CHAPTER 3 -- PROBLEMS AND EXERCISES

Problem 1: Prove that $u(t) = A\cos\omega_n t + B\sin\omega_n t$ can be also expressed as $u(t) = C\cos(\omega_n t + \psi)$, and find the relationship between A, B, C, and ψ

<u>Solution</u>

Expand $u(t) = C\cos(\omega_n t + \psi)$ to get

$$u(t) = C\cos(\omega_n t + \psi) = C\cos\omega_n t\cos\psi - C\sin\omega_n t\sin\psi$$

Group coefficients to get

$$u(t) = \left(C\cos\psi\right)\cos\omega_n t + \left(-C\sin\psi\right)\sin\omega_n t$$

Identify the coefficients of $\cos\omega_n t$ and $\sin\omega_n t$ to obtain:

$A = C\cos\psi$ and $B = -C\sin\psi$

Resolve to obtain

$$C = \sqrt{A^2 + B^2}$$

$\psi = \text{angle}(A, -B)$ or $\psi = \arg(A - iB)$

PROBLEM 3.1 SOLUTION

Here are some examples of using the absolute value and angle or arg functions:

$A := 3 \quad B := 4 \qquad \arg(A + -i\cdot B) = -53.13\,\mathrm{deg} \qquad |A + -i\cdot B| = 5$

$\mathrm{angle}(A, -B) = 306.87\,\mathrm{deg}$

$A := 3 \quad B := -4 \qquad \arg(A + -i\cdot B) = 53.13\,\mathrm{deg} \qquad |A + -i\cdot B| = 5$

$\mathrm{angle}(A, -B) = 53.13\,\mathrm{deg}$

$A := -3 \quad B := 4 \qquad \arg(A + -i\cdot B) = -126.87\,\mathrm{deg} \qquad |A + -i\cdot B| = 5$

$\mathrm{angle}(A, -B) = 233.13\,\mathrm{deg}$

$A := -3 \quad B := -4 \qquad \arg(A + -i\cdot B) = 126.87\,\mathrm{deg} \qquad |A + -i\cdot B| = 5$

$\mathrm{angle}(A, -B) = 126.87\,\mathrm{deg}$

Problem 2: Prove that $m\ddot{u}(t)+c\dot{u}(t)+ku(t)=0$ can be also expressed as $\ddot{u}(t)+2\zeta\omega_n\dot{u}(t)+\omega_n^2u(t)=0$ and derive the relations between the constants in the two equations

<u>Solution</u>

Start with

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=0$$

Divide by *m* to get

$$\ddot{u}(t)+\frac{c}{m}\dot{u}(t)+\frac{k}{m}u(t)=0$$

Recall Eq. (3.31), i.e., $c_{cr}=2\omega_n m=2\sqrt{mk}$ and $\zeta=c/c_{cr}$. express c as $c=\zeta c_{cr}=\zeta 2\omega_n m$. Then, recall Eq. (3.15), i.e., $\omega_n^2=\frac{k}{m}$. Upon substitution, get

$$\ddot{u}(t)+\frac{2\zeta\omega_n m}{m}\dot{u}(t)+\frac{k}{m}u(t)=0$$

and finally,

$$\ddot{u}(t)+2\zeta\omega_n\dot{u}(t)+\omega_n^2u(t)=0$$

Problem 3: Prove that $u(t) = C_1 e^{(-\zeta\omega_n + i\omega_d)t} + C_2 e^{(-\zeta\omega_n - i\omega_d)t}$ can be rewritten as $u(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t + \psi)$ and derive the relations between the constants in the two equations

<u>Solution</u>

Expand and group $u(t) = C_1 e^{(-\zeta\omega_n + i\omega_d)t} + C_2 e^{(-\zeta\omega_n - i\omega_d)t}$ to get

$$u(t) = C_1 e^{(-\zeta\omega_n + i\omega_d)t} + C_2 e^{(-\zeta\omega_n - i\omega_d)t} = e^{-\zeta\omega_n t}\left(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right)$$

Use Euler identity $e^{i\alpha} = \cos\alpha + i\sin\alpha$ to write

$$\begin{aligned} C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t} &= C_1(\cos\omega_d t + \sin\omega_d t) + C_2(\cos\omega_d t - \sin\omega_d t) \\ &= \left(C_1 + C_2\right)\cos\omega_d t + \left(C_1 - C_2\right)\sin\omega_d t \end{aligned}$$

Now, consider $u(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t + \psi)$ and expand it to get

$$\begin{aligned} u(t) &= Ce^{-\zeta\omega_n t}\cos(\omega_d t + \psi) = Ce^{-\zeta\omega_n t}\left(\cos\omega_d t\cos\psi - \sin\omega_d t\sin\psi\right) \\ &= e^{-\zeta\omega_n t}\left[\left(C\cos\psi\right)\cos\omega_d t + \left(-C\sin\psi\right)\sin\omega_d t\right] \end{aligned}$$

Identifying coefficients between the two expressions, we establish

$$\begin{aligned} C_1 + C_2 &= C\cos\psi \\ C_1 - C_2 &= -C\sin\psi \end{aligned}$$

Upon solution,

$$\begin{aligned} C_1 &= \tfrac{1}{2}C\left(\cos\psi - \sin\psi\right) \\ C_2 &= \tfrac{1}{2}C\left(\cos\psi + \sin\psi\right) \end{aligned}$$

Conversely,

$$C = \sqrt{\left(C_1 + C_2\right)^2 + \left(C_1 - C_2\right)^2}$$

$\psi = \text{angle}\left[\left(C_1 + C_2\right), \left(-C_1 + C_2\right)\right]$ or $\psi = \arg\left[\left(C_1 + C_2\right) + i\left(-C_1 + C_2\right)\right]$

Problem 4: Prove that when damping equals critical damping ($\zeta = 1$), the solution of $\ddot{u}(t) + 2\zeta\omega_n\dot{u}(t) + \omega_n^2 u(t) = 0$ is $u(t) = \left(C_1 + C_2 t\right)e^{-\omega_n t}$

Solution

Recall Eq. (3.30), $\ddot{u}(t) + 2\zeta\omega_n\dot{u}(t) + \omega_n^2 u(t) = 0$, and the characteristic equation (3.32), $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$. If $\zeta = 1$, then these two equations become

$$\ddot{u}(t) + 2\omega_n\dot{u}(t) + \omega_n^2 u(t) = 0$$

$$\lambda^2 + 2\omega_n\lambda + \omega_n^2 = 0$$

The characteristic equation has the double root, $\lambda_1 = \lambda_2 = -\omega_n$.

The general ODE theory shows that if the characteristic equation has a double root, say *a*, then both e^{at} and te^{at} are solutions of the ODE. Indeed, assume the ODE is $\ddot{u}(t) - 2a\dot{u}(t) + a^2 u(t) = 0$, which has the characteristic equation $\lambda^2 - 2a\lambda + a^2 = 0$ with the double root $\lambda_1 = \lambda_2 = a$. Let's verify that both $u_1 = e^{at}$ and $u_2 = te^{at}$ are solutions. The proof for u_1 is easily obtained through direct substitution and will not be elaborated here. The proof for u_2 is obtained by substitution as follows:

$$u_2 = te^{at}$$

$$u_2' = \left(te^{at}\right)' = e^{at} + tae^{at}$$

$$u_2'' = \left(e^{at} + tae^{at}\right)' = ae^{at} + ae^{at} + ta^2e^{at} = 2ae^{at} + ta^2e^{at}$$

The notations $(\ )'$ and $(\ )''$ were used to signify first and second derivatives. Upon substitution into the differential equation, we get

$$\left(2ae^{at} + ta^2e^{at}\right) - 2a\left(e^{at} + tae^{at}\right) + a^2te^{at}$$
$$= \cancel{2ae^{at}} + \cancel{ta^2e^{at}} - \cancel{2ae^{at}} - \cancel{2atae^{at}} + \cancel{a^2te^{at}} = 0$$

Thus we have proved that both $u_1 = e^{at}$ and $u_2 = te^{at}$ are solutions. Hence, the general solution is a linear combination of these two solutions, i.e.,

$$u(t) = \left(C_1 + C_2 t\right)e^{at}$$

To finalize the proof of the exercise, simply observer that $a = -\omega_n$. Hence, the general solution is

$$u(t) = \left(C_1 + C_2 t\right)e^{-\omega_n t}$$

Problem 5: Prove that the particular solution of $\ddot{u}(t)+\omega_n^2 u(t)=\hat{f}\cos\omega t$ is $u_p(t)=\frac{1}{-\omega^2+\omega_n^2}\hat{f}\cos\omega t$

<u>Solution</u>

By ODE theory, a particular solution is any solution that satisfies the inhomogeneous equation. One usually seeks particular solutions of the same for as the right hand side of the inhomogeneous equation. In our case, we seek a particular solution made up of trigonometric functions, i.e., of the form

$$u_p(t)=A\cos\omega t+B\sin\omega t$$

Upon substitution in the differential equation, we write

$$-\omega^2 A\cos\omega t-\omega^2 B\sin\omega t+\omega_n^2 A\cos\omega t+\omega_n^2 B\sin\omega t=\hat{f}\cos\omega t$$

Identifying coefficients of $\cos\omega t$ and $\sin\omega t$ we can solve for A,B to get

$$A=\frac{\hat{f}}{-\omega^2+\omega_n^2} \qquad B=0 \qquad (\omega\neq\omega_n)$$

Substitution of A,B gives the particular solution in the desired form, $u_p(t)=\frac{1}{-\omega^2+\omega_n^2}\hat{f}\cos\omega t$

Problem 6: Prove that using Eq. (3.106) in conjunction with Eqs. (3.15), (3.31), (3.97), (3.98), (3.100) yields the *response amplitude at the quadrature point* as $|\hat{u}_{90}| = \hat{F} / c\omega_n$

<u>Solution</u>

Recall

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{or} \quad \omega_n^2 = \frac{k}{m} \qquad (3.15)$$

$$\zeta = c / c_{cr} \qquad c_{cr} = 2\omega_n m = 2\sqrt{mk} \qquad (3.31)$$

$$u_{st} = \frac{\hat{F}}{k} \qquad (3.97)$$

$$p = \frac{\omega}{\omega_n} \qquad (3.98)$$

$$\hat{u}(p) = u_{st} H(p) \qquad (3.100)$$

$$|H(1)| = M_{90} = \frac{1}{2\zeta} \qquad (3.106)$$

Evaluating the magnitude of Eq. (3.100) at the quadrature point, $p = 1$, yields

$$|\hat{u}_{90}| = |\hat{u}(1)| = u_{st} |H(1)|$$

Using Eq. (3.106) gives

$$|\hat{u}_{90}| = |\hat{u}(1)| = u_{st} \frac{1}{2\zeta}$$

Substituting into Eq. (3.97) yields

$$|\hat{u}_{90}| = \frac{\hat{F}}{k} \frac{1}{2\zeta}$$

Using Eq. (3.31) gives

$$2\zeta k = 2\frac{c}{c_{cr}} k = 2\frac{c}{2\sqrt{mk}} k = c\sqrt{\frac{k}{m}} = c\omega_n$$

Upon substitution, we obtain the desired expression

$$|\hat{u}_{90}| = \frac{\hat{F}}{c\omega_n}$$

Problem 7: Prove that, for lightly damped systems, the bandwidth of the frequency response function $H(p)=\dfrac{1}{-p^2+i2\zeta p+1}$ takes the simple expression $\Delta\omega=\omega_U-\omega_L\cong 2\zeta\omega_n$.

Solution

Recall the bandwidth expression of Eq. (3.112), i.e., $\Delta\omega=\omega_U-\omega_L$ where ω_U and ω_L are the lower and upper *half-power frequencies* (3 *dB points*) located to the left and right of the resonance frequency. The half-power points correspond to points where the amplitude has decreased by 3 dB i.e., by a factor $\sqrt{2}$. The amplitude of the frequency response function $H(p)=\dfrac{1}{-p^2+i2\zeta p+1}$ is given by Eq. (3.101), i.e., $|H(p)|=\dfrac{1}{\sqrt{\left(1-p^2\right)^2+4\zeta^2p^2}}$.

For lightly damped systems, the amplitude at resonance is well approximated by the amplitude at $p=1$, which is $|H(1)|=1/2\zeta$. At the half-power points, the amplitude is decreased by a factor of $\sqrt{2}$, i.e., $|H(p_1)|=|H(p_2)|=\dfrac{1}{2\sqrt{2}\zeta}$

Imposing this condition, yields the equation

$$\sqrt{\left(1-p^2\right)^2+4\zeta^2p^2}=2\sqrt{2}\zeta$$

Hence, we have to solve the equation

$$\left(1-p^2\right)^2+4\zeta^2p^2=8\zeta^2$$

Upon expansion, we get

$$\left(1-p^2\right)^2+4\zeta^2p^2-8\zeta^2=0$$

or

$$1-2p^2+p^4+4\zeta^2p^2-8\zeta^2=0$$

or

$$p^4-2(1-2\zeta^2)p^2+1-8\zeta^2=0$$

We solve this quadratic equation in p^2, i.e.,

$$(p^2)_{1,2}=\left(1-2\zeta^2\right)\pm\sqrt{\left(1-2\zeta^2\right)^2-\left(1-8\zeta^2\right)}$$

Hence,

$$p_2^2-p_1^2=2\sqrt{\left(1-2\zeta^2\right)^2-\left(1-8\zeta^2\right)}$$

Upon expansion

$$p_2^2 - p_1^2 = 2\sqrt{\cancel{1} - 4\zeta^2 + 4\zeta^4 - \cancel{1} + 8\zeta^2}$$
$$= 2\sqrt{4\zeta^2 + 4\zeta^4}$$

Using the light damping approximation ($\zeta << 1$) we ignore the term in ζ^4 and write

$$p_2^2 - p_1^2 = 2\sqrt{4\zeta^2 + \cancel{4\zeta^4}} \cong 2\sqrt{4\zeta^2}$$

The left hand side can be expanded into sum and difference product, i.e.,

$$\left(p_2 + p_1\right)\left(p_2 - p_1\right) \cong 2\sqrt{4\zeta^2}$$

For light damping, the p_1 and p_2 values are approximately balanced about the resonance point $p = 1$, and thus $p_1 + p_2 \cong 2$. Hence, the above expression becomes

$$\left(p_2 - p_1\right) \cong \sqrt{4\zeta^2} = 2\zeta$$

In terms of physical frequencies, the above expression is written as

$$\Delta\omega = \omega_U - \omega_L \cong 2\zeta\omega_n$$

The proof is complete.

PROBLEM 3.7 SOLUTION

Here is an example of plotting the H function for various damping ratios

$I := 3 \qquad N := 100 \qquad emin := -2 \qquad emax := -emin \qquad de := \frac{emax - emin}{N \cdot I}$

$j := 0..3 \qquad \zeta_0 := 0.1 \qquad \zeta_1 := 0.2 \qquad \zeta_2 := 0.5 \qquad \zeta_3 := 0.7 \qquad \zeta = \begin{pmatrix} 0.2 \\ 0.5 \\ 0.7 \end{pmatrix}$

$i := 0..I \cdot N \qquad e_i := emin + i \cdot de \qquad p_i := 10^{e_i}$

create a power of 10 range of p values to get sufficient points around the peak

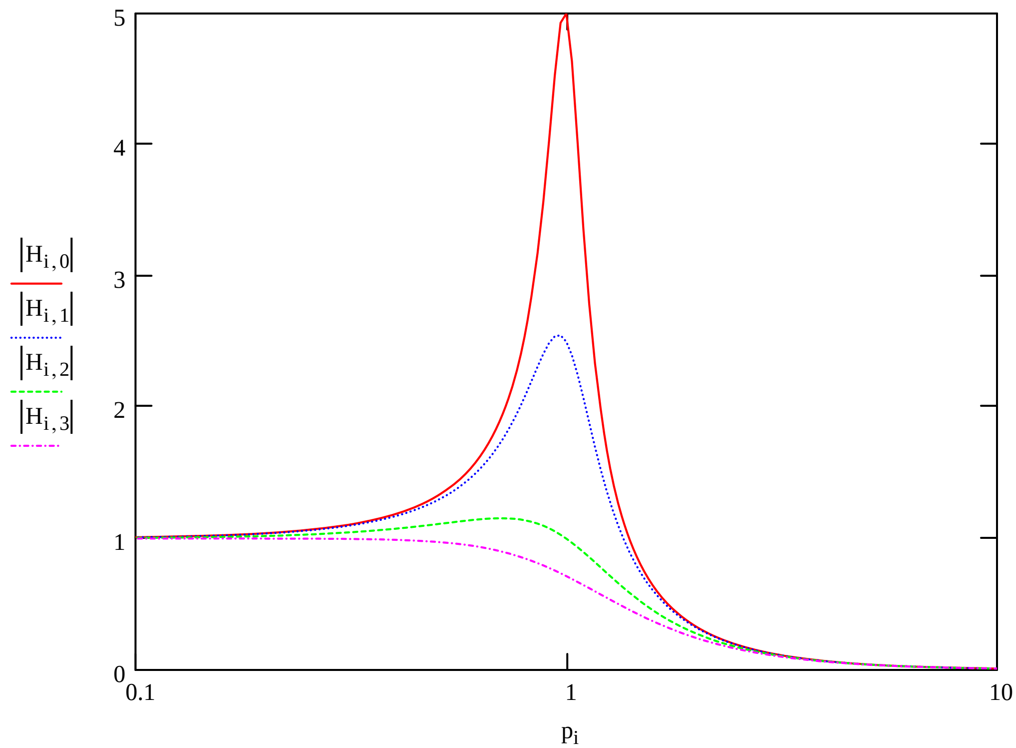
p =

	0
0	0.01
1	0.01
2	0.011
3	0.011
4	0.011
5	0.012
6	0.012
7	0.012

e =

	0
0	-2
1	-1.987
2	-1.973
3	-1.96
4	-1.947

$$H_{i,j} := \frac{1}{-(p_i)^2 + i \cdot 2 \cdot \zeta_j \cdot p_i + 1}$$



Problem 8: Prove that the power at resonance of a lightly-damped 1-dof system is given by $P_{\max} = \tfrac{1}{2}\hat{F}^2 \big/ c$

Solution

Recall Eq. (3.156) which gives the power at resonance as

$$P_{\max} = \tfrac{1}{2} c\omega_r^2 \left|\hat{u}_r\right|^2$$

For lightly damped systems, the resonance point can be sufficiently well approximated by the quadrature point, i.e., $\omega_r \cong \omega_{90} = \omega_n$, $\hat{u}_r \cong \hat{u}_{90}$. Hence,

$$P_{\max} \cong \tfrac{1}{2} c\omega_n^2 \left|\hat{u}_{90}\right|^2$$

The response at quadrature point, as proven in Problem 6 above, is given by

$$\left|\hat{u}_{90}\right| = \frac{\hat{F}}{c\omega_n}$$

Upon substitution, we get

$$P_{\max} \cong \tfrac{1}{2} c\omega_n^2 \left(\frac{\hat{F}}{c\omega_n}\right)^2 = \tfrac{1}{2} c\omega_n^2 \frac{\hat{F}^2}{c^2\omega_n^2} = \frac{1}{2}\frac{\hat{F}^2}{c}$$

Problem 9: Find the first, second, and third natural frequencies of in-plane axial vibration of a steel beam of thickness h_1 = 2.6 mm, width b_1 = 8 mm, length l = 100 mm, modulus E = 200 GPa, and density ρ = 7.750 g/cm^3. The beam is in free-free boundary conditions. Then, consider double the thickness (h_2 = 5.2 mm), wider width (b_2 = 19.6 mm), and then both. Recalculate the three frequencies for these other combinations of thickness and width. Discuss your results

Solution

Recall Eq. (3.192), i.e., $f_j = j\frac{1}{2l}\sqrt{\frac{EA}{m}}$, $j = 1,2,3$.

Use geometric dimensions and material properties to calculate

$A = 20.8\ \text{mm}^2$; $EA = 4.16\ \text{MN}$; $m = 0.161\ \text{kg/m}$

Substitute in the frequency equation to get

$f_1 = 25.4\ \text{kHz}$; $f_2 = 50.8\ \text{kHz}$; $f_3 = 76.2\ \text{kHz}$

Double the thickness

$A = 41.6\ \text{mm}^2$; $EA = 8.32\ \text{MN}$; $m = 0.322\ \text{kg/m}$

Substitute in the frequency equation to get

$f_1 = 25.4\ \text{kHz}$; $f_2 = 50.8\ \text{kHz}$; $f_3 = 76.2\ \text{kHz}$

The frequencies do not change because the changes in EA are compensated by the changes in m.

Double the width

$A = 41.6\ \text{mm}^2$; $EA = 8.32\ \text{MN}$; $m = 0.322\ \text{kg/m}$

Substitute in the frequency equation to get

$f_1 = 25.4\ \text{kHz}$; $f_2 = 50.8\ \text{kHz}$; $f_3 = 76.2\ \text{kHz}$

The frequencies do not change because the changes in EA are compensated by the changes in m.

Double the thickness and the width

$A = 83.2\ \text{mm}^2$; $EA = 16.64\ \text{MN}$; $m = 0.645\ \text{kg/m}$

Substitute in the frequency equation to get

$f_1 = 25.4\ \text{kHz}$; $f_2 = 50.8\ \text{kHz}$; $f_3 = 76.2\ \text{kHz}$

The frequencies do not change because the changes in EA are compensated by the changes in m.

PROBLEM 3.9 SOLUTION

AXIAL VIBRATION OF A STEEL BEAM

ORIGIN := 1

$h1 := 2.6\cdot 10^{-3}$ $\quad b1 := 8\cdot 10^{-3}$ $\quad L := 100\cdot 10^{-3}$ $\quad E := 200\cdot 10^{9}$ $\quad \rho := 7750$

$h := h1 \quad b := b1$

==========

$A := b\cdot h \qquad EA := E\cdot A \qquad m := \rho\cdot A$

$A = 20.800\,10^{-6} \qquad EA = 4.160\times 10^{6} \qquad m = 0.161$

$j := 1..3$

$$f_j := j\cdot\frac{1}{2\cdot L}\cdot\sqrt{\frac{EA}{m}} \qquad f = \begin{pmatrix}25.4\\50.8\\76.2\end{pmatrix}10^{3}$$

$h := 2\cdot h1 \quad b := b1$

$A := b\cdot h \qquad EA := E\cdot A \qquad m := \rho\cdot A$

$A = 41.600\,10^{-6} \qquad EA = 8.320\times 10^{6} \qquad m = 0.322$

$j := 1..3$

$$f_j := j\cdot\frac{1}{2\cdot L}\cdot\sqrt{\frac{EA}{m}} \qquad f = \begin{pmatrix}25.4\\50.8\\76.2\end{pmatrix}10^{3}$$

$h := h1 \quad b := 2\cdot b1$

$A := b\cdot h \qquad EA := E\cdot A \qquad m := \rho\cdot A$

$A = 41.600\,10^{-6} \qquad EA = 8.320\times 10^{6} \qquad m = 0.322$

$j := 1..3$

$$f_j := j\cdot\frac{1}{2\cdot L}\cdot\sqrt{\frac{EA}{m}} \qquad f = \begin{pmatrix}25.4\\50.8\\76.2\end{pmatrix}10^{3}$$

$h := 2\cdot h1 \quad b := 2\cdot b1$

$A := b\cdot h \qquad EA := E\cdot A \qquad m := \rho\cdot A$

$A = 83.200\,10^{-6} \qquad EA = 1.664\times 10^{7} \qquad m = 0.645$

$j := 1..3$

$$f_j := j\cdot\frac{1}{2\cdot L}\cdot\sqrt{\frac{EA}{m}} \qquad f = \begin{pmatrix}25.4\\50.8\\76.2\end{pmatrix}10^{3}$$

Problem 10: Find all the natural frequencies in the interval 1 kHz to 30 kHz of in-plane axial vibration of a steel beam of thickness h_1 = 2.6 mm, width b_1 = 8 mm, length l = 100 mm, modulus E = 200 GPa, and density ρ = 7.750 g/cm^3. The beam is in free-free boundary conditions. Then, consider double the thickness (h_2 = 5.2 mm), wider width (b_2 = 19.6 mm), and then both. Recalculate the frequencies for these other combinations of thickness and width. Discuss your results

<u>Solution</u>

In view of problem 10, the only axial frequency in the interval 1 kHz to 30 kHz is $f_1 = 25.4$ kHz . Since the axial frequencies are not affected by changes in thickness and width, this frequency is going to be the same whether one doubles the thickness, the width, or both.

Problem 11: Find the first, second, and third natural frequencies of out-of-plane flexural vibration of a steel beam of thickness h_1 = 2.6 mm, width b_1 = 8 mm, length l = 100 mm, modulus E = 200 GPa, and density ρ = 7.750 g/cm^3. The beam is in free-free boundary conditions. Then, consider double the thickness (h_2 = 5.2 mm), wider width (b_2 = 19.6 mm), and then both. Recalculate the three frequencies for these other combinations of thickness and width. Discuss your results

<u>Solution</u>

Recall Eq. (3.408), i.e., $f_j = \dfrac{1}{2\pi} z_j^2 \sqrt{\dfrac{EI}{ml^4}}$ $\quad j = 1,2,3$

Use geometric dimensions and material properties to calculate

$A = 20.8\ \text{mm}^2$; $I = 11.717\ \text{mm}^4$; $EI = 2.343\ \text{Nm}^2$; $m = 0.161$ kg/m

Get the values of γl from Table 3.5. Substitute in the frequency equation to get

$f_1 = 1.358$ kHz ; $f_2 = 3.742$ kHz ; $f_3 = 7.337$ kHz

<u>Double the thickness</u>

$A = 41.6\ \text{mm}^2$; $I = 93.74\ \text{mm}^4$; $EI = 18.748\ \text{Nm}^2$; $m = 0.322$ kg/m

Get the values of γl from Table 3.5. Substitute in the frequency equation to get

$f_1 = 2.715$ kHz ; $f_2 = 7.485$ kHz ; $f_3 = 14.674$ kHz

The frequencies have increased because *EI* increases as h^3 whereas *m* increases only as *h*. The faster increase in *EI* has produced increase in frequency.

<u>Double the width</u>

$A = 41.6\ \text{mm}^2$; $I = 23.43\ \text{mm}^4$; $EI = 4.687\ \text{Nm}^2$; $m = 0.322$ kg/m

Get the values of γl from Table 3.5. Substitute in the frequency equation to get

$f_1 = 1.358$ kHz ; $f_2 = 3.742$ kHz ; $f_3 = 7.337$ kHz

The frequencies have not increased because both *EI* and *m* increase as *b*.

<u>Double the thickness and the width</u>

$A = 83.2\ \text{mm}^2$; $I = 187.48\ \text{mm}^4$; $EI = 37.495\ \text{Nm}^2$; $m = 0.645$ kg/m

Get the values of γl from Table 3.5. Substitute in the frequency equation to get

$f_1 = 2.715$ kHz ; $f_2 = 7.485$ kHz ; $f_3 = 14.674$ kHz

The frequencies have increased in the same amount as for just double the thickness *h*. This is because *EI* increases as bh^3 whereas *m* increases as bh, indicating that thickness increase affects the flexural frequencies but width increase does not.

PROBLEM 3.11 SOLUTION

$ORIGIN := 1$

$h1 := 2.6\cdot10^{-3}$ $b1 := 8\cdot10^{-3}$ $L := 100\cdot10^{-3}$ $E := 200\cdot10^{9}$ $\rho := 7750$

$h := h1$ $b := b1$

$A := b\cdot h$ $I := \dfrac{b\cdot h^3}{12}$ $EI := E\cdot I$ $m := \rho\cdot A$

$A = 20.800\,10^{-6}$ $I = 11.717\,10^{-12}$ $EI = 2.343$ $m = 0.161$

$j := 1..3$ $\gamma L_1 := 4.73004074$ $\gamma L_2 := 7.85320462$ $\gamma L_3 := 10.9956078$

natural frequencies

$$f_j := \frac{1}{2\cdot\pi}\cdot\left(\gamma L_j\right)^2\cdot\sqrt{\frac{EI}{m\cdot L^4}} \qquad f = \begin{pmatrix} 1.358 \\ 3.742 \\ 7.337 \end{pmatrix} 10^3$$

$h := 2\cdot h1$ $b := b1$

$A := b\cdot h$ $I := \dfrac{b\cdot h^3}{12}$ $EI := E\cdot I$ $m := \rho\cdot A$

$A = 41.600\,10^{-6}$ $I = 93.74\,10^{-12}$ $EI = 18.748$ $m = 0.322$

$j := 1..3$ $\gamma L_1 := 4.73004074$ $\gamma L_2 := 7.85320462$ $\gamma L_3 := 10.9956078$

$$f_j := \frac{1}{2\cdot\pi}\cdot\left(\gamma L_j\right)^2\cdot\sqrt{\frac{EI}{m\cdot L^4}} \qquad f = \begin{pmatrix} 2.715 \\ 7.485 \\ 14.674 \end{pmatrix} 10^3$$

$h := h1 \qquad b := 2\cdot b1$

$A := b\cdot h \qquad I := \frac{b\cdot h^3}{12} \qquad EI := E\cdot I \qquad m := \rho\cdot A$

$A = 41.600\ 10^{-6} \qquad I = 23.43\ 10^{-12} \qquad EI = 4.687 \qquad m = 0.322$

$j := 1 .. 3 \qquad \gamma L_1 := 4.73004074 \qquad \gamma L_2 := 7.85320462 \qquad \gamma L_3 := 10.9956078$

$$f_j := \frac{1}{2\cdot\pi}\cdot\left(\gamma L_j\right)^2\cdot\sqrt{\frac{EI}{m\cdot L^4}} \qquad f = \begin{pmatrix} 1.358 \\ 3.742 \\ 7.337 \end{pmatrix} 10^3$$

$h := 2\cdot h1 \qquad b := 2\cdot b1$

$A := b\cdot h \qquad I := \frac{b\cdot h^3}{12} \qquad EI := E\cdot I \qquad m := \rho\cdot A$

$A = 83.200\ 10^{-6} \qquad I = 187.48\ 10^{-12} \qquad EI = 37.495 \qquad m = 0.645$

$j := 1 .. 3 \qquad \gamma L_1 := 4.73004074 \qquad \gamma L_2 := 7.85320462 \qquad \gamma L_3 := 10.9956078$

$$f_j := \frac{1}{2\cdot\pi}\cdot\left(\gamma L_j\right)^2\cdot\sqrt{\frac{EI}{m\cdot L^4}} \qquad f = \begin{pmatrix} 2.715 \\ 7.485 \\ 14.674 \end{pmatrix} 10^3$$

Problem 12: Find all the natural frequencies in the interval 1 KHz to 30 kHz of out-of-plane flexural vibration of a steel beam of thickness h_1 = 2.6 mm, width b_1 = 8 mm, length l = 100 mm, modulus E = 200 GPa, and density ρ = 7.750 g/cm^3. The beam is in free-free boundary conditions. Then, consider double the thickness (h_2 = 5.2 mm), wider width (b_2 = 19.6 mm), and then both. Recalculate the three frequencies for these other combinations of thickness and width. Discuss your results

Solution

Recall Eq. (3.408), i.e., $f_j = \dfrac{1}{2\pi} z_j^2 \sqrt{\dfrac{EI}{ml^4}}$ $\quad j = 1,2,3\ldots$

Use geometric dimensions and material properties to calculate

$A = 20.8\ \text{mm}^2$; $I = 11.717\ \text{mm}^4$; $EI = 2.343\ \text{Nm}^2$; $m = 0.161\ \text{kg/m}$

Get the values of γl from Table 3.5. Substitute in the frequency equation to get

$f_1 = 1.358\ \text{kHz}$; $f_2 = 3.742\ \text{kHz}$; $f_3 = 7.337\ \text{kHz}$

$f_4 = 12.128\ \text{kHz}$; $f_5 = 18.117\ \text{kHz}$; $f_6 = 25.304\ \text{kHz}$

Only the first six frequencies are in the bandwidth of interest (1-30 kHz). The next frequency, $f_7 = 33.689\ \text{kHz}$, is outside the bandwidth of interest.

Double the thickness

$A = 41.6\ \text{mm}^2$; $I = 93.74\ \text{mm}^4$; $EI = 18.748\ \text{Nm}^2$; $m = 0.322\ \text{kg/m}$

Get the values of γl from Table 3.5. Substitute in the frequency equation to get

$f_1 = 2.715\ \text{kHz}$; $f_2 = 7.485\ \text{kHz}$; $f_3 = 14.674\ \text{kHz}$

$f_4 = 24.256\ \text{kHz}$

Only the first four frequencies are in the bandwidth of interest (1-30 kHz). The next frequency, $f_5 = 36.234\ \text{kHz}$, is outside the bandwidth of interest.

Double the width

$A = 41.60\ \text{mm}^2$; $I = 23.43\ \text{mm}^4$; $EI = 4.687\ \text{Nm}^2$; $m = 0.322\ \text{kg/m}$

Get the values of γl from Table 3.5. Substitute in the frequency equation to get

$f_1 = 1.358\ \text{kHz}$; $f_2 = 3.742\ \text{kHz}$; $f_3 = 7.337\ \text{kHz}$

$f_4 = 12.128\ \text{kHz}$; $f_5 = 18.117\ \text{kHz}$; $f_6 = 25.304\ \text{kHz}$

Only the first six frequencies are in the bandwidth of interest (1-30 kHz). The next frequency, $f_7 = 33.689\ \text{kHz}$, is outside the bandwidth of interest. Note that the situation is similar with the original situation since doubling the width does not change the frequencies, as shown in problem 11.

Double the thickness and the with

$A = 83.2\ \text{mm}^2$; $I = 187.48\ \text{mm}^4$; $EI = 37.495\ \text{Nm}^2$; $m = 0.645\ \text{kg/m}$

Get the values of γl from Table 3.5. Substitute in the frequency equation to get

$f_1 = 2.715\ \text{kHz}$; $f_2 = 7.485\ \text{kHz}$; $f_3 = 14.674\ \text{kHz}$

$f_4 = 24.256\ \text{kHz}$

Only the first four frequencies are in the bandwidth of interest (1-30 kHz). The next frequency, $f_5 = 36.234\ \text{kHz}$, is outside the bandwidth of interest. Note that the situation is similar with the double the thickness situation since only the thickness influences the frequencies, as shown in Problem 11.

PROBLEM 3.12 SOLUTION

$ORIGIN := 1$

$h1 := 2.6\cdot 10^{-3}$ $b1 := 8\cdot 10^{-3}$ $L := 100\cdot 10^{-3}$ $E := 200\cdot 10^{9}$ $\rho := 7750$

$h := h1$ $b := b1$

$A := b\cdot h$ $I := \dfrac{b\cdot h^3}{12}$ $EI := E\cdot I$ $m := \rho\cdot A$

$A = 20.800\, 10^{-6}$ $I = 11.717\, 10^{-12}$ $EI = 2.343$ $m = 0.161$

$j := 1..7$ $\gamma LL(j) := (2\cdot j + 1)\cdot \dfrac{\pi}{2}$

$\gamma L_1 := 4.73004074$ $\gamma L_2 := 7.85320462$ $\gamma L_3 := 10.9956078$

$\gamma L_4 := 14.1371655$ $\gamma L_5 := 17.2787597$ $\gamma L_6 := \gamma LL(6)$ $\gamma L_7 := \gamma LL(7)$

$$f_j := \frac{1}{2\cdot\pi}\cdot\left(\gamma L_j\right)^2\cdot\sqrt{\frac{EI}{m\cdot L^4}}$$

$f_j =$ 10^3

1.358
3.742
7.337
12.128
18.117
25.304
33.689

$h := 2\cdot h1$ $b := b1$

$A := b\cdot h$ $I := \dfrac{b\cdot h^3}{12}$ $EI := E\cdot I$ $m := \rho\cdot A$

$A = 41.600\, 10^{-6}$ $I = 93.74\, 10^{-12}$ $EI = 18.748$ $m = 0.322$

$$f_j := \frac{1}{2\cdot\pi}\cdot\left(\gamma L_j\right)^2\cdot\sqrt{\frac{EI}{m\cdot L^4}} \qquad f = \begin{pmatrix} 2.715 \\ 7.485 \\ 14.674 \\ 24.256 \\ 36.234 \\ 50.608 \\ 67.378 \end{pmatrix} 10^3$$

$h := h1 \qquad b := 2 \cdot b1$

$A := b \cdot h \qquad I := \frac{b \cdot h^3}{12} \qquad EI := E \cdot I \qquad m := \rho \cdot A$

$A = 41.600\, 10^{-6} \qquad I = 23.43\, 10^{-12} \qquad EI = 4.687 \qquad m = 0.322$

$$f_j := \frac{1}{2 \cdot \pi} \cdot \left(\gamma L_j\right)^2 \cdot \sqrt{\frac{EI}{m \cdot L^4}} \qquad f = \begin{pmatrix} 1.358 \\ 3.742 \\ 7.337 \\ 12.128 \\ 18.117 \\ 25.304 \\ 33.689 \end{pmatrix} 10^3$$

$h := 2 \cdot h1 \qquad b := 2 \cdot b1$

$A := b \cdot h \qquad I := \frac{b \cdot h^3}{12} \qquad EI := E \cdot I \qquad m := \rho \cdot A$

$A = 83.200\, 10^{-6} \qquad I = 187.48\, 10^{-12} \qquad EI = 37.495 \qquad m = 0.645$

$$f_j := \frac{1}{2 \cdot \pi} \cdot \left(\gamma L_j\right)^2 \cdot \sqrt{\frac{EI}{m \cdot L^4}} \qquad f = \begin{pmatrix} 2.715 \\ 7.485 \\ 14.674 \\ 24.256 \\ 36.234 \\ 50.608 \\ 67.378 \end{pmatrix} 10^3$$

Problem 13: Consider SH vibration of a steel strip of thickness $h_1 = 2.6 \text{ mm}$, width $b_1 = 8 \text{ mm}$, length $l = 100 \text{ mm}$, elastic modulus $E = 200 \text{ GPa}$, Poisson ratio $\nu = 0.29$, mass density $\rho = 7,750 \text{ kg/m}^3$. The strip is in free-free boundary conditions. Find the first, second, and third natural frequencies of SH vibration. Sketch the modeshapes. Next, consider double the thickness ($h_2 = 5.2 \text{ mm}$), wider width ($b_2 = 19.6 \text{ mm}$), and then both. Recalculate the frequencies for these other combinations of thickness and width. Discuss your results

<u>Solution</u>

Recall Eq. (3.509), i.e., $f_j = j\dfrac{1}{2l}\sqrt{\dfrac{GA}{m}}$, $j = 1,2,3.$

Use geometric dimensions and material properties to calculate

$A = 20.8 \text{ mm}^2$; $GA = 1.612 \text{ MN}$; $m = 0.161 \text{ kg/m}$

Substitute in the frequency equation to get

$f_1 = 15.8 \text{ kHz}$; $f_2 = 31.6 \text{ kHz}$; $f_3 = 47.4 \text{ kHz}$

<u>Double the thickness</u>

$A = 41.6 \text{ mm}^2$; $GA = 3.225 \text{ MN}$; $m = 0.322 \text{ kg/m}$

Substitute in the frequency equation to get

$f_1 = 15.8 \text{ kHz}$; $f_2 = 31.6 \text{ kHz}$; $f_3 = 47.4 \text{ kHz}$

The frequencies do not change because the changes in *EA* are compensated by the changes in *m*.

<u>Double the width</u>

$A = 41.6 \text{ mm}^2$; $GA = 3.225 \text{ MN}$; $m = 0.322 \text{ kg/m}$

Substitute in the frequency equation to get

$f_1 = 15.8 \text{ kHz}$; $f_2 = 31.6 \text{ kHz}$; $f_3 = 47.4 \text{ kHz}$

The frequencies do not change because the changes in *EA* are compensated by the changes in *m*.

<u>Double the thickness and the width</u>

$A = 83.2 \text{ mm}^2$; $GA = 6.450 \text{ MN}$; $m = 0.645 \text{ kg/m}$

Substitute in the frequency equation to get

$f_1 = 15.8 \text{ kHz}$; $f_2 = 31.6 \text{ kHz}$; $f_3 = 47.4 \text{ kHz}$

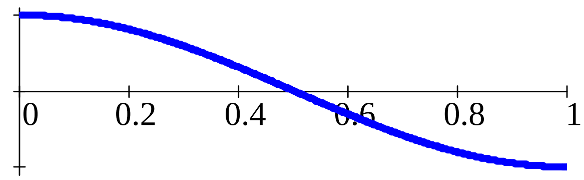
The frequencies do not change because the changes in *EA* are compensated by the changes in *m*.

The modeshapes are given by Eq. (3.515), i.e.,

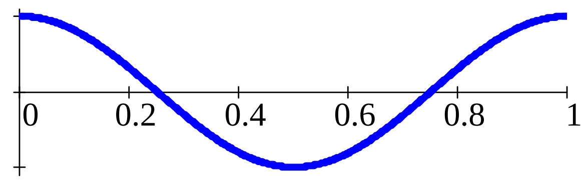
$$V_j(x)=\sqrt{\frac{2}{ml}}\cos\gamma_j x, \quad j=1,2,3,... \qquad (3.515)$$

The first, second, and third modeshapes are sketched below.

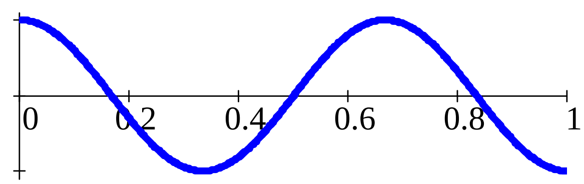
1st modeshape for $f_1 = 15.8$ kHz



2nd modeshape for $f_2 = 31.6$ kHz



3rd modeshape for $f_3 = 47.4$ kHz



PROBLEM 3.13

ORIGIN := 1

SH VIBRATION

$h1 := 2.6\cdot 10^{-3}$ $\quad b1 := 8\cdot 10^{-3}$ $\quad L := 100\cdot 10^{-3}$ $\quad E := 200\cdot 10^{9}$ $\quad \nu := 0.29$ $\quad \rho := 7750$

$G := \frac{E}{2\cdot(1+\nu)}$ $\quad G = 7.752\times 10^{10}$ $\quad cS := \sqrt{\frac{G}{\rho}}$ $\quad cS = 3.163\times 10^{3}$

$h := h1 \quad b := b1$

$A := b\cdot h \quad GA := G\cdot A \quad m := \rho\cdot A$

$A = 20.800\,10^{-6} \quad GA = 1.612\times 10^{6} \quad m = 0.161$

$j := 1..3$

$f_j := j\cdot\frac{1}{2\cdot L}\cdot\sqrt{\frac{GA}{m}}$ $\quad f = \begin{pmatrix} 15.8 \\ 31.6 \\ 47.4 \end{pmatrix} 10^{3}$ $\quad \omega_j := 2\cdot\pi\cdot f_j$ $\quad \gamma_j := \frac{\omega_j}{cS}$

$h := 2\cdot h1 \quad b := b1$

$A := b\cdot h \quad GA := G\cdot A \quad m := \rho\cdot A$

$A = 41.600\,10^{-6} \quad GA = 3.225\times 10^{6} \quad m = 0.322$

$j := 1..3$

$f_j := j\cdot\frac{1}{2\cdot L}\cdot\sqrt{\frac{GA}{m}}$ $\quad f = \begin{pmatrix} 15.8 \\ 31.6 \\ 47.4 \end{pmatrix} 10^{3}$

$h := h1 \qquad b := 2 \cdot b1$

$A := b \cdot h \qquad GA := G \cdot A \qquad m := \rho \cdot A$

$A = 41.600\,10^{-6} \qquad GA = 3.225 \times 10^{6} \qquad m = 0.322$

$j := 1 .. 3$

$$f_j := j \cdot \frac{1}{2 \cdot L} \cdot \sqrt{\frac{GA}{m}} \qquad f = \begin{pmatrix} 15.8 \\ 31.6 \\ 47.4 \end{pmatrix} 10^{3}$$

$h := 2 \cdot h1 \qquad b := 2 \cdot b1$

$A := b \cdot h \qquad GA := G \cdot A \qquad m := \rho \cdot A$

$A = 83.200\,10^{-6} \qquad GA = 6.450 \times 10^{6} \qquad m = 0.645$

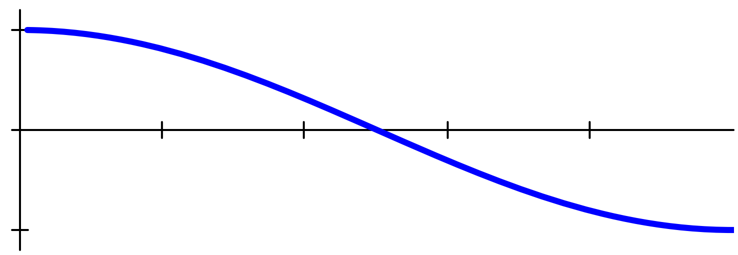
$j := 1 .. 3$

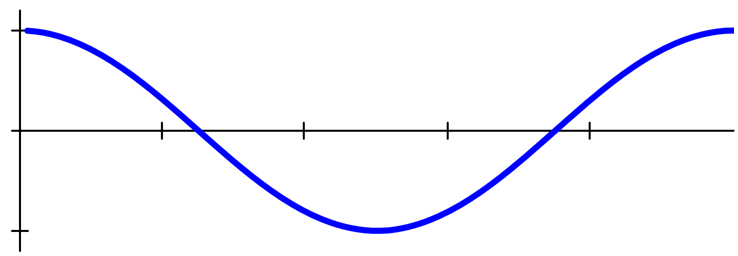
$$f_j := j \cdot \frac{1}{2 \cdot L} \cdot \sqrt{\frac{GA}{m}} \qquad f = \begin{pmatrix} 15.8 \\ 31.6 \\ 47.4 \end{pmatrix} 10^{3}$$

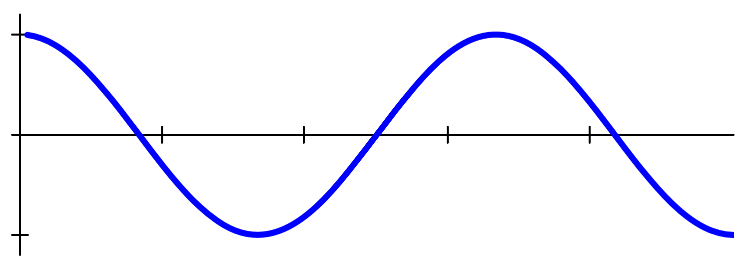
$Nx := 100 \quad nx := 1 .. Nx$

$xStart := 0 \quad xEnd := L \quad dx := \dfrac{xEnd - xStart}{Nx - 1} \quad x_{nx} := xStart + nx \cdot dx$

$V_{j,nx} := \cos\left(\gamma_j \cdot x_{nx}\right)$







Problem 14: Consider a steel bar of thickness $h = 2.6\text{ mm}$, width $b = 8\text{ mm}$, length $l = 100\text{ mm}$, elastic modulus $E = 200\text{ GPa}$, mass density $\rho = 7,750\text{ kg/m}^3$. The bar is excited by a pair of self-equilibrating harmonic forces of amplitude $\hat{F} = 100\text{ N}$ placed at $x_{\text{A}} = 40\text{ mm}$ and $x_{\text{B}} = 47\text{ mm}$; the forces act on the neutral axis, as shown in Figure 3.22. The excitation frequency varies in the range $f = 0 \ldots 100\text{ kHz}$ (consider 401 equally spaced values). Consider 1% modal damping in all modes. Find the index N_u of the axial frequency that brackets the frequency range of interest. Find and plot the response amplitudes of the displacements at x_{A} and x_{B}, i.e., $u_{\text{A}}(\omega)$, $u_{\text{B}}(\omega)$ as well as the difference $\Delta u(\omega) = u_{\text{B}}(\omega) - u_{\text{A}}(\omega)$. Use the four-quads plotting format of Figure 3.8 on page 70.

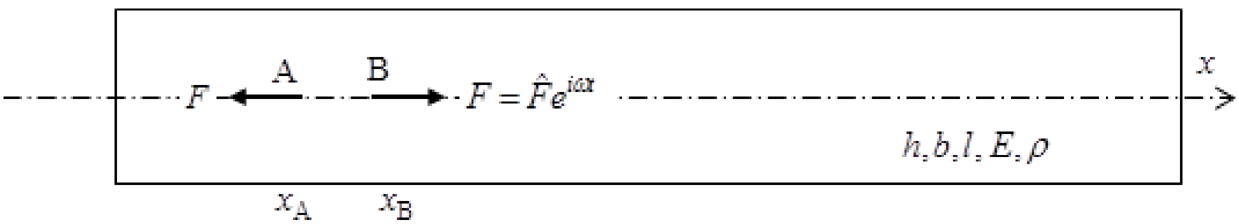


Figure 3.22 Bar undergoing axial vibration under the excitation of a pair of self-equilibrating axial forces

SOLUTION

The excitation forces acting upon the bar neutral axis are $F_A(t) = -\hat{F}e^{i\omega t}$, $F_B(t) = \hat{F}e^{i\omega t}$ (Figure 3.22). The corresponding distributed excitation axial force is expressed as

$$f_e(x,t) = \hat{f}_e(x)e^{i\omega t} = \hat{F}\left[-\delta(x - x_A) + \delta(x - x_B)\right]e^{i\omega t} \quad \text{(axial force excitation)} \tag{1}$$

where δ is Dirac's delta function. Recall from Chapter 3, Section 3.3.3 the equation of motion for forced axial vibration

$$\rho A\, \ddot{u}(x,t) - EA\, u''(x,t) = f_e(x,t) \tag{2}$$

Following the modal expansion method of Chapter 3, Section 3.3.3, we assume

$$u(x,t) = \sum_{j=1}^{N_u} \eta_j U_j(x)\, e^{i\omega t} \tag{3}$$

where N_u is the number of modes needed to bracket the frequency range of interest $f = 0 \ldots 100\text{ kHz}$. The coefficients η_j are the modal participation factors and the functions $U_j(x)$ are length-normalized orthonormal axial modes that satisfy the relation

$$\int_0^l U_p U_q dx = \delta_{pq} \tag{4}$$

with δ_{pq} being the Kronecker delta with the property $\delta_{pq} = 1$ for $p = q$, and 0 otherwise.

For free-free beams, the length-normalized axial modeshapes can be calculated with the formulae given in Chapter 3, Section 3.3.2.1, Eqs. (3.256), (3.259), i.e.,

$$U_j(x) = A_j \cos(\gamma_j x), \quad A_j = \sqrt{\frac{2}{l}}, \quad \gamma_j = \frac{j\pi}{l}, \quad \omega_j = \gamma_j \sqrt{\frac{E}{\rho}}, \quad j = 1,2,3,... \tag{5}$$

According to Chapter 3, Section 3.3.3.3, Eq. (3.295), the response by modal expansion is

$$u(x,t) = \frac{1}{\rho A} \sum_{j=1}^{N_u} \frac{f_j}{-\omega^2 + 2i\zeta_j \omega_j \omega + \omega_j^2} U_j(x)\, e^{\mathrm{i}\omega t} \tag{6}$$

where f_j is the modal excitation calculated as

$$f_j = \int_0^l \hat{f}(x) U_j(x) dx, \quad n = 1,2,3,... \tag{7}$$

Substitution of Eq. (1) into Eq. (7) yields

$$f_j = \int_0^l \hat{F}_{PWAS} \left[-\delta(x - x_A) + \delta(x - x_B) \right] U_j(x) dx = \hat{F}_{PWAS} \left[-U_j(x_A) + U_j(x_B) \right] \tag{8}$$

In resolving Eq. (8), the localization property of the Dirac delta function was used, i.e.,

$$\int \delta(x - x_0) f(x) dx = f(x_0) \tag{9}$$

Substitution of Eq. (7) into Eq. (6) yields the modal participation factor as

$$\eta_j = \frac{\hat{F}}{\rho A} \frac{U_j(x_B) - U_j(x_A)}{-\omega^2 + 2i\zeta_j \omega_j \omega + \omega_j^2}, \quad j = 1, 2, 3, \ldots N \tag{10}$$

Substitution of Eq. (10) into Eq. (3) followed by evaluation at x_A and x_B gives the amplitudes

$$u_A(\omega) = \hat{u}(x_A; \omega) = \sum_{j=1}^{N_u} \eta_j\, U_j(x_A) \tag{11}$$

$$\hat{u}_B(\omega) = u(x_B; \omega) = \sum_{j=1}^{N_u} \eta_j\, U_j(x_B) \tag{12}$$

$$\Delta u(\omega) = u_B(\omega) - u_A(\omega) \tag{13}$$

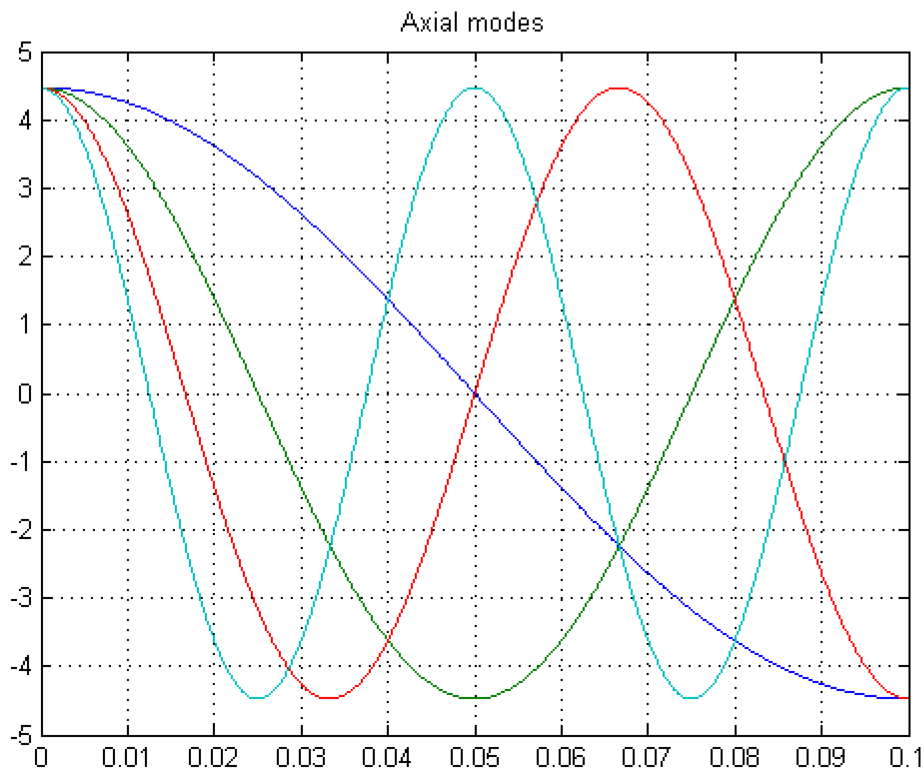
Note that substitution of Eqs. (10), (11), (12) into Eq. (13) and rearrangement yields

$$
\begin{aligned}
\Delta u(\omega) = u_B(\omega) - u_A(\omega) &= \frac{\hat{F}}{\rho A}\sum_{j=1}^{N_u} \frac{U_j(x_B) - U_j(x_A)}{-\omega^2 + 2i\zeta_j\omega_j\omega + \omega_j^2}\left[U_j(x_B) - U_j(x_A)\right] \\
&= \frac{\hat{F}}{\rho A}\sum_{j=1}^{N_u} \frac{\left[-U_j(x_A) + U_j(x_B)\right]^2}{-\omega^2 + 2i\zeta_j\omega_j\omega + \omega_j^2}
\end{aligned}
\tag{14}
$$

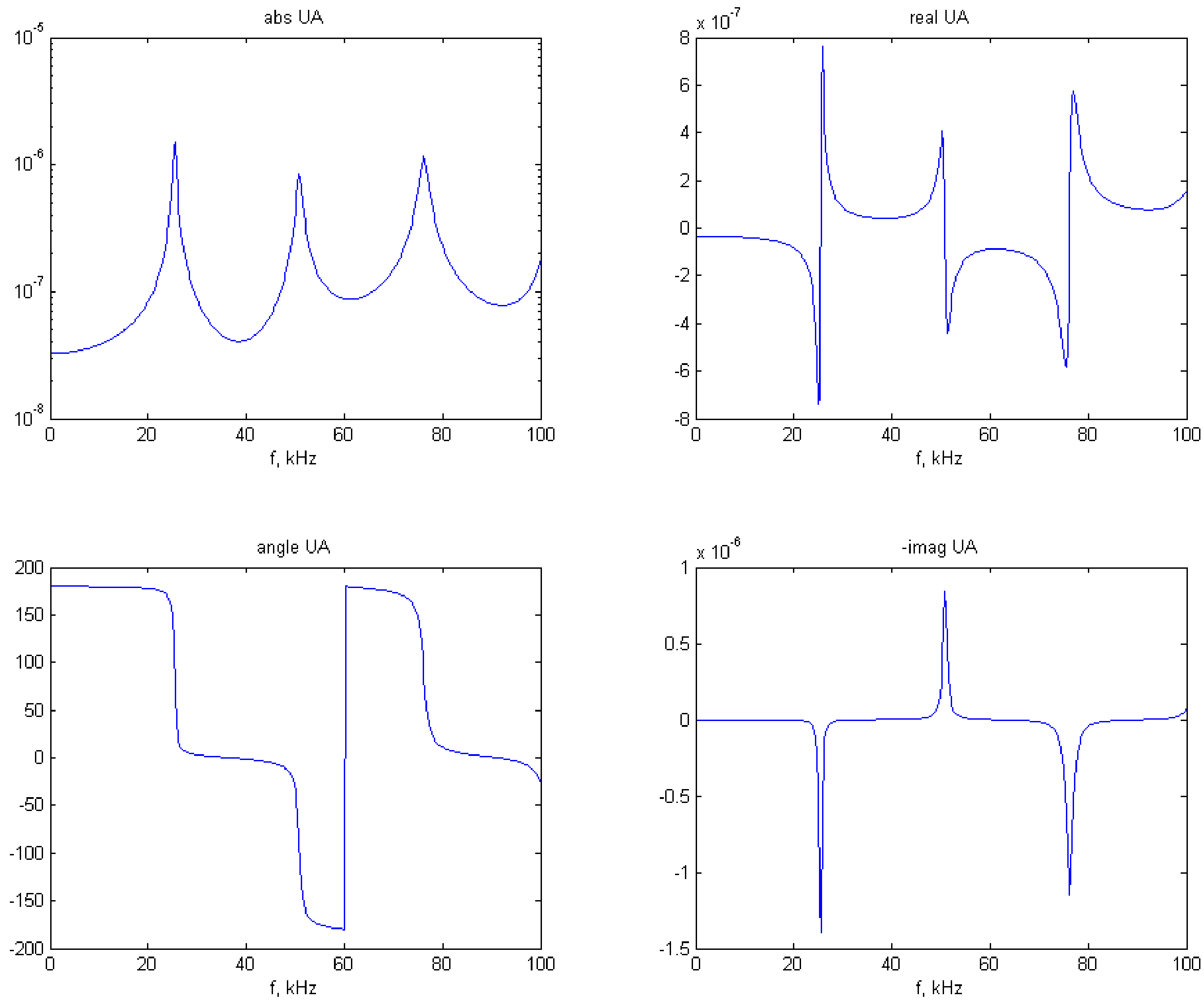
<u>The numerical results are as follows:</u>

The index N_u of the axial frequency that brackets the frequency range of interest is $N_u = 4$. The four axial frequencies that bracket the range $f = 0 \ldots 100\ \text{kHz}$ are 25.4, 50.8, 76.2, 101.6 kHz.

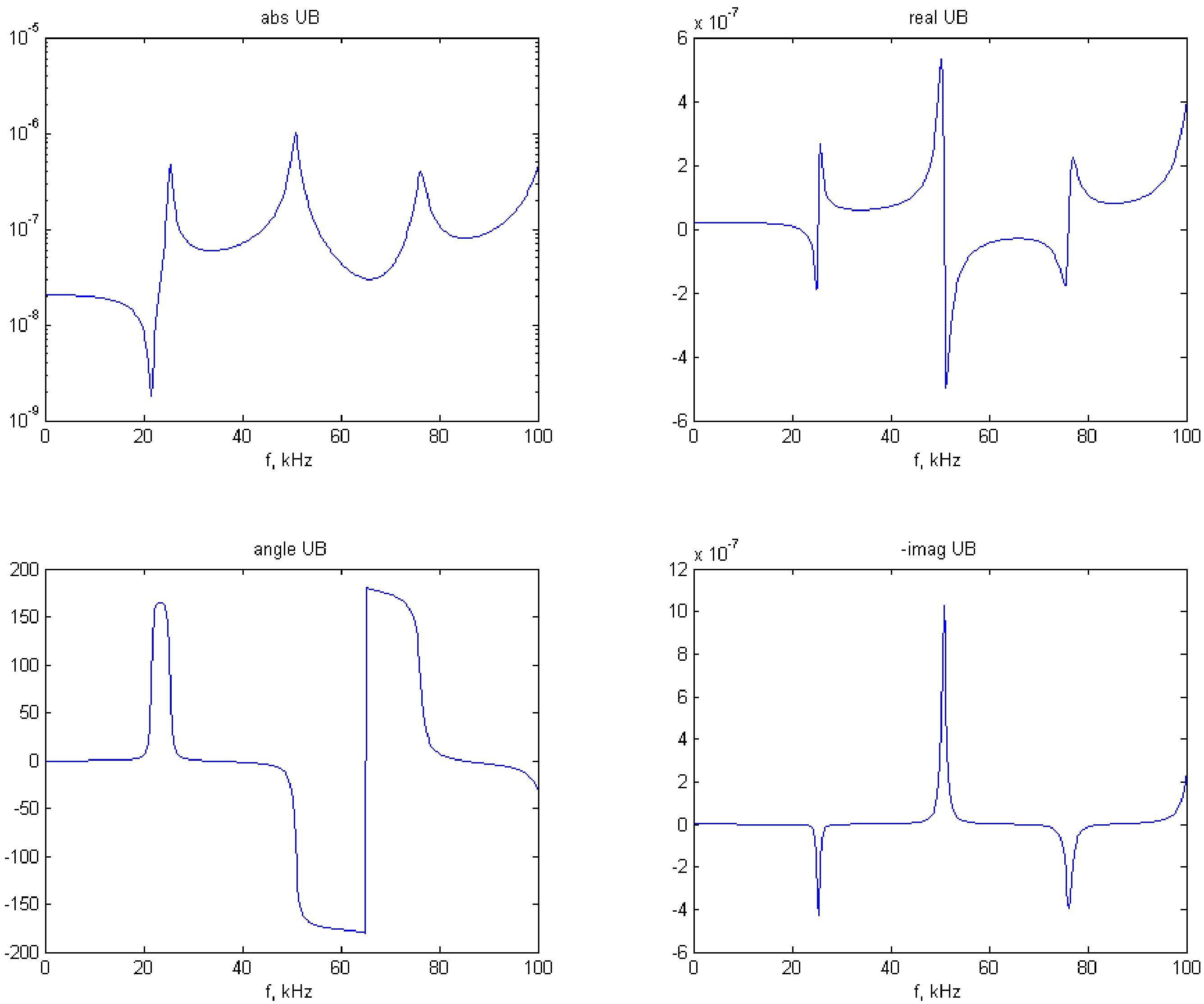
The four axial modes are shown below



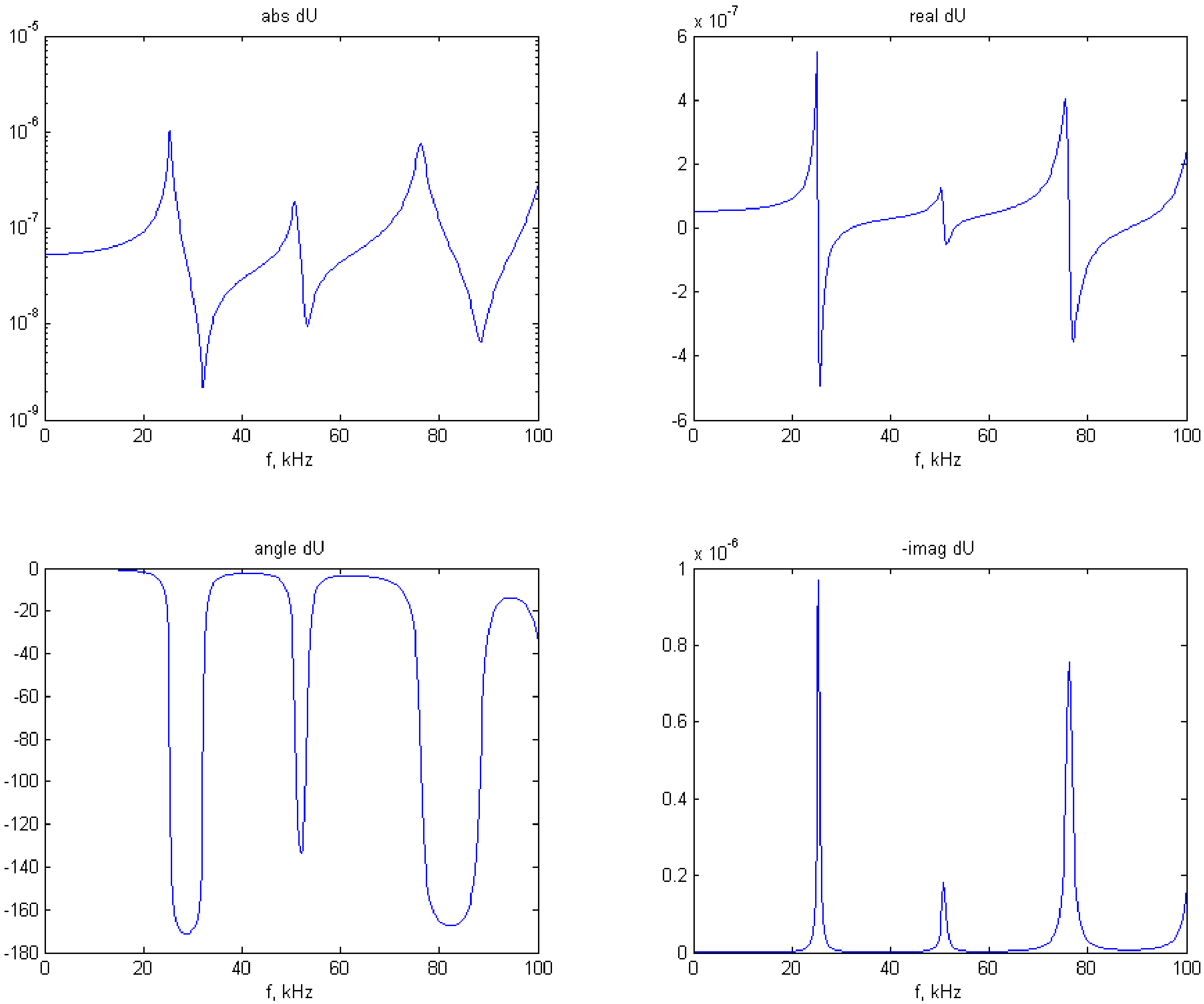
The response amplitudes of the displacements at x_A and x_B, i.e., $u_A(\omega)$, $u_B(\omega)$ as well as the difference $\Delta u(\omega) = u_B(\omega) - u_A(\omega)$ are given in the next three plots.


abs UA
10-5
10-6
10-7
10-8
0 20 40 60 80 100
f, kHz
real UA
x 10-7
8 6 4 2 0 -2 -4 -6 -8
0 20 40 60 80 100
f, kHz
angle UA
200 150 100 50 0 -50 -100 -150 -200
0 20 40 60 80 100
f, kHz
-imag UA
x 10-6
1 0.5 0 -0.5 -1 -1.5
0 20 40 60 80 100
f, kHz

20



21


abs dU
real dU
angle dU
-imag dU
f, kHz
x 10^-7
x 10^-6

```
% Ch.3 Problems 14, 15, 16
% Copyright Victor Giurgiutiu: SHM with PWAS book
clc
clear
%% DEFINE PROPERTIES
L=100e-3; b=8e-3; h=2.6e-3; E=200e9; I=(b*h^3)/12; A=b*h; p=7750;
zeta=1e-2;  % structural damping
xA=40e-3; xB=47e-3; % location of self-equilibrating forces and moments
F=100; % N, excitation force
M=100; %Nm, excitation moment
Nf=401; % number of frequencies in the spectrum
f_start=0e3; f_end=100e3; df=(f_end-f_start)/(Nf-1); f=f_start:df:f_end;
%% CALCULATE NATURAL FREQUENCIES AND MODESHAPES
Nx=1e3; dx=L/(Nx-1); x=0:dx:L;  %   discretize beam length

%%   Axial frequencies and modeshapes: Eqs.(3.255), (3.259) on pp. 95,96
ffU=0;jU=0;
while ffU<=f_end  % identify the required number of axial modes
    jU=jU+1;
    wU(jU)=jU*pi*sqrt(E/p)/L; % axial angular frequencies in rad/s
    ffU=wU(jU)/(2*pi); % axial frequency in Hz
    CU=sqrt(2/L); % length normalized axial modeshape amplitude
    U(:,jU)=CU*cos(jU*pi*x/L); % axial modeshapes
end
NU=jU; % number of required axial modes
fU=wU/(2*pi)*1e-3; % kHz axial frequencies
figure(1); plot(x,U);grid; title ('Axial modes')

%%   Flexural frequencies and modeshapes (pp 119)
% calculate flexural eigenvalues: solve Eq. (3.406), pp 119
% Calculate flexural frequencies
D=@(x)(cos(x)-1/cosh(x)); % characteristic eq-n for free-free flexural freq.
% f=@(x)(cos(x)*cosh(x)-1); this equation is less accurate - do not use
ffW=0; jW=0;
while ffW<=f_end % identified the required number of flexural modes
    jW=jW+1;
    ag(jW)=(2*jW+1)*pi/2; z=fzero(D,ag(jW));
    a(jW)=z/L; % flexural wave number
    wW(jW)=(a(jW))^2*sqrt(E*I/(p*A)); % flexural angular frequency in rad/s
    ffW=wW/(2*pi); % flexural frequency in Hz
    beta(jW)=(sinh(z)+sin(z))/(cosh(z)-cos(z));
end
NW=jW; % number of required flexural modeshapes

% Calculate modeshapes: Eqs. (3.411), (3.413), (3.431)
W=zeros(Nx,NW); AW=zeros(NW,1);
for jW=1:NW
AW(jW)=1/sqrt(L);   % length-normalized modeshape amplitude Eq. (3.413)
W(:,jW)=AW(jW)*((cos(a(jW)*x)-beta(jW)*sin(a(jW)*x)...
    +(1-beta(jW))/2*exp(a(jW)*x)+(1+beta(jW))/2*exp(-a(jW)*x))); % Eq.(4.431)
end
```

```
fW=wW/(2*pi)*1e-3; % kHz flexural frequencies
figure(2); plot(x,W); grid; title('Flexural modes')
% ======================================
%% START FREQUENCY LOOP TO CALCULUATE DISPLACEMENTS
ff=f/1e3; % freq in kHz
%% AXIAL RESPONSE FOR PROBLEM 3.14
for nf=1:Nf % frequency loop
    w=f(nf)*2*pi;
  % axial modes loop
    sum_UA=0; sum_UB=0; sum_dU=0;
    for jU=1:NU
        U_xA=CU*cos(jU*pi*xA/L);
        U_xB=CU*cos(jU*pi*xB/L);
        etaU=F/(p*A)*(U_xB-U_xA)/(wU(jU)^2+2*1i*zeta*wU(jU)*w-w^2);
        sum_UA=sum_UA+etaU*U_xA;
        sum_UB=sum_UB+etaU*U_xB;
    end
  UA(nf)=sum_UA; UB(nf)=sum_UB; dU(nf)=UB(nf)-UA(nf);
end
  %% PLOT AXIAL RESPONSE FOR PROBLEM 3.14
figure(3) % UA plot
subplot(2,2,1); plot(ff,abs(UA)); title('abs UA'); set(gca,'Yscale','log');...
     xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(UA));title('real UA'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(UA));title('angle UA'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(UA));title('-imag UA'); xlabel('f, kHz')
figure(4) % UB plot
subplot(2,2,1); plot(ff,abs(UB)); title('abs UB');set(gca,'Yscale','log');...
     xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(UB));title('real UB'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(UB));title('angle UB'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(UB));title('-imag UB'); xlabel('f, kHz')
figure(5) % dU plot
subplot(2,2,1); plot(ff,abs(dU)); title('abs dU');set(gca,'Yscale','log')...
    ; xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(dU));title('real dU'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(dU));title('angle dU'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(dU));title('-imag dU'); xlabel('f, kHz')

%% FLEXURAL RESPONSE FOR PROBLEM 3.15
for nf=1:Nf % frequency loop
    w=f(nf)*2*pi;
  % flexural modes loop
    sum_WA=0; sum_WB=0; sum_W1A=0; sum_W1B=0;
    for jW=1:NW
    % calculate the derivative (slope) of the flexural modes
    W_xA=AW(jW)*((cos(a(jW)*xA)-beta(jW)*sin(a(jW)*xA)...
    +(1-beta(jW))/2*exp(a(jW)*xA)+(1+beta(jW))/2*exp(-a(jW)*xA)));
    W_xB=AW(jW)*((cos(a(jW)*xB)-beta(jW)*sin(a(jW)*xB)...
    +(1-beta(jW))/2*exp(a(jW)*xB)+(1+beta(jW))/2*exp(-a(jW)*xB)));
    W1_xA=AW(jW)*a(jW)*((-sin(a(jW)*xA)-beta(jW)*cos(a(jW)*xA)...
```

```
    +(1-beta(jW))/2*exp(a(jW)*xA)-(1+beta(jW))/2*exp(-a(jW)*xA)));
    W1_xB=AW(jW)*a(jW)*((-sin(a(jW)*xB)-beta(jW)*cos(a(jW)*xB)...
    +(1-beta(jW))/2*exp(a(jW)*xB)-(1+beta(jW))/2*exp(-a(jW)*xB)));

etaW=-M/(p*A)*(W1_xB-W1_xA)/(wW(jW)^2+2*1i*zeta*wW(jW)*w-w^2);
    sum_WA=sum_WA+etaW*W_xA;
    sum_WB=sum_WB+etaW*W_xB;
    sum_W1A=sum_W1A+etaW*W1_xA;
    sum_W1B=sum_W1B+etaW*W1_xB;
    end
  WA(nf)=sum_WA; WB(nf)=sum_WB; dW(nf)=WB(nf)-WA(nf);
  W1A(nf)=sum_W1A; W1B(nf)=sum_W1B; dW1(nf)=W1B(nf)-W1A(nf);
end

  %% PLOT FLEXURAL RESPONSE FOR PROBLEM 3.15
figure(6) % WA plot
subplot(2,2,1); plot(ff,abs(WA)); title('abs WA');set(gca,'Yscale','log')...
    ; xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(WA));title('real WA'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(WA));title('angle WA'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(WA));title('-imag WA'); xlabel('f, kHz')
figure(7) % WB plot
subplot(2,2,1); plot(ff,abs(WB)); title('abs WB');set(gca,'Yscale','log')...
    ; xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(WB));title('real WB'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(WB));title('angle WB'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(WB));title('-imag WB'); xlabel('f, kHz')
figure(8) % dW plot
subplot(2,2,1); plot(ff,abs(dW)); title('abs dW');set(gca,'Yscale','log')...
    ; xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(dW));title('real dW'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(dW));title('angle dW'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(dW));title('-imag dW'); xlabel('f, kHz')
figure(9) % W1A plot
subplot(2,2,1); plot(ff,abs(W1A)); title('abs W1A');set(gca,'Yscale','log')...
    ; xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(W1A));title('real W1A'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(W1A));title('angle W1A'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(W1A));title('-imag W1A'); xlabel('f, kHz')
figure(10) % W1B plot
subplot(2,2,1); plot(ff,abs(W1B)); title('abs W1B');set(gca,'Yscale','log')...
    ; xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(W1B));title('real W1B'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(W1B));title('angle W1B'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(W1B));title('-imag W1B'); xlabel('f, kHz')
figure(11) % d1W plot
subplot(2,2,1); plot(ff,abs(dW1)); title('abs dW1');set(gca,'Yscale','log')...
    ; xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(dW1));title('real dW1'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(dW1));title('angle dW1'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,imag(dW1));title('imag dW1'); xlabel('f, kHz')
```

```
%% COMBINEa AXIAL AND FLEXURAL RESPONSES FOR PROBLEM 3.16
UUA=UA-h/2*(F*h/2/M)*W1A;
figure(12) % UUA plot
subplot(2,2,1); plot(ff,abs(UUA)); title('abs UUA');set(gca,'Yscale','log')...
    ; xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(UUA));title('real UUA'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(UUA));title('angle UUA'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(UUA));title('-imag UUA'); xlabel('f, kHz')
UUB=UB-h/2*(F*h/2/M)*W1B;
figure(13) % UUB plot
subplot(2,2,1); plot(ff,abs(UUB)); title('abs UUB');set(gca,'Yscale','log')...
    ; xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(UUB));title('real UUB'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(UUB));title('angle UUB'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(UUB));title('-imag UUB'); xlabel('f, kHz')
UUA=UA-h/2*(F*h/2/M)*W1A;
dUU=UUB-UUA;
figure(14) % dUU plot
subplot(2,2,1); plot(ff,abs(dUU)); title('abs dUU');set(gca,'Yscale','log')...
    ; xlabel('f, kHz')
subplot(2,2,2); plot(ff,real(dUU));title('real dUU'); xlabel('f, kHz')
subplot(2,2,3); plot(ff,180/pi*angle(dUU));title('angle dUU'); xlabel('f, kHz')
subplot(2,2,4); plot(ff,-imag(dUU));title('-imag dUU'); xlabel('f, kHz')
```

Problem 15: Consider a steel beam of thickness $h = 2.6\text{ mm}$, width $b = 8\text{ mm}$, length $l = 100\text{ mm}$, elastic modulus $E = 200\text{ GPa}$, mass density $\rho = 7,750\text{ kg/m}^3$. The beam is excited by a pair of self-equilibrating harmonic moments of amplitude $\hat{M} = 100\text{ N}\cdot\text{m}$ placed at $x_\text{A} = 40\text{ mm}$ and $x_\text{B} = 47\text{ mm}$, as shown in Figure 3.23. The excitation frequency varies in the range $f = 0 \ldots 40\text{ kHz}$ (consider 401 equally spaced values). Consider 1% modal damping in all modes. Find the index N_w of the flexural frequency that brackets the frequency range of interest. Find and plot the response amplitudes for displacements and slopes at x_A and x_B, i.e., $w_\text{A}(\omega)$, $w'_\text{A}(\omega)$; $w_\text{B}(\omega)$, $w'_\text{B}(\omega)$; as well as the differences $\Delta w(\omega) = w_\text{B}(\omega) - w_\text{A}(\omega)$, $\Delta w'(\omega) = w'_\text{B}(\omega) - w'_\text{A}(\omega)$. Use the four-quads plotting format of Figure 3.8 on page 70.

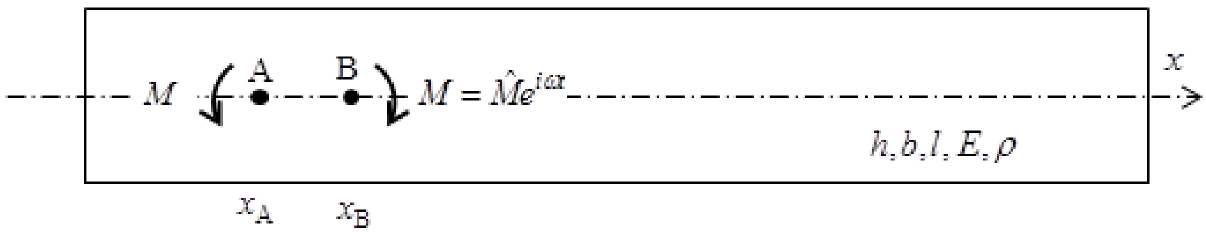


Figure 3.23 Beam undergoing flexural vibration under the excitation of a pair of self-equilibrating bending moments

SOLUTION

The excitation moments acting upon the beam are $M_A(t) = \hat{M}e^{i\omega t}$, $M_B(t) = -\hat{M}e^{i\omega t}$ (Figure 3.23). The corresponding distributed excitation moment is expressed as

$$m_e(x,t) = \hat{m}_e(x)e^{i\omega t} = \hat{M}\left[\delta(x - x_A) - \delta(x - x_B)\right]e^{i\omega t} \quad \text{(moment excitation)} \tag{15}$$

Recall from Chapter 3, Section 3.4.3.4, Eq. (3.461) the equation of motion for forced flexural vibration of a beam under distributed moment excitation, i.e.,

$$\rho A\,\ddot{w}(x,t) + EI\; w''''(x,t) = -m'_e(x,t) \tag{16}$$

Assume the modal expansion

$$w(x,t) = \sum_{j=1}^{N_w} \eta_j W_j(x)\, e^{i\omega t} \tag{17}$$

where N is the number of modes needed to bracket the frequency range of interest $f = 0 \ldots 100\text{ kHz}$. The coefficients η_j are the modal participation factors and the functions $W_j(x)$ are length-normalized orthonormal flexural modes that satisfy the relation

$$\int_0^l W_p W_q dx = \delta_{pq} \tag{18}$$

For free-free beams, the length-normalized flexural modeshapes can be calculated with the formulae given in Chapter 3, Section 3.4.2.1, Eqs. (3.409), (3.410) that dealt with vibration analysis, i.e.,

$$W_j(x) = \frac{1}{\sqrt{l}}\left[\left(\cosh\gamma_j x + \cos\gamma_j x\right) - \beta_j\left(\sinh\gamma_j x + \sin\gamma_j x\right)\right] \tag{19}$$

$$\gamma_j = \frac{z_j}{l}\ , \qquad \omega_j = \gamma_j^2\sqrt{\frac{EI}{\rho A}}\ , \qquad j = 1,2,3,... \tag{20}$$

with the eigenvalues z_j and the modeshape factors β_j being given in Chapter 3, Table 3.5. According to Chapter 3, Section 3.4.3.4, Eqs. (3.463), (3.464), the response by modal expansion is

$$w(x,t) = \frac{1}{\rho A}\sum_{j=1}^{\infty}\frac{f_j}{-\omega^2 + 2i\zeta_j\omega_j\omega + \omega_j^2}W_j(x)\, e^{i\omega t} \tag{21}$$

where the *modal excitation* f_j is given by

$$f_j = \int_0^l -\hat{m}_e'(x)W_j(x)dx\ , \qquad j = 1,2,3,... \tag{22}$$

Substitution of Eq. (15) into (22) gives

$$f_j = \int_0^l -\hat{m}_e'(x)W_j(x)dx = -\hat{M}\int_0^l\left[\delta'(x-x_A) - \delta'(x-x_B)\right]W_j(x)dx \tag{23}$$

The r.h.s. of Eq. (23) can be simplified through integration by parts, i.e.,

$$\int_0^l \delta'\left(x-x_0\right)W_j dx = \cancel{\left[\delta\left(x-x_0\right)W_j\right]_0^l} - \int_0^l \delta\left(x-x_0\right)W_j' dx = -W_j'(x_0) \tag{24}$$

Hence,

$$\int_0^l\left[\delta'\left(x-x_A\right) - \delta'\left(x-x_B\right)\right]W_j dx = -W_j'(x_A) + W_j'(x_B) \tag{25}$$

Substitution of Eq. (25) into Eq. (23) yields

$$f_j = -\hat{M}\left[-W_j'(x_A) + W_j'(x_B)\right] \tag{26}$$

Substitution of Eq. (26) into Eq. (21) yields the modal participation factor as

$$\eta_j = -\frac{\hat{M}}{\rho A}\,\frac{W_j'(x_B) - W_j'(x_A)}{-\omega^2 + 2i\zeta_j\omega_j\omega + \omega_j^2} \tag{27}$$

Substitution of Eq. (27) into Eq. (17) followed by evaluation at x_A and x_B gives the amplitudes

$$w_A(\omega) = \hat{w}\left(x_A;\omega\right) = \sum_{j=1}^{N_w} \eta_j W_j(x_A) \tag{28}$$

$$w_B(\omega) = \hat{w}\left(x_B;\omega\right) = \sum_{j=1}^{N_w} \eta_j W_j(x_B) \tag{29}$$

$$\Delta w(\omega) = w_B(\omega) - w_A(\omega) \tag{30}$$

Differentiation of Eq. (17) w.r.t. x gives the amplitude $\hat{w}'(x)$ as

$$\hat{w}'(x) = \sum_{j=1}^{N_w} \eta_j W_j'(x) \tag{31}$$

Evaluation of Eq. (31) at x_A and x_B yields

$$w_A'(\omega) = \hat{w}'\left(x_A;\omega\right) = \sum_{j=1}^{N_w} \eta_j W_j'(x_A) \tag{32}$$

$$w_B'(\omega) = \hat{w}'\left(x_B;\omega\right) = \sum_{j=1}^{N_w} \eta_j W_j'(x_B) \tag{33}$$

$$\Delta w'(\omega) = w_B'(\omega) - w_A'(\omega) \tag{34}$$

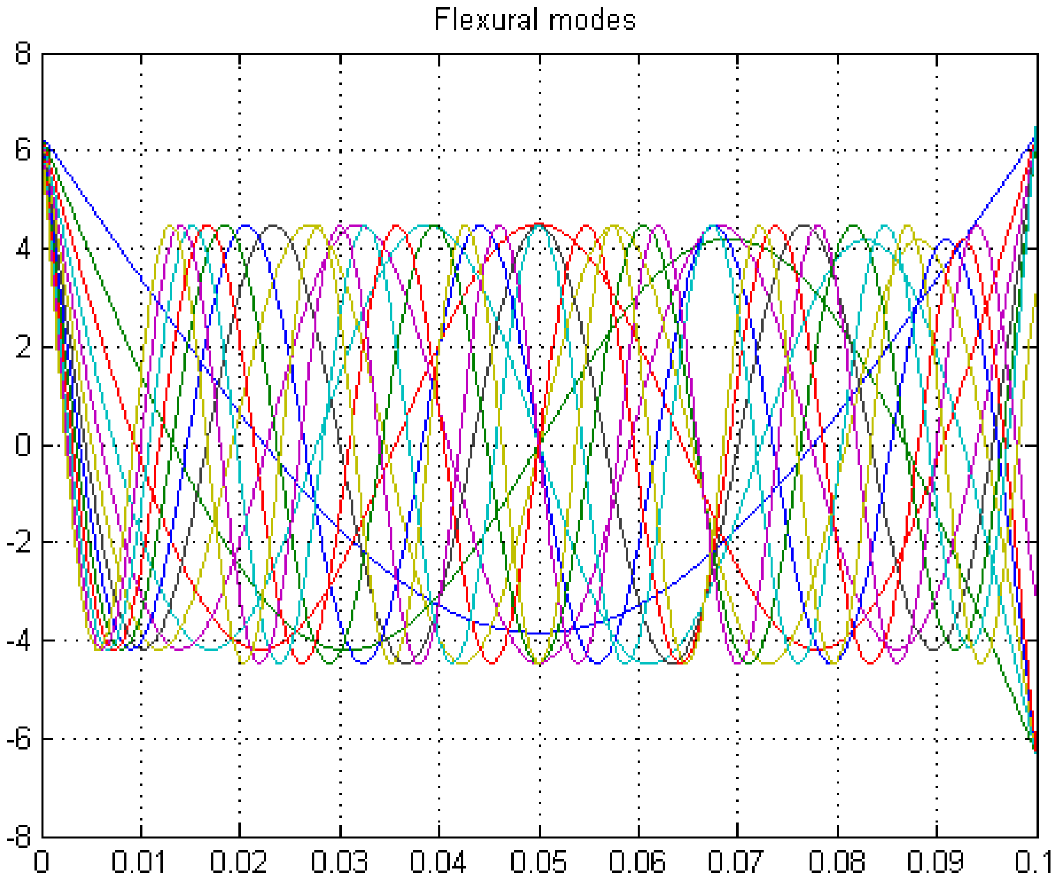
Note that substitution of Eqs. (27), (32), (33) into Eq. (34) and rearrangement yields

$$\begin{aligned} \Delta w'(\omega) = w_B'(\omega) - w_A'(\omega) &= -\frac{\hat{M}}{\rho A} \sum_{j=1}^{N_w} \frac{W_j'(x_B) - W_j'(x_A)}{-\omega^2 + 2i\zeta_j \omega_j \omega + \omega_j^2} \left[W_j'(x_B) - W_j'(x_A) \right] \\ &= -\frac{\hat{M}}{\rho A} \sum_{j=1}^{N_w} \frac{\left[W_j'(x_B) - W_j'(x_A) \right]^2}{-\omega^2 + 2i\zeta_j \omega_j \omega + \omega_j^2} \end{aligned} \tag{35}$$

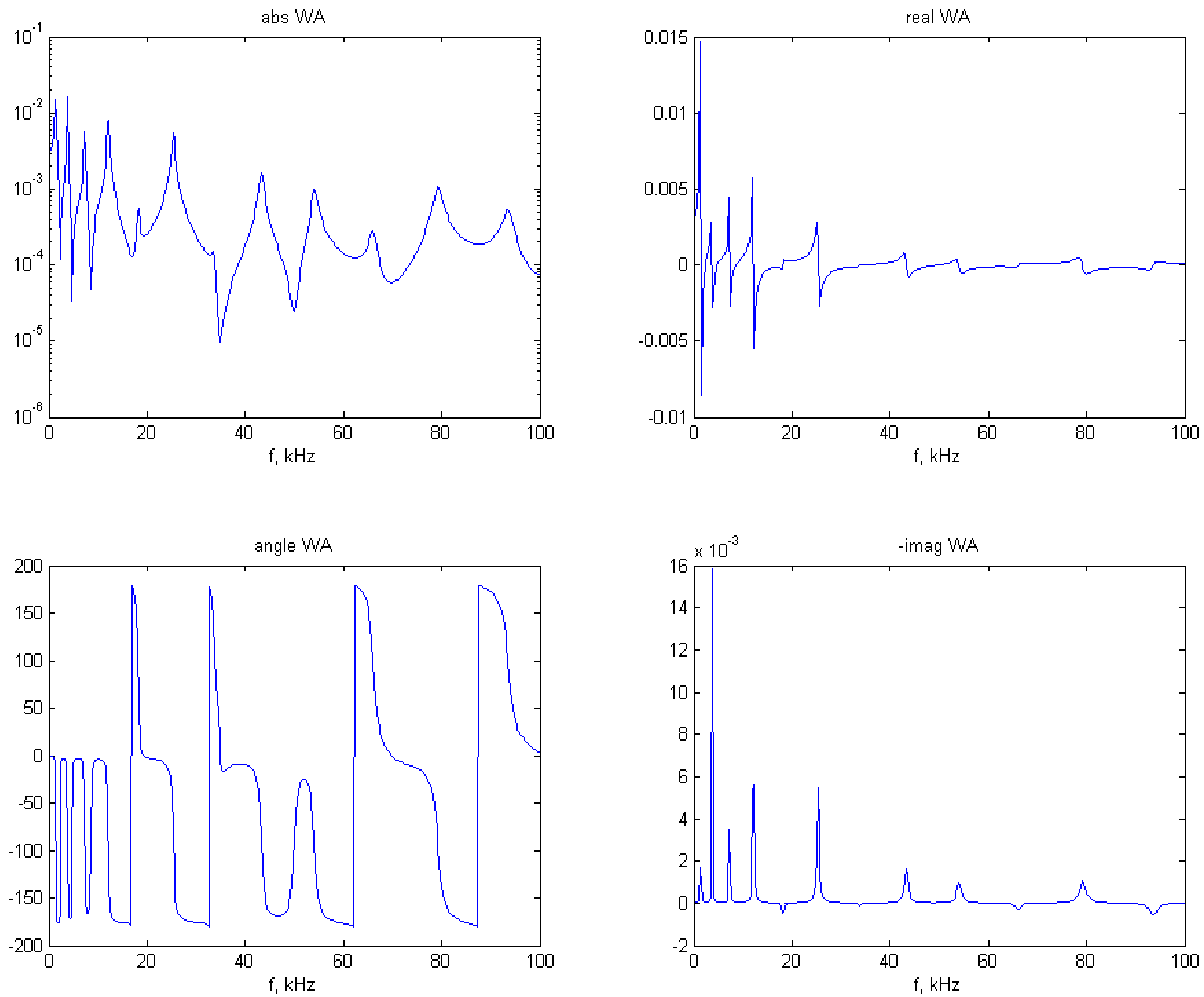
<u>The numerical results are as follows:</u>

The index N_w of the axial frequency that brackets the frequency range of interest is $N_w = 13$. The 13 flexural frequencies that bracket the range $f = 0 \ldots 100$ kHz are 1.358, 3.75, 7.34, 12.13, 18.12, 25.3, 33.7, 43.3, 54.1, 66.0, 79.2, 93.6, 109.2 kHz.

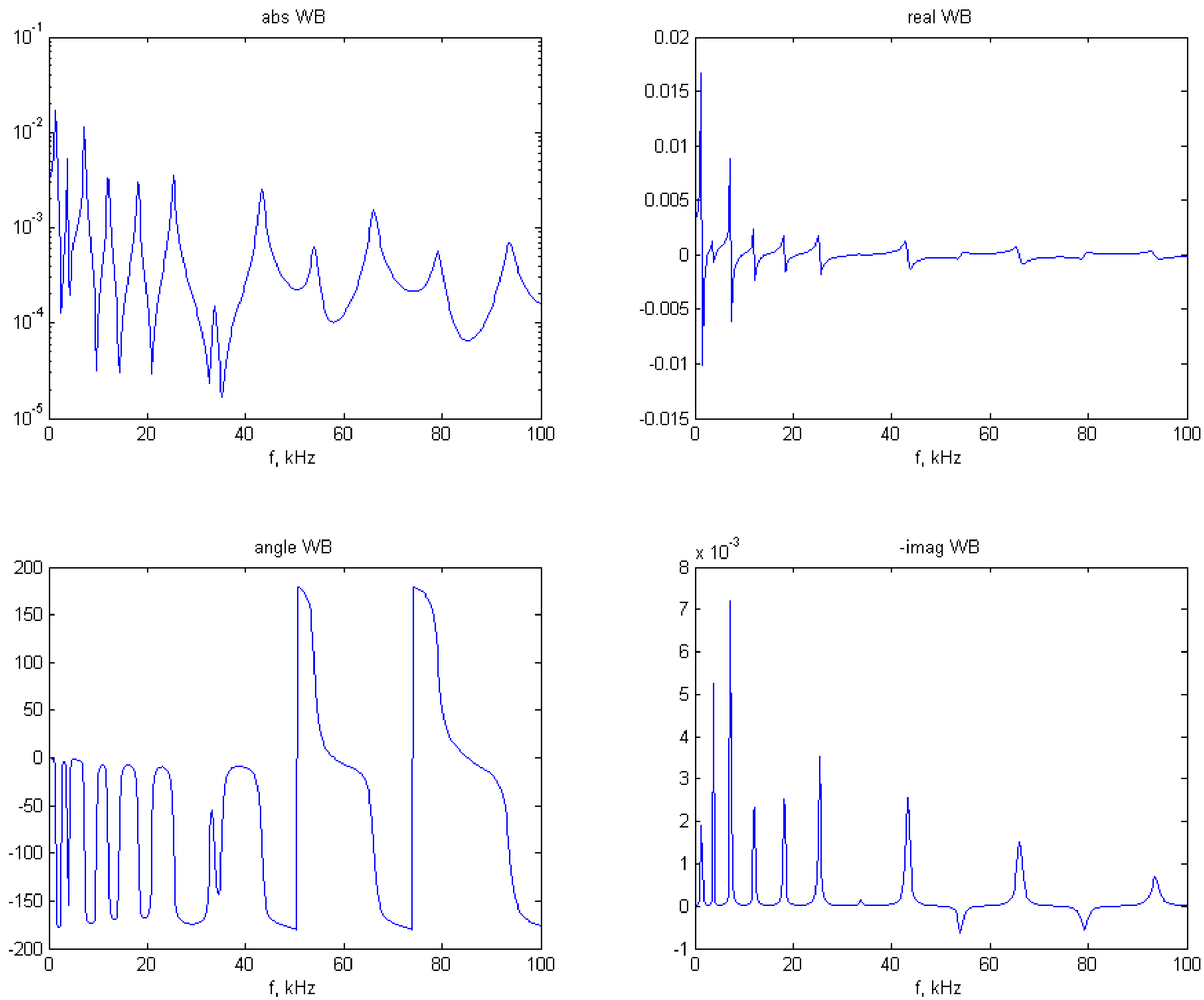
The thirteen flexural modes are shown below



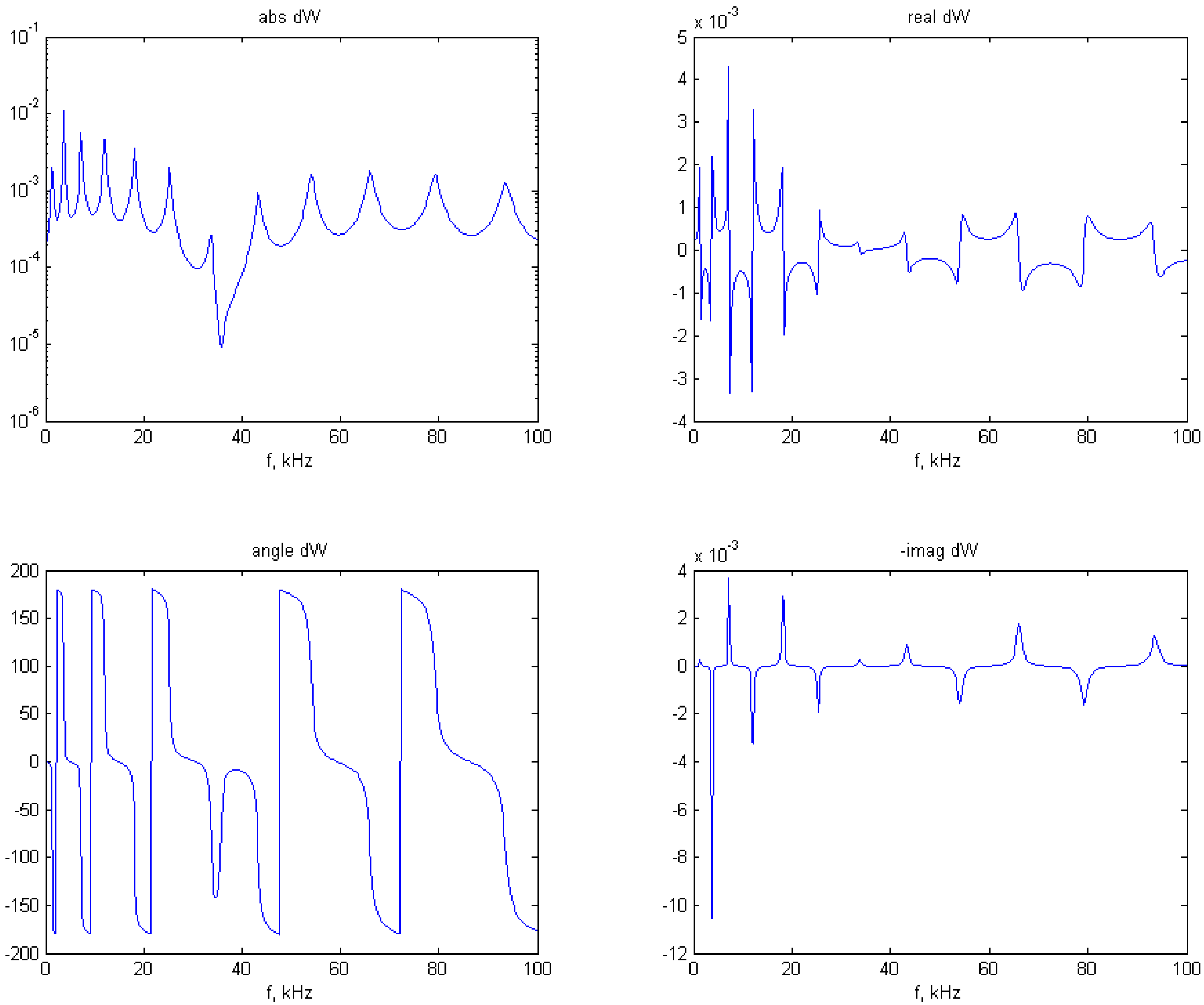
The response amplitudes for displacements and slopes at x_A and x_B, i.e., $w_A(\omega)$, $w'_A(\omega)$; $w_B(\omega)$, $w'_B(\omega)$; as well as the differences $\Delta w(\omega) = w_B(\omega) - w_A(\omega)$, $\Delta w'(\omega) = w'_B(\omega) - w'_A(\omega)$ are given in the next six plots.


abs WA
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0 20 40 60 80 100
f, kHz
real WA
0.015
0.01
0.005
0
-0.005
-0.01
0 20 40 60 80 100
f, kHz
angle WA
200
150
100
50
0
-50
-100
-150
-200
0 20 40 60 80 100
f, kHz
-imag WA
$x\ 10^{-3}$
16
14
12
10
8
6
4
2
0
-2
0 20 40 60 80 100
f, kHz

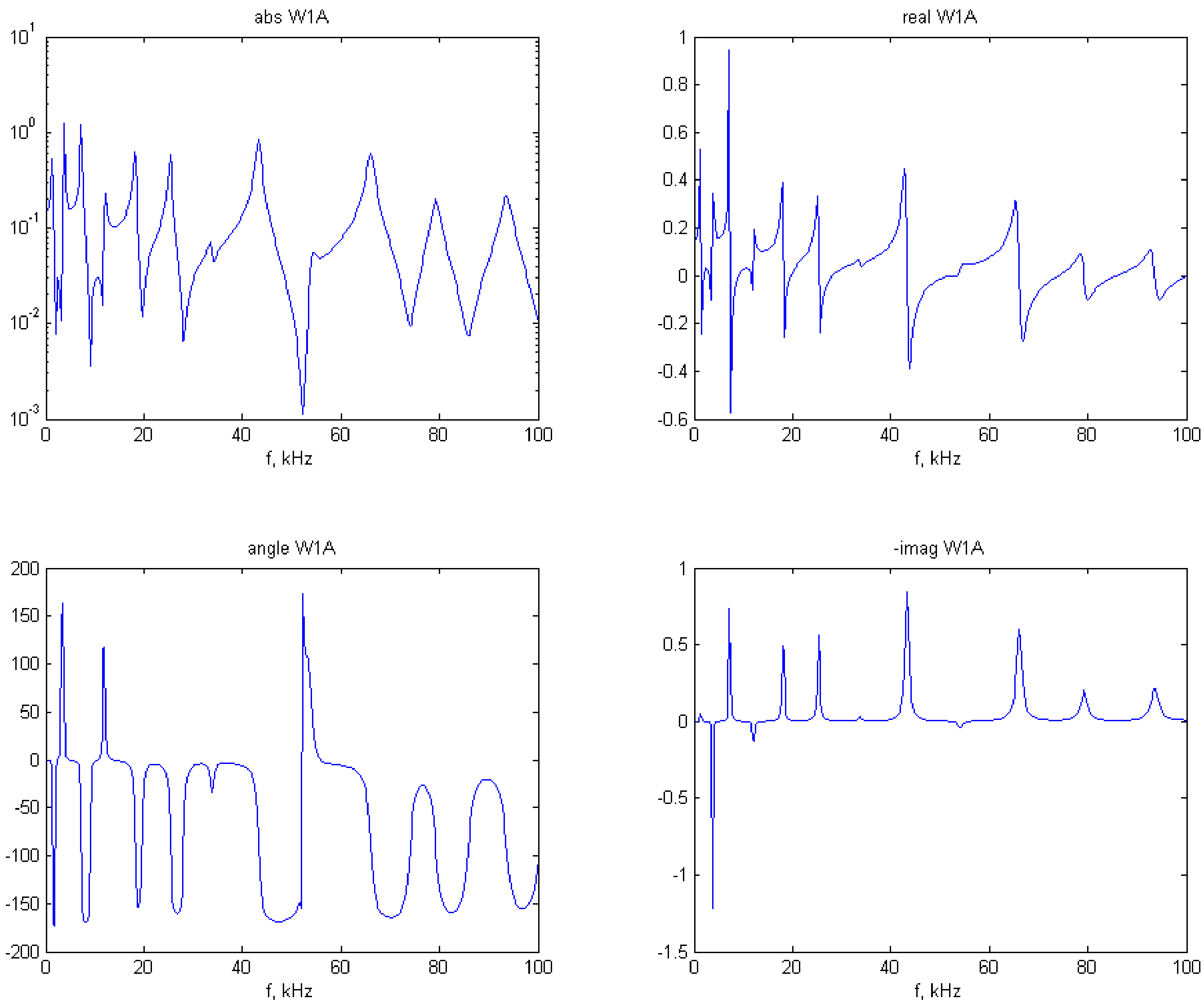
27

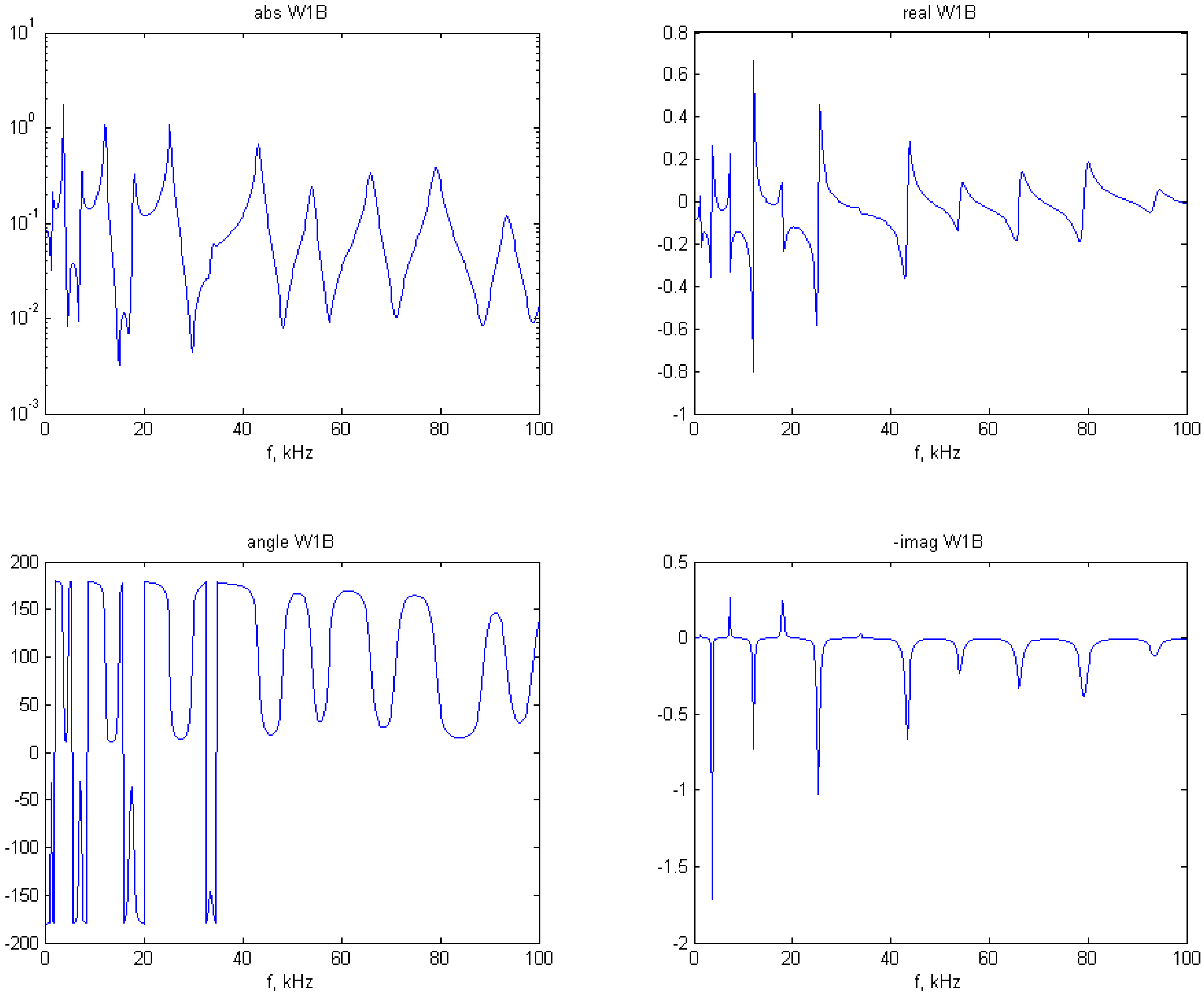


28

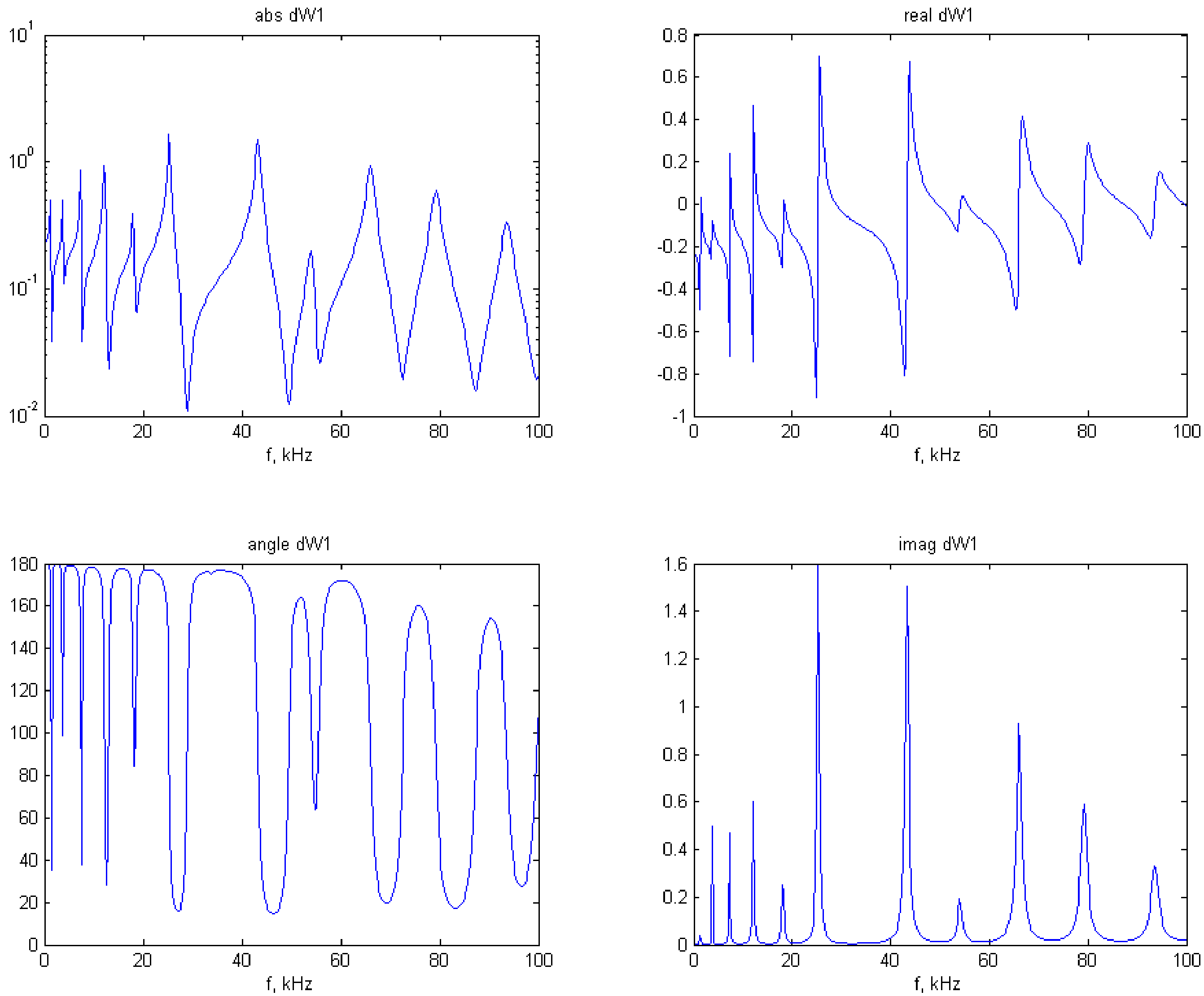

abs dW
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
0 20 40 60 80 100
f, kHz
x 10^-3
real dW
5 4 3 2 1 0 -1 -2 -3 -4
0 20 40 60 80 100
f, kHz
angle dW
200 150 100 50 0 -50 -100 -150 -200
0 20 40 60 80 100
f, kHz
x 10^-3
-imag dW
4 2 0 -2 -4 -6 -8 -10 -12
0 20 40 60 80 100
f, kHz

29


abs W1A
real W1A
angle W1A
-imag W1A
f, kHz
f, kHz
f, kHz
f, kHz


abs W1B
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
0
20
40
60
80
100
f, kHz
real W1B
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
20
40
60
80
100
f, kHz
angle W1B
200
150
100
50
0
-50
-100
-150
-200
0
20
40
60
80
100
f, kHz
-imag W1B
0.5
0
-0.5
-1
-1.5
-2
0
20
40
60
80
100
f, kHz

31


abs dW1
real dW1
angle dW1
imag dW1
f, kHz
f, kHz
f, kHz
f, kHz

Problem 16: Consider a steel beam of thickness $h = 2.6\text{ mm}$, width $b = 8\text{ mm}$, length $l = 100\text{ mm}$, elastic modulus $E = 200\text{ GPa}$, mass density $\rho = 7{,}750\text{ kg/m}^3$. The beam is excited by a pair of self-equilibrating harmonic forces of amplitude $\hat{F} = 100\text{ N}$ placed at $x_\text{A} = 40\text{ mm}$ and $x_\text{B} = 47\text{ mm}$. The forces act on the beam surface as shown in Figure 3.24. The excitation frequency varies in the range $f = 0 \ldots 100\text{ kHz}$ (consider 401 equally spaced values). Consider 1% modal damping in all modes. Find the index N_u of the axial frequency and the index N_w of the flexural frequency that bracket the frequency range of interest. Find and plot the surface response displacements $u_\text{A}(\omega)$ at x_A; $u_\text{B}(\omega)$ at x_B, and $\Delta u(\omega) = u_\text{B}(\omega) - u_\text{A}(\omega)$. Use the four-quads plotting format of Figure 3.8 on page 70. Hint: surface displacement u is calculated kinematically using the axial displacement u_0 and the flexural slope w' of the neutral axis, i.e., $u = u_0 - \dfrac{h}{2} w'$.

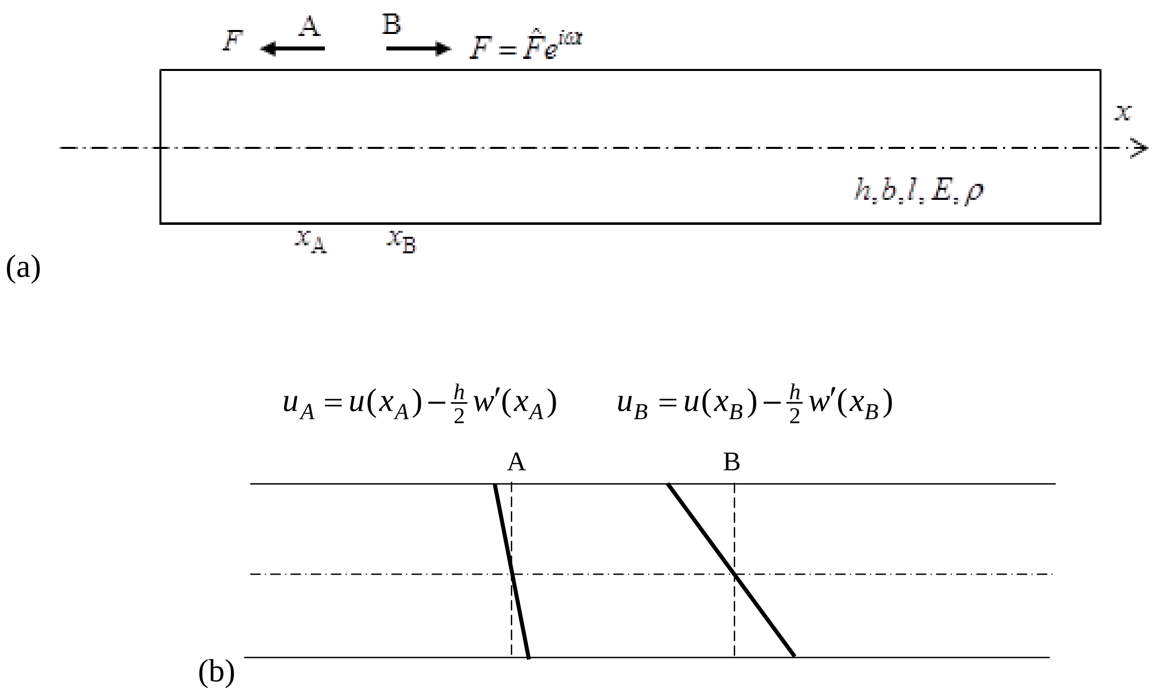


Figure 3.24 Beam undergoing combined axial and flexural vibration under the excitation of a pair of self-equilibrating forces place on the beam surface. The combined axial and flexural effect is created by the fact that the forces are offset from the neutral axis: (a) loading diagram; (b) surface displacements diagram

SOLUTION

The excitation forces acting upon the beam surface can be reduced at the neutral axis into a pair of axial forces $F_A(t) = -\hat{F}e^{i\omega t}$, $F_B(t) = \hat{F}e^{i\omega t}$ and a pair of bending moments $M_A(t) = \hat{M}e^{i\omega t}$, $M_B(t) = -\hat{M}e^{i\omega t}$ where

$$\hat{M} = \hat{F}\frac{h}{2} \tag{36}$$

The corresponding distributed excitation axial force and bending moment are expressed as

$$f_e(x,t) = \hat{f}_e(x)e^{i\omega t} = \hat{F}_{PWAS}\left[-\delta(x-x_A)+\delta(x-x_B)\right]e^{i\omega t} \quad \text{(axial force excitation)} \tag{37}$$

$$m_e(x,t) = \hat{m}_e(x)e^{i\omega t} = \hat{M}\left[\delta(x-x_A)-\delta(x-x_B)\right]e^{i\omega t} \quad \text{(moment excitation)} \tag{38}$$

where δ is Dirac's delta function. As shown in Figure 3.24b, the neutral axis displacements $\hat{u}(x_A)$, $\hat{u}(x_B)$ and $\hat{w}(x_A)$, $\hat{w}(x_B)$ combine to give the surface displacements u_A, u_B according to the kinematic formula

$$\begin{aligned} u_A &= \hat{u}(x_A) - \frac{h}{2}\hat{w}'(x_A) \\ u_B &= \hat{u}(x_B) - \frac{h}{2}\hat{w}'(x_B) \end{aligned} \tag{39}$$

The modal participation factors for axial and flexural motions are calculated by substituting Eqs. (36), (37), (38) into Eqs. (10), (27) to get

$$\eta_{j_u} = \frac{\hat{F}}{\rho A}\frac{U_{j_u}(x_B) - U_{j_u}(x_A)}{-\omega^2 + 2i\zeta_{j_u}\omega_{j_u}\omega + \omega_{j_u}^2}, \quad j_u = 1,2,3,\ldots N_u \tag{40}$$

$$\eta_{j_w} = -\frac{h}{2}\frac{\hat{F}}{\rho A}\frac{W'_{j_w}(x_B) - W'_{j_w}(x_A)}{-\omega^2 + 2i\zeta_{j_w}\omega_{j_w}\omega + \omega_{j_w}^2}, \quad j_w = 1,2,3,\ldots N_w \tag{41}$$

where the subscripts u and w signify axial and flexural modes, respectively. Substitution of Eqs. (40), (41) into Eq. (39) gives

$$u_A = \sum_{j_u=1}^{N_u}\eta_{j_u}U_{j_u}(x_A) - \frac{h}{2}\sum_{j_w=1}^{N_w}\eta_{j_w}W'_{j_w}(x_A) \tag{42}$$

$$u_B = \sum_{j_u=1}^{N_u}\eta_{j_u}U_{j_u}(x_B) - \frac{h}{2}\sum_{j_w=1}^{N_w}\eta_{j_w}W'_{j_w}(x_B) \tag{43}$$

Substitution of Eqs. (11), (12),(32), (33) into Eqs. (42), (43) yields

$$u_A = \frac{\hat{F}}{\rho A}\sum_{j_u=1}^{N_u}\frac{U_{j_u}(x_B) - U_{j_u}(x_A)}{-\omega^2 + 2i\zeta_{j_u}\omega_{j_u}\omega + \omega_{j_u}^2}U_{j_u}(x_A) + \left(\frac{h}{2}\right)^2\frac{\hat{F}}{\rho A}\sum_{j_w=1}^{N_w}\frac{-W'_{j_w}(x_A) + W'_{j_w}(x_B)}{-\omega^2 + 2i\zeta_{j_w}\omega_{j_w}\omega + \omega_{j_w}^2}W'_{j_w}(x_A) \tag{44}$$

$$u_B = \frac{\hat{F}}{\rho A}\sum_{j_u=1}^{N_u}\frac{U_{j_u}\left(x_B\right)-U_{j_u}\left(x_A\right)}{-\omega^2+2i\zeta_{j_u}\omega_{j_u}\omega+\omega_{j_u}^2}U_{j_u}\left(x_B\right)+\left(\frac{h}{2}\right)^2\frac{\hat{F}}{\rho A}\sum_{j_w=1}^{N_w}\frac{-W'_{j_w}(x_A)+W'_{j_w}(x_B)}{-\omega^2+2i\zeta_{j_w}\omega_{j_w}\omega+\omega_{j_w}^2}W'_{j_w}(x_B) \quad (45)$$

Subtraction of Eq. (44) from Eq. (45) yields

$$\begin{aligned}\Delta u = u_B - u_A = \frac{\hat{F}}{\rho A}\sum_{j_u=1}^{N_u}\frac{-U_{j_u}\left(x_A\right)+U_{j_u}\left(x_B\right)}{-\omega^2+2i\zeta_{j_u}\omega_{j_u}\omega+\omega_{j_u}^2}\left[U_{j_u}\left(x_B\right)-U_{j_u}\left(x_A\right)\right] \\ +\left(\frac{h}{2}\right)^2\frac{\hat{F}}{\rho A}\sum_{j_w=1}^{N_w}\frac{-W'_{j_w}(x_A)+W'_{j_w}(x_B)}{-\omega^2+2i\zeta_{j_w}\omega_{j_w}\omega+\omega_{j_w}^2}\left[W'_{j_w}(x_B)-W'_{j_w}(x_A)\right]\end{aligned} \quad (46)$$

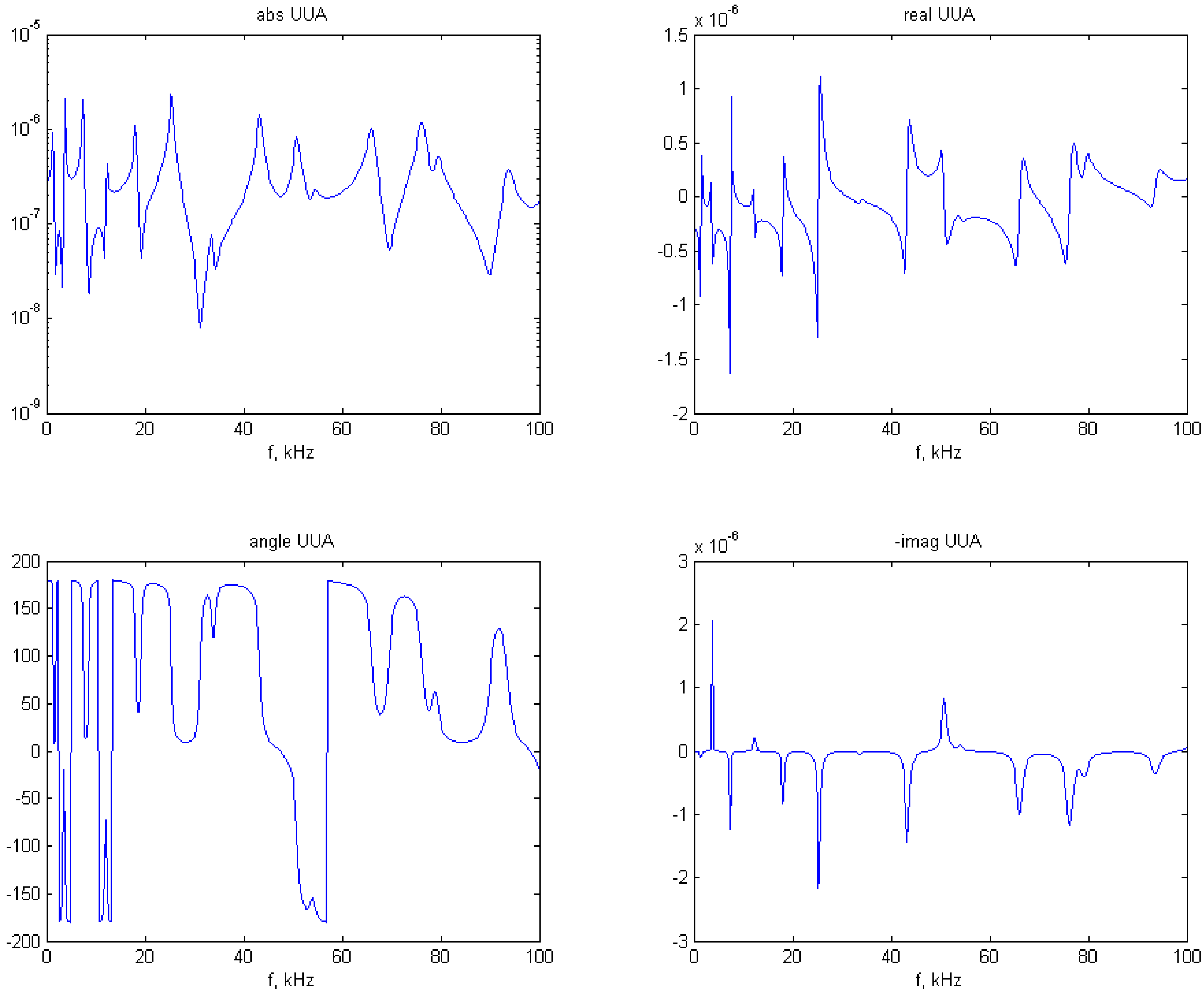
Upon rearrangement, Eq. (46) yields

$$\Delta u(\omega) = \frac{\hat{F}}{\rho A}\left\{\sum_{j_u=1}^{N_u}\frac{\left[U_{j_u}\left(x_B\right)-U_{j_u}\left(x_A\right)\right]^2}{-\omega^2+2i\zeta_{j_u}\omega_{j_u}\omega+\omega_{j_u}^2}+\left(\frac{h}{2}\right)^2\sum_{j_w=1}^{N_w}\frac{\left[W'_{j_w}(x_B)-W'_{j_w}(x_A)\right]^2}{-\omega^2+2i\zeta_{j_w}\omega_{j_w}\omega+\omega_{j_w}^2}\right\} \quad (47)$$

<u>The numerical results are as follows:</u>

The indices N_u of the axial frequency and N_w of the flexural frequency that bracket the frequency range of interest are $N_u = 4$ and $N_w = 13$.

The surface response displacements $u_A(\omega)$ at x_A; $u_B(\omega)$ at x_B, and $\Delta u(\omega) = u_B(\omega) - u_A(\omega)$ are given in the next three plots.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f, kHz
real UUA
x 10^-6
f, kHz
angle UUA
f, kHz
-imag UUA
x 10^-6
f, kHz

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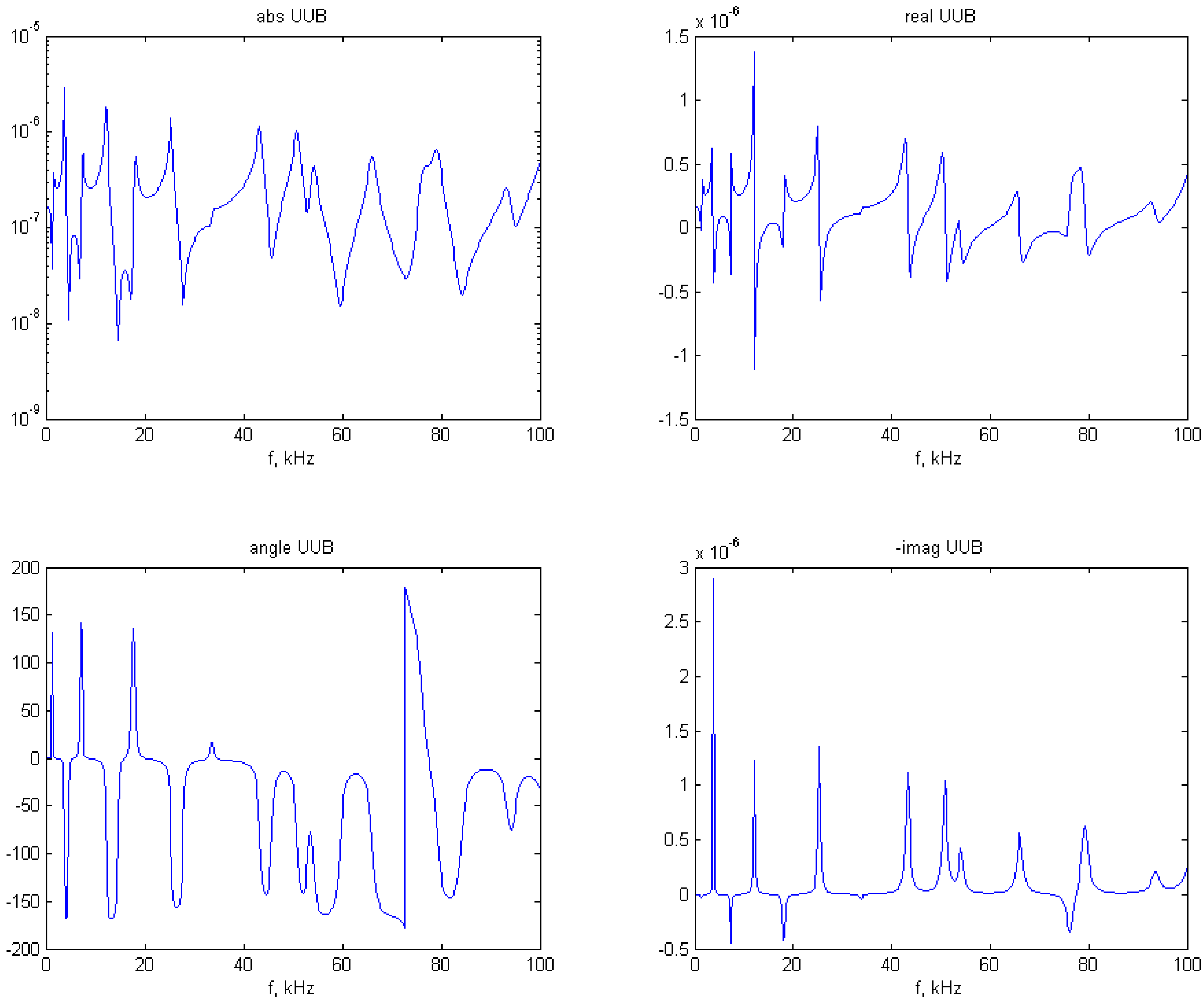
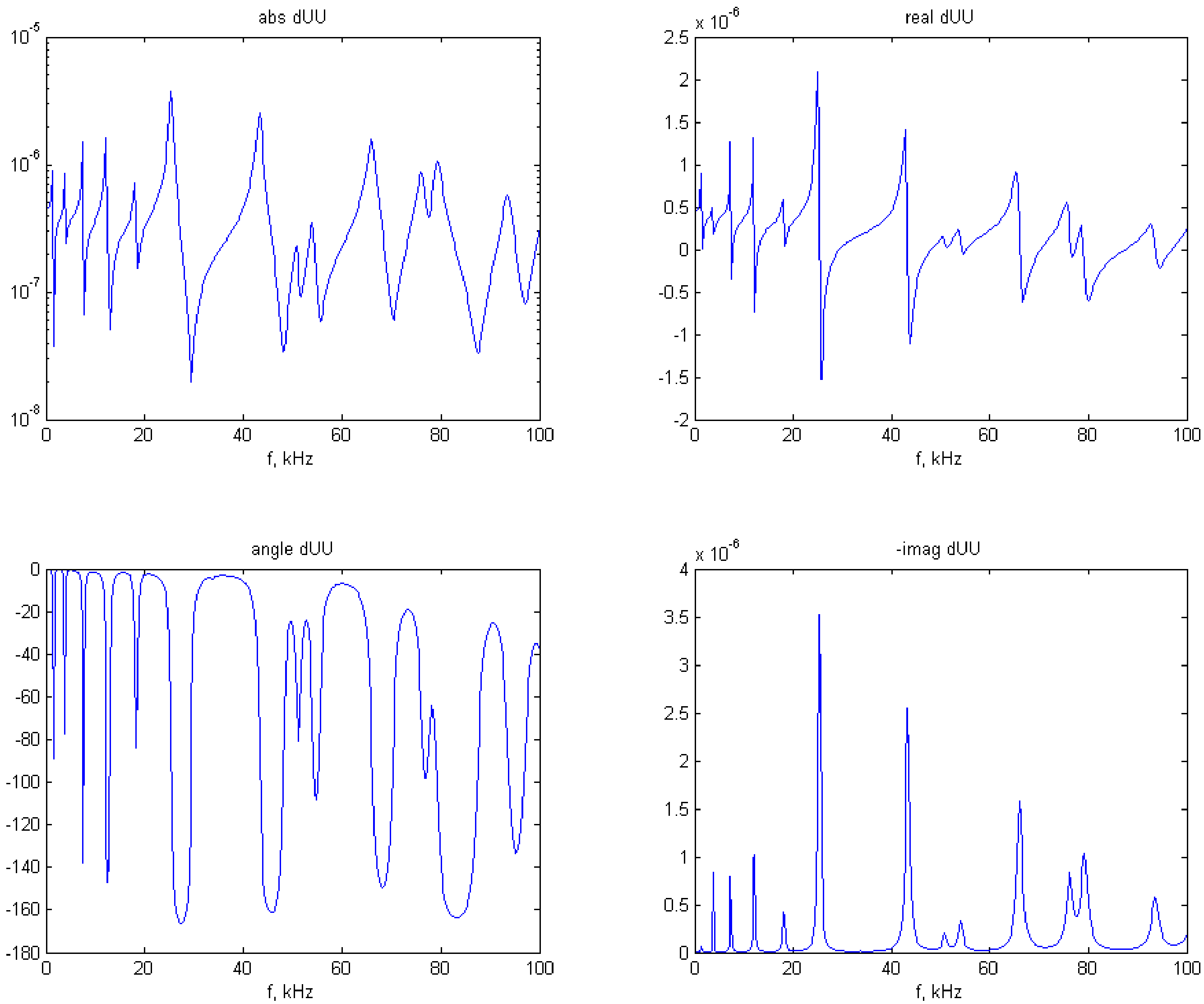

abs UUB
real UUB
angle UUB
-imag UUB
x 10^-6
f, kHz


abs dUU
10^{-5}
10^{-6}
10^{-7}
10^{-8}
0 20 40 60 80 100
f, kHz
real dUU
x 10^{-6}
2.5 2 1.5 1 0.5 0 -0.5 -1 -1.5 -2
0 20 40 60 80 100
f, kHz
angle dUU
0 -20 -40 -60 -80 -100 -120 -140 -160 -180
0 20 40 60 80 100
f, kHz
-imag dUU
x 10^{-6}
4 3.5 3 2.5 2 1.5 1 0.5 0
0 20 40 60 80 100
f, kHz

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