CHAPTER 11 PROBLEMS AND EXERCISES

Problem 1: Consider a PWAS of length $l = 7\ \text{mm}$, width $b = 1.65\ \text{mm}$, thickness $t = 0.2\ \text{mm}$, and material properties as given in Table 11.2. The PWAS is bonded to a 1-mm thick aluminum strip with material properties as given in Table 11.3. The PWAS length is oriented along the strip. (i) Calculate the first three frequencies at which the PWAS will tune into the axial waves propagating into the aluminum strip. (ii) Repeat the calculations considering the PWAS length oriented across the strip.

<u>Solution</u>

(i) To calculate the frequencies at which the PWAS will tune into the axial waves propagating into the aluminum strip, assume ideal bonding between the PWAS and the structural substrate and use the textbook Eq. (11.26), i.e.,

$$f_n = (2n-1)\frac{c}{2l_a}, \quad n = 1,2,3,... \qquad (11.26)\ (1)$$

where c is the axial wave speed in the aluminum strip, and l_a is the relevant dimension of the PWAS. As the PWAS has its length oriented to the direction of wave propagation, the relevant dimension is the PWAS length, i.e., $l_a = l = 7\ \text{mm}$. Upon calculation, the first three frequencies at which the PWAS will tune into the axial waves propagating into the aluminum strip are found to be $f_1 = 364\ \text{kHz}$, $f_2 = 1091\ \text{kHz}$, $f_3 = 1818\ \text{kHz}$. One notices that the axial tuning frequencies follow an odd-numbers proportionality rule (364/1091/1818 = 1/3/5).

(ii) If the PWAS length is oriented across the strip, then the PWAS dimension relevant to wave propagation is its width, i.e., $l_a = b$. Repeating the calculations with $l_a = b = 1.65\ \text{mm}$ yields $f_1 = 1543\ \text{kHz}$, $f_2 = 4629\ \text{kHz}$, $f_3 = 7715\ \text{kHz}$.

PROBLEM 11.1 SOLUTION

Axial wave tuning

Units $kHz := 1000 \cdot Hz$ $pF := 10^{-12} \cdot F$ $nF := 10^{-9} \cdot F$ $\mu m := 10^{-6} \cdot m$ $\mu\varepsilon := 10^{-6}$ $MPa := 10^{6} \cdot Pa$ $GPa := 10^{9} \cdot Pa$

Given:

$E := 70 \cdot GPa$ $\rho := 2700 \cdot \frac{kg}{m^3}$

PWAS geometry $L := 7 \cdot mm$ $b := 1.65 \cdot mm$ $t := 0.2 \cdot mm$

piezo material properties

Solution:

Axial wave speed in aluminum strip $c := \sqrt{\frac{E}{\rho}}$ $c = 5092 \frac{m}{s}$

(i) PWAS lenth along the strip

$La := L$

$fA(n) := (2 \cdot n - 1) \cdot \frac{c}{2La}$ $fA(1) = 364kHz$

$fA(2) = 1091kHz$

$fA(3) = 1818kHz$

(ii) PWAS lenth across the strip

$La := b$

$fA(n) := (2 \cdot n - 1) \cdot \frac{c}{2La}$ $fA(1) = 1543kHz$

$fA(2) = 4629kHz$

$fA(3) = 7715kHz$

Problem 2: Consider a PWAS of length $l = 7\text{ mm}$, width $b = 1.65\text{ mm}$, thickness $t = 0.2\text{ mm}$, and material properties as given in Table 11.2. The PWAS is bonded to a 1-mm thick aluminum strip with material properties as given in Table 11.3. The PWAS length is oriented along the strip. (i) Calculate the first three frequencies at which the PWAS will tune into the flexural waves propagating into the aluminum strip. (ii) Repeat the calculations considering the PWAS length oriented across the strip. (iii) Discuss the difference between these results and those of axial tuning calculated in the previous problem. Comment on the results of (i) and (ii).

<u>Solution</u>

(i) To calculate the frequencies at which the PWAS will tune into the flexural waves propagating into the aluminum strip, assume ideal bonding between the PWAS and the structural substrate and use the textbook Eq. (11.60), i.e.,

$$f_n = (2n-1)^2 \frac{\pi}{2l_a^2}\sqrt{\frac{Eh^2}{12\rho}}, \quad n = 1, 2, 3, \ldots \qquad (11.60)(1)$$

where h is the thickness of the aluminum strip, and l_a is the relevant dimension of the PWAS. As the PWAS has its length oriented to the direction of wave propagation, the relevant dimension is the PWAS length, i.e., $l_a = l = 7\text{ mm}$. Upon calculation, the first three frequencies at which the PWAS will tune into the flexural waves propagating into the aluminum strip are found to be $f_1 = 47\text{ kHz}$, $f_2 = 424\text{ kHz}$, $f_3 = 1178\text{ kHz}$.

(ii) If the PWAS length is oriented across the strip, then the PWAS dimension relevant to wave propagation is its width, i.e., $l_a = b$. Repeating the calculations with $l_a = b = 1.65\text{ mm}$ yields $f_1 = 848\text{ kHz}$, $f_2 = 7633\text{ kHz}$.

(iii) The first difference that one notices between these results and those of axial wave tuning is that, for length-wise orientation of the PWAS, the first tuning frequency is much lower than for the axial waves (47 vs 364 kHz). The second thing that one observes is that, whereas the axial tuning frequencies followed a linear rule, the flexural tuning frequencies follow an odd-numbers square rule. This fact makes the second and third tuning frequencies be much larger than the first tuning frequency.

Another important fact is revealed when one compares the results of (i) with those of (ii). Because in (ii) the relevant PWAS dimension is much smaller, the flexural tuning frequencies are much larger. In fact, the third flexural tuning frequency for $l_a = b$ is in the MHz range ($f_3 = 21202\text{ kHz}$). This value is, however, unrealistic, because in that frequency range the simple beam theory used in the deduction of the textbook formula Eq. (11.60) does no longer hold. At such high frequencies, the exact Lamb-waves theory must be used. The Lamb-waves theory predict that the "flexural" dispersion curve levels off and significantly departs from the simplistic flexural-wave theory for *fd* product values beyond 1000 kHz mm (Figure 6.17 in the textbook).

PROBLEM 11.2 SOLUTION

Flexural waves tuning

Units $kHz := 1000 \cdot Hz$ $pF := 10^{-12} \cdot F$ $nF := 10^{-9} \cdot F$ $\mu m := 10^{-6} \cdot m$ $\mu\varepsilon := 10^{-6}$ $MPa := 10^{6} \cdot Pa$ $GPa := 10^{9} \cdot Pa$

Given:

Aluminum strip $h := 1 \cdot mm$ $E := 70 \cdot GPa$ $\rho := 2700 \cdot \frac{kg}{m^3}$

PWAS geometry $L := 7 \cdot mm$ $b := 1.65 \cdot mm$ $t := 0.2 \cdot mm$

Solution:

(i) PWAS lenth along the strip

$La := L$

$La := L$

$$fA(n) := (2 \cdot n - 1)^2 \cdot \frac{\pi}{2 \cdot L^2} \left(\frac{E \cdot h^2}{12 \cdot \rho} \right)^{\frac{1}{2}}$$

$fA(1) = 47\,kHz$

$fA(2) = 424\,kHz$

$fA(3) = 1178\,kHz$

(ii) PWAS lenth across the strip

$La := b$

$$fA(n) := (2 \cdot n - 1)^2 \cdot \frac{\pi}{2 \cdot La^2} \left(\frac{E \cdot h^2}{12 \cdot \rho} \right)^{\frac{1}{2}}$$

$fA(1) = 848\,kHz$

$fA(2) = 7633\,kHz$

$fA(3) = 21202\,kHz$

Problem 3: Consider a PWAS of width $b = 1.65$ mm, thickness $t = 0.2$ mm, and material properties as given in Table 11.2. The PWAS is bonded to a 1-mm thick aluminum strip with material properties as given in Table 11.3. The PWAS length is oriented along the strip. (i) Calculate the smallest PWAS length *l* that will tune into the flexural waves propagating into the aluminum strip at 10 kHz. (ii) Calculate the smallest PWAS length *l* that will reject the flexural waves propagating into the aluminum strip at 10 kHz. (iii) Calculate the PWAS length *l* that will tune into the axial waves propagating into the aluminum strip at 100 kHz. (iv) Calculate the PWAS length *l* that will reject the axial waves propagating into the aluminum strip at 100 kHz.

<u>Solution</u>

(i) To calculate the PWAS length that will produce tuning into flexural waves, use the textbook Eqs. (11.56), i.e.,

$$l_a = (2n-1)\frac{\lambda_F}{2}, \quad n = 1,2,3,... \tag{11.56 (1)}$$

The wavelength λ_F is given by general relation

$$\lambda = \frac{c_F}{f} \tag{2}$$

where the flexural wavespeed c_F is given by the textbook Eq. (11.58), i.e.,

$$c_F(\omega) = \left(\frac{Eh^2}{12\rho}\right)^{1/4}\sqrt{\omega} \tag{11.58 (3)}$$

Upon calculation with $f = 10$ kHz, one gets $c_F = 304$ m/s, $\lambda_F = 30.4$ mm and hence the smallest PWAS length value that will tune into flexural waves at 10 kHz is $l_a =$ 15.2 mm.

(ii) To calculate the PWAS length that will effect rejection of the flexural waves propagating into the aluminum strip at 10 kHz, follow the general concept outlined in the textbook Section 11.3.3 and identify conditions for wave rejection. Specifically, recall Eq. (11.54), i.e.,

$$\varepsilon_x(x,t) = -i\frac{3}{2}\frac{a\tau_a}{Eh}\left(\sin\xi_F a\right)e^{i(\xi_F x - \omega t)} \tag{11.54 (4)}$$

Examination of Eq. (4) reveals that the response is proportional to the factor $\sin\xi_F a$, where ξ_F is the flexural wavenumber and *a* is the PWAS half length. The factor $\sin\xi_F a$ becomes zero when

$$\xi_F a = (n-1)\pi \qquad n = 1,2,3,... \tag{5}$$

Recalling that $\xi_F = 2\pi/\lambda_F$, and that the PWAS length is $l_a = 2a$, relation (5) implies that zero excitation (i.e., rejection) of flexural waves will happen when the PWAS length is an even integer multiple of the flexural half wavelength, i.e.,

$$l_a = 2n\frac{\lambda_F}{2} = n\lambda_F \qquad n = 1,2,3,... \tag{6}$$

The flexural wavelength at $f = 10\text{ kHz}$ has been already calculated in the previous paragraph as $\lambda_F = 30.4\text{ mm}$ Substitution into Eq. (11) yields the smallest value of PWAS length that will produce rejection of the flexural waves to be $l_a =$ 30.4 mm .

(iii) To calculate the PWAS length that will effect tuning into axial waves, use the textbook Eqs. (11.25), i.e.,

$$l_a = (2n-1)\frac{\lambda}{2} \qquad n = 1, 2, 3, \ldots \tag{11.25) (7}$$

where the wavelength λ is given by general relation

$$\lambda = \frac{c}{f} \tag{8}$$

with $c = \sqrt{E/\rho}$ being the axial wave speed in the aluminum strip. Upon calculation, one gets $c = 5092\text{ m/s}$; for $f = 100\text{ kHz}$, the axial wavelength is $\lambda = 50.9\text{ mm}$ and hence the smallest PWAS length value that will tune into axial waves at 100 kHz is $l_a =$ 25.5 mm .

(iv) To calculate the PWAS length that will effect rejection of the axial waves propagating into the aluminum strip at 100 kHz, follow the general concept outlined in the textbook Section 11.2 and identify conditions for wave rejection. Specifically, recall Eq. (11.23), i.e.,

$$\varepsilon(x) = i\frac{a\tau_a}{Eh}\left(\sin\xi_0 a\right)e^{i(\xi_0 x - \omega t)} \tag{9}$$

Equation (9) indicates that the response is proportional to the factor $\sin\xi_0 a$, where ξ_0 is the axial wavenumber and *a* is the PWAS half length. The factor $\sin\xi_0 a$ becomes zero when

$$\xi_0 a = (n-1)\pi\,, \quad n = 1, 2, 3, \ldots \tag{10}$$

Recalling that $\xi_0 = 2\pi/\lambda$, and that the PWAS length is $l_a = 2a$, relation (10) implies that zero excitation (i.e., rejection) of axial waves will happen when the PWAS length is an even integer multiple of the axial half wavelength, i.e.,

$$l_a = 2n\frac{\lambda}{2} = n\lambda\,, \quad n = 1, 2, 3, \ldots \tag{11}$$

The axial wavelength at $f = 100\text{ kHz}$ has been already calculated in the previous paragraph as $\lambda = 50.9\text{ mm}$. Substitution into Eq. (11) yields the smallest value of PWAS length that will produce rejection of the axial waves to be $l_a =$ 50.9 mm .

PROBLEM 11.3 SOLUTION

Axial and flexural waves tuning and rejection

Units $kHz := 1000 \cdot Hz$ $pF := 10^{-12} \cdot F$ $nF := 10^{-9} \cdot F$ $\mu m := 10^{-6} \cdot m$ $\mu\varepsilon := 10^{-6}$ $MPa := 10^{6} \cdot Pa$ $GPa := 10^{9} \cdot Pa$

Given:

Aluminum strip $h := 1 \cdot mm$ $E := 70 \cdot GPa$ $\rho := 2700 \cdot \frac{kg}{m^3}$

PWAS geometry $b := 1.65 \cdot mm$ $t := 0.2 \cdot mm$

Solution:

$f := 10 \cdot kHz$

(i) Tuning of flexural waves

Flexural wave speed

$$cF(f) := \left(\frac{E \cdot h^2}{12 \cdot \rho}\right)^{\frac{1}{4}} \cdot \sqrt{2 \cdot \pi \cdot f} \qquad cF(f) = 304 \frac{m}{s}$$

$$\lambda F(f) := \frac{cF(f)}{f} \qquad \lambda F(f) = 30.4mm$$

$$L(n) := (2n - 1) \cdot \frac{\lambda F(f)}{2} \qquad L(1) = 15.2mm$$

$$L(2) = 45.6mm$$

(ii) Rejection of flexural waves

$$La(n) := n \cdot \lambda F(f) \qquad La(1) = 30.4mm$$

$$La(2) = 60.8mm$$

(iii) Tuning of axial waves

Axial wave speed in aluminum strip $c := \sqrt{\frac{E}{\rho}}$ $c = 5092\,\frac{m}{s}$

$f := 100 \cdot kHz$ $\lambda(f) := \frac{c}{f}$ $\lambda(f) = 50.9\,mm$

$La(n) := (2n - 1) \cdot \frac{\lambda(f)}{2}$ $La(1) = 25.5\,mm$

$La(2) = 76.4\,mm$

(ii) Rejection of axial waves

$La(n) := n \cdot \lambda(f)$ $La(1) = 50.9\,mm$

$La(2) = 101.8\,mm$