})$

5. $s(s-1)Y(s)+y(0)-sy(0)-y'(0)=1/(s-1)$; $s((s-1)Y(s)-1/(s-1))=0$;
$Y(s)=\dfrac{-s-2}{s^2+3s-10}$; $y=te^t$

7. $\left(s^3+s\right)Y(s)-\left(s^2+1\right)y(0)-sy'(0)+\dfrac{1}{1-s}-y''(0)=0$; $\left(s^3+s\right)Y+\dfrac{1}{1-s}=0$; $Y(s)=\dfrac{1}{s\left(s^3-s^2+s-1\right)}$; $y=\frac{1}{2}(-2+e^t+\cos t-\sin t)$

9. $s^4y(0)+s^3y'(0)+s^2\left(y''(0)+y(0)\right)-\left(s^5+s^3\right)Y(s)+s\left(y^{(3)}(0)+y'(0)\right)+1=0$;
$\left(s^5+s^3\right)Y(s)-1=0$; $Y(s)=\dfrac{1}{s^5+s^3}$; $y=\frac{1}{2}(2\cos t+t^2-2)$

11. $-(s+2)y(0)+(s+1)^2Y(s)-\dfrac{1}{s+1}-y'(0)=0$; $(s+1)^2Y(s)-\dfrac{1}{s+1}=0$;
$Y(s)=\dfrac{1}{(s+1)^3}$; $y=\frac{1}{2}t^2e^{-t}$

13. $s^2(-y(0))+\left(s^2-1\right)sY(s)-sy'(0)+\dfrac{1}{1-s}-\dfrac{1}{s+1}-y''(0)+y(0)=0$;
$\left(s^2-1\right)^2Y(s)-2=0$; $Y(s)=\dfrac{2}{\left(s^2-1\right)^2}$; $y=\frac{1}{2}e^{-t}(1-e^{2t}+t+te^{2t})$

15. $18s^3y(0)+9s^2\left(2y'(0)+3y(0)\right)-\left(18s^3+27s^2+13s+2\right)sY(s)+s\left(18y''(0)+27y'(0)+13y(0)\right)+1=0$; $s\left(18s^3+27s^2+13s+2\right)Y(s)-1=0$; $Y(s)=\dfrac{1}{s\left(18s^3+27s^2+13s+2\right)}$;
$y=\frac{1}{2}-\frac{3}{2}e^{-2t/3}+4e^{-t/2}-3e^{-t/3}$

17. $s^2Y(s)-9Y(s)-3sY'(s)=\dfrac{3}{s}$; $Y(s)=3s^{-3}+Cs^{-3}e^{s^2/6}$. Because $\lim_{s\to\infty}Y(s)=0$, $C=0$ so $Y(s)=3s^{-3}$ and $y=\frac{3}{2}t^2$.

19. Taking the Laplace transform of $y''+ty'-2y=0$ and applying the initial conditions gives us $s^2Y(s)-s-3Y(s)-sY'(s)=0$ which has solution $Y(s)=2s^{-3}+s^{-1}+Cs^{-3}e^{s^2 2}$. Because $\lim_{s\to\infty}Y(s)=0$, $C=0$ so $Y(s)=2s^{-3}+s^{-1}$ and hence $y=t^2+1$

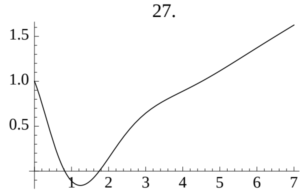
21. $s^2Y(s)-6Y(s)-2sY'(s)=\dfrac{2}{s}$ has solution $Y(s)=2s^{-3}+Cs^{-3}e^{s^2/4}$. Because $\lim_{s\to\infty}Y(s)=0$, $C=0$ so $Y(s)=2s^{-3}$ and hence $y=t^2$

23. $Y'(s)+s^2Y(s)=s$, $-(s^2+2s)Y'(s)+(s^2+n(n+1)Y(s)=1$, one

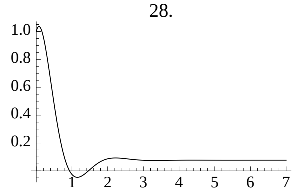
27. $y\left(t\right)=-1/24\,e^{-t}\sin\left(\sqrt{3}t\right)\sqrt{3}+\frac{35}{24}\,e^{-t}\cos\left(\sqrt{3}t\right)+1/24\,\left(-8+\left(6\,t-3\right)e^{t}\right)e^{-t}$

29. $y\left(t\right)=e^{-t}-1/6\,te^{-t}+1/6\,\left(-3\,t^2+t+2\,t^3\right)e^{-t}$

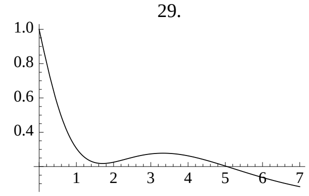
27.



28.



29.



Exercises 8.4

Throughout, our notation is that $\mathcal{L}\{f(t)\} = F(S)$; $\mathcal{L}^{-1}\{Y(s)\} = y(t)$ and $\mathcal{L}^{-1}\{X(s)\} = x(t)$.

1. $-28e^{-s}s^{-1}$
3. $(3e^{-8s} - e^{-4s})s^{-1}$
5. $-\dfrac{42e{-}4s}{s-1}$
7. $\dfrac{12e^{-2s}}{s^2-1}$
9. $-\dfrac{14e^{-\pi s/3}}{s^2+1}$
11. $\dfrac{e^s(e^s-1-s)}{s^2(e^{2s}-1)}$
13. $\dfrac{2e^s+3}{se^{2s}+se^s+s}$
15. $e^{-\pi s}$
17. $100e^{-2s} + e^{-s}$
19. $e^{-2\pi s}\left(100e^{3\pi s/2} + \dfrac{1}{s^2+1}\right)$
21. $\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-2s}}\int_0^2 e^{-st}\delta(t-1)\,dt = \dfrac{e^{-s}}{1-e^{-2s}}$
23. $-3\mathcal{U}(t-\pi)$
25. $-3\mathcal{U}(t-4) + 2\mathcal{U}(t-1)$
27. $-3\mathcal{U}(t-6) + 3\mathcal{U}(t-5) - 4\mathcal{U}(t-3)$
29. $e^{3t-6}\mathcal{U}(t-2)$
31. $\cos(2t-6)\,\mathcal{U}(t-3)$
33. $\frac{1}{2}\sin^2(t-5)\,\mathcal{U}(t-5)$
35. $\dfrac{2(1-2e^{2s}-e^{4s})}{s^2e^{4s}(1-e^{-4s})} = \dfrac{2(1-2e^{2s}-e^{-4s})}{s^2e^{4s}}\dfrac{1}{1-e^{-4s}} = \dfrac{2(1-2e^{2s}-e^{-4s})}{s^2e^{4s}}\sum_{k=0}^{\infty} e^{-4ks}$

so

$$\mathcal{L}^{-1}\left\{\frac{2(1-2e^{2s}-e^{4s})}{s^2e^{4s}(1-e^{-4s})}\right\} = \sum_{k=0}^{\infty}\mathcal{L}^{-1}\left\{\frac{2(1-2e^{2s}-e^{-4s})}{s^2e^{4s}}e^{-4ks}\right\}$$
$$= 2\sum_{k=0}^{\infty}\Big[(t-4k-4)\mathcal{U}(t-4k-4) - (2t-8k-4)\mathcal{U}(t-4k-2) - (t-4k)\mathcal{U}(t-4k)\Big]$$

37. $\cosh(6-2t)\mathcal{U}(t-3)$
39. $\dfrac{1}{s^2(1+e^{-4s})} = \dfrac{1}{s^2}\sum_{k=0}^{\infty} e^{-4ks}$. Then,

$$\sum_{k=0}^{\infty}\mathcal{L}^{-1}\left\{\frac{1}{s^2}e^{-4ks}\right\} = \sum_{k=0}^{\infty}(t-4k)\mathcal{U}(t-4k)$$

41. $\mathcal{L}^{-1}\left\{\sum_{k=0}^{\infty}\frac{1}{s^2+4}e^{-ks}\right\}=\frac{1}{2}\sum_{k=0}^{\infty}\sin(2t-2k)\mathcal{U}(t-k)$

43. $f(t)=-\delta(t-1/2)$, $0\le t<1$, $f(t)=f(t-1)$, $t\ge 1$

45. The Laplace transform of $f(t)$ is $F(s)=\dfrac{1-e^{-2s}}{s}$. Then, taking the Laplace transform of both sides of the equation and applying the initial condition gives us $sY(s)+3Y(s)-1=\dfrac{1-e^{-2s}}{s}$. Solving for $Y(s)$, $Y(s)=\dfrac{s-e^{-2s}+1}{s^2+3s}$, and taking the inverse Laplace transform results in $y=\frac{1}{3}e^{-3t}\left[2+e^3+(e^{3t}-e^3)\mathcal{U}(1-t)\right]$

47. The Laplace transform of $f(t)$ is $\mathcal{L}\{f(t)\}=\dfrac{e^{-2s}(e^s-1)^2}{s}$. Taking the Laplace transform of both sides of the equation, applying the initial conditions, and solving for $Y(s)$ gives us $Y(s)=\dfrac{e^{-2s}(e^s-1)^2}{s^2(s+4)}$. Taking the inverse Laplace transform results in $y=\begin{cases}\frac{1}{16}(4t-1+e^{-4t}),\ 0\le t\le 1\\ \frac{1}{16}e^{-4t}(1-2e^4+e^{4t}(9-4t)),\ 1<t\le 2\\ \frac{1}{16}e^{-4t}(e^4-1)^2,\ t>2\end{cases}$

49. Taking the Laplace transform of both sides of the equation $x''+x=1+\delta(t-\pi)$ and applying the initial conditions $x(0)=x'(0)=0$ gives us $X(s)+s^2X(s)=10e^{-\pi s}+1/s$. Solving for $X(s)$ we have $X(s)=\dfrac{1+10se^{-\pi s}}{s+s^3}=\dfrac{1}{s(s^2+1)}+\dfrac{10}{s^2+1}e^{-\pi s}$ and taking the inverse Laplace transform results in $x=1-\cos t-10\sin t\,\mathcal{U}(t-\pi)$

51. Taking the Laplace transform and applying the initial conditions gives us $-10+13X(s)+s^2X(s)+4sX(s)+s^2X(s)=20e^{-\pi s/2}$. Solving for $X(s)$ and simplifying the result gives us $X(s)=\dfrac{10+20e^{-\pi s/2}}{s^2+4s+13}$. Taking the inverse Laplace transform yields $x=\frac{10}{3}e^{-2t}\left[\sin 3t+2e^{\pi}\cos 3t\,\mathcal{U}(2t-\pi)\right]$

53. Taking the Laplace transform of both sides of the equation, applying the initial conditions and solving for $X(s)$ results in $X(s)=\dfrac{1+e^{-s}}{s(s^2+4s+13)}$. Taking the inverse Laplace transform yields $x=\frac{1}{39}e^{-2t}\begin{cases}3e^{2t}-3\cos 3t-2\sin 3t,\ 0\le t<1\\ 6e^{2t}-3\cos 3t+e^2(-3\cos(3t-3)-2\sin(3t-3))-2\sin 3t,\ t\ge 1\end{cases}$

55. $\mathcal{L}\{f(t)\}=\dfrac{10e^{-s}(e^{s/2}-1)}{s(1-e^{-s})}$ so

$$X(s)=\frac{10e^{-s}(e^{s/2}-1)}{s(s+2)(1-e^{-s})}=\frac{10e^{-s}(e^{s/2}-1)}{s(s+2)}\sum_{k=0}^{\infty}e^{-sk}.$$

Taking the inverse Laplace transform gives us

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} \mathcal{L}^{-1}\left\{\frac{10e^{-s}(e^{s/2}-1)}{s(s+2)}e^{-sk}\right\} \\ &= 5e^{-2t}\sum_{k=0}^{\infty}\left[(e^{2+2k}-e^{2t})\mathcal{U}(t-k-1)+(e^{2t}-e^{2k+1})\mathcal{U}(t-1/2-k)\right] \end{aligned}$$

57. $\mathcal{L}\{f(t)\} = \dfrac{e^{-2s}-e^{-s}+s}{s^2(1-e^{-2s})} = \dfrac{e^{-2s}-e^{-s}+s}{s^2}\sum_{k=0}^{\infty} e^{-2ks}$ so

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} \mathcal{L}^{-1}\left\{\frac{e^{-2s}-e^{-s}+s}{s^4}e^{-2ks}\right\} \\ &= \frac{1}{6}\sum_{k=0}^{\infty}\Big[(t-2-2k)^3\mathcal{U}(t-2-2k)+ \\ &\qquad (1-t+2k)^3\mathcal{U}(t-1-2k)+3(t-2k)^2\mathcal{U}(t-2k)\Big] \end{aligned}$$

59. First, $\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)\,dt = \dfrac{e^s}{s(e^s+1)}$. Then, $s(s+1)X(s) = \dfrac{e^s}{s(e^s+1)}$ so $X(s) = \dfrac{1-e^{-s}}{s^2(s+1)(1+e^{-s})} = \dfrac{1-e^{-s}}{s^2(s+1)}\sum_{k=0}^{\infty}(-1)^k e^{-sk}$ so

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} \mathcal{L}^{-1}\left\{(-1)^k\frac{1-e^{-s}}{s^2(s+1)}e^{-sk}\right\} \\ &= \sum_{k=0}^{\infty}(-1)^k\left[(2-e^{1+k+t}-k+t)\mathcal{U}(t-k-1)+(t-k+1+e^{kt})\mathcal{U}(t-k)\right] \end{aligned}$$

61. $\mathcal{L}\{f(t)\} = \dfrac{2e^{-\pi s/2}}{(s^2+4)(1-e^{-\pi s/2})} = \dfrac{2e^{-\pi s/2}}{s^2+4}\sum_{k=0}^{\infty} e^{-\pi ks/2}$ so

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} \mathcal{L}^{-1}\left\{\frac{2e^{-\pi s/2}}{(s^2+4)^2}e^{-\pi ks/2}\right\} \\ &= -\frac{1}{8}\sum_{k=0}^{\infty}\left[(\pi-2t+k\pi)\cos(2t-k\pi)+\sin(2t-k\pi)\right]\mathcal{U}(t-\pi/2-\pi k/2) \end{aligned}$$

63. $\mathcal{L}\{f(t)\} = \dfrac{e^{-s}}{1-e^{-2s}} = e^{-s}\sum_{k=0}^{\infty} e^{-2ks}$. Then,

$$\begin{aligned} s(s+2)Y(s) &= e^{-s}\sum_{k=0}^{\infty} e^{-2ks} \\ Y(s) &= \sum_{k=0}^{\infty} \frac{e^{-s}}{s(s+2)} e^{-2ks} \\ y(t) &= \frac{1}{2}\sum_{k=0}^{\infty}(1-e^{2+4k-2t})\mathcal{U}(t-1-2k) \end{aligned}$$

Exercises 8.5

1. $\frac{1}{3}t^3$
3. $\int_0^t (t-\nu)\nu^2\,d\nu = \frac{1}{12}t^4$
4. $\frac{1}{5}(2e^{-t} - 2\cos 2t + \sin 2t)$
5. $\int_0^t (t-\nu)^3 \sin 4\nu\,d\nu = \frac{1}{128}\left(3\left(-8t^2+16t\nu-8\nu^2+1\right)\sin(4\nu) - 4\left(8t^3-24t^2\nu+3t\left(8\nu^2-1\right)-8\nu^3+3\nu\right)\cos(4\nu)\right)\Big|_0^t = \frac{1}{128}(32t^3 - 12t + 3\sin 4t)$
7. $\dfrac{1}{s} - \dfrac{1}{1+s}$
9. $\dfrac{1}{s^2} - \dfrac{1}{s^2+1}$
10. $\dfrac{1}{s} - \dfrac{s}{s^2+1}$
11. $\dfrac{1}{s-1} - \dfrac{1}{s^2} - \dfrac{1}{s}$
13. $\dfrac{e^{-3s}}{s^2}$
15. $\dfrac{e^{-3\pi s/2}(e^{\pi s}-1)}{s^2+1}$
17. $\frac{1}{54}(2 - 2e^{-3t} - 6t + 9t^2)$
19. $\frac{1}{10}(2e^{-t} - 2\cos 2t + \sin 2t)$
21. $(1+\cos t)\mathcal{U}(t-\pi)$
23. $G(s) - \dfrac{1}{s^2} = -\dfrac{G(s)}{s^2} \Rightarrow G(s) = \dfrac{1}{s^2+1} \Rightarrow g(t) = \sin t$

25.

$$\begin{aligned}
\mathcal{L}\{h(t)\} - \frac{4}{s+2} &= \frac{s}{s^2+1} - \mathcal{L}\{\sin t * h(t)\} \\
H(s) - \frac{4}{s+2} &= \frac{s}{s^2+1} - \frac{1}{s^2+1}H(s) \\
H(s) &= \frac{s}{s^2+2} + \frac{10}{3(s+2)} + \frac{2(s-2)}{3(s^2+2)} \\
h(t) &= \frac{5}{3}\cos(\sqrt{2}t) - \frac{2\sqrt{2}}{3}\sin(\sqrt{2}t) + \frac{10}{3}e^{-2t}
\end{aligned}$$

27.

$$\begin{aligned}
sY(s) - y(0) - 4y(s) + \frac{4}{s}Y(s) &= \frac{3!}{(s-2)^4} \\
(s^2 - 4s + 4)Y(s) &= \frac{6s}{(s-2)^4} \\
Y(s) &= \frac{12}{(s-2)^6} + \frac{6}{(s-2)^5} \\
y(t) &= (\frac{1}{10}t^5 + \frac{1}{4}t^4)e^{2t}
\end{aligned}$$

Exercises 8.6

1. $sX(s) - 2X(s) + 3Y(s) = 0$, $9X(s) + sY(s) + 4Y(s) - 4 = 0$; $X(s) = -\frac{12}{s^2+2s-35}$, $Y(s) = \frac{4(s-2)}{s^2+2s-35}$; $x = e^{-7t} - e^{5t}$, $y = 3e^{-7t} + e^{5t}$

3. $sX(s) - 5X(s) - 5Y(s) - 2 = 0$, $4X(s) + sY(s) + 3Y(s) = 0$; $X(s) = \frac{2(s+3)}{s^2-2s+5}$, $Y(s) = -\frac{8}{s^2-2s+5}$; $x = 2e^t(\cos 2t + 2\sin 2t)$, $y = -4e^t \sin 2t$

5. $sX(s) - 1 = 2X(s) - Y(s) + 3Z(s)$, $sY(s) = 6X(s) - 2Y(s) + 6Z(s)$, $s\left(\mathcal{L}_t[z(t)](s)\right) = -2X(s) + Y(s) - 3Z(s)$; $X(s) = -\frac{-s-5}{(s+1)(s+2)}$, $Y(s) = \frac{6}{s(s+2)}$, $Z(s) = -\frac{2(s-1)}{s\,(s^2+3s+2)}$; $x = e^{-2t}(-3+4e^t)$, $y = 3e^{-2t}(-1+e^{2t})$, $z = e^{-2t}(3 - 4e^t + e^{2t})$

7. $sX(s) = 2X(s) - 3Y(s) + \frac{1}{s-1}$, $sY(s) = 4X(s) - 4Y(s)$; $X(s) = -\frac{-s-4}{s^3+s^2+2s-4}$, $Y(s) = \frac{4}{s^3+s^2+2s-4}$; $x = e^{-2t} - 2e^{-t} + e^t$, $y = \frac{2}{3}e^{-2t}(e^t-1)^2(2+e^t)$

9. $sX(s) = 2X(s) - 6Y(s)$, $sY(s) = X(s) - 3Y(s) + \frac{1}{s+1}$; $X(s) = -\frac{6}{s(s+1)^2}$, $Y(s) = -\frac{2-s}{s(s+1)^2}$; $x = e^{-t}(6 - 6e^t + 6t)$, $y' = e^{-t}(2 - 2e^t + 3t)$

11. $sX(s) = -2X(s) - 4Y(s) + \frac{1}{s+1}$, $sY(s) = 5X(s) + 2Y(s) - \frac{1}{s+1}$; $X(s) =$

$-\dfrac{-s-2}{(s+1)\,(s^2+16)}$, $Y(s)=-\dfrac{s-3}{(s+1)\,(s^2+16)}$; $x=\frac{1}{34}(2e^{-t}-2\cos 4t+9\sin 4t$, $y=\frac{1}{68}(16e^{t}-16\cos 4t-13\sin 4t)$

13. $sX(s)=X(s)-5Y(s)+\dfrac{1}{s}$, $sY(s)=X(s)-Y(s)\dfrac{1}{s^2}$; $X(s)=-\dfrac{-s^2-s+5}{s^2\,(s^2+4)}$, $Y(s)=-\dfrac{1-2s}{s^2\,(s^2+4)}$; $x=\frac{1}{8}(2-10t-2\cos 2t+9\sin 2t)$, $y=\frac{1}{8}(4-2t-4\cos 2t+\sin 2t)$

15. $x=\dfrac{1}{2}\begin{cases}-t\cos t+\sin t, & 0\le t<\pi\\ -\pi\cos t, & t\ge\pi\end{cases}$, $y=\dfrac{1}{2}\begin{cases}t\sin t, & 0\le t<\pi\\ \pi\sin t, & t\ge\pi\end{cases}$

17. $x=\cos 4t-\frac{1}{2}(\mathcal{U}(4t-\pi)-2\mathcal{U}(8t-\pi))\sin 4t$, $y=(\mathcal{U}(4t-\pi)-2\mathcal{U}(8t-\pi))\cos 4t+2\sin 4t$

19. $\mathcal{L}\{f(t)\}=\dfrac{1}{1-e^{-2s}}\left(\int_0^1 e^{-st}\,dt-\int_1^2 e^{-st}\,dt\right)=\dfrac{1-e^{-s}}{s(1+e^{-s})}=\sum_{k=0}^{\infty}(-1)^k\dfrac{1-e^{-s}}{s}e^{-ks}$. Take the Laplace transform of the system and apply the initial condition to obtain $sX(s)=X(s)+Y(s)+\mathcal{L}\{f(t)\}$, $sY(s)=-2X(s)-2Y(s)$. Solve for $X(s)$ and $Y(s)$ to get

$$X(s)=\frac{s+2}{s(s+1)}\mathcal{L}\{f(t)\}=\sum_{k=0}^{\infty}(-1)^k\frac{(s+2)(e^s-1)e^{-(1+k)s}}{s^2(s+1)}$$

and

$$Y(s)=-\frac{2}{s(s+1)}\mathcal{L}\{f(t)\}=2\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(e^s-1)e^{-(1+k)s}}{s^2(s+1)}.$$

Compute the inverse Laplace transform to obtain

$$x(t)=\sum_{k=0}^{\infty}(-1)^k e^{-t}\Big[\left(-e^{k+1}+e^t\,(3-2t+2k))\right)\mathcal{U}(t-1-k)+ \left(e^k+e^t\,(2t-1-2k)\right)\mathcal{U}(t-k)\Big]$$

and

$$y(t)=2\sum_{k=0}^{\infty}(-1)^{k+1}\Big[-\left(t-2-k+e^{1-t+k}\right)\mathcal{U}(t-1-k)+ \left(t-1-k+e^{k-t}\right)\mathcal{U}(t-k)\Big].$$

21. First, $\mathcal{L}\{f(t)\}=\dfrac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\,dt=\dfrac{1-e^{-\pi s}}{s(1-e^{-2\pi s})}=\dfrac{1-e^{-\pi s}}{s}\sum_{k=0}^{\infty}e^{-2sk}$. Then taking the Laplace transform, applying the initial conditions and solving

for $X(s)$ and $Y(s)$ results in

$$X(s) = \frac{s+1}{s^2+1}\mathcal{L}\{f(t)\} = \frac{s+1}{s^2+1}\frac{1-e^{-\pi s}}{s}\sum_{k=0}^{\infty} e^{-2sk}$$

and

$$Y(s) = \frac{2}{s^2+1}\mathcal{L}\{f(t)\} = \frac{2}{s^2+1}\frac{1-e^{-\pi s}}{s}\sum_{k=0}^{\infty} e^{-2sk}.$$

Taking the inverse Laplace transform gives us

$x(t) = (\sin(t-2k\pi) - \cos(t-2k\pi)+1)\mathcal{U}(t-2k\pi) - (\cos(t-2k\pi) - \sin(t-2k\pi+1)\mathcal{U}(t-$

and

$$y = -2(\cos(t-2k\pi)+1)\mathcal{U}(t-\pi-2k\pi) + 4\sin^2(t/2-k\pi)\,\mathcal{U}(t-2k\pi).$$

23. $s^2X(s) = 3sX(s) - 2X(s) - sY(s) + Y(s) - 1$, $sX(s) + sY(s) + 1 = 2X(s) - Y(s)$; $X(s) = -\dfrac{2}{(s-2)(s-1)s}$, $Y(s) = -\dfrac{s-2}{(s-1)s}$; $x(t) = -1 - e^{2t} + 2e^t$, $y(t) = -2 + e^t$

25. $s^2X(s) + s^2Y(s) = X(s) + \dfrac{1}{s^2}$, $sX(s) - X(s) + sY(s) = 0$; $X(s) = \dfrac{1}{(s-1)s^2}$, $Y(s) = -\dfrac{1}{s^3}$; $x = -1 + e^t - t$, $y = -\frac{1}{2}t^2$

Exercises 8.7

1. $Q(t) = 2 - e^{-t} - 2(1-e^{1-t})\mathcal{U}(t-1)$, $I(t) = 2e^{-t} - 2(1-e^{1-t})\delta(t-1) - 2e^{1-t}\mathcal{U}(t-1)$
3. $I(t) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(1-e^{-(t-n)})\mathcal{U}(t-n)$
5. $I(t) = (t-1+e^{-t}) - (t-1)\mathcal{U}(t-1) + (t-3+e^{-(t-2)})\mathcal{U}(t-2) - (t-3)\mathcal{U}(t-3) + (t-5+e^{-(t-4)})\mathcal{U}(t-4) - (t-5)\mathcal{U}(t-5) + \cdots$
7. $I(t) = 100te^{-3t}$
9. $I(t) = \frac{2}{3}\sqrt{3}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) - \frac{2}{3}\sqrt{3}e^{-t/2+\pi/2}\sin\left(\frac{\sqrt{3}}{2}(t-\pi)\right)\mathcal{U}(t-\pi)$
11. (a) $I(t) = e^{-4t+4}\mathcal{U}(t-1)$; (b) $I(t) = e^{-4t} + e^{-4t+4}\mathcal{U}(t-1)$
13. $I(t) = t - 1 + e^{-t} - (t-2+e^{-t+1})\mathcal{U}(t-1)$
15. $I(t) = e^{2-t}\mathcal{U}(t-2) + e^{6-t}\mathcal{U}(t-6)$
19. $x(t) = \frac{1}{2}((-t\cos t + \sin t)\mathcal{U}(t) - ((\pi - t)\cos t + \sin t)\mathcal{U}(t-\pi))$
21. $x(t) = 3\cos 3t - \frac{2}{3}\sin 3t$
23. $x(t) = e^{-2t}(2t+1)$
25. $x(t) = \frac{1}{3}e^{-4t}(4e^{3t} - 1)$
27. $x(t) = e^{-2t}(4 - 4e^t + 2t + 2te^t)$
29. $x(t) = \frac{1}{9}e^{-4t}(1 - e^{3t} - 3t + 6te^{3t})$
31. $x(t) = \frac{1}{36+13\pi^2+\pi^4}[((\pi^2-6)\cos\pi t - e^{-3t}(-12e^3 + 18e^{t+2} - 3e^3\pi^2 + 2\pi^2e^{t+2} + 5\pi e^{3t}\sin\pi t)\mathcal{U}(t-1)) - e^{-3t}(-12 + 18e^t - 3\pi^2 + 2\pi^2e^t + e^{3t}(\pi^2-6)\cos\pi t - 5\pi e^{3t}\sin\pi t)\mathcal{U}(t)]$
33. $x(t) = -\frac{1}{2}e^{-2t}((3e^4 + e^{2t} - 2te^4)\mathcal{U}(t-2) - (-2 + e^2 + e^{2t} + 4t - 2te^2 - 2t^2)\mathcal{U}(t-$

$1) - 2t^2\mathcal{U}(t))$

35. $x(t) = \frac{1}{18}e^{-2t}(3t\sin 3t\,\mathcal{U}(t) - (6\pi\cos 3t - 3t\cos 3t + \sin 3t)\mathcal{U}(t-2\pi) + (3\pi\cos 3t - 3t\cos 3t + \sin 3t + 3\pi\sin 3t - 3t\sin 3t)\mathcal{U}(t-\pi))$

37. $x(t) = \frac{1}{3(\pi^4+17\pi^2+16)}[(3(\pi^2-4)\cos\pi t - e^{-4t}(15\pi e^{4t}\sin\pi t + 16e^{3t+1} - 4e^4 + \pi^2 e^{3t+1} - 4\pi^2 e^4))\mathcal{U}(t-1) - e^{-4t}(3e^{4t}(\pi^2-4)\cos\pi t + 16e^{3t} + \pi^2 e^{3t} - 4\pi^2 - 15\pi e^{4t}\sin\pi t - 4)\mathcal{U}(t)]$

39. $x(t) = \frac{1}{15}(\sin t - \frac{1}{4}\sin 4t - 15\sin(4-4t)\mathcal{U}(t-1))$

41. $x(t) = \frac{1}{3}e^{-2t}(e^6\sin(3t-9)\mathcal{U}(t-3) + e^2\sin(3t-3)\mathcal{U}(t-1))$

43. $x(t) = 200 + \frac{127150}{13}e^{-5t} + \frac{250}{13}\cos t - \frac{1250}{13}\sin t$, bounded

45. $x(t) = 200 + \frac{63650}{13}e^{-5t} - \frac{1250}{3}\cos t - \frac{250}{13}\sin t$, bounded

47. $x(t) = -250 + 5350e^{2t} - 200\sin t - 100\cos t$, unbounded

49. $x(t) = -10000 + 17500e^t + 2500\cos t + 2500\sin t + (10000 - 10000e^{t-5} + 2500e^{t-5}\cos 5 - 2500\cos 5\,\cos(t-5) + 2500e^{t-5}\sin 5 - 2500\sin 5\,\cos(t-5) - 2500\cos 5\,\sin(t-5) + 2500\sin 5\,\sin(t-5))\mathcal{U}(t-5)$

51. $x(t) = -5000\Big\{1 - \frac{5}{2}e^t + \frac{1}{2}(\cos t - \sin t) + \frac{1}{2}(-2 + 2e^{t-1} + e^{t-1}\cos 1 - \cos t - e^{t-1}\sin 1 + \sin t)\mathcal{U}(t-1) + \big(1 - \frac{3}{2}e^{t-2} + \frac{1}{2}(\cos(t-2) - \sin(t-2))\big)\mathcal{U}(t-2) + \cdots\Big\}$

53. $x(t) = e^{t-2}(100e^2 + 200\mathcal{U}(t-2))$ so $x(5) = e^3(200 + 100e^2) \approx 18858$

55. $x(t) = \frac{1}{k+k^3}e^{-kt}(-c + ce^{kt} + ck - ck^2 + ck^2e^{kt} + kx_0 + k^3x_0 - cke^{kt}\cos t + ck^2e^{kt}\sin t)$

57. (b) $x(t) = \sum_{n=0}^{6}\frac{c_0}{a}\Big[(1 - e^{-a(t-4n)})\mathcal{U}(t-4n) - (1 - e^{-a(t-4n-1/2)})\mathcal{U}(t-4n-1/2)\Big]$; (c) $y(t) = \sum_{n=0}^{6} c_0\Big[\frac{1}{b}(1 - e^{-b(t-4n)})\mathcal{U}(t-4n) - \frac{1}{b-a}(e^{-a(t-4n)} + e^{-b(t-4n)})\mathcal{U}(t-4n)\Big] - \sum_{n=0}^{6} c_0\Big[\frac{1}{b}(1 - e^{-b(t-4n-1/2})\mathcal{U}(t-4n-1/2) - \frac{1}{b-a}(e^{-a(t-4n-1/2)} + e^{-b(t-4n-1/2)})\mathcal{U}(t-4n-1/2)\Big]$

59. (d) $x(t) = \cos 5t - \cos 5\sqrt{3}t$, $y(t) = 2(\cos 5t + \cos 5\sqrt{3}t)$

63. $x(t) = \frac{2}{5}(\cos t - \cos\sqrt{6}t)$, $y(t) = \frac{1}{5}(4\cos t + \cos\sqrt{6}t)$

65. $x(t) = \frac{1}{3}(\cos t + 2\cos 2t)$, $y(t) = \frac{2}{3}(\cos t - \cos 2t)$

67. $x(t) = \frac{1}{5}\left(4\cos\sqrt{3}t + \cos\left(\frac{1}{\sqrt{2}}t\right)\right)$, $y(t) = \frac{2}{5}\left(-\cos\sqrt{3}t + \cos\left(\frac{1}{\sqrt{2}}t\right)\right)$

69. $x(t) = \frac{1}{4} + \frac{1}{18}\sin 2t + \frac{7}{12}\cos 2t + \frac{1}{18}\sin t + \frac{1}{6}\cos t - \frac{1}{6}t\cos t$, $y(t) = \frac{1}{4} - \frac{1}{18}\sin 2t - \frac{7}{12}\cos 2t + \frac{4}{9}\sin t + \frac{1}{3}\cos t - \frac{1}{3}t\cos t$

71. $x(t) = \frac{7}{18}\cos t + \frac{11}{18}\cos 2t + \frac{1}{12}t\sin 2t$, $y(t) = \frac{7}{9}\cos t - \frac{7}{9}\cos 2t - \frac{1}{12}t\sin 2t$

73. $\theta_1(t) = -\frac{1}{8}\sqrt{3}\sin\left(\frac{2}{\sqrt{3}}t\right) + \frac{1}{8}\sin 2t$, $\theta_2(t) = -\frac{1}{4}\sqrt{3}\sin\left(\frac{2}{\sqrt{3}}t\right) - \frac{1}{4}\sin 2t$

75. $\theta_1(t) = \frac{1}{2}\cos\left(\frac{2}{\sqrt{3}}t\right) + \frac{1}{2}\cos 2t$, $\theta_2(t) = \cos\left(\frac{2}{\sqrt{3}}t\right) - \cos 2t$

77. $\theta_1(t) = \frac{1}{4}\sqrt{3}\sin\left(\frac{2}{\sqrt{3}}t\right) + \frac{1}{4}\sin 2t$, $\theta_2(t) = \frac{1}{2}\sqrt{3}\sin\left(\frac{2}{\sqrt{3}}t\right) - \frac{1}{2}\sin 2t$

79. $\theta_1(t) = -\frac{1}{4}\cos\left(\frac{2}{\sqrt{3}}t\right) + \frac{1}{4}\cos 2t$, $\theta_2(t) = -\frac{1}{2}\cos\left(\frac{2}{\sqrt{3}}t\right) - \frac{1}{2}\cos 2t$

81. $x(t) = -\cos\left(\frac{1}{\sqrt{m}}t\sqrt{2k + k\omega^2}\right)$, $y(t) = \cos\left(\frac{1}{\sqrt{m}}t\sqrt{2k + k\omega^2}\right)$

83. $x(t)$ has frequency ω_1, where ${\omega_1}^2 = 9k/(2m) - k\sqrt{17}/(2m)$; $y(t)$ has frequency ω_2, where ${\omega_2}^2 = 9k/(2m) + k\sqrt{17}/(2m)$

87. (a) $Q(t) = \frac{292175000}{159587072641} e^{-100(5+2\sqrt{6})t} - \frac{7387691623}{1276696581128\sqrt{6}} e^{-100(5+2\sqrt{6})t} + \frac{292175000}{159587072641} e^{400\sqrt{6}t-1}$ $\frac{7387691623}{1276696581128\sqrt{6}} e^{400\sqrt{6}t-100(5+2\sqrt{6})t} - \frac{584350000}{159587072641}\cos 377t - \frac{204799950}{159587072641}\sin 377t$;

(c) $I(t) = -\frac{584350000}{159587072641}\cos 377t - \frac{204799950}{159587072641}\sin 377t$

91. $x(t) = -\frac{1}{3}\sqrt{3}\sin\sqrt{3}t$, $y(t) = \frac{1}{3}\sqrt{3}\sin\sqrt{3}t$

93. $\theta_1(t) = \frac{1}{8}\left[4\cos 2t + 4\cos\left(\frac{2}{\sqrt{3}}t\right) + \sin 2t - \sqrt{3}\sin\left(\frac{2}{\sqrt{3}}t\right)\right]$, $\theta_2(t) = -\cos 2t + \cos\left(\frac{2}{\sqrt{3}}t\right) - \frac{1}{4}\sin 2t - \frac{\sqrt{3}}{4}\sin\left(\frac{2}{\sqrt{3}}t\right)$

97. $x_1(t) = -2\sin t$, $x_2(t) = \sin t$, $x_3(t) = 2\sin t$

Chapter 8 Review Exercises

1. $\dfrac{s-1}{s^2}$

3. $\dfrac{1-e^{-5\,s}}{s}$

5. $2\,s^{-3} + 5\,s^{-1}$

7. $(s-2)^{-2}$

9. $6\,(s-1)^{-4}$

11. $\dfrac{s^2-9}{(s^2+9)^2}$

13. $\dfrac{s+5}{s^2+10s+34}$

15. $e^{-3/2\,s\pi}$

17. $-2\,\dfrac{-3\,e^{-7\,s}+2\,e^{-4\,s}}{s}$

19. $-42\,\dfrac{e^{5-s}}{s-5}$

21. $2\,\dfrac{e^{-2\,s}\left(2\,s^2+2\,s+1\right)}{s^3}$

23. $\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-s}}\int_0^1 (1-t)e^{-st}\,dt = \dfrac{(s-1)e^s+1}{s^2(e^s-1)}$

25. $\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-2s}}\int_0^1 \cos\frac{\pi}{2}t\,e^{-st}\,dt = \dfrac{2e^2(2se^s+\pi)}{(4s^2+\pi^2)(e^{2s}-1)}$

27. $\mathcal{L}\left\{\int_0^t \dfrac{\sin\tau}{\tau}\,d\tau\right\} = \dfrac{1}{s}\tan^{-1}\dfrac{1}{s}$

29. $\mathcal{L}\left\{\dfrac{1-e^{-t}}{t}\right\} = \ln\left(1+\dfrac{1}{s}\right)$

31. $\mathcal{L}^{-1}\left\{-\dfrac{2s}{(s^2+1)^2}\right\} = -2\sin t$

33. $\mathcal{L}^{-1}\left\{\dfrac{2s^2-7s+20}{s(s^2-2s+10)}\right\} = 2 - e^t\sin 3t$

35. $\mathcal{L}^{-1}\left\{-\dfrac{14}{se^{2s}}\right\} = -14\mathcal{U}(t-2)$
37. $\mathcal{L}^{-1}\left\{\dfrac{3e^{6-s}}{6-s}\right\} = -3e^{6t}\mathcal{U}(t-1)$
39. $\mathcal{L}^{-1}\left\{\dfrac{-18}{(s^2+1)(1-e^{-3s})}\right\} = \mathcal{L}^{-1}\left\{\dfrac{-18}{s^2+1}\sum_{k=0}^{\infty} e^{-3ks}\right\} = \sum_{k=0}^{\infty} 18\sin(3k-t)\mathcal{U}(t-3k)$
41. $\mathcal{L}^{-1}\left\{\dfrac{s\cos\phi - \omega\sin\phi}{s^2+\omega^2}\right\} = \cos(\omega t+\phi)$
43. $\frac{2}{3}(\cos t - \cos 2t)$
45. $y = e^{2t}(\cos t - 2\sin t)$
47. $y = e^{-2t}(1 + e^2\,\mathcal{U}(t-1))$
49. First, $\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-2s}}\left(\int_0^1 te^{-st}\,dt + \int_1^2 (2-t)e^{-st}\,dt\right) = \dfrac{e^s-1}{s^2(e^s+1)} = \dfrac{1-e^{-s}}{s^2}\dfrac{1}{1+e^{-s}} = \dfrac{1-e^{-s}}{s^2}\sum_{k=0}^{\infty}(-1)^k e^{-sk}$. Then, taking the Laplace transform of both sides of the differential equation, applying the initial conditions and solving for $Y(s)$ gives us

$$(s+4)(s+8)Y(s) = \frac{e^s-1}{s^2(e^s+1)} = \frac{1-e^{-s}}{s^2}\sum_{k=0}^{\infty}(-1)^k e^{-sk}$$
$$Y(s) = \frac{1-e^{-s}}{s^2(s+4)(s+8)}\sum_{k=0}^{\infty}(-1)^k e^{-sk}.$$

Taking the inverse Laplace transform gives us

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+4)(s+8)}\sum_{k=0}^{\infty}(-1)^k e^{-sk}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+4)(s+8)}\sum_{k=0}^{\infty}(-1)^k e^{-sk-s}\right\} \\ &= \frac{1}{256}e^{-8t}\sum_{k=0}^{\infty}(-1)^k\Big[\Big(e^{8+8k} - 4e^{4(t+1+k)} + e^{8t}(11+8k-8t)\Big)\mathcal{U}(t-k-1) - \\ &\qquad \Big(e^{8k} - 4e^{4(t+k)} + e^{8t}(3+8k-8t)\Big)\mathcal{U}(t-k)\Big]. \end{aligned}$$

51. $x(t) = e^{1-2t}\left[e(-e^2+e^t)\mathcal{U}(t-2) + (-e+e^t)\mathcal{U}(t-1)\right]$
53. $g(t) = 1 + \sin t - \cos t$
57. $x(t) = \begin{cases} \frac{1}{15}(\cos(t/2) - \cos 2t - 30\sin(t/2)),\ 0 \le t < \pi \\ \frac{1}{15}(\cos(t/2) - 31\sin(t/2),\ t \ge \pi \end{cases}$
59. $Q(t) = \begin{cases} \frac{11}{250}\sin^2 50t,\ 0 \le t < 2 \\ \frac{11}{500}((-1+\cos 200)\cos 100t + \sin 200\ \sin 200t),\ t \ge 2 \end{cases}$
61. $x(t) = 100e^{2t-2}\left(100e^2 + \mathcal{U}(t-1)\right)$
65. $x = \frac{1}{2}(t + \cos t\ \sin t)$, $y = \frac{1}{8}(-2t + \sin 2t)$
67. $x = 4e^{-t}\left((e^t - e^2(-1+t))\mathcal{U}(t-2) + (1-e^t+t)\mathcal{U}(t)\right)$,
$y = e^{-t}\left((-2e^t + te^2)\mathcal{U}(t-2) + (-2+2e^t-t)\mathcal{U}(t)\right)$

69. $x(t) = \frac{2}{5}\left(2\sqrt{2}\sin(\sqrt{2}t) + \sqrt{3}\sin\left(\frac{1}{\sqrt{3}}t\right)\right)$, $y(t) = \frac{1}{5}\left(-2\sqrt{2}\sin(\sqrt{2}t) + 4\sqrt{3}\sin\left(\frac{1}{\sqrt{3}}t\right)\right)$
71. $x(t) = \frac{1}{2}(\cos t - \cos\sqrt{3}t + t\sin t)$, $y(t) = \frac{1}{2}(-\cos t + \cos\sqrt{3}t + t\sin t)$
73. $x(t) = \frac{1}{8}\left(-3\sin 2t - \sqrt{3}\sin\left(\frac{2}{\sqrt{3}}t\right)\right)$, $y(t) = \frac{1}{4}\left(3\sin 2t - \sqrt{3}\sin\left(\frac{2}{\sqrt{3}}t\right)\right)$

Differential Equations at Work

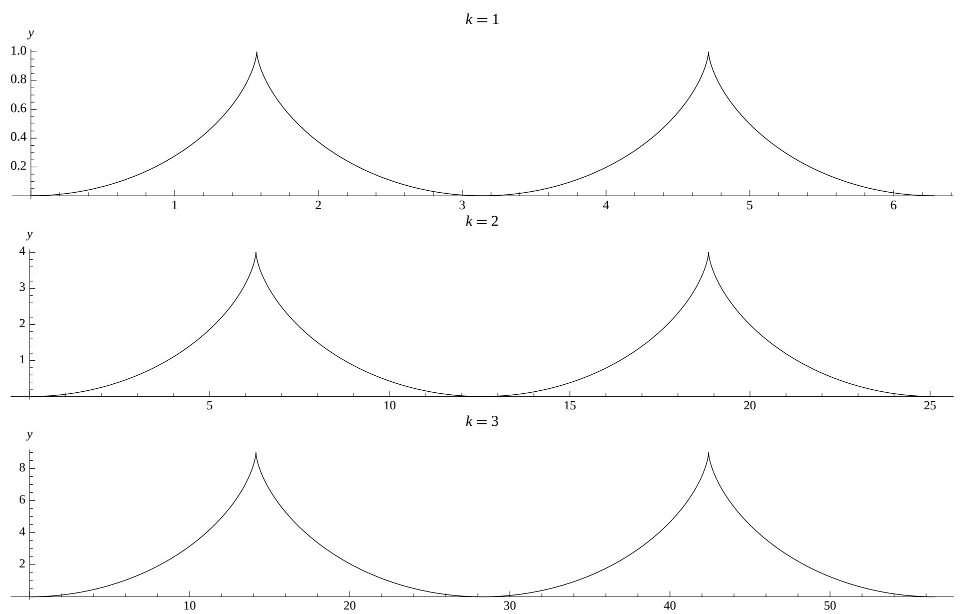
A. The Tautochrone

1.

$$dx = \sqrt{\frac{k^2}{k^2y^2} - 1}\, dy$$

$$dx = \sqrt{\frac{k^2}{k^2\sin^2\theta} - 1} \cdot 2k^2\sin\theta\,\cos\theta\, d\theta = \sqrt{\frac{k^2(1-\sin^2\theta)}{k^2\sin^2\theta}} \cdot 2k^2\sin\theta\,\cos\theta\, d\theta$$
$$= \frac{\cos\theta}{\sin\theta} \cdot 2k^2\sin\theta\,\cos\theta\, d\theta = 2k^2\cos^2\theta\, d\theta.$$

Then, $x(\theta) = \frac{1}{2}\left(2k^2\theta + k^2\sin 2\theta\right) + C_1$ so $x(0) = 0$ means that $C_1 = 0$ so that $x(\theta) = \frac{1}{2}\left(2k^2\theta + k^2\sin 2\theta\right)$.
2. $y(\theta) = k^2\sin^2\theta = \frac{1}{2}k^2(1 - \cos 2\theta)$.
3. Increasing the value of k increases the length of the curve.



4. The time is independent of the choice of y (that, is the choice of θ). Therefore,

$$\text{time} = \int_0^y \frac{\phi(z)}{\sqrt{y-z}}\,dz = \int_0^y \frac{ky^{-1/2}}{\sqrt{y-z}}\,dz = -2k\left[\sqrt{\frac{y-z}{y}}\right]_0^y = -2k\cdot -1 = 2k.$$

B. Vibration Absorbers

1. First, we apply the Laplace transform to the system

$$\begin{aligned} m_1 s\left(X_1(s)-x_1(0)\right) - x_1'(0) + (k_1+k_2)X_1(s) - k_2X_2(s) &= \frac{F_0\omega}{s^2+\omega^2} \\ m_2 s\left(X_2(s)-x_2(0)\right) - x_2'(0) - k_2X_1(s) + k_2X_2(s) &= 0 \end{aligned}$$

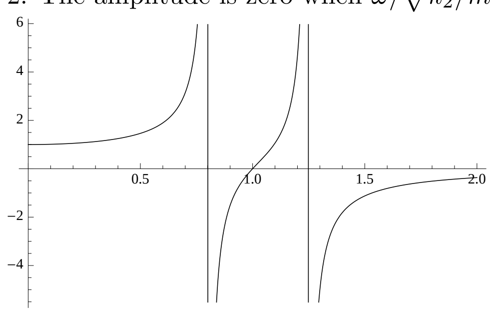
and then rearrange the equations and apply the initial conditions to obtain

$$\begin{aligned} (m_1s^2+k_2+k_2)X_1(s) - k_2X_2(s) &= \frac{F_0\omega}{s^2+\omega^2} \\ -k_2X_1(s) + (m_2s^2+k_2)X_2(s) &= 0. \end{aligned}$$

We eliminate $X_2(s)$ from the system by multiplying the first equation by $m_2s^2+k_2$ and the second equation by k_2 and then adding

$$\begin{aligned} \left[\left(m_2s^2+k_2\right)\left(m_1s^2+k_1+k_2\right)-k_2^2\right]X_1(s) &= \left(m_2s^2+k_2\right)\frac{F_0\omega}{s^2+\omega^2} \\ \left[m_1m_2s^4+(m_2k_1+m_2k_2+m_1k_2)s^2+k_1k_2\right]X_1(s) &= \left(m_2s^2+k_2\right)\frac{F_0\omega}{s^2+\omega^2}. \end{aligned}$$

2. The amplitude is zero when $\omega/\sqrt{k_2/m_2}=1$.



C. Airplane Wing

1. $x_1(t) = \frac{\sqrt{2}}{3\sqrt{c}}\sin\left(\frac{3}{2}\sqrt{2c}\,t\right)$, $x_2(t) = -\frac{\sqrt{2}}{6\sqrt{c}}\sin\left(\frac{3}{2}\sqrt{2c}\,t\right)$, $x_3(t) = -\frac{\sqrt{2}}{6\sqrt{c}}\sin\left(\frac{3}{2}\sqrt{2c}\,t\right)$, $c = \frac{EI}{m_1\ell^3}$,; Period: $p(c) = 2\pi/(3\sqrt{2c}/2)$, $p(0.0001)\approx 296.19$, $p(0.01)\approx 29.62$, $p(0.1)\approx 9.37$, $p(1.)\approx 2.96$, $p(2.)\approx 2.09$

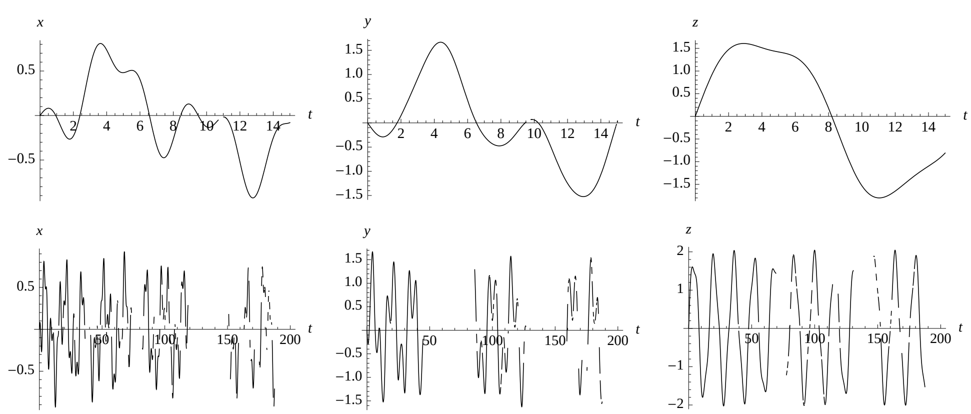
D. Free Vibration of a Three-Story Building

1. Notice that

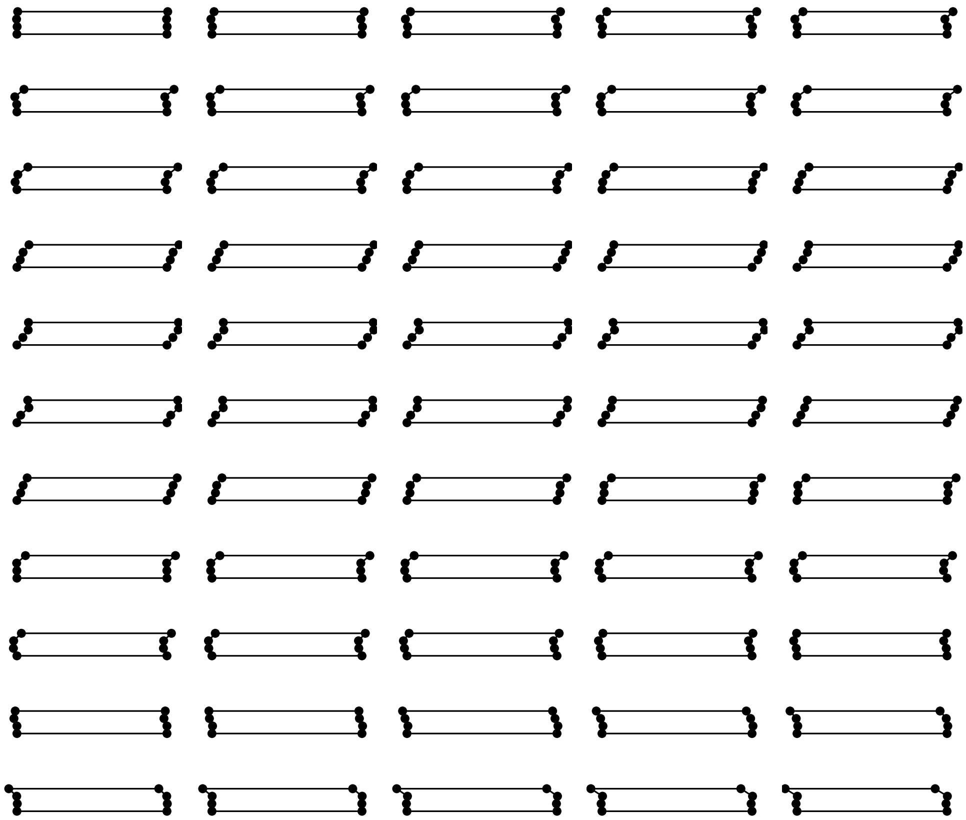
$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}'' + \begin{pmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$\begin{pmatrix} m_1x_1'' \\ m_2x_2'' \\ m_3x_3'' \end{pmatrix} + \begin{pmatrix} (k_1+k_2)x_1 - k_2x_2 \\ -k_2x_1 + (k_2+k_3)x_2 - k_3x_3 \\ -k_3x_2 + k_3x_3 \end{pmatrix} = \begin{pmatrix} m_1x_1'' + (k_1+k_2)x_1 - k_2x_2 \\ m_2x_2'' - k_2x_1 + (k_2+k_3)x_2 - k_3x_3 \\ m_3x_3'' - k_3x_2 + k_3x_3 \end{pmatrix}.$$

2-4.



5.



7. Increasing the number of stories increases the number of equations in the system of differential equations. A five-story building leads to a system of five second-order differential equations; a fifty-story building leads to a system of fifty second-order differential equations.