CHAPTER 5 PROBLEMS AND EXERCISES

Problem 1: Consider the 1-D wave equation $c^2u'' = \ddot{u}$ and recall the D'Alembert solution. (a) Verify by direct substitution that the functions $f(x-ct)$ and $g(x+ct)$ of the D'Alembert solution satisfy the wave equation. (b) Find the expressions of f and g for the following initial conditions at $t=0$, $u(x,0)=u_0(x)$, $\dot{u}(x,0)=0$. (c) Sketch the behavior of $f(x-ct)$ and $g(x+ct)$ for various times $t>0$ and identify the forward wave and which is the backward wave.

Solution

(a) Assume $u(x,t) = f(x-ct)$. Then

$$\frac{\partial}{\partial x}u(x,t) = f'(x-ct), \quad \frac{\partial^2}{\partial x^2}u(x,t) = f''(x-ct) \tag{1}$$

$$\frac{\partial}{\partial t}u(x,t) = -cf'(x-ct) \quad \frac{\partial^2}{\partial t^2}u(x,t) = (-c)^2 f''(x-ct) \tag{2}$$

Recall the wave equation $c^2u'' = \ddot{u}$ and write it explicitly as

$$c^2 \frac{\partial^2}{\partial x^2}u = \frac{\partial^2}{\partial t^2}u \tag{3}$$

Substitution of Eqs. (1), (2) into Eq. (3) yields

$$c^2 f''(x-ct) = (-c)^2 f''(x-ct) = c^2 f''(x-ct) \tag{4}$$

Equation. (4) proves that the function $f(x-ct)$ satisfies the wave equation!

By a similar argument, if we assume $u(x,t) = g(x+ct)$, then we get

$$\frac{\partial}{\partial x}u(x,t) = g'(x+ct), \quad \frac{\partial^2}{\partial x^2}u(x,t) = g''(x+ct) \tag{5}$$

$$\frac{\partial}{\partial t}u(x,t) = cg'(x+ct) \quad \frac{\partial^2}{\partial t^2}u(x,t) = c^2g''(x+ct) \tag{6}$$

Substitution into Eq. (3) yields

$$c^2g''(x+ct) = c^2g''(x+ct) \tag{7}$$

Equation (7) proves that the function $g(x+ct)$ satisfies the wave equation. QED

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(b) Recall Eq. (5.20) representing the general expression of the D'Alembert solution, i.e.,

$$u(x,t) = g(x+ct) + f(x-ct) \tag{8}$$

Correspondingly,

$$\dot{u}(x,t) = cg'(x+ct) + (-c)f'(x-ct) \tag{9}$$

Substitution in the initial conditions $u(x,0) = u_0(x)$, $\dot{u}(x,0) = 0$ yields

$$u(x,0) = g(x) + f(x) = u_0(x) \tag{10}$$

$$\dot{u}(x,0) = cg'(x) - cf'(x) = 0 \tag{11}$$

Equation (11) yields

$$g'(x) = f'(x) \tag{12}$$

Upon integration,

$$g(x) = f(x) + C \tag{13}$$

where C is an arbitrary constant. Substitution of Eq. (13) into Eq. (10) yields

$$2f(x) + C = u_0(x) \tag{14}$$

Upon solution,

$$f(x) = \tfrac{1}{2}u_0(x) - \tfrac{1}{2}C \tag{15}$$

Substitution of Eq. (8) into Eq. (6) gives

$$g(x) = \tfrac{1}{2}u_0(x) + \tfrac{1}{2}C \tag{16}$$

To keep the discussion generic, we use the unspecified variable z and write

$$\begin{aligned} f(z) &= \tfrac{1}{2}u_0(z) - \tfrac{1}{2}C \\ g(z) &= \tfrac{1}{2}u_0(z) + \tfrac{1}{2}C \end{aligned} \tag{17}$$

Equation (17) gives the general form of functions $f(\ )$ and $g(\ )$. When substituting Eq. (17) into Eq. (8), we will make $z = x - ct$ for $f(\ )$ and $z = x + ct$ for $g(\ )$; hence,

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}u_0(x+ct) + \cancel{\tfrac{1}{2}C} + \tfrac{1}{2}u_0(x-ct) - \cancel{\tfrac{1}{2}C} \\ &= \tfrac{1}{2}u_0(x+ct) + \tfrac{1}{2}u_0(x-ct) \end{aligned} \tag{18}$$

Thus, the answer to the problem is

$$f(x,t) = \tfrac{1}{2}u_0(x+ct) \tag{19}$$

$$g(x,t) = \tfrac{1}{2}u_0(x-ct) \tag{20}$$

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(c)The sketch of the behavior of $f(x-ct)$ for various times $t>0$ is given in Figure 1. It is apparent that $f(x-ct)$ is moving forward, i.e., it is a forward wave. A similar exercise can be done for the backward wave $g(x+ct)$

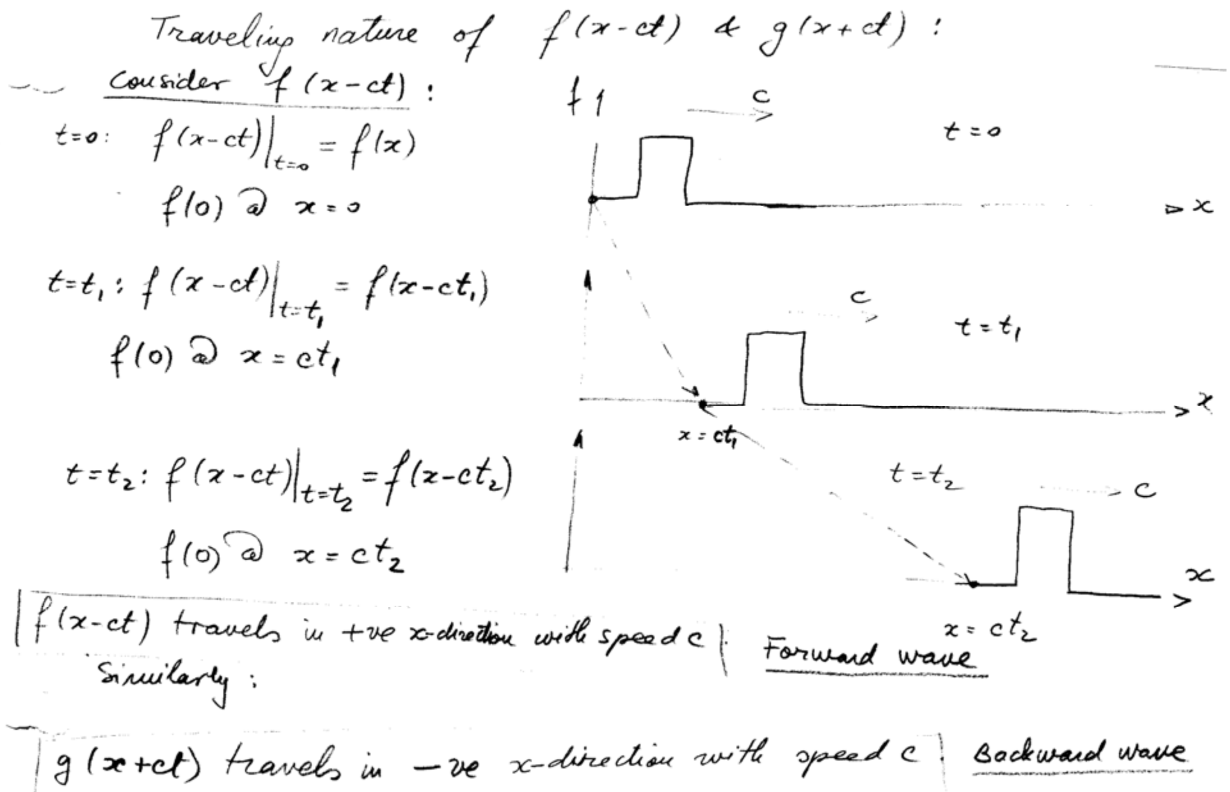

Figure 1 Sketch explaining the traveling nature of the forward wave $f(x-ct)$; A similar sketch can be done for the backward wave $g(x+ct)$

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Problem 2: Verify the alternative forms of the d'Alembert solution given by Eqs. (5.21), (5.22), (5.23)

Solution

Recall Eqs. (5.21), (5.22), (5.23), i.e.,

$$u_1(x,t) = g(ct+x) + f(ct-x) \tag{5.21}$$

$$u_2(x,t) = g\left(\frac{x}{c}+t\right) + f\left(\frac{x}{c}-t\right) \tag{5.22}$$

$$u_3(x,t) = g\left(t+\frac{x}{c}\right) + f\left(t-\frac{x}{c}\right) \tag{5.23}$$

Recall the wave equation, i.e.,

$$c^2 u'' = \ddot{u} \tag{1}$$

Write the derivatives of Eqs. (5.21), i.e.,

$$\begin{aligned} u_1'(x,t) &= g'(ct+x) - f'(ct-x) \qquad & \dot{u}_1(x,t) &= cg'(ct+x) + cf'(ct-x) \\ u_1''(x,t) &= g''(ct+x) + f''(ct-x) \qquad & \ddot{u}_1(x,t) &= c^2 g''(ct+x) + c^2 f''(ct-x) \end{aligned} \tag{2}$$

Substitution of Eq. (2) into Eq. (1) yields

$$c^2 u_1''(x,t) = c^2\left[g''(ct+x) + f''(ct-x)\right] = c^2 g''(ct+x) + c^2 f''(ct-x) = \ddot{u}_1(x,t) \tag{3}$$

Write the derivatives of Eqs. (5.22), i.e.,

$$\begin{aligned} u_2'(x,t) &= \frac{1}{c}g'\left(\frac{x}{c}+t\right) + \frac{1}{c}f'\left(\frac{x}{c}-t\right) \qquad & \dot{u}_2(x,t) &= g'\left(\frac{x}{c}+t\right) - f'\left(\frac{x}{c}-t\right) \\ u_2''(x,t) &= \frac{1}{c^2}g''\left(\frac{x}{c}+t\right) + \frac{1}{c^2}f''\left(\frac{x}{c}-t\right) \qquad & \ddot{u}_2(x,t) &= g''\left(\frac{x}{c}+t\right) + f''\left(\frac{x}{c}-t\right) \end{aligned} \tag{4}$$

Substitution of Eq. (4) into Eq. (1) yields

$$\begin{aligned} c^2 u_1''(x,t) &= c^2\left[\frac{1}{c^2}g''\left(\frac{x}{c}+t\right) + \frac{1}{c^2}f''\left(\frac{x}{c}-t\right)\right] \\ &= g''\left(\frac{x}{c}+t\right) + f''\left(\frac{x}{c}-t\right) = \ddot{u}_1(x,t) \end{aligned} \tag{5}$$

Write the derivatives of Eqs. (5.23), i.e.,

$$\begin{aligned} u_3'(x,t) &= \frac{1}{c}g'\left(t+\frac{x}{c}\right) - \frac{1}{c}f'\left(t-\frac{x}{c}\right) \qquad & \dot{u}_3(x,t) &= g'\left(t+\frac{x}{c}\right) + f'\left(t-\frac{x}{c}\right) \\ u_3''(x,t) &= \frac{1}{c^2}g''\left(t+\frac{x}{c}\right) + \frac{1}{c^2}f''\left(t-\frac{x}{c}\right) \qquad & \ddot{u}_3(x,t) &= g''\left(t+\frac{x}{c}\right) + f''\left(t-\frac{x}{c}\right) \end{aligned} \tag{6}$$

Substitution of Eq. (6) into Eq. (1) yields

$$\begin{aligned} c^2 u_1''(x,t) &= c^2\left[\frac{1}{c^2}g''\left(t+\frac{x}{c}\right) + \frac{1}{c^2}f''\left(t-\frac{x}{c}\right)\right] \\ &= g''\left(t+\frac{x}{c}\right) + f''\left(t-\frac{x}{c}\right) = \ddot{u}_1(x,t) \end{aligned} \tag{7}$$

Problem 3: For each material given in Table 5.3, find: (a) the wave speed. How long does it take the wave to travel 10 m? (b) the pressure wave amplitude (expressed in absolute value and as percentage of yield stress) needed to achieve a maximum strain of 1000 με[1]. Is this feasible? (c) the acoustic impedance; (d) the particle velocity for a pressure wave with amplitude $p_{\max} = \frac{1}{2}Y$. Comment on the result. (e) the pressure wave amplitude (expressed in absolute value and as percentage of yield stress) needed to achieve a maximum particle velocity of 20 m/s. Is this feasible? (f) the displacement wave amplitude for a 100 kHz harmonic pressure wave with amplitude $p_{\max} = \frac{1}{2}Y$.

Table 5.3 Typical material properties of aluminum and steel

	Aluminum (7075 T6)	Steel (AISI 4340 normalized)
Modulus, E	70 GPa	200 GPa
Poisson ratio	0.33	0.3
Density, ρ	2700 kg/m^3	7750 kg/m^3
Yield stress, Y	500 MPa	860 MPa

Solution

(a) Recall Eq. (5.10) giving the wave speed as $c = \sqrt{E/\rho}$. Upon substitution,

$c_{Al} = 5092$ m/s, $c_{Steel} = 5080$ m/s

The time to travel 10 m is $t_{Al} = 1.964$ ms, $t_{Steel} = 1.969$ ms; the times are comparable because the wavespeeds are comparable.

(b) Recall the stress-strain relation $\sigma = E\varepsilon$. The stress corresponding to $\varepsilon = 1000\ \mu\varepsilon$ is:

$\sigma_{Al} = 70$ MPa = 14% of Y, $\sigma_{Steel} = 200$ MPa = 23% of Y.

The stresses are feasible because they do not exceed the yield values

(c) Recall Eq. (5.46) giving the expression of the acoustic impedance, i.e., $Z = \rho c = \sqrt{\rho E} = E/c$.

Upon calculation, $Z_{Al} = 13.75\ \text{MPa} \cdot \text{m}^{-1} \cdot \text{s}$, $Z_{Al} = 39.4\ \text{MPa} \cdot \text{m}^{-1} \cdot \text{s}$.

Comment: the aluminum acoustic impedance is significantly lower than that of steel because aluminum has a lower density while having almost the same wavespeed.

(d) Recall Eq. (5.45) giving the relation between stress and particle velocity amplitudes, i.e.,

$$\sigma_{\max} = \rho c\, \dot{u}_{\max} \tag{1}$$

Recalling from item (c) above that $\rho c = Z$, it follows that

$$\sigma_{\max} = Z\, \dot{u}_{\max} \tag{2}$$

Upon solution, Eq. (2) yields

$$\dot{u}_{\max} = \sigma_{\max} / Z \tag{3}$$

[1] 1 με = 1 micro-strain = 10^{-6} units of strain

For a pressure wave with amplitude $\sigma_{\max} = \frac{1}{2}Y$, we get $\dot{u}^{Al}_{\max} = 18.18$ m/s, $\dot{u}^{Steel}_{\max} = 10.92$ m/s
Comment: For the same stress level relative to yield stress, the particle velocity in aluminum is significantly higher than in steel because aluminum has a lower acoustic impedance.

(e) Recall again Eq. (5.45) $\sigma_{\max} = \rho c\, \dot{u}_{\max} = Z\, \dot{u}_{\max}$ and calculate the stress amplitude corresponding to particle velocity amplitude of 25 m/s. Upon calculation, we get
$|p_{Al}| = 344$ MPa = 69% of Y, $|p_{Steel}| = 984$ MPa = 114% of Y
Comment:

- (i) To achieve the same particle velocity one has to apply stress levels in steel that are almost double those required in aluminum.
- (ii) The stress level in steel would exceed the yield value, hence the linear analysis used here would no longer apply.

(f) the displacement wave amplitude for a 100 kHz harmonic pressure wave with amplitude $p_{\max} = \frac{1}{2}Y$ is calculated as follows. Recall Eq. (3) and express it in terms of amplitudes

$$\hat{\dot{u}} = \hat{\sigma}/Z \tag{4}$$

To calculate displacement wave amplitude, recall Eq. (5.61) giving the expression of a harmonic wave, i.e.,

$$u(x,t) = \hat{u}\, e^{i(\gamma x - \omega t)} \qquad (5.61)\ (5)$$

The particle velocity is obtained by differentiation of Eq. (5) w.r.t. time t, i.e.,

$$\dot{u}(x,t) = \frac{\partial}{\partial t}\hat{u}\, e^{i(\gamma x-\omega t)} = -i\omega\hat{u}\, e^{i(\gamma x-\omega t)} = \hat{\dot{u}}\, e^{i(\gamma x-\omega t)} \tag{6}$$

It is apparent from examination of Eq. (6) that the particle velocity amplitude $\hat{\dot{u}}$ is

$$\hat{\dot{u}} = -i\omega\hat{u} \qquad \text{(particle velocity amplitude)} \tag{7}$$

Upon solution, Eq. (7) yields

$$\hat{u} = i\hat{\dot{u}}/\omega \qquad \text{(displacement amplitude in terms of particle velocity amplitude)} \tag{8}$$

Equations (4),(8) give

$$|\hat{u}| = |i\hat{\dot{u}}/\omega| = |\hat{\dot{u}}|/\omega = |\hat{\sigma}|/\omega Z \tag{9}$$

where $\omega = 2\pi f$. For $p_{\max} = \frac{1}{2}Y$, i.e., $\hat{\sigma} = -\frac{1}{2}Y$, $|\hat{\sigma}| = \frac{1}{2}Y$, Eq. (9) yields:
$\hat{u}_{Al} = 28.9\,\mu\text{m}$, $\hat{u}_{Steel} = 17.4\,\mu\text{m}$

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PROBLEM 5.3 SOLUTION

Units: $GPa := 10^{9} Pa$ $MPa := 10^{6} Pa$ $ms := 10^{-3} s$ $\mu\varepsilon := 10^{-6}$ $kHz := 10^{3} \cdot Hz$ $\mu m := 10^{-6} \cdot m$

For each material in Table 5.3:

(a) Find the wave speed

aluminum $E\_al := 70GPa$ $\rho\_al := 2700 kg \cdot m^{-3}$ $c\_al := \sqrt{\frac{E\_al}{\rho\_al}}$ $c\_al = 5092 \frac{m}{s}$

steel $E\_st := 200GPa$ $\rho\_st := 7750 kg \cdot m^{-3}$ $c\_st := \sqrt{\frac{E\_st}{\rho\_st}}$ $c\_st = 5080 \frac{m}{s}$

How long does it take the wave to travel 10 m? $L := 10 \cdot m$ $\frac{L}{c\_al} = 1.964\, ms$ aluminum

$\frac{L}{c\_st} = 1.969\, ms$ steel

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(b) the pressure wave amplituder (expressed in absolute value and as percentage of yield) needed to achieve a maximum strain of 1000 $\mu\varepsilon$ is calculated as follows

$\varepsilon := 1000 \cdot \mu\varepsilon$ $\sigma\_al := E\_al \cdot \varepsilon$ $\sigma\_al = 70.0\, MPa$ $Y\_al := 500 \cdot MPa$ $\frac{\sigma\_al}{Y\_al} = 14\%$

$\sigma\_st := E\_st \cdot \varepsilon$ $\sigma\_st = 200.0\, MPa$ $Y\_st := 860 \cdot MPa$ $\frac{\sigma\_st}{Y\_st} = 23\%$

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(c) the acoustic impedance is calculated as follows

$Z\_al := \rho\_al \cdot c\_al$ $Z\_al = 13.75\, MPa \cdot m^{-1} \cdot s$

$Z\_st := \rho\_st \cdot c\_st$ $Z\_st = 39.37\, MPa \cdot m^{-1} \cdot s$

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(d) the particle velocity is calculated as follows

$\sigma\_max\_al := \frac{Y\_al}{2}$ $u\_dot\_al := \frac{\sigma\_max\_al}{Z\_al}$ $u\_dot\_al = 18.18 \frac{m}{s}$

$\sigma\_max\_st := \frac{Y\_st}{2}$ $u\_dot\_st := \frac{\sigma\_max\_st}{Z\_st}$ $u\_dot\_st = 10.92 \frac{m}{s}$

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(e) the pressure wave amplitude needed to achieve 25 m/s is calculated as follows

$$u\_dot := 25 \cdot \frac{m}{s}$$

$\sigma1\_al := Z\_al \cdot u\_dot$ $\qquad$ $\sigma1\_al = 344\,MPa$ $\qquad$ $\dfrac{\sigma1\_al}{Y\_al} = 69\,\%$

$\sigma1\_st := Z\_st \cdot u\_dot$ $\qquad$ $\sigma1\_st = \blacksquare\,MPa$ $\qquad$ $\dfrac{\sigma1\_st}{Y\_st} = \blacksquare\,\%$

(f) the displacement wave amplitude for a 100 kHz harmonic pressure wave with amplitude p=Y/2

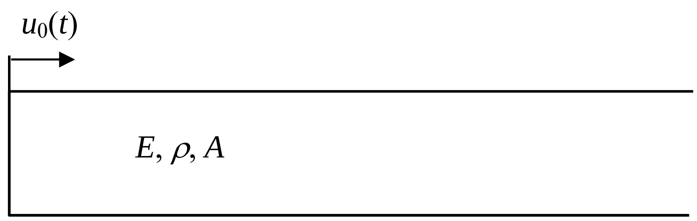
$f := 100 \cdot kHz$ $\qquad$ $\omega := 2 \cdot \pi \cdot f$ $\qquad$ $\omega = \blacksquare\,\dfrac{rad}{s}$

$\sigma\_hat\_al := \dfrac{Y\_al}{2}$ $\qquad$ $u\_hat\_al := \dfrac{\sigma\_hat\_al}{\omega \cdot Z\_al}$ $\qquad$ $\sigma\_hat\_al = \blacksquare\,MPa$ $\qquad$ $u\_hat\_al = \blacksquare\,\mu m$

$\sigma\_hat\_st := \dfrac{Y\_st}{2}$ $\qquad$ $u\_hat\_st := \dfrac{\sigma\_hat\_st}{\omega \cdot Z\_st}$ $\qquad$ $\sigma\_hat\_st = \blacksquare\,MPa$ $\qquad$ $u\_hat\_st = \blacksquare\,\mu m$

Problem 4: Consider a semi-infinite slender bar subjected to end displacement excitation $u_0(t)$ of the form: $u(0,t) = u_0(t) = bt\exp(-t/\tau)$, $t > 0$, where $\tau = 50$ ms, $b = 10^{-4}c$, and c is the wave speed in the bar. The bar is made of aluminum (Table). The cross sectional area is $A = 25\text{ mm}^2$. Plot the value of u_0 at 1 ms intervals up to $t_{max} = 10\tau$. Predict the value of time when the maximum value of u_0 occurs and verify this value on the plot. Find the solution $u(t,x)$. Follow the wave propagation up to $x_{\max} = 10\tau c$. Sketch the solution at times $t = 0.1\tau,\ 0.5\tau,\ 1\tau,\ 2\tau,\ 3\tau,\ 4\tau,\ 6\tau,\ 8\tau$. Present results in the form of subsequent plots, one below the other. Describe what you see.

Solution



(a) Consider the expression

$$u_0(t) = bte^{-\frac{t}{\tau}} \tag{1}$$

Calculate $t_{\max} = 10\tau = 500$ ms and use appropriate software to obtain the plot in Figure 2 below.

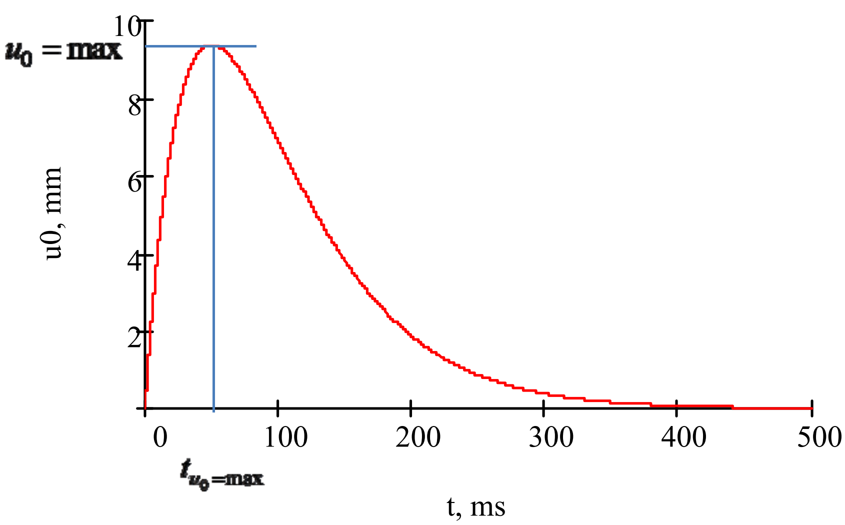


Figure 2 Plot of $u_0(t)$ indicating the maximum value and its position in time.

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(b) To obtain the maximum value, differentiate $u_0(t)$ and set the result to zero, i.e.,

$$\dot{u}_0(t) = be^{-\frac{t}{\tau}} + bt\left(-\frac{1}{\tau}\right)e^{-\frac{t}{\tau}} = b\left(1-\frac{1}{\tau}t\right)e^{-\frac{t}{\tau}} = 0 \tag{2}$$

Hence,

$$t_{u_0=\max} = \tau = 50 \text{ ms}, \qquad u_{0\max} = u_0(\tau) = b\tau e^{-1} = 9.366 \text{ mm} \tag{3}$$

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(c) Refer to Section 5.3.2 and assume that the solution has the general expression

$$u(x,t) = f(ct-x) \tag{4}$$

Impose boundary condition

$$u(0,t) = f(ct) = u_0(t) \tag{5}$$

i.e.,

$$f(ct) = \frac{b}{c}(ct)e^{-\frac{(ct)}{c\tau}} \tag{6}$$

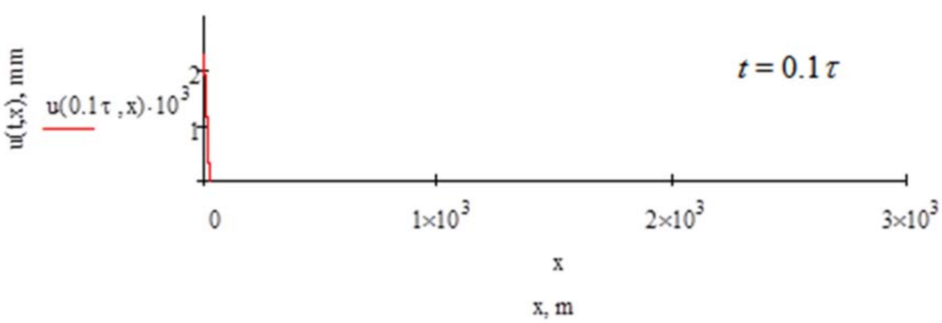
This means that the general form of $f(\ )$ is

$$f(z) = \frac{b}{c}ze^{-\frac{z}{c\tau}} \tag{7}$$

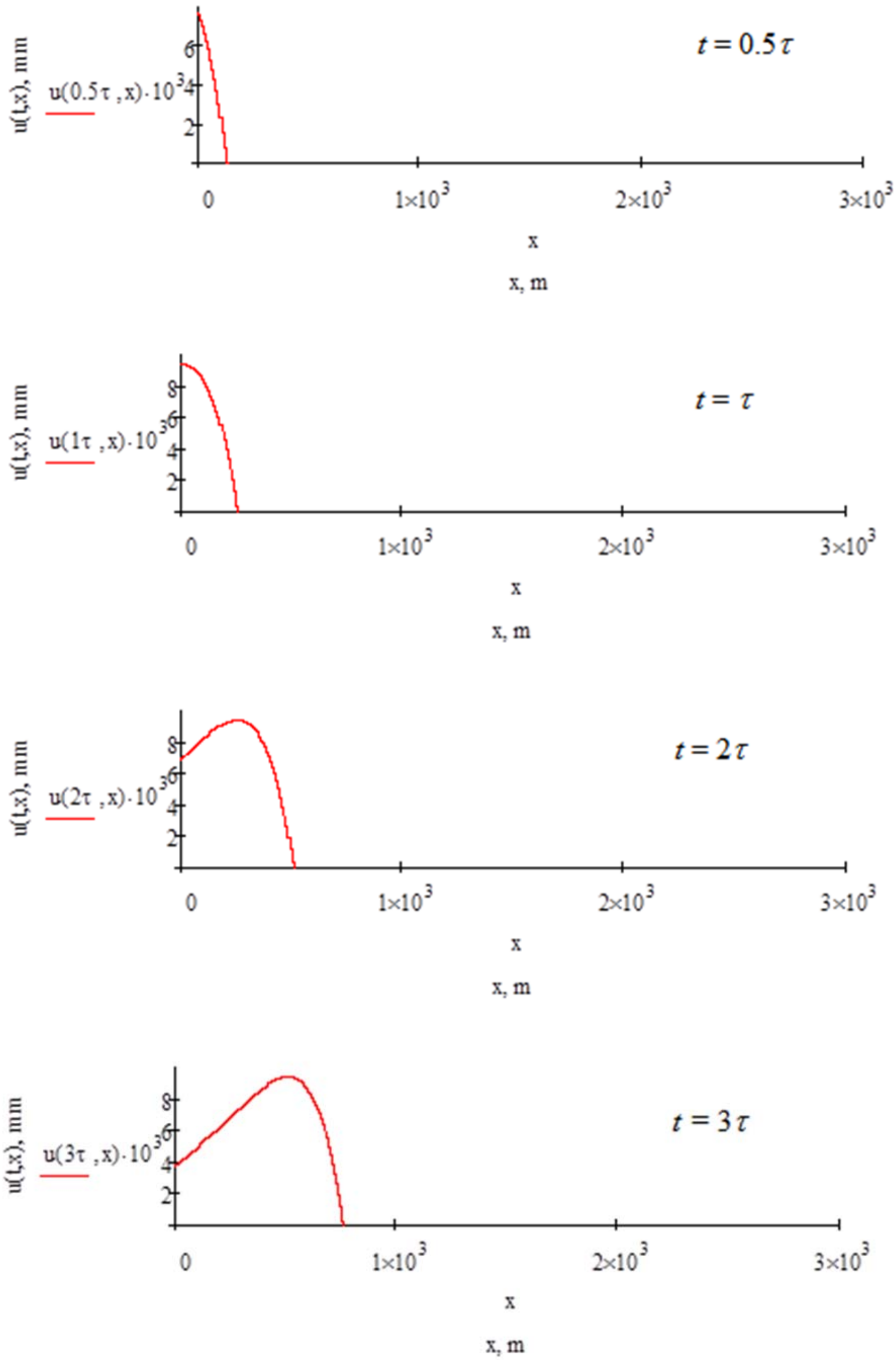
Substitute $z = ct - x$ into Eq. (7) to get

$$u(x,t) = f(ct-x) = \frac{b}{c}(ct-x)e^{-\frac{ct-x}{c\tau}} \tag{8}$$

(d) Calculate the material wave speed $c = \sqrt{E/\rho} = 5092$ m/s. Then, calculate $x_{\max} = 10\tau c = 2546$ m. Substitute these values in Eq. (8) and set the time to the values t = 0.1 τ, 0.5 τ, 1 τ, 2 τ, 3 τ, 4 τ, 6 τ, 8 τ. and $t_{\max} = x_{\max}/c$. The resulting plots are show in Figure 3.



(Figure 3 continued)f



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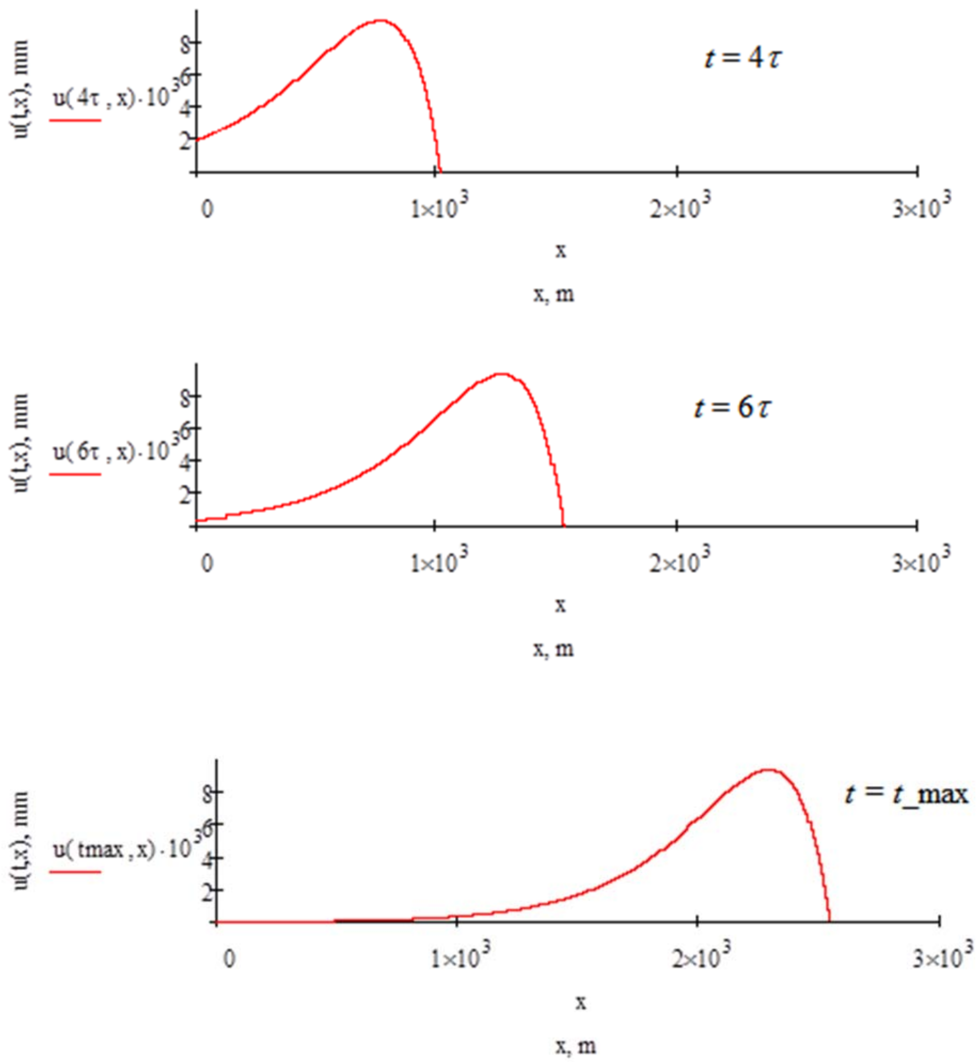


Figure 3 Plot of $u(x,t)$ at various times

Discussion: We see how the wave progresses inside the bar. We see the wave emerging from the excitation end and propagating forward. Note that the plotting on the time axis and on the space axis are reversed in shape. The wave shape in the space domain is flipped in comparison with the wave shape in the time domain (i.e., the wave shape corresponding to smaller times progresses first into the material bar.) This is OK, since the head of the wave must correspond to small time values.

PROBLEM 5.4 SOLUTION

Units: $GPa := 10^{9} Pa$ $ms := 10^{-3} s$

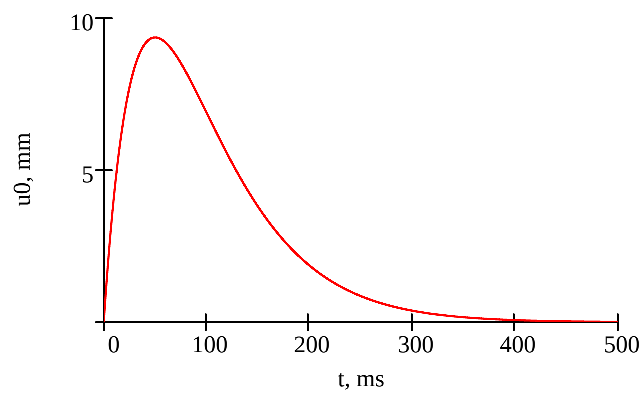
Given $E := 70GPa$ $\rho := 2700 kg \cdot m^{-3}$ $A := 25mm$ $\tau := 50ms$

$c := \sqrt{\dfrac{E}{\rho}}$ $b := 10^{-4} c$

$c = 5092 \dfrac{m}{s}$ $b = 0.509 \dfrac{m}{s}$ $u0(t) := b \cdot t \cdot e^{-\frac{t}{\tau}}$

Ch5-4(a)

$tmin := 0 \cdot \tau$ $dt := 0.01ms$ $tmax := 10 \cdot \tau$ $t := tmin, (tmin + dt) .. tmax$



Ch5-4(b)

$$u0(t) := b \cdot t \cdot e^{-\frac{t}{\tau}} \qquad u0(\tau) = 9.366\,\text{mm}$$

Ch5-4(c)

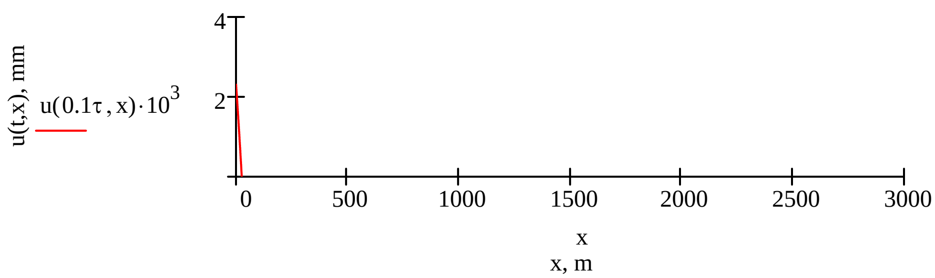
$$u(t,x) := \frac{b}{c} \cdot (c \cdot t - x)\, e^{-\frac{c \cdot t - x}{c \cdot \tau}}$$

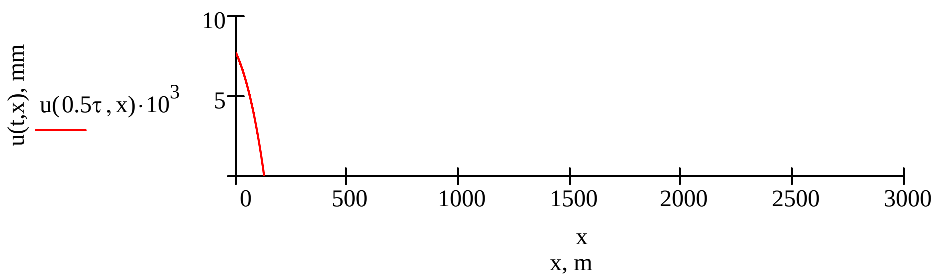
Ch5-4(d)

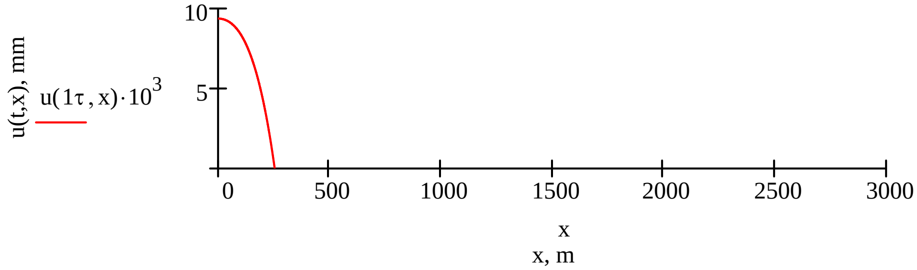
$$xmax := 10 \cdot (\tau \cdot c) \qquad dx := 100\text{mm} \qquad x := 0, dx \,..\, xmax$$

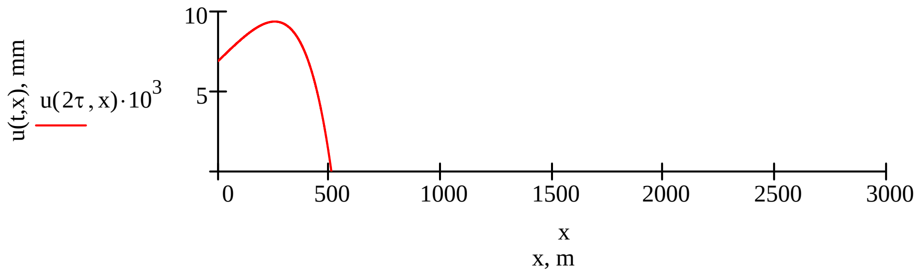
$$xmax = 2546\,\text{m} \qquad tmax := \frac{xmax}{c}$$

$$b = 509 \times 10^{-3}\,\frac{\text{m}}{\text{s}} \quad c = 5092\,\frac{\text{m}}{\text{s}} \qquad \tau = 50 \times 10^{-3}\,\text{s}$$






u(t,x), mm
u(1τ , x)·10^3
10
5
0
500
1000
1500
2000
2500
3000
x
x, m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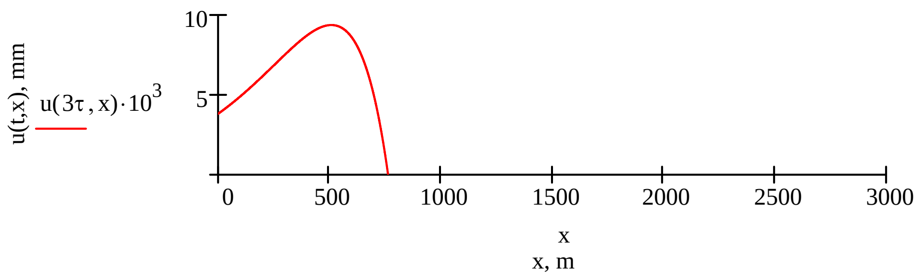
u(t,x), mm
u(3τ , x)·10^3
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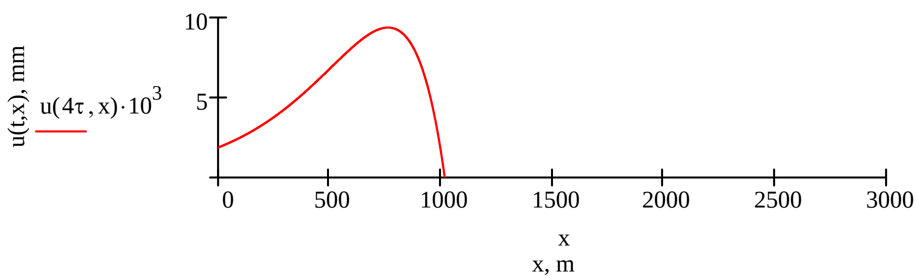
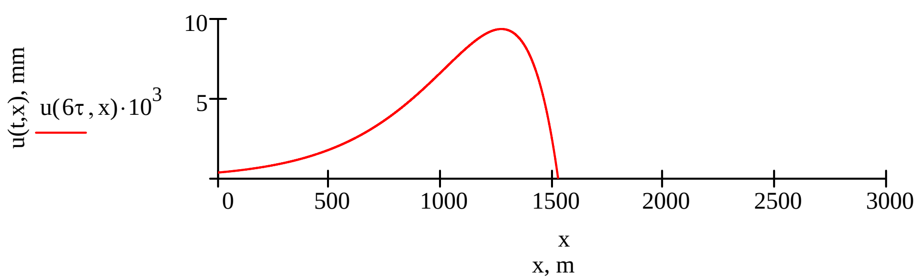
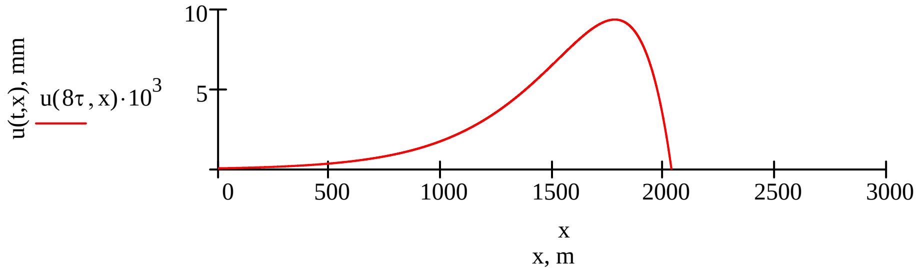
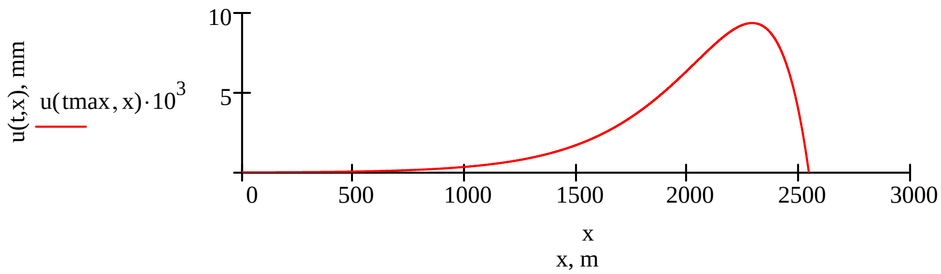
x
x, m


u(t,x), mm
u(4τ , x)·10^3
10
5
0
500
1000
1500
2000
2500
3000
x
x, m


u(t,x), mm
u(6τ , x)·10^3
10
5
0
500
1000
1500
2000
2500
3000
x
x, m



10
5
u(t,x), mm
u(8τ , x)·10^3
0
500
1000
1500
2000
2500
3000
x
x, m



10
5
u(t,x), mm
u(tmax, x)·10^3
0
500
1000
1500
2000
2500
3000
x
x, m

Problem 5: Consider the split Hopkinson bar. A long and slender aluminum bar ($r_1 = r_3 = 12\text{ mm}$) is split at the center and a steel piece ($r_2 = 10\text{ mm}$) is inserted in between as shown in Figure 24 (here, Figure 4). The contact at the two interfaces is assumed perfect. Assume that an incident compressive stress wave pulse of 400 MPa is traveling forward in bar #1. At the interface with bar #2 (the steel piece), some of the wave will be transmitted, and some will be reflected. The transmitted wave will hit the second interface. Again, part of the incident wave will be reflected and part will be transmitted.

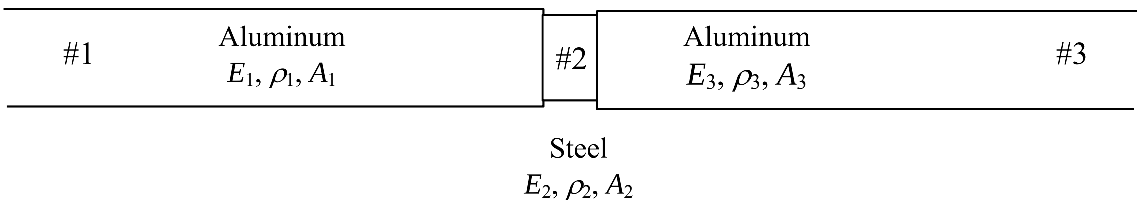


Figure 4 Split Hopkinson bar

<u>Find</u>: (a) the amplitude of the waves transmitted into bars #2 and #3. Comment on the stress values in comparison with the yield stress of the materials and explain what this means. (b) The amplitude of the waves reflected at the 1-2 and 2-3 interfaces. Comment on negative stress values (if they appear) and explain what this means. (c) What radius should the steel piece have such that no reflection occurs at the two interfaces, and the amplitude of the wave transmitted in bar #3 is the same as that of the incident wave in bar #1. After finding the value, verify that indeed no reflection takes place..

<u>Solution</u>

The problem is solved using the theory of wave propagation at interfaces discussed in Section 5.3.10. In particular, one uses Eq. (5.143), i.e.,

$$\begin{aligned} \sigma_t &= \frac{2A_1Z_2}{A_1Z_1 + A_2Z_2}\sigma_i \\ \sigma_r &= -\frac{A_1Z_1 - A_2Z_2}{A_1Z_1 + A_2Z_2}\sigma_i \end{aligned} \qquad (5.143)\ (1)$$

where Z is the acoustic impedance, $Z = \rho c$.

To solve Eq. (1), one first calculates the following numerical values:

$$\begin{aligned} &c_{Al} = 5092\text{ m/s}, \quad Z_{Al}=13.75\text{ MPa}\cdot\text{m}^{-1}\cdot\text{s}, \quad A_{Al}=452\text{ mm}^2 \\ &c_{St} = 5080\text{ m/s}, \quad Z_{St}=39.37\text{ MPa}\cdot\text{m}^{-1}\cdot\text{s}, \quad A_{St}=314\text{ mm}^2 \end{aligned} \qquad (2)$$

(a) The amplitudes of the stress waves transmitted in bars #2 and #3 for an input stress wave $\sigma_1 = 400$ MPa in bar #1 are calculated as

$$\sigma2 := \frac{2A1 \cdot Z2}{A1 \cdot Z1 + A2 \cdot Z2} \cdot \sigma1 \qquad \sigma3 := \frac{2A2 \cdot Z3}{A2 \cdot Z2 + A3 \cdot Z3} \cdot \sigma2$$

$$\sigma2 = -767\text{MPa} \qquad \sigma3 = -356\text{MPa}$$

The stress in steel bar #2 is at 89% of yield, whereas the stress in the aluminum bar #3 is at 71% of yield

(b) The amplitudes of the stress waves reflected at the 1-2 and 2-3 interfaces are calculated as

$$\sigma r12 := -\frac{A1 \cdot Z1 - A2 \cdot Z2}{A1 \cdot Z1 + A2 \cdot Z2} \cdot \sigma1 \qquad \sigma r23 := -\frac{A2 \cdot Z2 - A3 \cdot Z3}{A2 \cdot Z2 + A3 \cdot Z3} \cdot \sigma2$$

$$\sigma r12 = -132\text{MPa} \qquad \sigma r23 = 254\text{MPa}$$

The negative stress values of $\sigma_{r12} = -132$ MPa indicates that the incident stress $\sigma_2 = -767$ MPa did not change sign upon reflection at the 12 interface. The positive stress value of $\sigma_{r23} = 254$ MPa signifies that the incident stress $\sigma_3 = -356$ MPa changed sign upon reflection at the 23 interface. These facts illustrate the difference between the two interfaces.

(c) has two parts, (c1) and (c2) as follows:
(c1) To find the radius of the steel piece that would produce no reflection at the interfaces we set the condition that the denominator in the expression for σ_r in Eq. (1) is zero. This is obtained when $A_1Z_1 - A_2Z_2 = 0$, i.e., the area of the second bar is related to the area of the first bar by the expression

$$A2' = \frac{Z1}{Z2} A1 \tag{3}$$

Upon calculation, one finds $A2' = 158\ \text{mm}^2$. Hence, the radius of the second bar should be

$$r2' := \sqrt{\frac{A2'}{\pi}} \qquad r2' = 7.09\text{mm}$$

(c2) After finding the value, we now verify that indeed no reflection takes place and the full wave amplitude is recovered in bar #3. The amplitudes of the wave transmitted in bar #2 and #3 are

$$\sigma 2' := \frac{2A1 \cdot Z2}{A1 \cdot Z1 + A2' \cdot Z2} \cdot \sigma 1 \qquad \sigma 3' := \frac{2A2' \cdot Z3}{A2' \cdot Z2 + A3 \cdot Z3} \cdot \sigma 2'$$

$$\sigma 2' = -1145 \mathrm{MPa} \qquad \sigma 3' = -400 \mathrm{MPa}$$

It is apparent that the wave transmitted in bar #3 is the same as that of the incident wave in bar #1. Under this *interface matching conditions*, the steel bar #2 is not visible as far as wave propagation is concerned.

The amplitude of the stress waves reflected at the two interfaces are

$$\sigma r12' := -\frac{A1 \cdot Z1 - A2' \cdot Z2}{A1 \cdot Z1 + A2' \cdot Z2} \cdot \sigma 1 \qquad \sigma r23' := -\frac{A2' \cdot Z2 - A3 \cdot Z3}{A2' \cdot Z2 + A3 \cdot Z3} \cdot \sigma 2'$$

$$\sigma r12' = 0 \mathrm{MPa} \qquad \sigma r23' = 0 \mathrm{MPa}$$

Indeed, no reflections take place under the interface matching conditions.

PROBLEM 5.5 SOLUTION

Units: $GPa := 10^{9}Pa$ $MPa := 10^{6}Pa$ $ms := 10^{-3}s$

$E\_al := 70GPa$ $\rho\_al := 2700kg\cdot m^{-3}$ $c\_al := \sqrt{\frac{E\_al}{\rho\_al}}$ $c\_al = 5092\,\frac{m}{s}$ $Z\_al := \rho\_al\cdot c\_al$ $Z\_al = 13.75\,MPa\cdot m^{-1}\cdot s$

$E\_st := 200GPa$ $\rho\_st := 7750kg\cdot m^{-3}$ $c\_st := \sqrt{\frac{E\_st}{\rho\_st}}$ $c\_st = 5080\,\frac{m}{s}$ $Z\_st := \rho\_st\cdot c\_st$ $Z\_st = 39.37\,MPa\cdot m^{-1}\cdot s$

$Y\_al := 500\cdot MPa$

$Y\_st := 860\cdot MPa$

$r\_al := 12mm$ $A\_al := \pi\cdot r\_al^{2}$ $A\_al = 452\,mm^{2}$

$r\_st := 10mm$ $A\_st := \pi\cdot r\_st^{2}$ $A\_st = 314\,mm^{2}$

$\sigma1 := -400MPa$

$E1 := E\_al$ $\rho1 := \rho\_al$ $c1 := c\_al$ $Z1 := Z\_al$ $A1 := A\_al$

$E2 := E\_st$ $\rho2 := \rho\_st$ $c2 := c\_st$ $Z2 := Z\_st$ $A2 := A\_st$

$E3 := E1$ $\rho3 := \rho1$ $c3 := c1$ $Z3 := Z\_al$ $A3 := A1$

Transmitted waves $\sigma2 := \frac{2A1\cdot Z2}{A1\cdot Z1 + A2\cdot Z2}\cdot\sigma1$ $\sigma3 := \frac{2A2\cdot Z3}{A2\cdot Z2 + A3\cdot Z3}\cdot\sigma2$

$\sigma2 = -767\,MPa$ $\sigma3 = -356\,MPa$

$\frac{\sigma2}{Y\_st} = -89\,\%$ $\frac{\sigma3}{Y\_al} = -71\,\%$

Reflected waves $\sigma r12 := -\frac{A1\cdot Z1 - A2\cdot Z2}{A1\cdot Z1 + A2\cdot Z2}\cdot\sigma1$ $\sigma r23 := -\frac{A2\cdot Z2 - A3\cdot Z3}{A2\cdot Z2 + A3\cdot Z3}\cdot\sigma2$

$\sigma r12 = -132\,MPa$ $\sigma r23 = 254\,MPa$

$A2' := \frac{Z1}{Z2}\cdot A1$ $A2' = 158\,mm^{2}$ $r2' := \sqrt{\frac{A2'}{\pi}}$ $r2' = 7.09\,mm$

Verify that the reflected waves are zero

Transmitted waves $\sigma2' := \frac{2A1\cdot Z2}{A1\cdot Z1 + A2'\cdot Z2}\cdot\sigma1$ $\sigma3' := \frac{2A2'\cdot Z3}{A2'\cdot Z2 + A3\cdot Z3}\cdot\sigma2'$

$\sigma2' = -1145\,MPa$ $\sigma3' = -400\,MPa$

$$\sigma r12' := -\frac{A1 \cdot Z1 - A2' \cdot Z2}{A1 \cdot Z1 + A2' \cdot Z2} \cdot \sigma 1 \qquad \sigma r23' := -\frac{A2' \cdot Z2 - A3 \cdot Z3}{A2' \cdot Z2 + A3 \cdot Z3} \cdot \sigma 2'$$

Reflected waves $\sigma r12' = 0\,\mathrm{MPa}$ $\sigma r23' = 0\,\mathrm{MPa}$

Problem 6: Consider a semi-infinite aluminum bar starting at $x=0$. A rectangular pressure pulse is applied at the $x=0$ end, $p(t)=p_0$ for $0<t<T$; $p(t)=0$ otherwise. The bar has the cross sectional area $A=10\ \text{mm}^2$ and material properties listed in Table 5.3. The pressure pulse has $p_0=-Y/2$, $T=10\ \text{ms}$. Find: (a) stress wave expression and its maximum value; (b) particle velocity expression and its maximum value; (c) power flow in the bar (expression and maximum value); kinetic, elastic, and total energy density per unit length of the bar (expression and maximum value)

Solution

A sketch of the problem setup is shown in Figure 5. The expression of the pressure pulse is

$$p(t)=\begin{cases} p_0, & 0<t<T \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $p_0=Y/2=250\ \text{MPa}$, $T=10\ \text{ms}$.

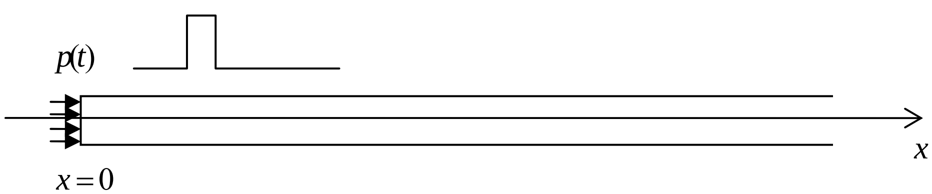


Figure 5 Pressure pulse applied to the $x=0$ end of an elastic bar

(a) The general solution is given by Eq. (5.20) in the form $u(x,t)=f(x-ct)+g(x+ct)$. However, the backward wave $g(ct+x)$ cannot propagate because there is no physical medium for $x<0$ since the bar only exists for $0\le x$. This is the *radiation condition*, which indicates that radiation of the wave is not possible in the direction where the physical medium is missing. For this reason, we disregard the term $g(x+ct)$ and assume the solution in the form

$$u(x,t)=f(x-ct) \tag{2}$$

The boundary condition imposed at the $x=0$ end is given in the form

$$\sigma(0,t)=-p(t) \tag{3}$$

Using Eqs. (5.4) and (5.5) from the textbook and Eq. (2) above, we calculate the stress as

$$\sigma(x,t)=Eu'(x,t)=Ef'(x-ct) \tag{4}$$

Substitution of Eq. (4) into Eq. (3) yields

$$\sigma(0,t)=Eu'(0,t)=Ef'(0-ct)=-p(t) \tag{5}$$

Upon rearrangement, Eq. (5) becomes

$$f'(z)=-\frac{1}{E}p(-z/c) \tag{6}$$

Substituting $z = ct - x$, Eq. (6) becomes

$$f'(x-ct) = -\frac{1}{E}\, p\left(t - \frac{x}{c}\right) \tag{7}$$

Substitution of Eq. (7) into Eq. (4) yields the solution

$$\sigma(x,t) = -p\left(t - \frac{x}{c}\right) \tag{8}$$

Equation (9) indicates that the stress wave follows the shape of the pressure pulse, i.e.,

$$\sigma(x,t) = \begin{cases} -p_0, & 0 < t - \dfrac{x}{c} < T \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

The maximum absolute value of the pressure wave is $\sigma_{\max} = p_0 = 250\ \text{MPa}$ compression.

(b) The particle velocity is calculated with Eq. (5.43), i.e.,

$$\sigma(x,t) = -\rho c\, \dot{u}(x,t) \qquad (5.43)\ (10)$$

Upon solution, Eq. (10) gives

$$\dot{u}(x,t) = -\frac{1}{\rho c}\sigma(x,t) \tag{11}$$

Substitution of Eq. (8) into Eq. (11) yields the particle velocity in the form

$$\dot{u}(x,t) = \frac{1}{\rho c}\, p\left(t - \frac{x}{c}\right) \tag{12}$$

i.e.,

$$\dot{u}(t,x) = \begin{cases} \dfrac{p_0}{\rho c}, & 0 < t - \dfrac{x}{c} < T \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

Maximum value of the particle velocity is $\dot{u}_{\max} = \dfrac{p_0}{\rho c}$, i.e., $\dot{u}_{\max} = 18.18\ \text{m/s}$.

(c) Power flow in the bar (expression and maximum value) is calculated in accordance with Section 5.3.9.2, Eq. (5.82), i.e.,

$$P(x,t) = mc^3 f'^2(x-ct) = A\rho c^3\left[\frac{1}{E}\, p\left(t - \frac{x}{c}\right)\right]^2 = A\frac{1}{\rho c}\, p^2\left(t - \frac{x}{c}\right) \tag{14}$$

$$P(t,x) = \begin{cases} \dfrac{A}{\rho c}\, p_0^2, & 0 < t - \dfrac{x}{c} < T \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

Maximum value is given by $P_{\max} = \dfrac{A}{\rho c}\, p_0^2$, i.e., $P_{\max} = 45.5\ \text{kW}$

Kinetic, elastic, and total energy density per unit length of the bar (expression and maximum value) are calculated in accordance with Section 5.3.9.1, Eqs. (5.79), (5.80), i.e.,

$$k(t,x) = v(t,x) = \frac{1}{2} mc^2 f'^2(x-ct) = A\rho c^2 \left[\frac{1}{E} p\left(t - \frac{x}{c}\right) \right]^2 = \frac{1}{2}\frac{A}{E} p^2\left(t - \frac{x}{c}\right) \tag{16}$$

$$e(t,x) = mc^2 f'^2(x-ct) = A\rho c^2 \left[\frac{1}{E} p\left(t - \frac{x}{c}\right) \right]^2 = \frac{A}{E} p^2\left(t - \frac{x}{c}\right) \tag{17}$$

Maximum values are given by $k_{\max} = v_{\max} = \frac{1}{2}\frac{A}{E} p_0^2$, i.e., $k_{\max} = v_{\max} = 4.46 \text{ J/m}$, and $e_{\max} = \frac{A}{E} p_0^2$, i.e., $e_{\max} = 8.83 \text{ J/m}$

PROBLEM 5.6 SOLUTION

Units: $GPa := 10^{9} Pa$ $\quad MPa := 10^{6} Pa$ $\quad ms := 10^{-3} s$

$E := 70GPa$ $\quad \rho := 2700 kg \cdot m^{-3}$ $\quad c := \sqrt{\frac{E}{\rho}}$ $\quad c = 5.092 \times 10^{3} \frac{m}{s}$

$A := 10 \cdot mm^{2}$ $\quad Y := 500 \cdot MPa$

$p0 := \frac{Y}{2}$ $\quad p0 = 250\,MPa$

(a) $\sigma := p0$ $\quad \sigma = 250\,MPa$

(b) $u\_dot := \frac{\sigma}{\rho \cdot c}$ $\quad u\_dot = 18.18 \frac{m}{s}$

(c) $P := \frac{A}{\rho \cdot c} \cdot p0^{2}$ $\quad P = 45.5\,kW$

$k := \frac{1}{2} \cdot \frac{A}{E} p0^{2}$ $\quad k = 4.46 \frac{J}{m}$

$e := \frac{A}{E} p0^{2}$ $\quad e = 8.93 \frac{J}{m}$

Problem 7: Consider a semi-infinite aluminum bar subjected to 100 kHz harmonic pressure excitation at one end. The bar has cross sectional area $A = 10\ \text{mm}^2$ and material properties listed in Table 5.3. The pressure amplitude is $p_0 = -Y/2$. Find: (a) Stress and particle displacement wave expressions and its maximum values. Phase relation between stress and displacement: (b) Particle velocity expression and numerical value of its amplitude. Phase relation between stress and particle velocity; (c) The power flow in the bar (expression and numerical value of its amplitude); (d) The kinetic, elastic, and total energy density per unit length of the bar (expression and numerical value of its amplitude)

<u>Solution</u>

The problem definition is sketched in Figure 6.

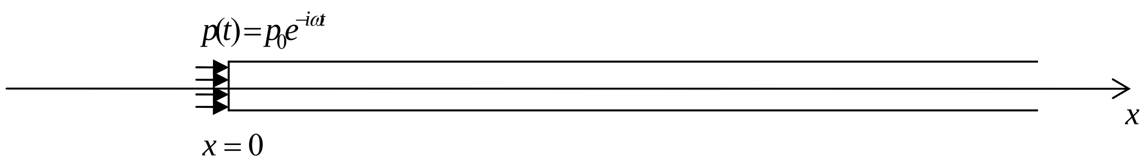


Figure 6 Harmonic pressure applied to the $x = 0$ end of an elastic bar

The pressure amplitude is $p_0 = Y/2 = 250$ MPa and the angular frequency is $\omega = 2\pi \times 100$ kHz.

(a) The general solution for harmonic waves is given by Eq. (5.59) in the form $u(x,t) = Ae^{i(\gamma x - \omega t)} + Be^{-i(\gamma x + \omega t)}$. However, the backward wave $Ae^{-i(\gamma x + \omega t)}$ cannot propagate because there is no physical medium for $x < 0$ since the bar only exists for $0 \le x$. This is the *radiation condition*, which indicates that radiation of the wave is not possible in the direction where the physical medium is missing. For this reason, we disregard the term $Ae^{-i(\gamma x + \omega t)}$ and assume the solution in the form

$$u(x,t) = Be^{i(\gamma x - \omega t)} \tag{1}$$

The boundary condition is given by

$$\sigma(0,t) = -p_0 e^{-i\omega t} \tag{2}$$

Using Eqs. (5.4) and (5.5) from the textbook and Eq. (1) above, we calculate the stress as

$$\sigma(x,t) = Eu'(x,t) = E(i\gamma)Be^{i(\gamma x - \omega t)} \tag{3}$$

Substitution of Eq. (3) into Eq. (2) gives

$$\sigma(0,t) = E(i\gamma)Be^{-i\omega t} = -p_0 e^{-i\omega t} \tag{4}$$

Upon solution, Eq. (4) yields

$$B = \frac{-p_0}{iE\gamma} = \frac{p_0}{E\gamma} e^{i\left(\pi - \frac{\pi}{2}\right)} = \frac{p_0}{E\gamma} e^{i\frac{\pi}{2}} = \frac{p_0}{\rho c \omega} e^{i\frac{\pi}{2}} \tag{5}$$

(a1) Stress wave expression is calculated by substitution of Eq. (5) into Eq. (3); the result is

$$\sigma(x,t) = iE\gamma Be^{i(\gamma x-\omega t)} = i\,\cancel{E\gamma}\,\frac{p_0}{\cancel{E\gamma}}e^{i\frac{\pi}{2}}e^{i(\gamma x-\omega t)} = p_0 e^{i(\gamma x-\omega t+\pi)} = -p_0 e^{i(\gamma x-\omega t)} \quad (6)$$

The stress-wave amplitude is

$$\hat{\sigma} = p_0 = 250 \text{ MPa} \quad (7)$$

(a2) Displacement wave expression is calculated by substitution of Eq. (5) into Eq. (1); the result is

$$u(x,t) = \frac{p_0}{\rho c\omega}e^{i(\gamma x-\omega t+\frac{\pi}{2})} \quad (8)$$

The displacement-wave amplitude is

$$\hat{u} = \frac{p_0}{\rho c\omega} \quad (9)$$

Upon calculation, $\hat{u} = 28.9\ \mu\text{m}$.

Comparison of Eqs. (8) and (6) indicates that the phases of the stress and displacement waves are different by $\pi/2$. This means that the stress and displacement waves are in quadrature.

(b) The particle velocity expression is calculated with Eq. (5.43), i.e.,

$$\dot{u}(x,t) = -\frac{1}{\rho c}\sigma(x,t) \quad (10)$$

Upon substitution of Eq. (6) into Eq. (10) we get

$$\dot{u}(x,t) = \frac{p_0}{\rho c}e^{i(\gamma x-\omega t)} \quad (11)$$

The displacement wave amplitude is $\hat{\dot{u}} = \frac{p_0}{\rho c}$, i.e., $\hat{\dot{u}} = 18.2$ m/s. Comparison of expressions (11) and (6) indicates that the particle velocity is in antiphase with the stress.

(c) The power flow in the bar is calculated with Eq. (5.102), i.e.,

$$P(x,t) = mc\omega^2\hat{u}^2\sin^2(\gamma x-\omega t) = \rho Ac\frac{1}{\rho^2c^2}p_0^2\sin^2(\gamma x-\omega t) = \frac{A}{\rho c}p_0^2\sin^2(\gamma x-\omega t) \quad (12)$$

The power amplitude is

$$\hat{P} = \frac{A}{\rho c}p_0^2 \qquad \hat{P} = 45.5 \text{ kW} \quad (13)$$

The time-averaged power is calculated with Eq. (5.107) using Eq. (5.103), i.e.,

$$\langle P\rangle = \frac{1}{2}mc\omega^2\hat{u}^2 = \frac{1}{2}\hat{P} \qquad \langle P\rangle = 22.75 \text{ kW} \quad (14)$$

(d) The kinetic, elastic, and total energy densities per unit length of the bar (expression and numerical value of its amplitude) are calculated using Eqs. (5.96) and (5.97), i.e.,

$$v(x,t) = k(x,t) = \tfrac{1}{2} m\omega^2 \hat{u}^2 \sin^2(\gamma x - \omega t) \tag{15}$$

$$e(x,t) = k(x,t) + v(x,t) = m\omega^2 \hat{u}^2 \sin^2(\gamma x - \omega t) \tag{16}$$

Substituting Eq. (9) into Eqs. (15) and (16) yields

$$\begin{aligned} v(x,t) = k(x,t) &= \frac{1}{2} m\omega^2 \hat{u}^2 \sin^2(\gamma x - \omega t) \\ &= \frac{1}{2} \cancel{\rho} A \cancel{\omega^2} \frac{p_0^2}{\rho^{\cancel{2}} c^2 \cancel{\omega^2}} \sin^2(\gamma x - \omega t) = \frac{1}{2} \frac{A}{\rho c^2} p_0^2 \sin^2(\gamma x - \omega t) \end{aligned} \tag{17}$$

$$e(x,t) = 2v(x,t) = \frac{A}{\rho c^2} p_0^2 \sin^2(\gamma x - \omega t) \tag{18}$$

The amplitude of the kinetic, elastic, and total energy densities are

$$\hat{v} = \hat{k} = \frac{1}{2} \frac{A}{\rho c^2} p_0^2, \quad \hat{v} = \hat{k} = 4.465 \text{ J m}^{-1} \tag{19}$$

$$\hat{e} = \frac{A}{\rho c^2} p_0^2, \quad \hat{e} = 8.929 \text{ J m}^{-1} \tag{20}$$

The time-averaged total energy density is calculated with Eq. (5.101), i.e.,

$$\langle e \rangle = \tfrac{1}{2} m\omega^2 \hat{u}^2 = \frac{1}{2} m\omega^2 \frac{p_0^2}{\rho^2 c^2 \omega^2} = \frac{1}{2} \frac{A}{\rho c^2} p_0^2, \text{ i.e., } \langle e \rangle = 4.465 \text{ J m}^{-1} \tag{21}$$

The time-averaged kinetic and elastic energy densities are calculated by simply taking half of Eq. (21), i.e.,

$$\langle v \rangle = \langle k \rangle = \frac{1}{4} \frac{A}{\rho c^2} p_0^2, \quad \langle v \rangle = \langle k \rangle = 2.232 \text{ J m}^{-1} \tag{22}$$

PROBLEM 5.7 SOLUTION

Units: $GPa := 10^{9} Pa$ $MPa := 10^{6} Pa$ $ms := 10^{-3} s$ $kHz := 10^{3} \cdot Hz$ $\mu m := 10^{-6} \cdot m$

$E := 70 GPa$ $\rho := 2700 kg \cdot m^{-3}$ $c := \sqrt{\dfrac{E}{\rho}}$ $c = 5.092 \times 10^{3} \dfrac{m}{s}$

$A := 10 \cdot mm^{2}$ $Y := 500 \cdot MPa$ $f := 100 \cdot kHz$ $\omega := 2 \cdot \pi \cdot f$

$p0 := \dfrac{Y}{2}$ $p0 = 250\, MPa$

(a) $\sigma := p0$ $\sigma = 250\, MPa$

$u := \dfrac{p0}{\rho \cdot c \cdot \omega}$ $u = 28.942\, \mu m$

(b) $u\_dot := \dfrac{\sigma}{\rho \cdot c}$ $u\_dot = 18.185 \dfrac{m}{s}$

(c) $P := \dfrac{A}{\rho \cdot c} \cdot p0^{2}$ $P = 45.5\, kW$ $\dfrac{P}{2} = 22.7\, kW$

(d) $k := \dfrac{1}{2} \cdot \dfrac{A}{\rho \cdot c^{2}} p0^{2}$ $k = 4.464 \dfrac{J}{m}$ $\dfrac{k}{2} = 2.232 \dfrac{J}{m}$

$e := \dfrac{A}{\rho \cdot c^{2}} p0^{2}$ $e = 8.929 \dfrac{J}{m}$ $\dfrac{e}{2} = 4.464 \dfrac{J}{m}$

Problem 8: Consider elastic waves in a finite slender bar of length *L* with free ends generated by a harmonic pressure excitation applied at the left hand end. Analyze the wave reflection process in the bar (boundary condition at $x=0$; the reflection at $x=L$).Analyze the resonance condition and calculate the eigenvalues, eigen frequencies, eigen lengths. Calculate the resonance solution in the bar: (a) the stress wave solution; sketch first, second, and third modeshapes; (b) the displacement wave solution; sketch first, second, and third mode shapes

Solution

The problem definition is sketched in Figure 7

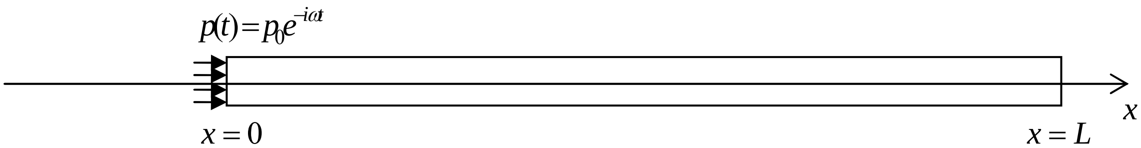


Figure 7 Finite-length bar under the action of a harmonic pressure excitation at the $x=0$ end

The material properties are given in Table 5.3.

Incident wave $u_i(x,t)$ is generated at $x=0$ and travels towards $x=L$ where is reflected and travels back as the reflected wave $u_r(x,t)$. Since the excitation $p(t)$ is continuous, the process is steady-state and both the incident and the reflected waves coexist in the bar.

$$u_i(x,t) = Be^{i(\gamma x-\omega t)} \tag{1}$$

$$u_r(x,t) = Ae^{-i(\gamma x+\omega t)} \tag{2}$$

The total wave in the bar results from the superposition of the incident and reflected waves, i.e.,

$$u(x,t) = u_i(x,t) + u_r(x,t) = Ae^{-i(\gamma x+\omega t)} + Be^{i(\gamma x-\omega t)} \tag{3}$$

Differentiation of Eq. (3) w.r.t. *x*, and multiplication by the elastic modulus *E* yields the stress wave

$$\sigma(x,t) = E\left(i\gamma Ae^{-i(\gamma x+\omega t)} - i\gamma Be^{i(\gamma x-\omega t)}\right) \tag{4}$$

Imposing the free boundary condition at $x=L$, we write

$$\sigma(L,t) = E\left(i\gamma Ae^{-i(\gamma L+\omega t)} - i\gamma Be^{i(\gamma L-\omega t)}\right) = 0 \tag{5}$$

or

$$\left(i\gamma Ae^{-i\gamma L} - i\gamma Be^{i\gamma L}\right)e^{-i\omega t} = 0 \tag{6}$$

Upon solution, Eq. (6) yields

$$A = Be^{i2\gamma L} \tag{7}$$

and

$$\sigma(x,t) = i\gamma E\, B\left(e^{i2\gamma L}e^{-i(\gamma x+\omega t)} - e^{i(\gamma x-\omega t)}\right) \tag{8}$$

Imposing now the boundary condition at $x=0$ gives

$$\sigma(0,t) = i\gamma E\, B\left(e^{i2\gamma L}e^{-i\omega t} - e^{-i\omega t}\right) = i\gamma E\, Be^{-i\omega t}\left(e^{i2\gamma L} - 1\right) = -p_0e^{-i\omega t} \tag{9}$$

Upon solution, Eq. (9) gives

$$B=\frac{-p_0}{i\gamma E}\frac{1}{e^{i2\gamma L}-1}=i\frac{p_0}{\gamma E}\frac{1}{e^{i2\gamma L}-1} \tag{10}$$

Substitution of Eq. (10) into Eq. (8) yields

$$\begin{aligned}\sigma(x,t)&=i\gamma E\,B\left(e^{i2\gamma L}e^{-i(\gamma x+\omega t)}-e^{i(\gamma x-\omega t)}\right)=i\gamma Ei\frac{p_0}{\gamma E}\frac{1}{\left(e^{i2\gamma L}-1\right)}\left(e^{i2\gamma L}e^{-i(\gamma x+\omega t)}-e^{i(\gamma x-\omega t)}\right)\\&=p_0\frac{1}{\left(1-e^{i2\gamma L}\right)}\left(e^{i2\gamma L}e^{-i(\gamma x+\omega t)}-e^{i(\gamma x-\omega t)}\right)\end{aligned} \tag{11}$$

Upon simplification, the stress wave solution becomes

$$\sigma(x,t)=\frac{p_0}{1-e^{i2\gamma L}}\left(e^{i\gamma(2L-x)}-e^{i\gamma x}\right)e^{-i\omega t} \qquad \text{(stress wave solution)} \tag{12}$$

The first factor in Eq. (12) is the amplitude; the second factor is the spatial variation; while the third factor is the temporal variation.

Correspondingly, the displacement wave solution is obtain by substituting Eqs. (7), (10) into Eq. (3), i.e.,

$$\begin{aligned}u(x,t)&=Ae^{-i(\gamma x+\omega t)}+Be^{i(\gamma x-\omega t)}=Be^{i2\gamma L}e^{-i(\gamma x+\omega t)}+Be^{i(\gamma x-\omega t)}\\&=B\left(e^{i\gamma(2L-x)}+e^{i\gamma x}\right)e^{-i\omega t}=-i\frac{p_0}{\gamma E}\frac{1}{1-e^{i2\gamma L}}\left(e^{i\gamma(2L-x)}+e^{i\gamma x}\right)e^{-i\omega t}\end{aligned} \tag{13}$$

Equation (13) indicates that the displacement wave solution is

$$u(x,t)=-i\frac{p_0}{\gamma E}\frac{1}{1-e^{i2\gamma L}}\left(e^{i\gamma(2L-x)}+e^{i\gamma x}\right)e^{-i\omega t} \qquad \text{(displacement wave solution)} \tag{14}$$

Resonance occurs when the wave amplitude becomes very large. Examination of Eqs. (12), (13) indicates that very large amplitudes are obtained as the denominator tends towards zero. Hence the resonance condition can be expressed as

$$1-e^{-i2\gamma L}=0, \text{ or } e^{i2\gamma L}=1 \tag{15}$$

Solution of Eq. (15) yields the eigenvalues. Note that

$$1=e^{i2\pi}=e^{i4\pi}=e^{i6\pi}=...=e^{i2\pi j} \qquad \text{i}=\sqrt{-1} \qquad j=1,2,3... \tag{16}$$

Substitution of Eq. (16) into Eq. (15) yields the eigenvalues

$$2\gamma L=2\pi j \tag{17}$$

i.e.,

$$\left(\gamma L\right)_j=\pi j=\pi,2\pi,3\pi,... \tag{18}$$

To calculate the eigen frequencies, recall that $\gamma=\omega/c$ and $\omega=2\pi f$; hence

$$\gamma L=\frac{2\pi f}{c}L \qquad \text{i.e.,} \qquad f_j=\frac{c}{2\pi L}\left(\gamma L\right)_j \tag{19}$$

Substitution of Eq. (18) into Eq. (19) yields the eigen frequencies

$$f_j=\frac{c}{2\pi L}\pi j=j\frac{c}{2L}=\frac{c}{2L},\frac{2c}{2L},\frac{3c}{2L},... \tag{20}$$

For example, if $L=100$ mm, then $f_1=25.5$ kHz, $f_2=50.9$ kHz, $f_3=76.4$ kHz.

(c) To calculate the eigen lengths, assume the frequency is constant and hence the wavelength is fixed to the value

$$\lambda = cT = \frac{c}{f}, \tag{21}$$

The wavenumber can be expressed in terms of the wavelength as

$$\gamma = \frac{2\pi}{\lambda} \tag{22}$$

Substitution of Eq. (22) into Eq. (18) yields

$$\frac{2\pi}{\lambda} L_j = \pi j = \pi, 2\pi, 3\pi, ... \tag{23}$$

or

$$L_j = j\frac{\lambda}{2} = \frac{\lambda}{2}, 2\frac{\lambda}{2}, 3\frac{\lambda}{2}, ... \tag{24}$$

For $f = 50\text{ kHz}$, $\lambda = 101.8\text{ mm}$ and $L_1 = 50.9\text{ mm}$ $L_2 = 101.8\text{ mm}$, $L_3 = 153\text{ mm}$.

(a) The stress wave solution at resonance is calculated as follows. At resonance, Eq. (15) is satisfied, i.e., $e^{i2\gamma L} = 1$, and hence the amplitude in becomes infinity. In practice, material damping prevents the amplitude from going to infinity; however it is very large when resonance conditions happen. Now, recall Eq. (12), i.e.,

$$\sigma(x,t) = \frac{p_0}{1 - e^{i2\gamma L}}\left(e^{i\gamma(2L-x)} - e^{i\gamma x}\right)e^{-i\omega t} \tag{25}$$

Consider the part of Eq. (25) that represents the spatial variation, i.e., $\left(e^{i\gamma(2L-x)} - e^{i\gamma x}\right)$. At resonance, $e^{i2\gamma L} = 1$, and hence this part simplifies as follows:

$$\left(e^{i\gamma(2L-x)} - e^{i\gamma x}\right) = \left(\cancel{e^{i2\gamma L}} e^{-i\gamma x} - e^{i\gamma x}\right)\Big|_{e^{i2\gamma L}=1} = -\left(e^{i\gamma x} - e^{-i\gamma x}\right) = -2i\sin\gamma x \tag{26}$$

The factor -2i may be absorbed into the amplitude term; the remaining spatial variation $\sin\gamma x$ is the modeshape, denoted by

$$\bar{\sigma}(x) = \sin\gamma x \qquad \text{(stress modeshape)} \tag{27}$$

The first, second, and third modeshapes are calculated with Eq. (27) by giving to γ the corresponding values from Eq. (18), i.e.,

$$\bar{\sigma}_1(x) = \sin\pi\frac{x}{L} \qquad \bar{\sigma}_2(x) = \sin 2\pi\frac{x}{L} \qquad \bar{\sigma}_3(x) = \sin 3\pi\frac{x}{L} \tag{28}$$

Sketches of the first three stress wave modeshapes are shown in Figure 8.

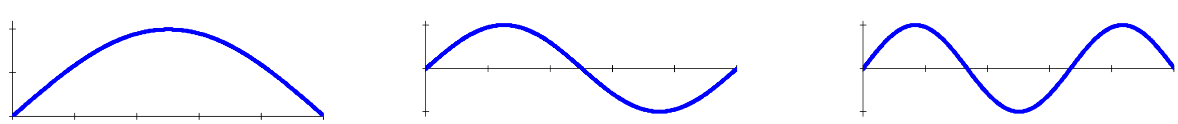


Figure 8 First, second, and third stress modeshapes of a free bar

(b) The displacement wave solution at resonance is calculated as follows. Recall the displacement wave solution of Eq. (14), i.e.,

$$u(x,t) = -i\frac{p_0}{\gamma E}\frac{1}{1-e^{i2\gamma L}}\left(e^{i\gamma(2L-x)} + e^{i\gamma x}\right)e^{-i\omega t} \tag{29}$$

At resonance, Eq. (15) is satisfied, i.e., $e^{i2\gamma L} = 1$, and the space varying term in Eq. (29) becomes

$$\left(e^{i\gamma(2L-x)} + e^{i\gamma x}\right) = \left(\cancel{e^{i2\gamma L}} e^{-i\gamma x} + e^{i\gamma x}\right)\Big|_{e^{i2\gamma L}=1} = -\left(e^{i\gamma x} + e^{-i\gamma x}\right) = -2\cos\gamma x \tag{30}$$

The factor -2 is absorbed into the amplitude term, and the remaining spatial variation is the modeshape

$$\bar{u}(x) = \cos\gamma x \tag{31}$$

The first, second, and third modeshapes are calculated with Eq. (31) by giving to γ the corresponding values from Eq. (18), i.e.,

$$\begin{aligned} \bar{u}_1(x) &= \cos\pi\frac{x}{L} \\ \bar{u}_2(x) &= \cos 2\pi\frac{x}{L} \\ \bar{u}_3(x) &= \cos 3\pi\frac{x}{L} \end{aligned} \tag{32}$$

Sketches of the first three stress wave modeshapes are shown in Figure 9.

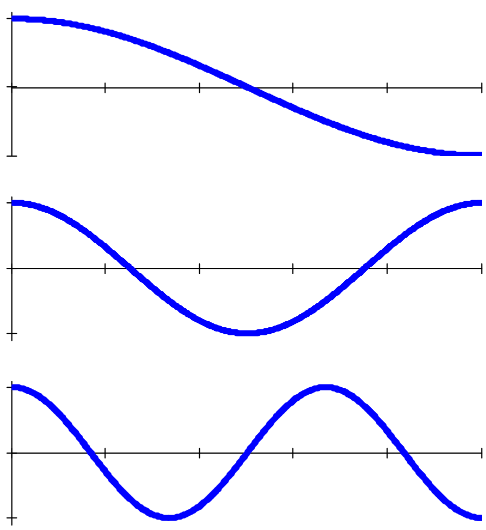


Figure 9 First, second, and third displacement modeshapes of a free bar

PROBLEM 5.8 SOLUTION

Units: $GPa := 10^{9}Pa$ $MPa := 10^{6}Pa$ $ms := 10^{-3}s$ $kHz := 10^{3}\cdot Hz$

$E := 70GPa$ $\rho := 2700kg\cdot m^{-3}$ $c := \sqrt{\frac{E}{\rho}}$ $c = 5.092 \times 10^{3}\,\frac{m}{s}$

$A := 10\cdot mm^{2}$ $Y := 500\cdot MPa$ $L := 100\cdot mm$

$p0 := \frac{Y}{2}$ $p0 = 250\,MPa$ $\rho c := \rho\cdot c$

$n := 1..3$

$f_{n} := \frac{c}{2\cdot\pi\cdot L}\cdot n\cdot\pi$ $f_{n} =$

f_n (kHz)
25.5
50.9
76.4

$f := 50\cdot kHz$ $\omega := 2\cdot\pi\cdot f$

$\lambda := \frac{c}{f}$ $\lambda = 101.8\,mm$ $\frac{\lambda}{2} = 50.9\,mm$ $\frac{3\cdot\lambda}{2} = 152.8\,mm$

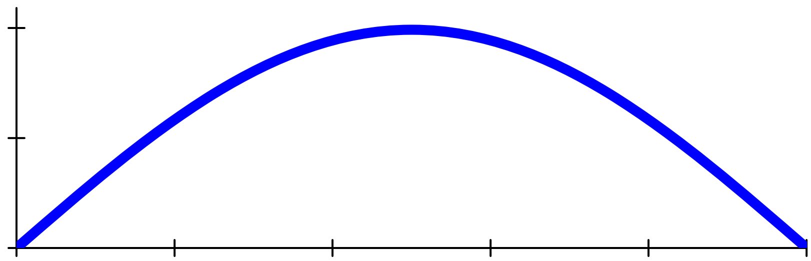
$L1 := \frac{\lambda}{2}$ $L1 = 50.9\,mm$

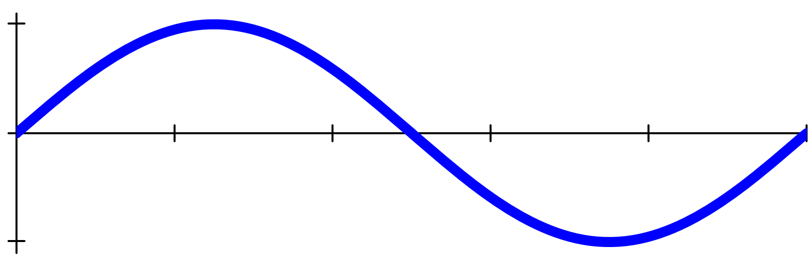
$L2 := 2\cdot\frac{\lambda}{2}$ $L2 = 101.8\,mm$

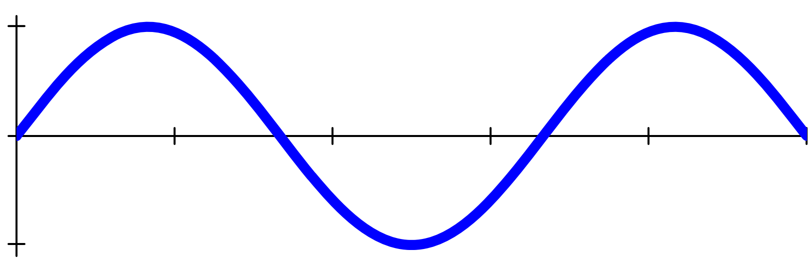
$L3 := 3\cdot\frac{\lambda}{2}$ $L3 = 152.8\,mm$

$j := 1..3$ $\quad dx := 0.001$ $\quad x := 0, dx..1$

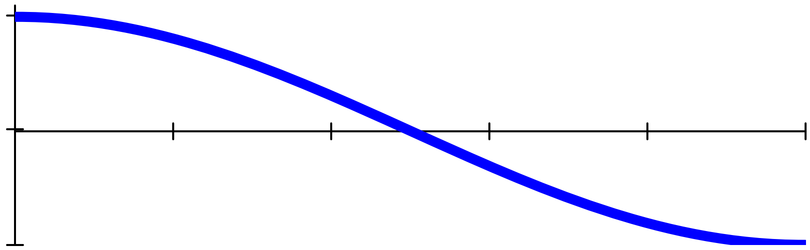
$$U(j, x) := \sin(\pi \cdot j \cdot x)$$

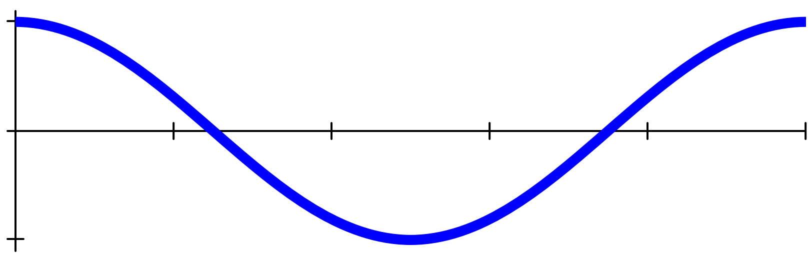


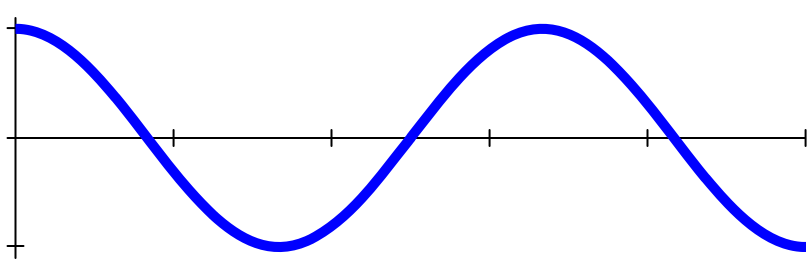




$V(j, x) := \cos(\pi \cdot j \cdot x)$







Problem 9: Consider harmonic elastic waves in an infinite slender bar. (a) Derive the expression for standing waves considering the superposition of two identical harmonic waves traveling in opposite directions; accompany your derivation with the appropriate sketches. (b) Determine the expression for the bar length L, such that will permit the generation of standing waves of a given frequency, f; consider the following cases: fundamental; overtone; second overtone. (c) Determine and expression for the frequency that will generate standing waves in a bar of a given length, L; consider the following cases: fundamental; overtone; second overtone.

Solution

The solution to this problem can be found in textbook Chapter 5, Section 5.3.8 and need not be repeated here.

Problem 10: The wave speed (phase velocity) for flexural waves in a plate is given by $c_F = a\sqrt{\omega}$ where $a = \left[Ed^2/3\rho\left(1-\nu^2\right)\right]^{1/4}$. (a) Show that the group velocity is twice the wave speed, i.e., $c_{gF} = 2c_F$. (b) Calculate the energy velocity.

Solution

(a) To calculate the group velocity, recall the definition of group velocity given in the textbook Eq. (5.214) in the form

$$c_g = \frac{d\omega}{d\gamma} \tag{1}$$

To express ω as function of γ, recall the definition of wavenumber given in the textbook by Eq. (3.172), i.e.,

$$\gamma = \frac{\omega}{c} \tag{2}$$

In our case, the generic notation c is replaced by the specific notation c_F where

$$c_F = a\sqrt{\omega} \tag{3}$$

and $a = \left[Ed^2/3\rho\left(1-\nu^2\right)\right]^{1/4}$. Substitution of Eq. (3) into Eq. (2) yields

$$\gamma = \frac{\omega}{c} = \frac{\omega}{a\sqrt{\omega}} = \frac{1}{a}\sqrt{\omega} \tag{4}$$

Use Eq. (4) to express ω as function of γ, i.e.,

$$\omega = a^2\gamma^2 \tag{5}$$

Differentiate Eq. (5) w.r.t. γ to get

$$\frac{d\omega}{d\gamma} = 2a^2\gamma \tag{6}$$

Substitute Eqs. (4) and (6) into Eq. (1) and get

$$c_{gF} = \frac{d\omega}{d\gamma} = 2a^2\gamma = 2a^2\frac{1}{a}\sqrt{\omega} = 2a\sqrt{\omega} \tag{7}$$

Substitute Eq. (3) into Eq. (7) to get, as required,

$$c_{gF}(\omega) = 2c_F(\omega) \tag{8}$$

(b) To calculate the energy velocity, recall the definition of energy velocity given in the textbook Eq. (5.243) which indicates that the energy velocity is equal to the group velocity. Hence, the answer to this question is

$$c_e = c_{gF} = 2c_F \tag{9}$$

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Problem 11: Consider straight-crested SH waves in an aluminum plate with material properties given in Table 5.3. Calculate the speed of propagation of the SH wave front. Sketch on the same picture the particle motion and wave velocity vectors. How is the particle motion of the straight-crested SH waves different from the particle motion of the straight-crested axial waves in a plate?

Solution

(a) The SH wave front propagates with the wavespeed of the SH waves, which is the shear wave speed given by

$$c_S = \sqrt{\frac{G}{\rho}} \qquad \text{(shear wave speed)} \tag{1}$$

where the shear modulus G is calculated with the formula

$$G = \frac{E}{2(1+\nu)} = 26.3 \text{ GPa} \qquad \text{(shear modulus)} \tag{2}$$

Upon calculation, Eq. (1) yields

$$c_S = \sqrt{\frac{G}{\rho}} = 3122 \ \mathrm{m/s} \qquad \text{(shear wave speed)} \tag{3}$$

(b) Figure 10 shows the particle motion $\vec{u}_{SH}$ and wavefront normal $\vec{n}_{SH}$ for an SH wave in a strip. The wave velocity vector is parallel to the wavefront normal $\vec{n}$.

$$\vec{v}_{SH} = c_S \vec{n}_{SH} \qquad \text{(wave velocity vector)} \tag{4}$$

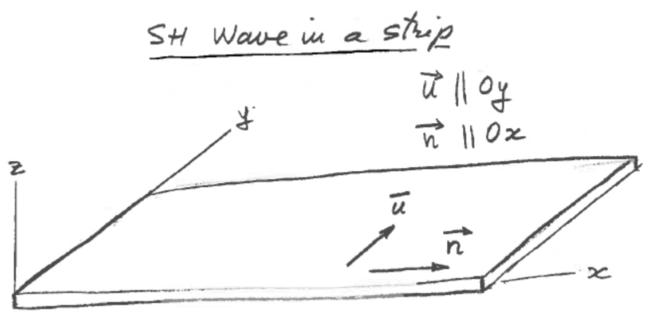

Figure 10 Particle motion $\vec{u}_{SH}$ and wavefront normal $\vec{n}_{SH}$ for an SH wave in a strip. The wave velocity vector $\vec{v}_{SH} = c_S \vec{n}_{SH}$ is parallel to the wavefront normal $\vec{n}$.

(c) The particle motion of the SH wave is perpendicular to the wave velocity vector whereas the particle motion of the axial wave is parallel to the wave velocity vector.

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Units: $GPa := 10^{9} Pa$ $\quad MPa := 10^{6} Pa$ $\quad ms := 10^{-3} s$ $\quad \mu\varepsilon := 10^{-6}$ $\quad kHz := 10^{3} \cdot Hz$ $\quad \mu m := 10^{-6} \cdot m$

Aluminum material properties from Table 5.3:

$E := 70GPa \qquad \rho := 2700 kg \cdot m^{-3} \qquad \nu := 0.33 \qquad G := \frac{E}{2 \cdot (1 + \nu)} \qquad G = 26.3\, GPa$

wave speed

$c := \sqrt{\frac{G}{\rho}} \qquad c = 3122 \frac{m}{s}$

Problem 12: Consider_the properties of aluminum and steel show in Table 5.3. (a) Write the expressions for the elastic constants λ, μ, and bulk modulus, B; calculate their values. (b) Calculate the following wave speeds values: $c_P = \sqrt{(\lambda + 2\mu)/\rho}$ (3-D pressure wave); $c_S = \sqrt{\mu/\rho}$ (shear wave); $c = \sqrt{E/\rho}$ (1-D pressure wave); $c_L = \sqrt{E/\rho(1-\nu^2)}$ (axial waves in plates). Comment on why c and c_P have different values, while both are "pressure" wave speeds.

Solution

(a)

	Aluminum	Steel
λ	51.1 GPa	107.1 GPa
μ	26.3 GPa	77.5GPa
Bulk modulus, B	68.6 GPa	158.7 GPa

(b)

	Aluminum	Steel
$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ (3-D pressure wave)	6175 m/s	5778 m/s
$c_S = \sqrt{\frac{\mu}{\rho}}$ (Shear wave)	3110 m/s	3142 m/s
$c = \sqrt{\frac{E}{\rho}}$ (1-D pressure wave)	5073 m/s	5048 m/s
$c_L = \sqrt{\frac{E}{\rho(1-\nu^2)}}$ (Axial waves in plates)	5374 m/s	5274 m/s

The reason that c and c_P are different while both are "pressure" wave speeds is that the corresponding "pressure" waves propagate in different ways. The wave speed c corresponds to a very simplified pressure wave that propagates in a 1-dimensional slender rod. The wave speed c_P corresponds to the true pressure wave that propagates in an infinite 3-dimensional medium.

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PROBLEM 5.12 SOLUTION

$$GPa := 10^{9} \cdot Pa$$

<u>Aluminum</u>

$$E := 70 \cdot GPa \qquad \nu := 0.33 \qquad \rho := 2700 \cdot \frac{kg}{m^{3}}$$

$$\lambda := \frac{\nu}{(1+\nu)\cdot(1-2\cdot\nu)} \cdot E \qquad \lambda = 51.1\,GPa$$

$$\mu := \frac{1}{2\cdot(1+\nu)} \cdot E \qquad \mu = 26.3\,GPa$$

$$B := \frac{3\cdot\lambda + 2\cdot\mu}{3} \qquad B = 68.6\,GPa$$

$$EE := \frac{\mu\cdot(3\cdot\lambda + 2\cdot\mu)}{\lambda + \mu} \qquad EE = 70\,GPa$$

$$\nu\nu := \frac{\lambda}{2\cdot(\lambda + \mu)} \qquad \nu\nu = 0.33$$

$$cP := \sqrt{\frac{(\lambda + 2\cdot\mu)}{\rho}} \qquad cP = 6198\,\frac{m}{s}$$

$$cS := \sqrt{\frac{\mu}{\rho}} \qquad cS = 3122\,\frac{m}{s}$$

$$c := \sqrt{\frac{E}{\rho}} \qquad c = 5092\,\frac{m}{s}$$

$$cL := \sqrt{\frac{E}{\rho\cdot(1-\nu^{2})}} \qquad cL = 5394\,\frac{m}{s}$$

Steel

$$E := 200 \cdot GPa \qquad \nu := 0.3 \qquad \rho := 7750 \cdot \frac{kg}{m}$$

$$\lambda := \frac{\nu}{(1+\nu)\cdot(1-2\cdot\nu)} \cdot E \qquad \lambda = 115.4\,GPa$$

$$\mu := \frac{1}{2\cdot(1+\nu)} \cdot E \qquad \mu = 76.9\,GPa$$

$$EE := \frac{\mu\cdot(3\cdot\lambda + 2\cdot\mu)}{\lambda + \mu} \qquad EE = 200\,GPa$$

$$\nu\nu := \frac{\lambda}{2\cdot(\lambda + \mu)} \qquad \nu\nu = 0.3$$

$$cP := \sqrt{\frac{(\lambda + 2\cdot\mu)}{\rho}} \qquad cP = 5894\,Hz$$

$$cS := \sqrt{\frac{\mu}{\rho}} \qquad cS = 3150\,Hz$$

$$c := \sqrt{\frac{E}{\rho}} \qquad c = 5080\,Hz$$

$$cL := \sqrt{\frac{E}{\rho\cdot(1-\nu^2)}} \qquad cL = 5325\,Hz$$

Problem 13: Consider a 3-D plane wave traveling with speed c along an arbitrary direction $\vec{n}$. State the general expression of the particle motion, $\vec{u}$. Substitute it into the Navier equations and deduce the characteristic equation for the wave speed c. Solve the characteristic equation. Explain how many types of waves can travel in the 3-D material. Give the appropriate wave speeds. Sketch the particle motion for each wave type.

Solution

Solution to this problem can be found in the textbook Chapter 5, sections 5.10, 5.10.1, 5.10.2 and Figure 5.26.

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Problem 14: Consider the z-invariant 3-D conditions. Explain the meaning of the "z-invariant" condition and write out all the related conditions that apply. Derive the expressions of strains in terms of displacements for the z-invariant case starting from the general 3-D strain-displacement relations. Derive the stress-strain expressions for the z-invariant case starting from the general stress-strain expressions (use expressions in terms of Lame constants). Using these results, derive the expressions of stress in terms of displacements for the z-invariant case.

Solution

The meaning of the "z-invariant" condition is explained in the textbook Chapter 5, Section 5.10.6.

(a) The derivation of strains in terms of displacements for the z-invariant case starting from the general 3-D strain-displacement relations is as follows: Recall the strain-displacement relation from Appendix B, Eq. (B.19), i.e.,

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} & \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} & \varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \varepsilon_{zz} &= \frac{\partial u_z}{\partial z} & \varepsilon_{zx} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \end{aligned} \tag{1}$$

Recall the z-invariant condition, i.e.,

$$\frac{\partial}{\partial z} \equiv 0 \tag{2}$$

Substitution of Eq. (2) into Eq. (1) gives

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} & \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} & \varepsilon_{yz} &= \frac{1}{2}\left(\cancel{\frac{\partial u_y}{\partial z}} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2}\frac{\partial u_z}{\partial y} \\ \varepsilon_{zz} &= \cancel{\frac{\partial u_z}{\partial z}} = 0 & \varepsilon_{zx} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \cancel{\frac{\partial u_x}{\partial z}}\right) = \frac{1}{2}\frac{\partial u_z}{\partial x} \end{aligned} \tag{3}$$

(b) The derivation of stress-strain expressions for the z-invariant case starting from the general stress-strain expressions is done as follows: Recall the stress-strain expressions in terms of Lame constants, i.e.,

$$\begin{aligned} \sigma_{xx} &= (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz} & \sigma_{xy} &= 2\mu\varepsilon_{xy} \\ \sigma_{yy} &= \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda\varepsilon_{zz} & \sigma_{yz} &= 2\mu\varepsilon_{yz} \\ \sigma_{zz} &= \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz} & \sigma_{zx} &= 2\mu\varepsilon_{zx} \end{aligned} \tag{4}$$

Substitution of Eq. (3) into Eq. (4) yields

$$
\begin{aligned}
\sigma_{xx} &= (\lambda+2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\cancel{\varepsilon_{zz}} = (\lambda+2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} & \sigma_{xy} &= 2\mu\varepsilon_{xy} \\
\sigma_{yy} &= \lambda\varepsilon_{xx} + (\lambda+2\mu)\varepsilon_{yy} + \lambda\cancel{\varepsilon_{zz}} = \lambda\varepsilon_{xx} + (\lambda+2\mu)\varepsilon_{yy} & \sigma_{yz} &= 2\mu\varepsilon_{yz} \qquad (5) \\
\sigma_{zz} &= \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda+2\mu)\cancel{\varepsilon_{zz}} = \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} & \sigma_{zx} &= 2\mu\varepsilon_{zx}
\end{aligned}
$$

(c) The derivation of stress in terms of displacements for the z-invariant case proceeds as follows: substitute Eq. (3) into Eq. (5) to get

$$
\begin{aligned}
\sigma_{xx} &= (\lambda+2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} = (\lambda+2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_y}{\partial y} & \sigma_{xy} &= 2\mu\varepsilon_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\
\sigma_{yy} &= \lambda\varepsilon_{xx} + (\lambda+2\mu)\varepsilon_{yy} = \lambda\frac{\partial u_x}{\partial x} + (\lambda+2\mu)\frac{\partial u_y}{\partial y} & \sigma_{yz} &= 2\mu\varepsilon_{yz} = \mu\frac{\partial u_z}{\partial y} \qquad (6) \\
\sigma_{zz} &= \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) & \sigma_{zx} &= 2\mu\varepsilon_{zx} = \mu\frac{\partial u_z}{\partial x}
\end{aligned}
$$

============

Problem 15: Consider 3-D plane wave traveling in a direction $\vec{n} \perp Oz$. Assume the motion *z*-invariant. State the general solution of wave propagation in a 3-D medium using Φ and $\vec{H}$ potentials. Particularize the general solution to the case of *z*-invariant plane waves. State what type of plane waves can travel in the 3-D material under the *z*-invariance assumption. Give the expressions for the particle motion and stresses for each of these wave types in terms of the Φ and $\vec{H}$ potentials.

Solution

(a) Particularization of the general solution to the case of z-invariant plane waves is done as follows. Recall the general solution of wave propagation in a 3-D medium using Φ and $\vec{H}$ potentials is

$$\vec{u} = \vec{\nabla}\Phi + \vec{\nabla} \times \vec{H} \tag{1}$$

where $\vec{H} = H_x \vec{e}_x + H_y \vec{e}_y + H_z \vec{e}_z$. The potentials Φ, H_x, H_y, H_z satisfy the wave equations and the uniqueness condition, i.e.,

$$c_P^2 \nabla^2 \Phi = \ddot{\Phi} \qquad \begin{cases} c_S^2 \nabla^2 H_x = \ddot{H}_x \\ c_S^2 \nabla^2 H_y = \ddot{H}_y \\ c_S^2 \nabla^2 H_z = \ddot{H}_z \end{cases} \quad \text{(wave equations)} \tag{2}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \qquad \text{(uniqueness condition)} \tag{3}$$

The *z*-invariant condition is

$$\frac{\partial}{\partial z} \equiv 0 \tag{4}$$

When the z-invariant condition applies, the differential operators $\vec{\nabla}$ and ∇^2 becomes

$$\begin{aligned} \vec{\nabla} &= \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \cancel{\frac{\partial}{\partial z}} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \cancel{\frac{\partial^2}{\partial z^2}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned} \tag{5}$$

Substitution of Eq. (5) into Eq. (2) yields

$$c_P^2 \nabla^2 \Phi = c_P^2 \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) = \ddot{\Phi} \qquad \begin{cases} c_S^2 \nabla^2 H_x = c_S^2 \left(\dfrac{\partial^2 H_x}{\partial x^2} + \dfrac{\partial^2 H_x}{\partial y^2} \right) = \ddot{H}_x \\ c_S^2 \nabla^2 H_y = c_S^2 \left(\dfrac{\partial^2 H_y}{\partial x^2} + \dfrac{\partial^2 H_y}{\partial y^2} \right) = \ddot{H}_y \\ c_S^2 \nabla^2 H_z = c_S^2 \left(\dfrac{\partial^2 H_z}{\partial x^2} + \dfrac{\partial^2 H_z}{\partial y^2} \right) = \ddot{H}_z \end{cases} \tag{6}$$

It is apparent that all three types of plane waves (pressure, P; shear vertical, SV; shear horizontal, SH) can travel in the 3-D material under the z-invariant assumption.

(b) The expression of the particle motion is obtained by substitution of Eq. (5) into Eq. (1), i.e.,

$$
\begin{aligned}
\vec{u} &= \left(\vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} \right) \Phi + \left(\vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} \right) \times \left(H_x \vec{e}_x + H_y \vec{e}_y + H_z \vec{e}_z \right) \\
&= \frac{\partial \Phi}{\partial x} \vec{e}_x + \frac{\partial \Phi}{\partial y} \vec{e}_y + \cancel{\vec{e}_x \frac{\partial}{\partial x} \times H_x \vec{e}_x} + \vec{e}_x \frac{\partial}{\partial x} \times H_y \vec{e}_y + \vec{e}_x \frac{\partial}{\partial x} \times H_z \vec{e}_z \\
&\quad + \vec{e}_y \frac{\partial}{\partial y} \times H_x \vec{e}_x + \cancel{\vec{e}_y \frac{\partial}{\partial y} \times H_y \vec{e}_y} + \vec{e}_y \frac{\partial}{\partial y} \times H_z \vec{e}_z \\
&= \frac{\partial \Phi}{\partial x} \vec{e}_x + \frac{\partial \Phi}{\partial y} \vec{e}_y + \frac{\partial H_y}{\partial x} \vec{e}_z - \frac{\partial H_z}{\partial x} \vec{e}_y - \frac{\partial H_x}{\partial y} \vec{e}_z + \frac{\partial H_z}{\partial y} \vec{e}_x \\
&= \left(\frac{\partial \Phi}{\partial x} + \frac{\partial H_z}{\partial y} \right) \vec{e}_x + \left(\frac{\partial \Phi}{\partial y} - \frac{\partial H_z}{\partial x} \right) \vec{e}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \vec{e}_z
\end{aligned} \tag{7}
$$

Equation (7) yields the displacement components

$$
\begin{aligned}
u_x &= \frac{\partial \Phi}{\partial x} + \frac{\partial H_z}{\partial y} \\
u_y &= \frac{\partial \Phi}{\partial y} - \frac{\partial H_z}{\partial x} \\
u_z &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}
\end{aligned} \tag{8}
$$

Examinations of Eq. (8) indicates that it is possible to partition the solution into two parts:

(i) a solution for u_z which depends only on the two potentials, H_x and H_y

(ii) a separate solution for u_x and u_y which depend on the other two potentials, Φ and H_z.

The first solution, which accepts only the u_z displacement, will be a shear motion polarized in the horizontal plane Oxz, i.e., a shear-horizontal wave, SH. This SH motion is described in terms of the two potentials, H_x and H_y. The second solution, which accepts u_x and u_y displacements, will be the combination of a pressure wave P represented by the potential Φ, and a shear vertical wave SV represented by the potential H_z. This second solution is denoted P+SV. Note that the particle motion of this second solution is constrained to the vertical plane; hence, the associated shear wave is a shear vertical wave, SV. The two solutions are treated separately.

For <u>SH waves</u>, the motion is contained in the horizontal plane and the relevant potentials are H_x and H_y, i.e.,

$$u_x = u_y = 0, \quad u_z \neq 0, \quad H_x \text{ and } H_y \text{ only} \qquad \text{(SH waves)} \tag{9}$$

To calculate the stresses, recall the stress-displacement relation of Problem 14, i.e.,

$$
\begin{aligned}
\sigma_{xx} &= (\lambda+2\mu)\frac{\partial u_x}{\partial x}+\lambda\frac{\partial u_y}{\partial y} & \sigma_{xy} &= \mu\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\\
\sigma_{yy} &= \lambda\frac{\partial u_x}{\partial x}+(\lambda+2\mu)\frac{\partial u_y}{\partial y} & \sigma_{yz} &= \mu\frac{\partial u_z}{\partial y}\\
\sigma_{zz} &= \lambda\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right) & \sigma_{zx} &= \mu\frac{\partial u_z}{\partial x}
\end{aligned}
\tag{10}
$$

Substitution of Eq. (9) into Eq. (10) yields

$$
\begin{aligned}
\sigma_{xx} &= (\lambda+2\mu)\frac{\partial \cancel{u_x}}{\partial x}+\lambda\frac{\partial \cancel{u_y}}{\partial y}=0 & \sigma_{xy} &= \mu\left(\frac{\partial \cancel{u_x}}{\partial y}+\frac{\partial \cancel{u_y}}{\partial x}\right)=0\\
\sigma_{yy} &= \lambda\frac{\partial \cancel{u_x}}{\partial x}+(\lambda+2\mu)\frac{\partial \cancel{u_y}}{\partial y}=0 & \sigma_{yz} &= \mu\frac{\partial u_z}{\partial y} \qquad \text{(SH waves)}\\
\sigma_{zz} &= \lambda\left(\frac{\partial \cancel{u_x}}{\partial x}+\frac{\partial \cancel{u_y}}{\partial y}\right)=0 & \sigma_{zx} &= \mu\frac{\partial u_z}{\partial x}
\end{aligned}
\tag{11}
$$

Examination of Eq. (11) reveals that the only nonzero stresses are σ_{xz}, σ_{yz}, i.e.,

$$
\begin{aligned}
\sigma_{yz} &= \mu\frac{\partial u_z}{\partial y}\\
\sigma_{zx} &= \mu\frac{\partial u_z}{\partial x}
\end{aligned}
\qquad \text{(SH waves)}
\tag{12}
$$

For <u>P+SV waves</u>, the motion is contained in the vertical plane, and the relevant potentials are Φ and H_z, i.e.,

$$
u_x \neq 0, \quad u_y \neq 0, \quad u_z = 0, \quad \frac{\partial}{\partial z}=0, \qquad \Phi \text{ and } H_z \text{ only} \quad \text{(P+SV waves)}
\tag{13}
$$

The non-zero u_x, u_y displacements and their derivatives are

$$
\begin{aligned}
u_x &= \frac{\partial \Phi}{\partial x}+\frac{\partial H_z}{\partial y} & \frac{\partial u_x}{\partial x} &= \frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 H_z}{\partial x \partial y} & \frac{\partial u_x}{\partial y} &= \frac{\partial^2 \Phi}{\partial x \partial y}+\frac{\partial^2 H_z}{\partial y^2}\\
u_y &= \frac{\partial \Phi}{\partial y}-\frac{\partial H_z}{\partial x} & \frac{\partial u_y}{\partial x} &= \frac{\partial^2 \Phi}{\partial x \partial y}-\frac{\partial^2 H_z}{\partial x^2} & \frac{\partial u_y}{\partial y} &= \frac{\partial^2 \Phi}{\partial y^2}-\frac{\partial^2 H_z}{\partial x \partial y}
\end{aligned}
\tag{14}
$$

To calculate the stresses, recall the stress-displacement relation of Eq. (10), i.e.,

$$
\begin{aligned}
\sigma_{xx} &= (\lambda+2\mu)\frac{\partial u_x}{\partial x}+\lambda\frac{\partial u_y}{\partial y} & \sigma_{xy} &= \mu\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\\
\sigma_{yy} &= \lambda\frac{\partial u_x}{\partial x}+(\lambda+2\mu)\frac{\partial u_y}{\partial y} & \sigma_{yz} &= \mu\frac{\partial u_z}{\partial y}\\
\sigma_{zz} &= \lambda\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right) & \sigma_{zx} &= \mu\frac{\partial u_z}{\partial x}
\end{aligned}
\tag{15}
$$

Substitution of Eq. (13) into Eq. (15) yields

$$\sigma_{xx} = (\lambda + 2\mu)\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 H_z}{\partial x \partial y}\right) + \lambda\left(\frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 H_z}{\partial x \partial y}\right) = (\lambda + 2\mu)\frac{\partial^2 \Phi}{\partial x^2} + \lambda\frac{\partial^2 \Phi}{\partial y^2} + 2\mu\frac{\partial^2 H_z}{\partial x \partial y}$$

$$\sigma_{yy} = \lambda\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 H_z}{\partial x \partial y}\right) + (\lambda + 2\mu)\left(\frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 H_z}{\partial x \partial y}\right) = \lambda\frac{\partial^2 \Phi}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 \Phi}{\partial y^2} - 2\mu\frac{\partial^2 H_z}{\partial x \partial y} \tag{16}$$

$$\sigma_{zz} = \lambda\left(\frac{\partial^2 \Phi}{\partial x^2} + \cancel{\frac{\partial^2 H_z}{\partial x \partial y}} + \frac{\partial^2 \Phi}{\partial y^2} - \cancel{\frac{\partial^2 H_z}{\partial x \partial y}}\right) = \lambda\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right)$$

$$\sigma_{xy} = \mu\left(\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 H_z}{\partial x^2}\right) = \mu\left(2\frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2}\right)$$

$$\sigma_{yz} = \mu\frac{\partial \cancel{u_z}}{\partial y} = 0 \tag{17}$$

$$\sigma_{zx} = \mu\frac{\partial \cancel{u_z}}{\partial x} = 0$$

==================

Problem 16: Consider the definition of group velocity in the form $c_g = \mathrm{d}\omega/\mathrm{d}\gamma$. Prove the following equivalent formulae for calculating group velocity: (a) $c_g = c + \gamma \dfrac{\mathrm{d}c}{\mathrm{d}\gamma}$;

(b) $c_g = c^2 \Big/ \left(c - \omega \dfrac{\mathrm{d}c}{\mathrm{d}\omega} \right)$ and $c_g = c^2 \Big/ \left(c - f \dfrac{\mathrm{d}c}{\mathrm{d}f} \right)$; (c) $c_g = c^2 \Big/ \left(c - (fd) \dfrac{\mathrm{d}c}{\mathrm{d}(fd)} \right)$

<u>Solution</u>

Recall Eq. (5.214) in textbook Chapter 5, Section 5.4.3.1 giving the definition of group velocity as

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}\gamma} \tag{1}$$

(a) Write

$$\omega = c\gamma$$

$$\frac{d\omega}{d\gamma} = c\frac{d\gamma}{d\gamma} + \gamma\frac{d\,c}{d\gamma} = c + \gamma\frac{d\,c}{d\gamma} \tag{2}$$

Comparing Eqs. (1) and (2) it becomes apparent that

$$c_g = c + \gamma\frac{d\,c}{d\gamma} \tag{3}$$

(b) Recall

$$\gamma = \frac{\omega}{c} = \omega c^{-1} \tag{4}$$

Upon differentiation, we write

$$\begin{aligned} \frac{d\gamma}{d\omega} &= \frac{d\omega}{d\omega}c^{-1} + \omega\frac{d}{d\omega}\left(c^{-1}\right) = c^{-1} + \omega(-1)c^{-2}\frac{d\,c}{d\omega} \\ &= c^{-2}\left(c - \omega\frac{d\,c}{d\omega}\right) \end{aligned} \tag{5}$$

Recall Eq. (1) and express it as

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}\gamma} = \left(\frac{\mathrm{d}\gamma}{\mathrm{d}\omega}\right)^{-1} \tag{6}$$

Substituting Eq. (5) into Eq. (6) gives the required expression

$$c_g = \left[c^{-2}\left(c - \omega\frac{d\,c}{d\omega}\right)\right]^{-1} = \frac{c^2}{\left(c - \omega\dfrac{d\,c}{d\omega}\right)} \quad (7)$$

Substituting $\omega = 2\pi f$ into Eq. (7) yields

$$c_g = \frac{c^2}{\left(c - f\,\dfrac{d\,c}{df}\right)} \qquad (8)$$

(c) Note that

$$f\,\frac{\mathrm{d}\,c}{\mathrm{d}\,f} = (fd)\frac{\mathrm{d}\,c}{\mathrm{d}(fd)} \qquad (9)$$

Substituting Eq. (9) into Eq. (8) gives

$$c_g = \frac{c^2}{\left(c - (fd)\dfrac{\mathrm{d}\,c}{\mathrm{d}(fd)}\right)} \qquad (10)$$

Eq. (10) is useful in calculating the group velocity in terms of the *fd* product.

==================

Problem 17: Consider a generic spherical wave, $\Phi(r, t)$, of energy E_0 emanating from a point source. At $r = r_0$, the wave amplitude is $A(r_0) = A_0$. Starting with the generic wave equation $\ddot{\Phi} = c^2\nabla^2\Phi$, deduce: (a) the form of the wave equation that applies to this situation; (b) the D'Alembert solution for $\Phi(r, t)$; (c) the wave amplitude expression, $A(r)$, as function of r and A_0; (d) the wave energy density expression, $e(r)$, as function of r and E_0

Solution

Refer to textbook Chapter 5, Section 5.9.2. Consider the wave equation in a 3-D medium in the generic form

$$\ddot{\Phi} = c^2\nabla^2\Phi \tag{1}$$

where ∇^2 is the Laplacian in 3-D coordinates. In order to analyze a spherical wave emanating from a point source, we will choose a spherical coordinate system with the origin in the wave source (Figure 11).

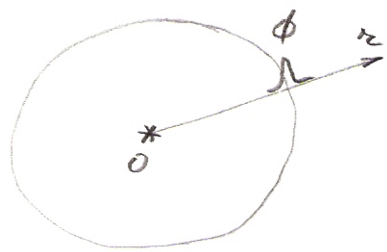


Figure 11 Generic spherical wave propagating outwards from a point source at origin

(a) Due to spherical symmetry, the spatial dependence of the spherical wave is restricted to only the radial coordinate, r. Hence, a spherical wave will have the general expression

$$\Phi = \Phi(r,t) \tag{2}$$

According to the Appendix, the Laplacian in spherical coordinates has the expression

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \tag{3}$$

However, by virtue of spherical symmetry, the derivatives with respect to ϕ and θ vanish, and Equation (3) becomes

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) \tag{4}$$

Equation (4) can be also expressed in the more convenient form

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2(r\Phi)}{\partial r^2} \tag{5}$$

Proof: Expand each equation and show that one arrives at the same expression. Expansion of Eq. (4) gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \frac{1}{r^2}\left(2r\frac{\partial\Phi}{\partial r} + r^2\frac{\partial^2\Phi}{\partial r^2}\right) = \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial r^2} \tag{6}$$

Whereas expansion of Eq. (5) gives

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)=\frac{1}{r}\frac{\partial}{\partial r}\left(\Phi+r\frac{\partial}{\partial r}\Phi\right)=\frac{1}{r}\left(\frac{\partial}{\partial r}\Phi+\frac{\partial}{\partial r}\Phi+r\frac{\partial^2}{\partial r^2}\Phi\right)=\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{\partial^2\Phi}{\partial r^2} \tag{7}$$

Since Eqs. (6) and(7) give the same result, it follows that Eq. (5) holds. Substitution of Eq. (5) into Eq. (1) yields

$$\frac{\partial^2\Phi}{\partial t^2}=c^2\frac{1}{r}\frac{\partial^2(r\Phi)}{\partial r^2} \tag{8}$$

Since r is independent of t, we can rewrite Eq. (8) in the form

$$\frac{\partial^2(r\Phi)}{\partial t^2}=c^2\frac{\partial^2(r\Phi)}{\partial r^2} \tag{9}$$

(b) Equation (9) is a 1-D wave equation in the function $r\Phi$ and accepts the D'Alembert solution

$$r\Phi(r,t)=f\left(t-\frac{r}{c}\right)+g\left(t+\frac{r}{c}\right) \tag{10}$$

where

- $f\left(t-\frac{r}{c}\right)$ is a diverging spherical wave emanating from origin
- $g\left(t+\frac{r}{c}\right)$ is a converging spherical wave sinking into the origin

For our case of a point source at the origin, only the emanating solution $f(t-r/c)$ applies. This is the radiation condition in spherical coordinates. Hence, Eq. (10) gives

$$\Phi(r,t)=\frac{1}{r}f\left(t-\frac{r}{c}\right) \tag{11}$$

The function in Eq. (11) is singular at $r=0$. However, the function exists for all $r>0$.
Assume the function $f(t)$ represents a generic disturbance as shown in Figure 12. Then, $\Phi(r,t)$ will have the same shape, but scaled by $1/r$.

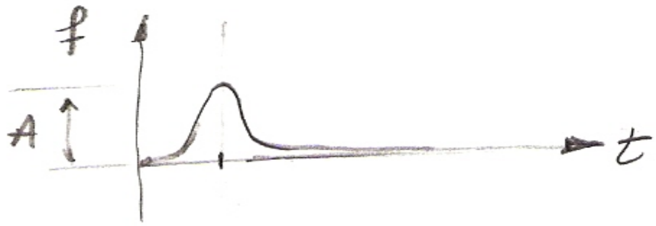


Figure 12 Generic disturbance

(c) Assume that the wave amplitude at a location r_0 is A_0. Then, in virtue of Eq. (11), the wave amplitude at any other location r is given by

$$A(r)=\frac{r_0}{r}A_0 \tag{12}$$

It is apparent from Eq. (12) that the amplitude of a spherical wave is inverse proportional with the radial distance from the source and that the wave decreases as it propagates outwards. For illustration, consider three points in space at increasing distance from the origin, i.e., $r_1 < r_2 < r_3$,specifically $r_1 = R,\ r_2 = 2R, r_3 = 3R$, as indicated in Figure 13. The corresponding amplitudes are $A_1 = A_0,\ A_2 = A_0/2,\ A_3 = A_0/3$.

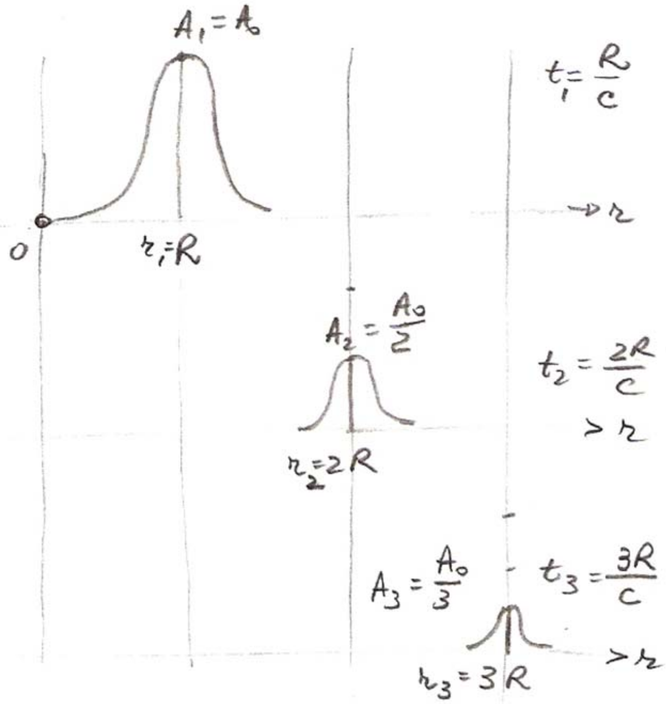


Figure 13 Amplitude decrease with radial distance

(d) The wave energy density is calculated as follows. Recall from textbook Eq. (5.94) that the wave energy density is proportional to the square of the wave amplitude, i.e.,

$$e(r) = K\ A^2(r) \tag{13}$$

where K is an arbitrary constant. As before, assume that at a reference location r_0 the wave amplitude is A_0; the corresponding energy density is $e_0 = KA_0^2$. The total energy contained in the wavefront is the product between the energy density and the area of the wavefront. One gets

$$E_0 = 4\pi r_0^2 e_0 = 4\pi r_0^2 K\ A_0^2 \tag{14}$$

Solving Eq. (14) for K and substituting in Eq. (13) one gets

$$K = \frac{E_0}{4\pi r_0^2 A_0^2}\ \text{, and } e(r) = \frac{E_0}{4\pi r_0^2}\frac{A^2(r)}{A_0^2} \tag{15}$$

Recalling Eq. (12), one writes the second part of Eq. (15) as

$$e(r) = \frac{E_0}{4\pi r^2} \tag{16}$$

Equation (16) indicates that the energy density diminishes proportional with r^2. In fact, Eq. (16) states the law of energy conservation: during the propagation of a spherical wave, the energy contained in the wavefront is being smeared thinner and thinner over an ever increasing spherical wavefront area, which is proportional with r^2 (see

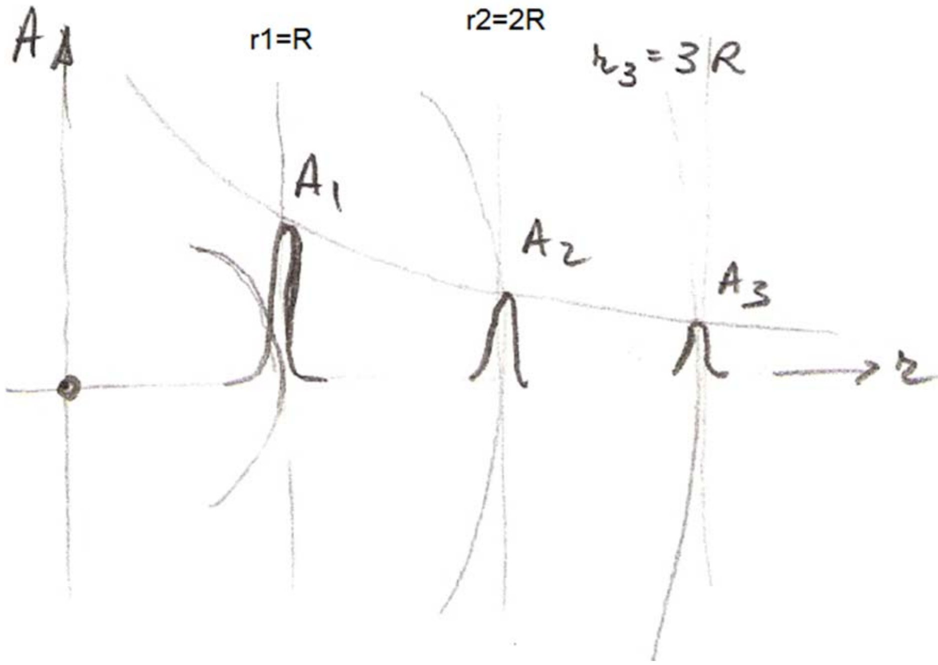
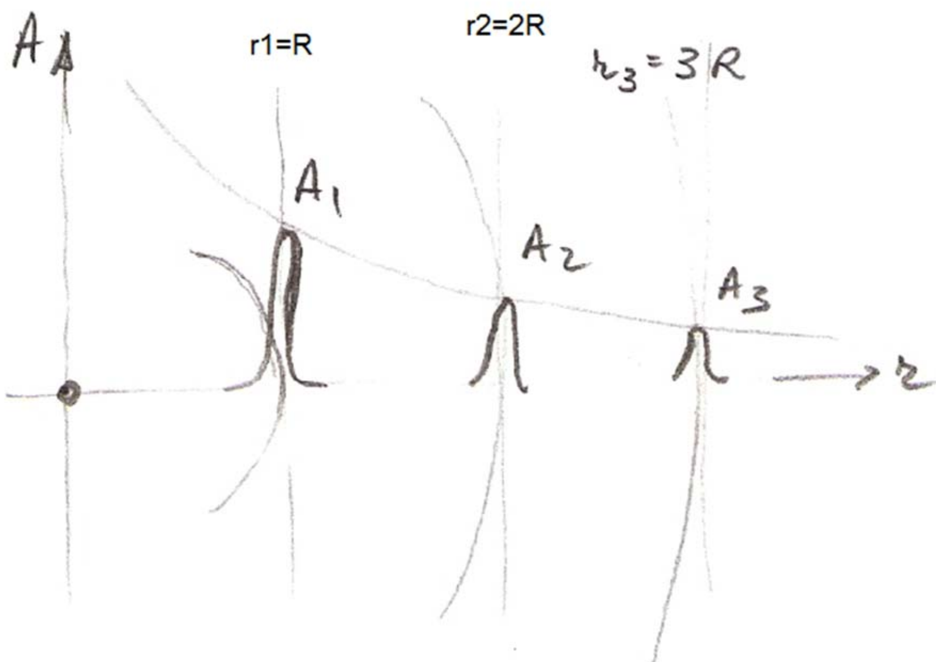


Figure 14).



A
r1=R
r2=2R
$r_3 = 3R$
A_1
A_2
A_3
r

Figure 14   Propagation of a spherical wave

Problem 18: Consider a generic circular wave, $\Phi(r, t)$, of energy E_0 emanating from a point source. At $r = r_0$, the wave amplitude is $A(r_0) = A_0$. Consider the problem to be 2-D. Starting with the generic wave equation $\ddot{\Phi} = c^2\nabla^2\Phi$, deduce: (a) the form of the wave equation that applies to this situation; (b) the D'Alembert solution for $\Phi(r, t)$; (c) the wave amplitude expression, $A(r)$, as function of r and A_0; (d) the wave energy density expression, $e(r)$, as function of r and E_0

Solution

This problem is similar with the previous problem 14, only that the propagation takes place in a 2-D medium. Consider the wave equation in a 3-D medium in the generic form

$$\ddot{\Phi} = c^2\nabla^2\Phi \tag{1}$$

where ∇^2 is the Laplacian in 2-D coordinates. In order to analyze a circular wave emanating from a point source, we will choose a polar coordinate system with the origin in the wave source (Figure 15).

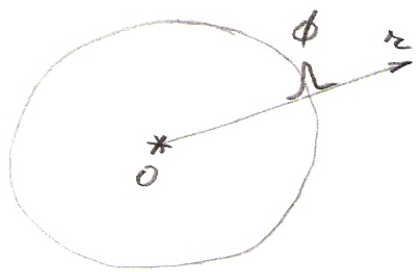


Figure 15 Generic circular wave propagating outwards from a point source at origin

(a) Due to polar symmetry, the spatial dependence of the circular wave is restricted to only the radial coordinate, r. Hence, a spherical wave will have the general expression

$$\Phi = \Phi(r,t) \tag{2}$$

According to the Appendix, the Laplacian in polar coordinates has the expression

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \tag{3}$$

However, by virtue of spherical symmetry, the derivatives with respect to ϕ vanish, and Eq. (3) becomes

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} \tag{4}$$

Equation (4) can be also expressed in the more convenient form

$$\nabla^2\Phi = \frac{1}{\sqrt{r}}\frac{\partial^2\left(\Phi\sqrt{r}\right)}{\partial r^2} + \frac{1}{4r^2}\Phi \tag{5}$$

Proof: Expand each equation and show that one arrives at the same expression. Expansion of Eq. (5) gives

$$\frac{\partial\left(\Phi\sqrt{r}\right)}{\partial r} = \frac{\partial\left(\Phi\, r^{1/2}\right)}{\partial r} = \frac{\partial\Phi}{\partial r}r^{1/2} + \Phi\frac{1}{2}r^{-1/2} \tag{6}$$

$$\frac{\partial^2\left(\Phi\sqrt{r}\right)}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial\Phi}{\partial r}r^{1/2}+\Phi\frac{1}{2}r^{-1/2}\right)=...=r^{1/2}\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}-\frac{1}{4r^2}\Phi\right) \tag{7}$$

Substitution of Eq. (7) into Eq. (5) yields Eq. (4). However, Eq. (5) does not readily accept closed form solution. But, at large values of *r*, the second term in Eq. (5) vanishes $\left(\frac{1}{4r^2}\Phi\to 0 \text{ as } r\to\infty\right)$, and hence

$$\nabla^2\Phi\cong\frac{1}{\sqrt{r}}\frac{\partial^2\left(\Phi\sqrt{r}\right)}{\partial r^2}+\frac{1}{4r^2}\Phi \ \text{ for } r\to\infty \tag{8}$$

Equation (8) accepts closed-form D'Alembert solutions in the form

$$r^{1/2}\Phi(r,t)=f\left(t-\frac{r}{c}\right)+g\left(t+\frac{r}{c}\right) \tag{9}$$

where

- $f\left(t-\frac{r}{c}\right)$ is a diverging spherical wave emanating from origin
- $g\left(t+\frac{r}{c}\right)$ is a converging spherical wave sinking into the origin

For our case of a point source at the origin, only the emanating solution $f\left(t-r/c\right)$ applies. This is the radiation condition in spherical coordinates. Hence, Eq. (10) gives

$$\Phi(r,t)=\frac{1}{\sqrt{r}}f\left(t-\frac{r}{c}\right) \tag{10}$$

The function in Eq. (11) is singular at $r=0$. However, the function exists for all $r>0$.
Assume the function $f\left(t\right)$ represents a generic disturbance as shown in Figure 12. Then, $\Phi(r,t)$ will have the same shape, but scaled by $1/\sqrt{r}$.
(c) Assume that the wave amplitude at a location r_0 is A_0. Then, in virtue of Eq. (11), the wave amplitude at any other location *r* is given by

$$A(r)=A_0\sqrt{\frac{r_0}{r}} \tag{11}$$

It is apparent from Eq. (12) that the amplitude of a circular wave is inverse proportional with the square root of the radial distance from the source and that the wave decreases as it propagates outwards. For illustration, consider three points in space at increasing distance from the origin, i.e., $r_1<r_2<r_3$,specifically $r_1=R,\ r_2=2R,\ r_3=3R$, as indicated in Figure 16. The corresponding amplitudes are $A_1=A_0,\ A_2=A_0/\sqrt{2},\ A_3=A_0/\sqrt{3}$.

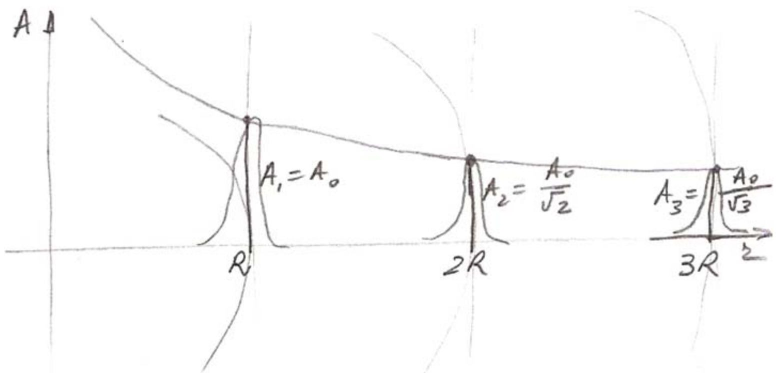


Figure 16 Propagation of a circular wave

(d) The wave energy density is calculated as follows. Recall from textbook Eq. (5.94) that the wave energy density is proportional to the square of the wave amplitude, i.e.,

$$e(r) = K\ A^2(r) \tag{12}$$

where K is an arbitrary constant. As before, assume that at a reference location r_0 the wave amplitude is A_0; the corresponding energy density is $e_0 = KA_0^2$. The total energy contained in the wavefront is the product between the energy density and the circumferential length of the wavefront. One gets

$$E_0 = 2\pi r_0 e_0 = 2\pi r_0 K\ A_0^2 \tag{13}$$

Solving Eq. (14) for K and substituting in Eq. (13) one gets

$$K = \frac{E_0}{2\pi r_0 A_0^2}, \text{ and } e(r) = \frac{E_0}{2\pi r_0}\frac{A^2(r)}{A_0^2} \tag{14}$$

Recalling Eq. (12), one writes the second part of Eq. (15) as

$$e(r) = \frac{E_0}{2\pi r} \tag{15}$$

Equation (16) indicates that the energy density diminishes proportional with r. In fact, Eq. (16) states the law of energy conservation: during the propagation of a circular wave, the energy contained in the wavefront is being smeared thinner and thinner over an ever increasing circular wavefront length, which is proportional with r.