CHAPTER 13 PROBLEMS AND EXERCISES

Problem 1: What is the most important phenomenon that enables the successful use of PWAS phased arrays in conjunction with multi-modal guided waves in thin wall structures?

Solution

The answer to this question can be found in the textbook Section 13.3. The single most important phenomenon that enables the use of PWAS phased arrays in conjunction with multi-modal guided waves in thin wall structures is the PWAS-Lamb wave tuning, as described in Chapter 11 of this book. The PWAS-Lamb wave tuning principle allows one to find convenient combinations of PWAS dimensions and excitation frequency that permit the preferential excitation of just one Lamb-wave mode, preferably one of minimal dispersion. In the following developments, we will assume that such tuning is possible and that a minimally dispersive Lamb wave can be tuned into. In this way, the situation depicted in textbook Figure 13.4 can be achieved in spite of the generally multi-modal character of the Lamb waves.

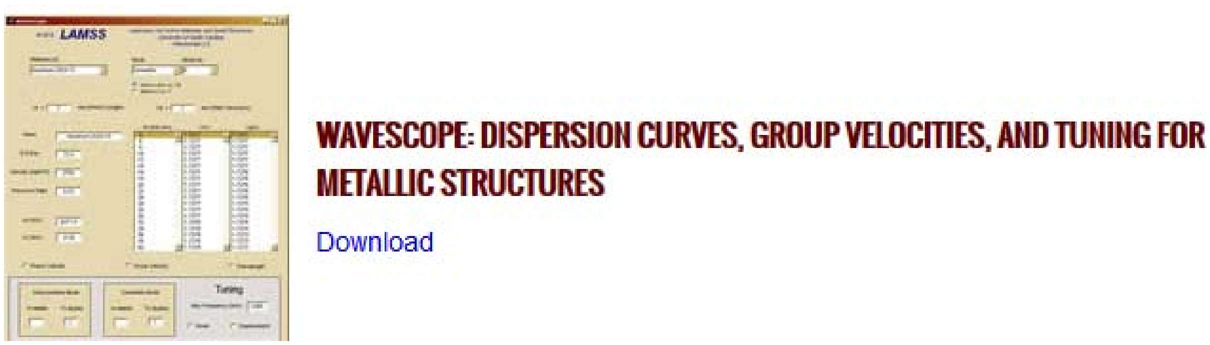
Problem 2: (i) Using the straight-wavefront assumption, calculate the optimum phased-array pitch for a PWAS phased-array made up of 10-mm PWAS ideally bonded to a 1-mm thick aluminum plate ($E = 72.4\text{ GPa}$, $\rho = 2780\text{ kg/m}^2$, $\nu = 0.33$). (ii) At what frequency would this PWAS array would operated?

Solution

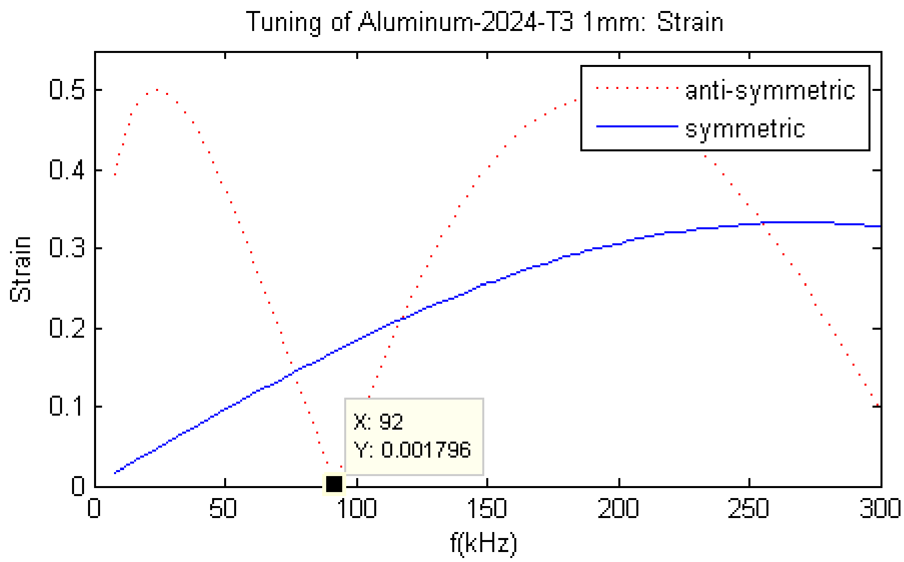
(i) As indicated in the textbook Section 13.5.3.2, the ideal pitch for any phased array is half the wavelength, i.e., $d = 0.5\lambda$. However, the wavelength is dictated by the tuning principles developed in the textbook Section 11.4.5, Eq. (11.108), i.e.,

$$\varepsilon_x(x,t) = -i\frac{a\tau_a}{\mu}\sum_{j=0}^{J_S}\left(\sin\xi_j^S a\right)\frac{N_S(\xi_j^S)}{D'_S(\xi_j^S)}e^{i(\xi_j^S x-\omega t)} - i\frac{a\tau_a}{\mu}\sum_{j=0}^{J_A}\left(\sin\xi_j^A a\right)\frac{N_A(\xi_j^A)}{D'_A(\xi_j^A)}e^{i(\xi_j^A x-\omega t)} \qquad (11.108)\ (1)$$

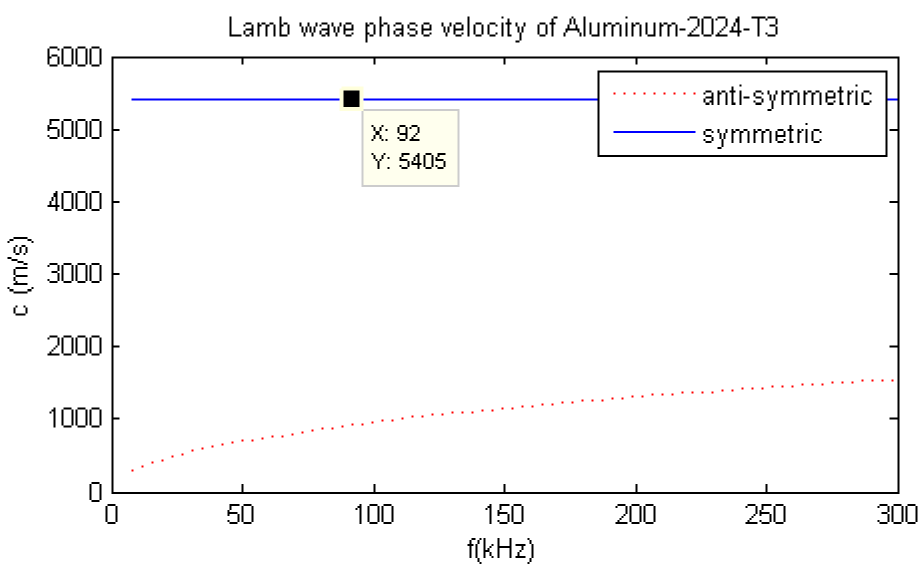
One could code Eq. (1) or could use the software programs posted on the LAMSS website http://www.me.sc.edu/research/lamss/html/software.html . In particular, download and activate the following program:



After activating the program, one obtains the chart below.



The problem at hand is to find the first tuning frequency for the S0 mode, which is a low dispersive mode ideally suited for phased array processing. In the chart above, the S0 mode is presented in solid line ('symmetric"), whereas the A0 mode is presented in dashed line ("anti-symmetric"). Examination of this tuning chart indicates that the A0 mode goes through a minimum (mode rejection) at 92 kHz. When one mode is rejected, the other becomes dominant. Hence, the frequency at which the S0 mode is dominant is $f^{S0} = 92\text{ kHz}$. At this tuning frequency, the S0 mode is indeed very little dispersive, as shown by the chart below.



As indicated on the chart, the actual value of the S0 wavespeed is $c^{S0}\big|_{f^{S0}=92\text{ kHz}} = 5405\text{ m/s} \simeq 5.4\text{ mm}/\mu\text{s}$. The associated wavelength is $\lambda = c/f \simeq 60\text{ mm}$. Hence, the optimum phased array pitch is $d = \lambda/2 \simeq 30\text{ mm}$.

(ii) The PWAS phased array will operate at the tuning frequency, $f^{S0} = 92\text{ kHz}$.

PROBLEM 13.2 SOLUTION

Units

Aluminum plate $E := 72.4 \cdot GPa$ $\rho := 2780 \cdot \frac{kg}{m^3}$ $\nu := 0.33$ $d := \frac{1}{2} \cdot mm$

Axial wave speed in aluminum strip $c := 5.4 \cdot \frac{mm}{\mu s}$ $c = 5.4 \frac{mm}{\mu s}$

$f := 92 \cdot kHz$

Wavelength $\lambda := \frac{c}{f}$ $d := \frac{\lambda}{2}$

$\lambda = 59\,mm$ $d = 29\,mm$

Problem 3: Outline the main advantages and disadvantages of the usage of 1-D and 2-D PWAS phased arrays

Solution

The 1-D PWAS phased arrays are discussed in the textbook Section 13.3 through 13.5. The 2-D PWAS phased arrays are discussed in the textbook Sections 13.7 and 13.8. The reader is expected to read these sections and extract the advantages and disadvantages of the usage of 1-D and 2-D PWAS phased arrays from this reading.

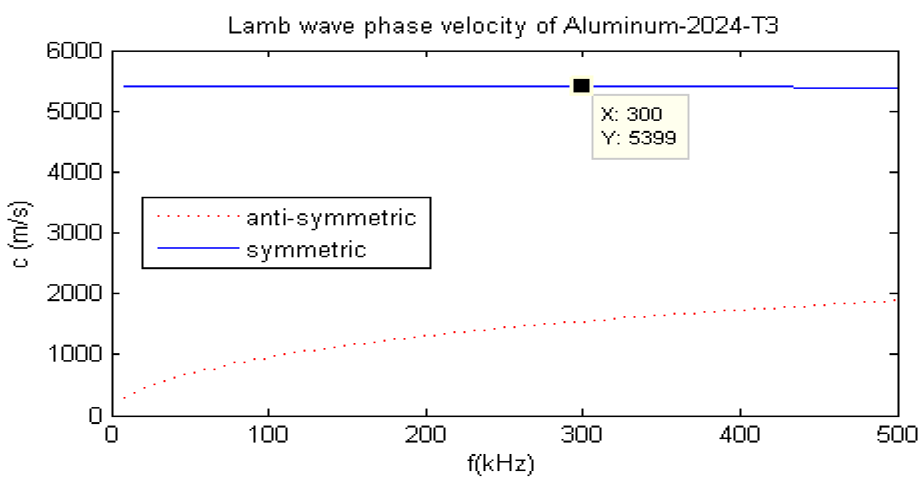
Problem 4: Assume a PWAS phased array tuned to a 300-kHz S0 Lamb wave in a 1-mm thick aluminum plate ($E = 72.4$ GPa, $\rho = 2780$ kg/m^2, $\nu = 0.33$). The array contains eight PWAS elements placed linearly at 7-mm pitch. The PWAS phased array is used to detect far-field damage. Calculate the array delay times required to steer the array beam in the 40-deg direction

Solution

The PWAS elements of the array are placed linearly, hence the 1-D array principles apply. The number of elements in the array is $M = 8$. The elements are placed at $d = 7$ mm pitch. The array is used to detect far-field damage; hence the simplified analysis of the textbook Section 13.3 applies. Under these assumptions, the beamforming delay is calculated with the textbook Eq. (13.11), i.e.,

$$\delta_m(\phi) = m\frac{d\cos\phi}{c} \qquad (13.11)\ (2)$$

where $m = 1, 2, ... M$ and c is the wave speed. To calculate the wave speed, we need to analyze the S0 Lamb wave in the plate. Using again the Wavescope program from the LAMSS website http://www.me.sc.edu/research/lamss/html/software.html , we get the chart below from which we read $c^{S0}\big|_{f^{S0}=300\text{ kHz}} = 5399$ m/s $\simeq$ 5.4 mm/μs



Upon calculation, we get:

$$\Delta^{T} = (0 \quad 0.992 \quad 1.984 \quad 2.976 \quad 3.968 \quad 4.96 \quad 5.951 \quad 6.943)\ \mu\text{s}$$

PROBLEM 13.4 SOLUTION

Units

Aluminum plate $E := 72.4 \cdot \mathrm{GPa}$ $\rho := 2780 \cdot \frac{\mathrm{kg}}{\mathrm{m}^3}$ $\nu := 0.33$ $d := \frac{1}{2} \cdot \mathrm{mm}$

Axial wave speed in aluminum strip $c := \sqrt{\frac{E}{\rho \cdot (1 - \nu^2)}}$ $c = 5406 \frac{\mathrm{m}}{\mathrm{s}}$

$d := 7 \cdot \mathrm{mm}$ $M := 8$ $im := 0 .. M - 1$

$\phi 0 := 40 \cdot \mathrm{deg}$

$\Delta_{im} := im \cdot \frac{d}{c} \cdot \cos(\phi 0)$

$\Delta_{im} =$

0.000	μs
0.992	
1.984	
2.976	
3.968	
4.960	
5.951	
6.943	

$\Delta^T = (0 \quad 0.992 \quad 1.984 \quad 2.976 \quad 3.968 \quad 4.96 \quad 5.951 \quad 6.943)\ \mu\mathrm{s}$

Problem 5: What are grating lobes and how do they appear?

Solution

The appearance and cause of the grating lobes is discussed in the textbook Section 13.10.2.6. The reader is encouraged to study this section and compose an answer showing understanding of the subject.

Problem 6: Describe the effect of spatial aliasing and give a numerical example.

Solution

The effect of spatial aliasing is discussed in the textbook Section 13.10.2. The reader is encouraged to study this section and compose an answer showing understanding of the subject. A numerical example of spatial aliasing is offered in the textbook Figure 13.68, which is reproduced below, i.e.,

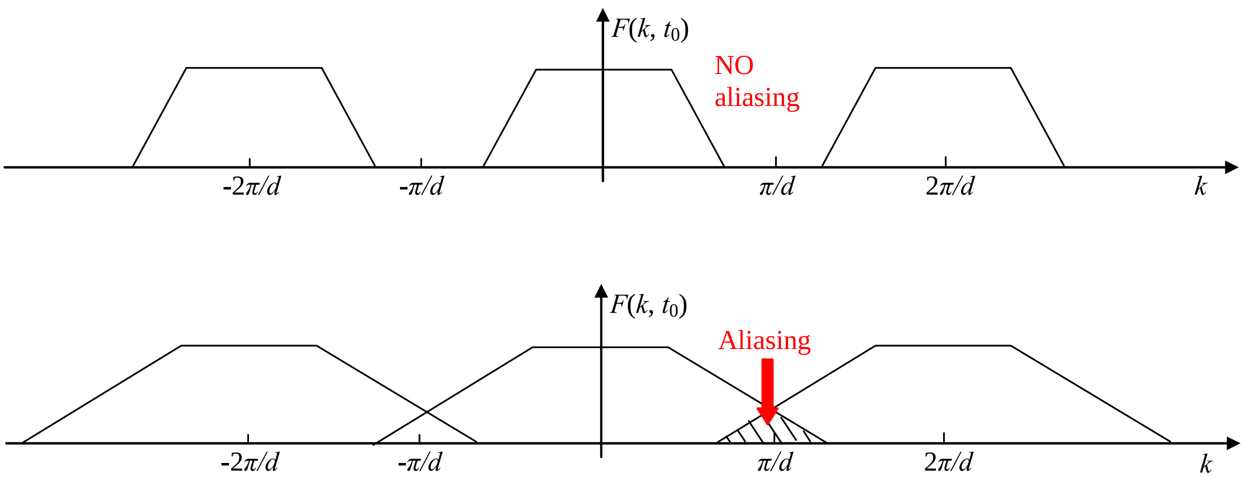


Figure 13.68 Aliasing occurs if the spatial signal is not spatially bandlimited below k_{NQ}

As a numerical example, consider first the situation discussed in Problem 13.1 above in which a phased array of 10-mm PWAS is bonded a 1-mm aluminum plate. The tuning frequency for S0 mode was found to be $f^{S0} = 92\text{ kHz}$ with a corresponding frequency $c^{S0}\big|_{f^{S0}=92\text{ kHz}} \simeq 5.4\text{ mm}/\mu\text{s}$. The associated S0 wavelength is $\lambda = c/f \simeq 60\text{ mm}$. The array pitch is $d = \lambda/2 \simeq 30\text{ mm}$. The spatial sampling frequency is $k_s = 2\pi/d = 209/\text{m}$. The Nyquist spatial frequency is $k_{NQ} = \pi/d \simeq 107/\text{m}$. Therefore, the space-time signal to be sampled with this phased-array should have wavenumber bandwidth $k_B < k_{NQ}$ i.e., $k_B < 107/\text{m}$.

In the limit, let's take k_B at it maximum permissible value, i.e., $k_B = 107/\text{m}$. For this value, we calculate the maximum permissible frequency

$$f_B = \frac{c\,k_B}{2\pi} \tag{3}$$

Upon numerical calculations, we get $f_B = 92\text{ kHz}$, which is the tuning frequency. This is not surprising, since we chose on purpose the phased array pitch to be at its optimal value, i.e., half wavelength, $d = \lambda/2 = 30\text{ mm}$. This fact highlights a shortcoming of using optimally designed phased arrays: the design is not robust, and is producing aliasing at slightly higher frequencies.

However, we can make the phased array less optimal, by taking, say, $d^* = 0.4\lambda = 23\text{ mm}$. In this case, the Nyquist frequency becomes $k^*_{NQ} = \pi / d^* = 134/\text{m}$. Subsequently, the permissible frequency bandwidth becomes $f^*_B = \frac{c\,k^*_B}{2\pi} = 115\text{ kHz}$. Thus, the aliasing frequency has been pushed up and the problem has been become more robust.

Another example to consider is that of using a phased array which was constructed around a certain wave type to detect other wave types. For example, the phase array discussed at the beginning was constructed for S0 waves at 92 kHz and had a wavenumber Nyquist frequency $k_{NQ} = 107/\text{m}$. However, the same phased array would be sensitive to A0 waves. The question is: *what is the frequency bandwidth limitation on A0 waves to avoid aliasing?*. To address this question, we have to calculate the maximum frequency that would produce a wavenumber k_{NQ}. Since A0 waves are dispersive, the wavespeed depends on the frequency. At this relatively low frequencies, the A0 mode is very dispersive and varies approximately like $\sqrt{f}$; hence we will take a $\sqrt{\omega}$ approximation of its variation over the frequency range of interest using as reference the A0 wavespeed at 100 kHz, i.e., $c^{A0}\big|_{f^{A0}=100\text{ kHz}} = 955\text{ m/s}$, i.e.,

$$c_{A0}(f) = \left(\sqrt{\frac{f}{100\text{ kHz}}}\right) 955\text{ m/s} \qquad\qquad 0 \le f \le 100\text{ kHz} \tag{4}$$

Substituting Eq. (4) into Eq. (3) we get

$$f_B = \frac{k_B}{2\pi} c_{A0}(f_B) \tag{5}$$

Upon solution, we find $f^{A0}_B \simeq 2.65\text{ kHz}$, which is much lower than the S0 frequency $f^{S0}_B = 92\text{ kHz}$. It is apparent that, if a phased array designed for S0 waves at 90 kHz were to be used with A0 waves, than its operating frequency would have to be limited to no more than $f^{A0}_B \simeq 2.65\text{ kHz}$ in order to avoid spatial aliasing.

PROBLEM 13.6 SOLUTION

Units

Aluminum plate $E := 72.4 \cdot \mathrm{GPa}$ $\rho := 2780 \cdot \frac{\mathrm{kg}}{\mathrm{m}^3}$ $\nu := 0.33$ $d1 := \frac{1}{2} \cdot \mathrm{mm}$

S0 wave speed in aluminum strip at A0 rejection frequency of 92 kHz is $c := 5.4 \cdot \frac{\mathrm{mm}}{\mu s}$ $c = 5.4 \frac{\mathrm{mm}}{\mu s}$

$fS0 := 92 \cdot \mathrm{kHz}$ $f := fS0$ $\lambda := \frac{c}{f}$ $d := \frac{\lambda}{2}$

$\lambda = 59\,\mathrm{mm}$ $d = 29\,\mathrm{mm}$

$ks := \frac{2 \cdot \pi}{d}$ $kNQ := \frac{\pi}{d}$

$ks = 214 \frac{1}{\mathrm{m}}$ $kNQ = 107 \frac{1}{\mathrm{m}}$

In the limit, kB is kNQ:

$kB := kNQ$ $\lambda B := \frac{2 \cdot \pi}{kB}$ $\lambda B = 59\,\mathrm{mm}$

If we consider the axial (S0) waves only, then the upper frequency at which this may work with axial waves is:

$fB := \frac{c \cdot kB}{2 \cdot \pi}$ $fB = 92\,\mathrm{kHz}$

It is apparent that this upper frequency for the S0 waves is the same as the tuning frequency, which is not surprising since the phased array was designed for S0 waves tuned at 90 kHz.

This aspect highlights an inherent shortcomings of building an optimal array, i.e. taking the optimal half wavelength value for the array pitch d. If one selects the optimum pitch equal to half the tuning wavelength, then one is limited by the fact that frequencies above the tuning frequency will be aliased. To overcome this effect, it would be more prudent to keep the pitch below the optimum pitch (sacrifizing some aperture in the process), and thus obtain a higher sampling and hence Nyquist frequency. For example, if we take the pitch to be 0.4 λ, then we get:

$d' := 0.4 \cdot \lambda$ $kNQ' := \frac{\pi}{d'}$ $kB' := kNQ'$ $fB' := \frac{c \cdot kB'}{2 \cdot \pi}$

$d' = 23\,\mathrm{mm}$ $kNQ' = 134 \frac{1}{\mathrm{m}}$ $fB' = 115\,\mathrm{kHz}$

However, the same phased array will also sense other waves, e.g., A0 waves.

$$cA0\_100kHz := 955 \cdot \frac{m}{s}$$

$$cA0(f) := cA0\_100kHz \cdot \sqrt{\frac{f}{100 \cdot kHz}}$$

Solving the wavenumber equation for f, one gets:

$$fB\_A0(f) := \frac{kB}{2 \cdot \pi} \cdot cA0(f) \qquad fB\_A0(2.65 \cdot kHz) = 2.65\,kHz$$