CHAPTER 8 PROBLEMS AND EXERCISES

Problem 1: Consider a linear PWAS transducer mounted on a thin-wall aluminum structure; the PWAS has $E_a = 63\text{ GPa}$, $t_a = 0.2\text{ mm}$, $l_a = 7\text{ mm}$, $d_{31} = -175\text{ mm/kV}$; the structure has $E = 70\text{ GPa}$, $t = 2\text{ mm}$; the adhesive has $G_b = 2\text{ GPa}$ $t_b = 100\ \mu\text{m}$; the applied voltage is $V = 10\text{ V}$.

(a) Calculate and plot the shear lag distribution of the shear stress in the adhesive layer.
(b) Find the maximum shear stress and its location
(c) Find the required adhesive thickness to decrease the maximum shear stress by a factor of 2 and give the new shear stress value
(d) Find the effective PWAS size in physical units and as percentage of the actual PWAS size
(e) Find the required adhesive thickness to bring the effective PWAS size within 5% of the actual PWAS size and give the new value of the effective PWAS size

Solution

(a) To calculate and plot the shear lag distribution of the shear stress in the adhesive layer, recall Eq. (8.54) of the textbook Chapter 8, i.e.,

$$\tau(x) = \frac{t_a}{a}\frac{\psi}{\alpha+\psi}E_a\varepsilon_{ISA}\left(\Gamma a\frac{\sinh\Gamma x}{\cosh\Gamma a}\right) \qquad \text{(shear stress in adhesive)} \qquad (8.54)\ (1)$$

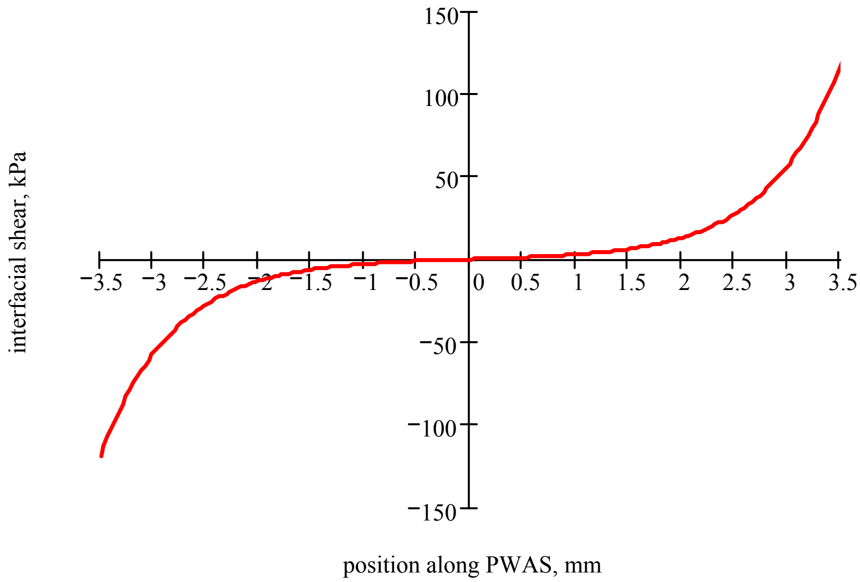
where $\alpha = 4$, ε_{ISA} is given by Eq. (8.1), ψ by Eq. (8.32), and Γ by Eq. (8.33), i.e.,

$$\varepsilon_{ISA} = d_{31}\frac{V}{t_a} \qquad (8.1)\ (2)$$

$$\psi = \frac{Et}{E_a t_a} \qquad \text{(relative stiffness coefficient)} \qquad (8.32)\ (3)$$

$$\Gamma^2 = \frac{G_b}{E_a}\frac{1}{t_a t_b}\frac{\alpha+\psi}{\psi} \qquad \text{(shear-lag parameter)} \qquad (8.33)\ (4)$$

Upon calculation, one gets $\varepsilon_{ISA} = 8.75\ \mu\text{m}$, $\psi = 11.11$, $\Gamma = 1.469/\text{mm}$ and the following plot:



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(b) The maximum shear stress is obtained at the ends of the PWAS, i.e., at $x = \pm a = \pm 3.5$ mm

$$\tau_{\max} = \tau(x)\big|_{x=a} = \frac{t_a}{a}\frac{\psi}{\alpha+\psi} E_a \varepsilon_{ISA} \left(\Gamma a \frac{\sinh \Gamma a}{\cosh \Gamma a} \right) \qquad \text{(max shear stress in adhesive)} \qquad (5)$$

The calculated value is $\tau_{\max} = 119.1\ \mathrm{kPa}$. This is a small value because the adhesive layer is already relatively thick ($t_b = 100\ \mu\mathrm{m}$); as indicated in the textbook Chapter 8, Section 8.2.2, Figure 8.7, the maximum shear stress is very high for very thin bonding layer and decreases rapidly beyond a bonding layer of approx. $20\ \mu\mathrm{m}$

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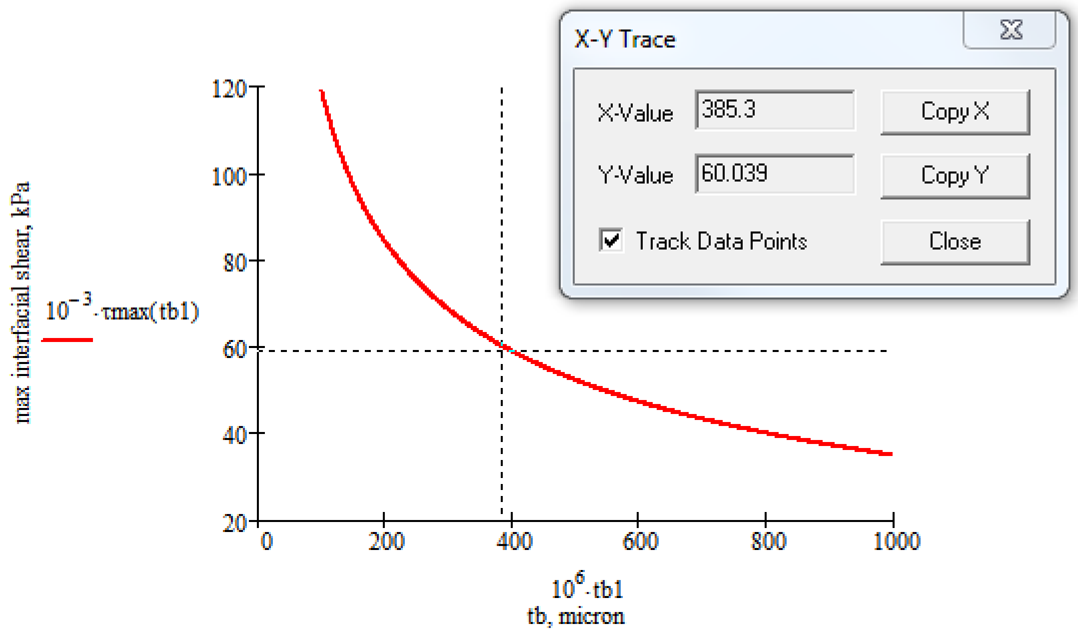
(c) To find the required adhesive thickness to decrease the maximum shear stress by a factor of 2 we proceed as follows: explore the dependence of the maximum shear stress with adhesive thickness, write the maximum shear stress as a function of t_b and plot to identify the value of t_b that will give the required reduction in the shear stress. Define the shear lag parameter as a function of t_b, i.e.,

$$\Gamma(t_b) = \sqrt{\frac{G_b}{E_a}\frac{1}{t_a t_b}\frac{\alpha+\psi}{\psi}} \qquad \text{(shear-lag parameter)} \tag{6}$$

Using Eq. (6), write Eq. (5) as

$$\tau_{\max}(t_b) = \frac{t_a}{a}\frac{\psi}{\alpha+\psi}E_a\varepsilon_{ISA}\left(a\Gamma(t_b)\frac{\sinh a\Gamma(t_b)}{\cosh a\Gamma(t_b)}\right) \qquad \text{(max shear stress as function of } t_b\text{)} \tag{7}$$

The plot of $\tau_{\max}(t_b)$ for various values of t_b is shown below. It is apparent that it $\tau_{\max}(t_b)$ decreases rapidly with t_b.



The target shear stress value is half of the value found in item (b) above, i.e., $\tau_{\max}^{(b)}/2 \simeq 60\text{ kPa}$. As shown in the above plot, the approximate value of t_b that will give a shear stress of approximately 60 kPa is $t_{b1} = 385\ \mu\text{m}$. If more accuracy is needed, then, by iterations, we find that $t_{b1} = 390\ \mu\text{m}$ gives $\tau_{\max}(t_b) = 59.5\text{ kPa}$, which exactly half of the one found in item (b) above.

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(d) To find the effective PWAS size, use Eq. (8.85) of textbook Chapter 8, i.e.,

$$a_e = a\left(\frac{1-\dfrac{\sinh \Gamma a}{\Gamma a \cosh \Gamma a}}{1-\dfrac{1}{\cosh \Gamma a}}\right) \quad \text{(location of the effective shear stress)} \qquad (8.85)\ (8)$$

Upon calculation, one gets $a_e = 2.85\ \text{mm}$. The effective PWAS size is twice a_e, i.e., $l_e = 2a_e = 5.70\ \text{mm} = 81.5\%$ of l_a.

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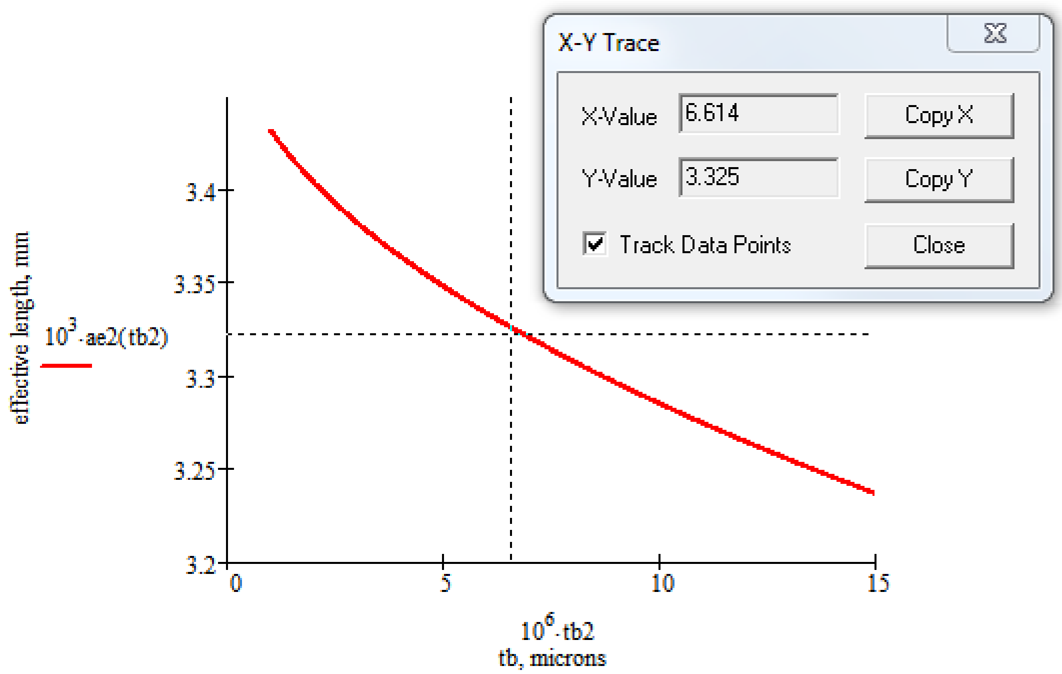
(e) To calculate the adhesive thickness to bring the PWAS size within 5% of the actual PWAS size we proceed as follows: Calculate the new value of the effective PWAS size and get $l_{e2} = 6.65\ \text{mm}$. The corresponding location of the effective shear stress is $a_{e2} = 3.325\ \text{mm}$. Express Eq. (8) as function of t_b, i.e.,

$$a_e(t_b) = a\left(\frac{1-\dfrac{\sinh a\Gamma(t_b)}{a\Gamma(t_b)\cosh a\Gamma(t_b)}}{1-\dfrac{1}{\cosh a\Gamma(t_b)}}\right) \qquad (8.85)\ (9)$$

Plot $a(t_b)$ vs. t_b and estimate the required value of t_{b2} to get $a_{e2} = 3.325\ \text{mm}$, i.e.,



The approximate value is $t_{b2} \simeq 6.6\ \mu\text{m}$. This value should be accurate enough for practical purposes because adhesive thickness is not easy to control very precisely.

PROBLEM 8.1 SOLUTION

scale units $mm := 10^{-3}$ $\mu m := 10^{-6}$ $GPa := 10^{9}$ $MPa := 10^{6}$ $kPa := 10^{3}$

applied voltage $V := 10$

Actuator (APC 850): $Ea := 63 \cdot GPa$ $ta := 0.2 \cdot mm$ $La := 7 \cdot mm$ $d31 := 175 \cdot 10^{-12}$

$a := \frac{La}{2}$ $\varepsilon ISA := d31 \cdot \frac{V}{ta}$

$a = 3.5\,10^{-3}$ $\varepsilon ISA = 8.75\,10^{-6}$

Structure (Aluminum): $E := 70 \cdot 10^{9}$ $t := 2 \cdot mm$ $t = 2 \times 10^{-3}$

Bond (super glue): $Gb := 2 \cdot 10^{9}$ $tb := 100 \cdot \mu m$ $tb = 1 \times 10^{-4}$

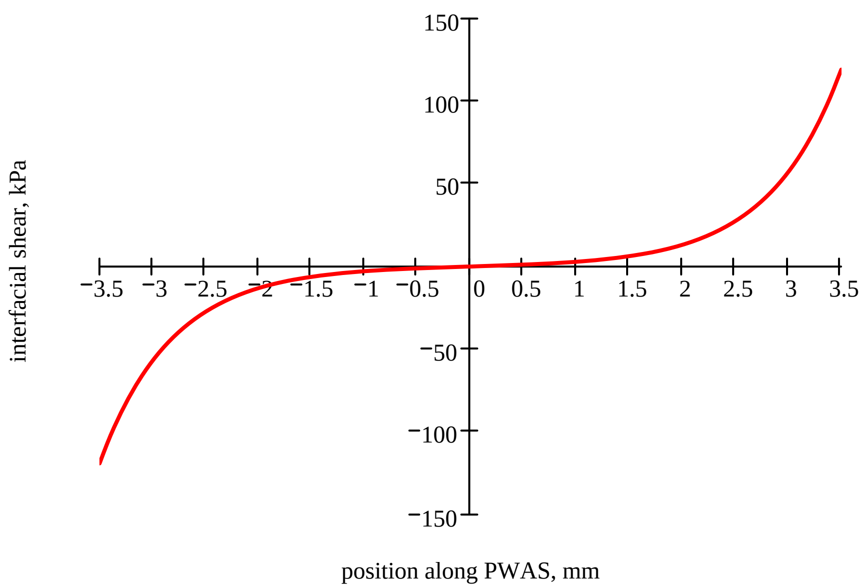
distribution factor $\alpha := 4$

(a) Calculate and plot the shear lag distribution of the shear stress in the adhesive layer

$$\psi := \frac{E \cdot t}{Ea \cdot ta} \qquad \Gamma := \sqrt{\frac{Gb}{Ea} \cdot \frac{1}{ta \cdot tb} \cdot \frac{\psi + \alpha}{\psi}} \qquad \tau(x) := \frac{ta}{a} \cdot \frac{\psi}{\alpha + \psi} \cdot Ea \cdot \left(\Gamma \cdot a \cdot \frac{\sinh(\Gamma \cdot x)}{\cosh(\Gamma \cdot a)} \right) \cdot \varepsilon ISA$$

$\psi = 11.111$ $\Gamma = 1.469 \times 10^{3}$

$x := -a, -0.99 \cdot a \,..\, a$



(b) The maximum shear stress is found at the tip of the PWAS, i.e., x=a; its value is

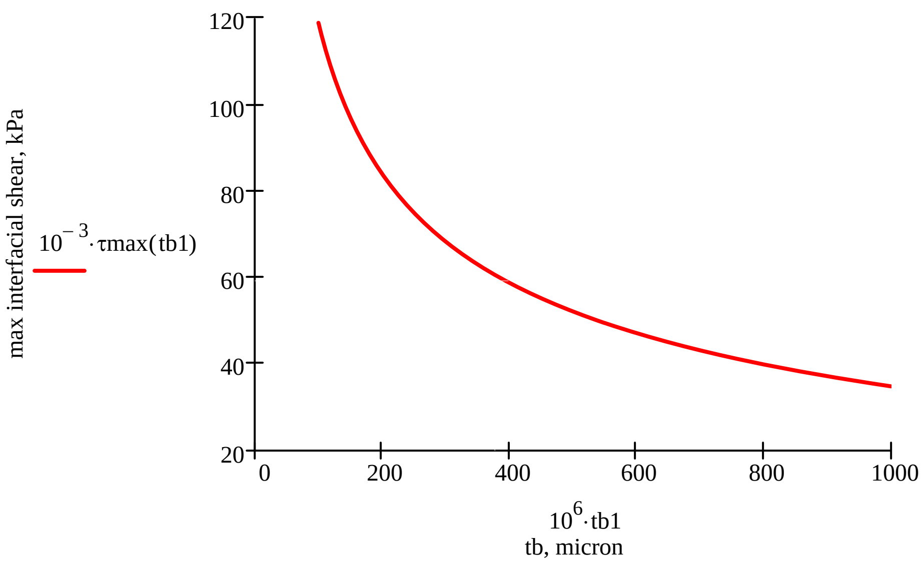
$$\tau(a) = 119.1\,\mathrm{kPa}$$

(c) To explore the dependence of maximum shear stress with adhesive thickness write it as function of tb, i.e.,

$$\Gamma1(tb1) := \sqrt{\frac{Gb}{Ea}\cdot\frac{1}{ta\cdot tb1}\cdot\frac{\psi+\alpha}{\psi}}$$

$$\tau max(tb1) := \frac{ta}{a}\cdot\frac{\psi}{\alpha+\psi}\cdot Ea\cdot\left(\Gamma1(tb1)\cdot a\cdot\frac{\sinh(\Gamma1(tb1)\cdot a)}{\cosh(\Gamma1(tb1)\cdot a)}\right)\cdot\varepsilon ISA$$

$tbmin := 100\cdot 10^{-6}$ $\quad tbmax := 10\cdot tbmin$ $\quad dtb := \dfrac{(tbmax - tbmin)}{1000}$ $\quad tb1 := tbmin, tbmin + dtb .. tbmax$



$$\tau max(392\cdot\mu m) = 59.5\,\mathrm{kPa}$$

(d) The effective PWAS size is calculated as follows.

Calculate ae with Eq. (8.85) of the textbook Chapter 8, i.e.:

$$ae := a \cdot \left(\frac{1 - \dfrac{\sinh(\Gamma \cdot a)}{\Gamma \cdot a \cdot \cosh(\Gamma \cdot a)}}{1 - \dfrac{1}{\cosh(\Gamma \cdot a)}} \right) \qquad ae = 2.85\,\text{mm}$$

$$Le := 2 \cdot ae \qquad Le = 5.71\,\text{mm}$$

$$\frac{Le}{La} = 81.5\,\%$$

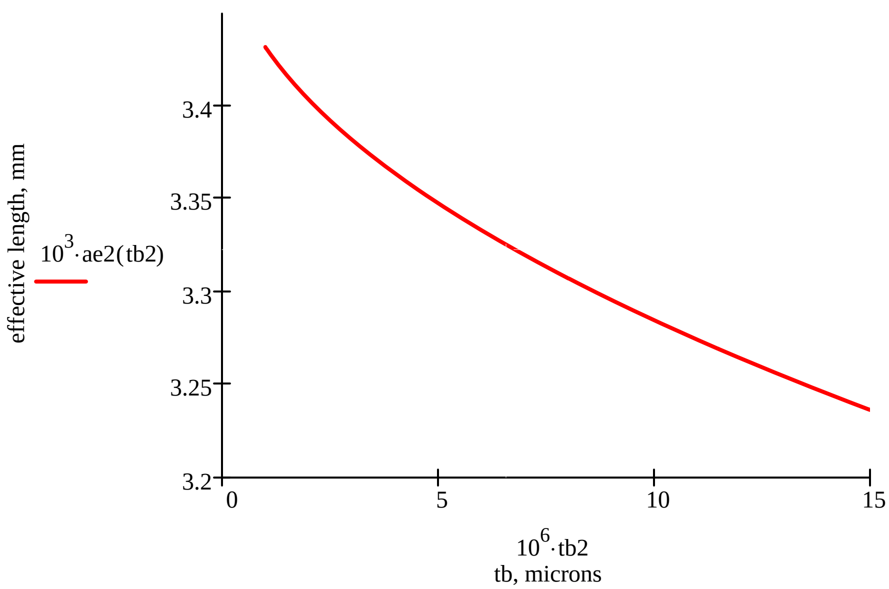
(e) To calculate the adhesive thickness to bring the PWAS size within 5% of the actual PWAS size we proceed as follows:

Calculate the new value of the effective PWAS size $\quad Le2 := (1 - 5 \cdot \%) \cdot La \qquad Le2 = 6.65\,\text{mm}$

$$a2e := \frac{Le2}{2} \qquad a2e = 3.325\,\text{mm}$$

$$tbmin := 1 \cdot 10^{-6} \qquad tbmax := 15 \cdot tbmin \qquad dtb := \frac{(tbmax - tbmin)}{1000} \qquad tb2 := tbmin, tbmin + dtb \,..\, tbmax$$

$$\Gamma 2(tb2) := \sqrt{\frac{Gb}{Ea} \cdot \frac{1}{ta \cdot tb2} \cdot \frac{\psi + \alpha}{\psi}} \qquad ae2(tb2) := a \cdot \left(\frac{1 - \dfrac{\sinh(\Gamma 2(tb2) \cdot a)}{\Gamma 2(tb2) \cdot a \cdot \cosh(\Gamma 2(tb2) \cdot a)}}{1 - \dfrac{1}{\cosh(\Gamma 2(tb2) \cdot a)}} \right)$$



Problem 2: Consider a circular PWAS transducer mounted on a thin wall structure; the PWAS has $E_a = 63\ \text{GPa}$, $\nu_a = 0.35$, $t_a = 0.2\ \text{mm}$, $d_{31} = -175\ \text{mm/kV}$, the PWAS diameter is $2a = 7\ \text{mm}$; the structure has $E = 70\ \text{GPa}$, $\nu = 0.33$, $t = 2\ \text{mm}$; the adhesive has $G_b = 2\ \text{GPa}$ $t_b = 100\ \mu\text{m}$; the applied voltage is $V = 10\ \text{V}$.

(a) Calculate and plot the radial and circumferential shear lag distribution of the shear stress in the adhesive layer.
(b) Find the maximum shear stress and its location
(c) Find the required adhesive thickness to decrease the maximum shear stress by a factor of 2 and give the new shear stress value
(d) Find the effective PWAS size in physical units and as percentage of the actual PWAS size
(e) Find the required adhesive thickness to bring the effective PWAS size within 5% of the actual PWAS size and give the new value of the effective PWAS size

Solution

(a) To calculate and plot the radial and circumferential shear lag distribution of the shear stress in the adhesive layer, proceed as follows:

The circumferentially-oriented shear stress is zero because the problem is axisymmetric. The radial-oriented shear stress $\tau(r)$ is given by Eq. (8.184), i.e.,

$$\tau(r) = C\, I_1(\Gamma r) \qquad r \le a \qquad (8.184)\ (1)$$

where

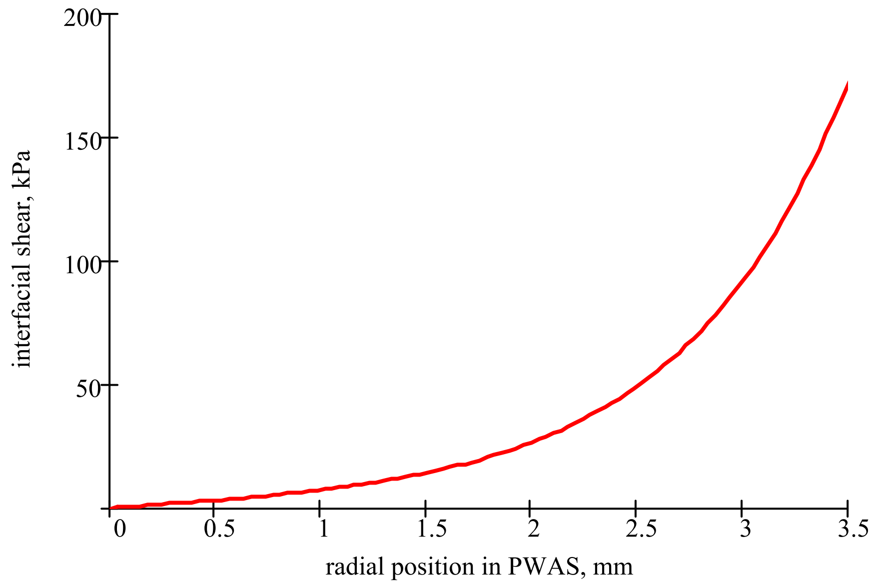
$$\varepsilon_{ISA} = d_{31}\frac{V}{t_a} \qquad (8.1)\ (2)$$

$$\Gamma^2 = \frac{G_b}{t_b}\left(\frac{1-\nu_a^2}{E_a t_a} + \alpha\frac{1-\nu^2}{Et}\right) = \frac{G_b}{t_b}\frac{1-\nu_a^2}{E_a t_a}\frac{\alpha+\psi}{\psi} \qquad \text{(2-D shear lag parameter)} \qquad (8.178)\ (3)$$

$$\alpha = 4 \qquad \psi = \frac{Et}{1-\nu^2}\Big/\frac{E_a t_a}{1-\nu_a^2} \qquad \text{(2-D relative stiffness coefficient)} \qquad (8.179)\ (4)$$

$$C = \left[\Gamma I_0(\Gamma a)\left(\frac{1-\nu_a}{E_a t_a} + \alpha\frac{1-\nu}{Et}\right) - \left(\frac{(1-\nu_a)^2}{E_a t_a} + \alpha\frac{(1-\nu)^2}{Et}\right)\frac{I_1(\Gamma a)}{a}\right]^{-1}\Gamma^2\varepsilon_{ISA} \qquad (8.216)\ (5)$$

Upon calculation, one gets $\varepsilon_{ISA} = 8.75\ \mu\text{m}$, $\psi = 10.94$, $\Gamma = 1.379/\text{mm}$, $K = 499\ \text{N}$, and the following plot:



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(b) The maximum shear stress is obtained at the edge of the PWAS, i.e., at $r = a = 3.5$ mm

$$\tau_{\max} = \tau(r)\big|_{r=a} = C\, I_1(\Gamma a) \qquad \text{(max shear stress in adhesive)} \qquad (6)$$

The calculated value is $\tau_{\max} = 172.5$ kPa . This is a small value because the adhesive layer is already relatively thick ($t_b = 100$ μm); as indicated in the textbook Chapter 8, Section 8.2.2, Figure 8.7, the maximum shear stress is very high for very thin bonding layer and decreases rapidly beyond a bonding layer of approx. 20 μm

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(c) To find the required adhesive thickness to decrease the maximum shear stress by a factor of 2 we proceed as follows: Explore the dependence of the maximum shear stress with adhesive thickness, write the maximum shear stress as a function of t_b and plot to identify the value of t_b that will give the required reduction in the shear stress. Define the shear lag parameter as a function of t_b, i.e.,

$$\Gamma(t_b) = \sqrt{\frac{G_b}{t_b}\frac{1-\nu^2}{E_a t_a}\frac{\alpha+\psi}{\psi}} \qquad \text{(shear-lag parameter)} \qquad (7)$$

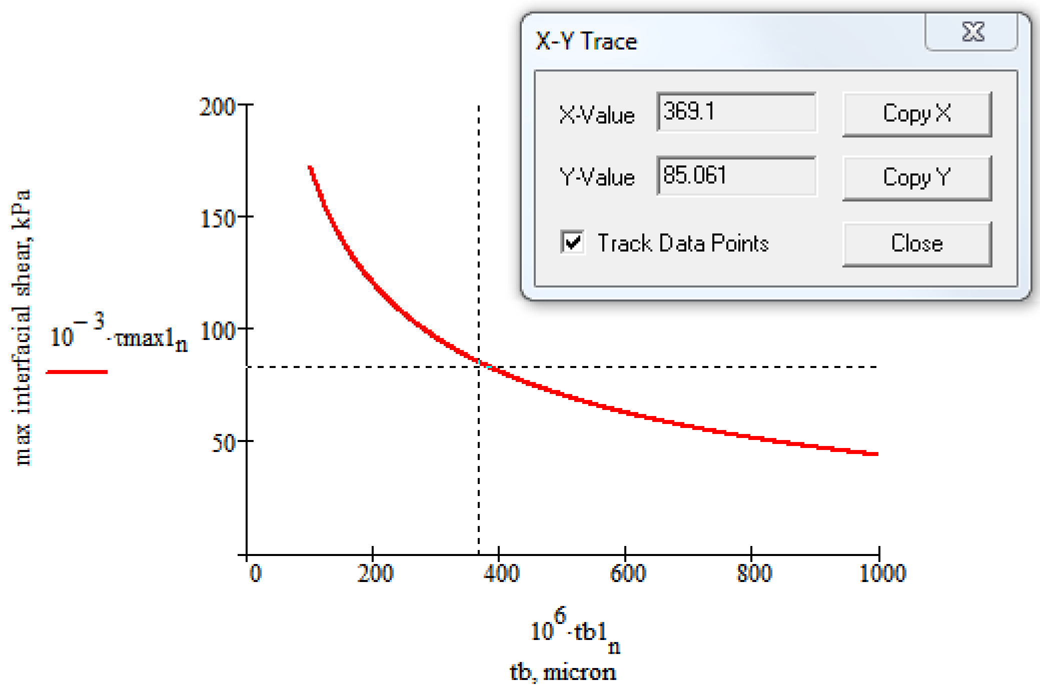
Also define the constant K as a function of t_b, i.e.,

$$C(t_b) = \left[\Gamma(t_b) I_0\left(\Gamma(t_b)a\right)\left(\frac{1-\nu_a}{E_a t_a} + \alpha\frac{1-\nu}{Et}\right) - \left(\frac{(1-\nu_a)^2}{E_a t_a} + \alpha\frac{(1-\nu)^2}{Et}\right)\frac{I_1\left(\Gamma(t_b)a\right)}{a}\right]^{-1} \Gamma^2(t_b)\ \varepsilon_{ISA} \quad (8)$$

Using Eqs. (7), (8), write Eq. (6) as

$$\tau_{\max}(t_b) = C(t_b)\ I_1\left(\Gamma(t_b)a\right) \quad \text{(max shear stress as function of } t_b\text{)} \qquad (9)$$

The plot of $\tau_{\max}(t_b)$ for various values of t_b is shown below. It is apparent that it $\tau_{\max}(t_b)$ decreases rapidly with t_b.



The target shear stress value is half of the value found in item (b) above, i.e., $\tau_{\max}^{(b)}/2 \simeq 85$ kPa. As shown in the above plot, the approximate value of t_b that will give a shear stress of approximately 185 kPa is $t_{b1} = 369$ μm. If more accuracy is needed, then, by iterations, we find that $t_{b1} = 360$ μm gives $\tau_{\max}(t_{b1}) = 86.3$ kPa, which exactly half of the one found in item (b) above.

(d) To find the effective PWAS size, use the second part of Eq. (8.226) of textbook Chapter 8, i.e.,

$$a_e = \frac{\int_0^a I_1(\Gamma r) r^2 dr}{\int_0^a I_1(\Gamma r) r dr} \qquad (8.226)\ (10)$$

The integrals in Eq. (10) can be found numerically. Upon calculation, one gets $a_e = 2.92$ mm. The effective PWAS diameter is $2a_e = 5.84$ mm, which is 83.5% of the actual PWAS size.

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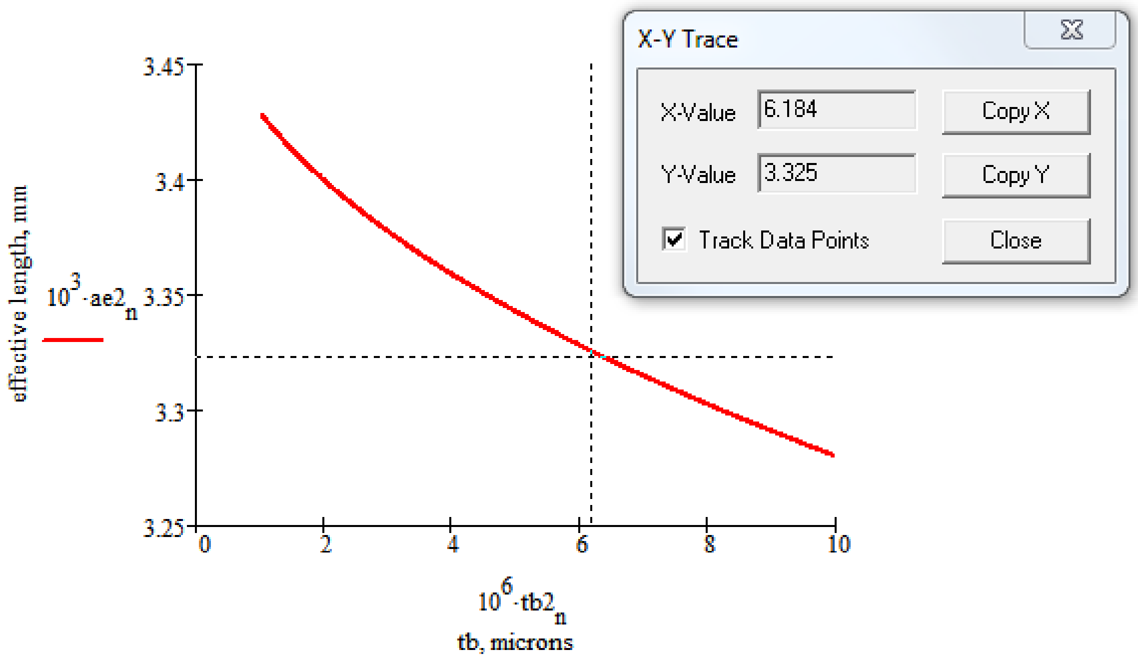
(e) To calculate the adhesive thickness to bring the PWAS size within 5% of the actual PWAS size we proceed as follows: Calculate the new value of the effective PWAS size and get $a_{e2} = 3.325$ mm. Express Eq. (8) as function of t_b, i.e.,

$$a_e(t_b) = \frac{\int_0^a I_1\left(\Gamma(t_b) r\right) r^2 dr}{\int_0^a I_1\left(\Gamma(t_b) r\right) r dr} \qquad (8.85)\ (11)$$

Plot $a(t_b)$ vs. t_b and estimate the required value of t_{b2} to get $a_{e2} = 3.325$ mm, i.e.,



The approximate value is $t_{b2} \simeq 6.2\ \mu\text{m}$. This value should be accurate enough for practical purposes because adhesive thickness is not easy to control very precisely.

PROBLEM 8.2 SOLUTION

$ORIGIN := 1$

scale units $mm := 10^{-3}$ $\mu m := 10^{-6}$ $GPa := 10^{9}$ $MPa := 10^{6}$ $kPa := 10^{3}$

applied voltage $V := 10$

Actuator (APC 850): $Ea := 63 \cdot GPa$ $ta := 0.2 \cdot mm$ $Da := 7 \cdot mm$ $d31 := 175 \cdot 10^{-12}$ $\nu a := 0.35$

$a := \frac{Da}{2}$ $\varepsilon ISA := d31 \cdot \frac{V}{ta}$

$a = 3.5\,10^{-3}$ $\varepsilon ISA = 8.75\,10^{-6}$

Structure (Aluminum): $E := 70 \cdot GPa$ $t := 2 \cdot mm$ $\nu := 0.33$

Bond (super glue): $Gb := 2 \cdot GPa$ $tb := 100 \cdot \mu m$

distribution factor $\alpha := 4$

(a) Calculate and plot the shear lag distribution of the shear stress in the adhesive layer

$$\psi := \frac{\dfrac{E \cdot t}{1-\nu^2}}{\dfrac{Ea \cdot ta}{1-\nu a^2}} \qquad \Gamma := \sqrt{\left(\frac{Gb}{tb} \cdot \frac{1-\nu a^2}{Ea \cdot ta} \cdot \frac{\alpha + \psi}{\psi}\right)}$$

$\psi = 10.942$ $\Gamma = 1.379 \times 10^{3}$

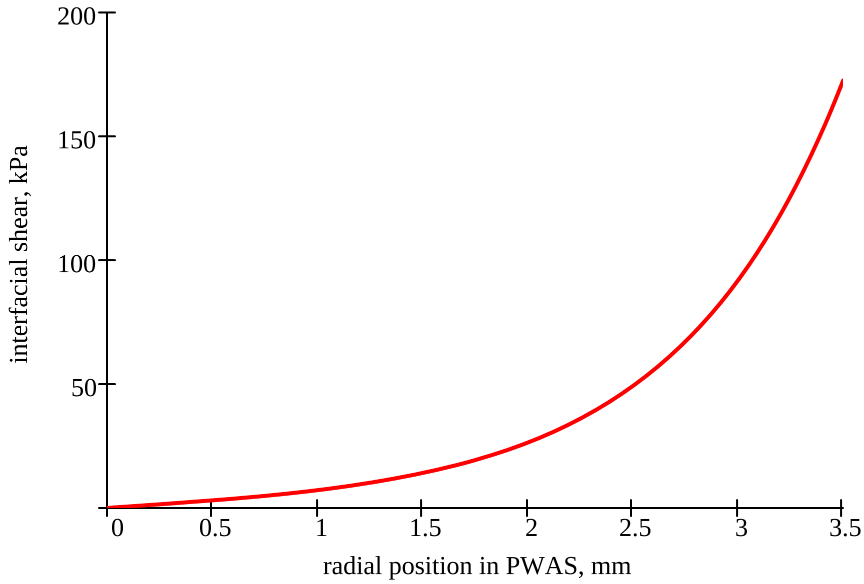
$$T1 := \Gamma \cdot I0(\Gamma \cdot a) \cdot \left(\frac{1-\nu a}{Ea \cdot ta} + \alpha \cdot \frac{1-\nu}{E \cdot t}\right) \qquad T2 := \left[\frac{(1-\nu a)^2}{Ea \cdot ta} + \alpha \cdot \frac{(1-\nu)^2}{E \cdot t}\right] \cdot \frac{I1(\Gamma \cdot a)}{a} \qquad K := (T1 - T2)^{-1}$$

$T1 = 2.278 \times 10^{-3}$ $T2 = 2.75 \times 10^{-4}$ $K = 499.363$

$$C := \left[\Gamma \cdot I0(\Gamma \cdot a) \cdot \left(\frac{1-\nu a}{Ea \cdot ta} + \alpha \cdot \frac{1-\nu}{E \cdot t}\right) - \left[\frac{(1-\nu a)^2}{Ea \cdot ta} + \alpha \cdot \frac{(1-\nu)^2}{E \cdot t}\right] \cdot \frac{I1(\Gamma \cdot a)}{a}\right]^{-1} \cdot \Gamma^2 \cdot \varepsilon ISA$$

$\tau(r) := C \cdot I1(\Gamma \cdot r)$

$r := 0, 0.01 \cdot a .. a$



(b) The maximum shear stress is found at the PWAS edge, i.e., r=a; its value is

$\tau(a) = 172.5\,\text{kPa}$ $\qquad \dfrac{\tau(a)}{2} = 86.272\,\text{kPa}$

(c) To explore the dependence of maximum shear stress with adhesive thickness write it as function of tb, i.e.,

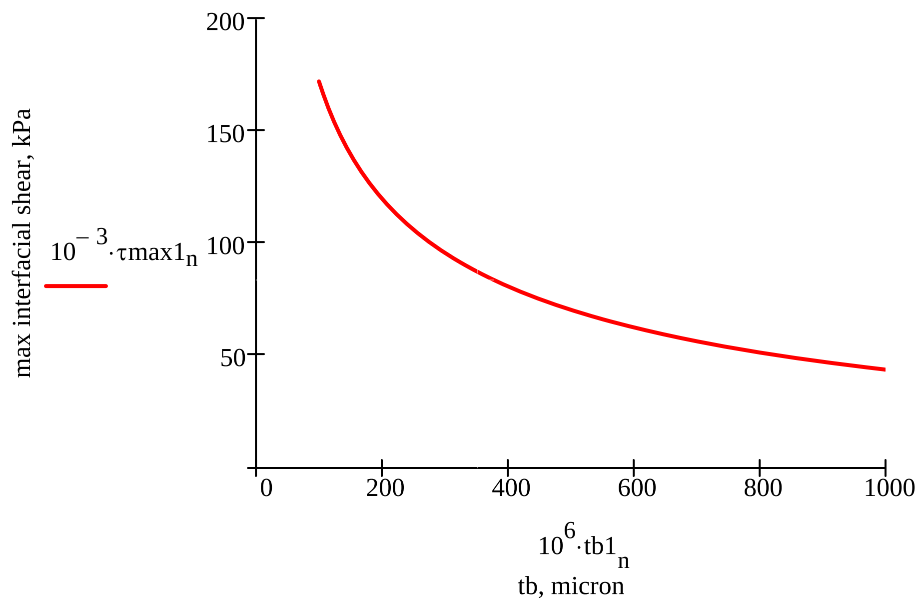
$N := 1000 \qquad tbmin := 100 \cdot 10^{-6} \qquad tbmax := 10 \cdot tbmin \qquad dtb := \dfrac{(tbmax - tbmin)}{N}$

$n := 1 .. N \qquad tb1_n := tbmin + (n-1) \cdot dtb$

$$\Gamma1_n := \sqrt{\frac{Gb}{Ea} \cdot \frac{1 - \nu a^2}{ta \cdot tb1_n} \cdot \frac{\alpha + \psi}{\psi}} \qquad \Gamma1_1 = 1.379 \times 10^3$$

$$C1_n := \left[\Gamma1_n \cdot I0(\Gamma1_n \cdot a) \cdot \left(\frac{1 - \nu a}{Ea \cdot ta} + \alpha \cdot \frac{1 - \nu}{E \cdot t} \right) - \left[\frac{(1 - \nu a)^2}{Ea \cdot ta} + \alpha \cdot \frac{(1 - \nu)^2}{E \cdot t} \right] \cdot \frac{I1(\Gamma1_n \cdot a)}{a} \right]^{-1} \cdot (\Gamma1_n)^2 \cdot \varepsilon ISA$$

$$\tau max1_n := C1_n \cdot I1(\Gamma1_n \cdot a)$$



$10^{-3} \cdot \tau max1_n =$

86.704
86.575
86.446
86.318
86.191
86.063
85.937

$10^{-3} \cdot \tau max1_{290} = 86.318 \qquad tb1_{290} = 360.1\ \mu m$

(d) The effective PWAS size is calculated as follows.

Calculate ae with Eq. (8.226) of the textbook Chapter 8, i.e.:

$$num := \int_0^a I1(\Gamma \cdot r) \cdot r^2 \, dr \qquad den := \int_0^a I1(\Gamma \cdot r) \cdot r \, dr \qquad ae := \frac{num}{den}$$

$$num = 1.31 \times 10^{-7} \qquad den = 4.48 \times 10^{-5} \qquad ae = 2.92\,mm$$

$$2 \cdot ae = 5.847\,mm$$

$$\frac{ae}{a} = 83.5\,\%$$

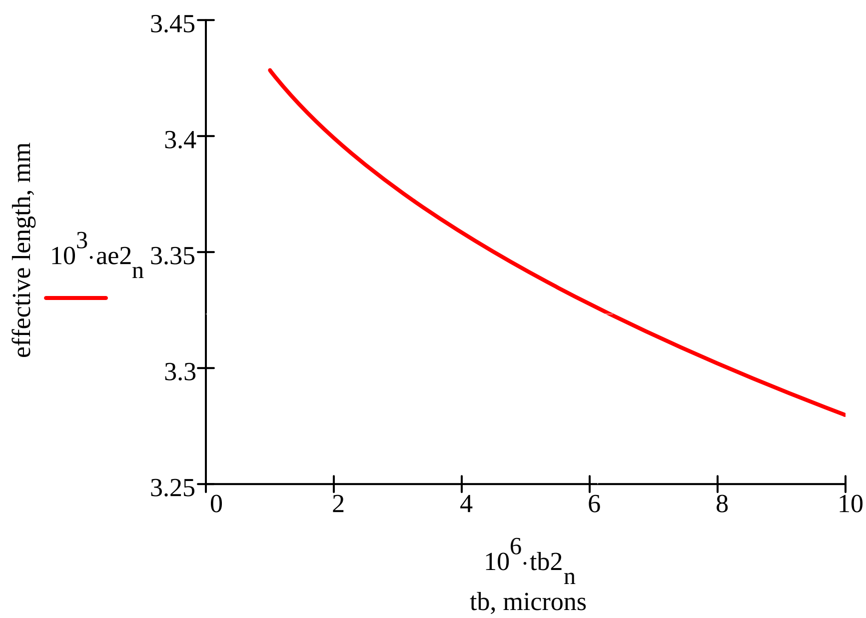
(e) To calculate the adhesive thickness to bring the PWAS size within 5% of the actual PWAS size we proceed as follows:

Calculate the new value of the effective PWAS size $ae2\_ := (1 - 5 \cdot \%) \cdot a$ $ae2\_ = 3.325\,mm$

$$tbmin := 1 \cdot 10^{-6} \qquad tbmax := 10 \cdot tbmin \qquad dtb := \frac{(tbmax - tbmin)}{N} \qquad tb2_n := tbmin + (n-1) \cdot dtb$$

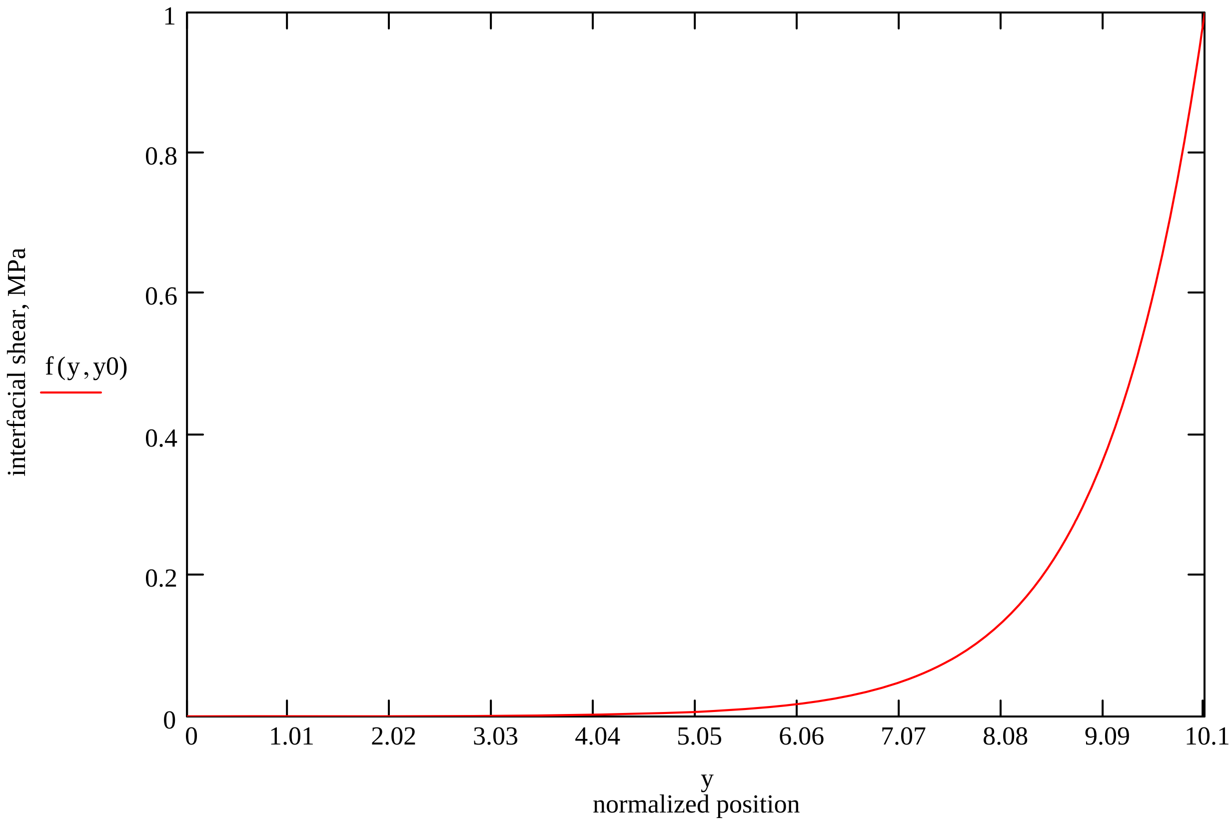
$$\Gamma 2_n := \sqrt{\frac{Gb}{Ea} \cdot \frac{1 - va^2}{ta \cdot tb2_n} \cdot \frac{\alpha + \psi}{\psi}} \qquad \Gamma 2_1 = 1.379 \times 10^4$$

$$num2_n := \int_0^a I1(\Gamma 2_n \cdot r) \cdot r^2 \, dr \qquad den2_n := \int_0^a I1(\Gamma 2_n \cdot r) \cdot r \, dr \qquad ae2_n := \frac{num2_n}{den2_n}$$



$y0 := 1.01 \cdot 10$

$$f(y, y0) := \frac{\sinh(y)}{\cosh(y0)}$$



Problem 3: Obtain Eqs. (8.314) through (8.320) by using the solution Eq. (8.313) and the appropriate equations from the range (8.285) through (8.306).

Given:

$$\tau(x) = \frac{G_b \varepsilon_{ISA} a}{t_b} \frac{\sinh \Gamma x}{\Gamma a \cosh \Gamma a} = \frac{t_a}{a} \frac{\psi}{\alpha + \psi} E_a \varepsilon_{ISA} \left(\Gamma a \frac{\sinh \Gamma x}{\cosh \Gamma a} \right) \qquad (8.313)\ (1)$$

$$t_a \sigma_a' - \tau = 0 \qquad \text{(PWAS)} \qquad (8.285)\ (2)$$

$$t\sigma' + \alpha \tau = 0 \qquad \text{(Structure)} \qquad (8.286)\ (3)$$

$$\varepsilon_a = \frac{du_a}{dx} \qquad \text{(PWAS)} \qquad (8.287)\ (4)$$

$$\gamma = \frac{u_a - u}{t_b} \qquad \text{(bonding layer)} \qquad (8.288)\ (5)$$

$$\varepsilon = \frac{du}{dx} \qquad \text{(structure)} \qquad (8.289)\ (6)$$

$$\sigma_a = E_a(\varepsilon_a - \varepsilon_{ISA}) \qquad \text{(PWAS)} \qquad (8.290)\ (7)$$

$$\tau = G_b \gamma \qquad \text{(bonding layer)} \qquad (8.291)\ (8)$$

$$\sigma = E\varepsilon \qquad \text{(structure)} \qquad (8.292)\ (9)$$

$$t_a E_a \varepsilon_a' - \tau = 0 \qquad (8.293)\ (10)$$

$$tE\varepsilon' + \alpha \tau = 0 \qquad (8.294)\ (11)$$

$$\psi = \frac{Et}{E_a t_a} \qquad \text{(relative stiffness coefficient)} \qquad (8.300)\ (12)$$

$$\Gamma^2 = \frac{G_b}{E_a} \frac{1}{t_a t_b} \frac{\alpha + \psi}{\psi} \qquad \text{(shear–lag parameter)} \qquad (8.301)\ (13)$$

$$\begin{cases} \sigma_a(\pm a) = 0 \\ \sigma(\pm a) = 0 \end{cases} \qquad (8.304)\ (14)$$

$$\begin{cases} \varepsilon_a(\pm a) = \varepsilon_{ISA} \\ \varepsilon(\pm a) = 0 \end{cases} \qquad (8.305)\ (15)$$

$$\begin{cases} u_a'(\pm a) = \varepsilon_{ISA} \\ u'(\pm a) = 0 \end{cases} \qquad (8.306)\ (16)$$

Find: We need to achieve the following results:

$$\varepsilon_a(x) = \frac{\alpha}{\alpha + \psi} \varepsilon_{ISA} \left(1 + \frac{\psi}{\alpha} \frac{\cosh \Gamma x}{\cosh \Gamma a} \right) \qquad \text{(PWAS actuation strain)} \qquad (8.314)\ (17)$$

$$\sigma_a(x) = -\frac{\psi}{\alpha + \psi} E_a \varepsilon_{ISA} \left(1 - \frac{\cosh \Gamma x}{\cosh \Gamma a} \right) \qquad \text{(PWAS stress)} \qquad (8.315)\ (18)$$

$$u_a(x) = \frac{\alpha}{\alpha + \psi} \varepsilon_{ISA} a \left(\frac{x}{a} + \frac{\psi}{\alpha} \frac{\sinh \Gamma x}{(\Gamma a) \cosh \Gamma a} \right) \qquad \text{(PWAS displacement)} \qquad (8.316)\ (19)$$

$$\tau(x)=\frac{t_a}{a}\frac{\psi}{\alpha+\psi}E_a\varepsilon_{ISA}\left(\Gamma a\frac{\sinh\Gamma x}{\cosh\Gamma a}\right)$$ (interfacial shear stress in bonding layer) (8.317) (20)

$$\varepsilon(x)=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(1-\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)$$ (structure strain at the surface) (8.318) (21)

$$\sigma(x)=\frac{\alpha}{\alpha+\psi}E\varepsilon_{ISA}\left(1-\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)$$ (structure stress) (8.319) (22)

$$u(x)=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}a\left(\frac{x}{a}-\frac{\sinh\Gamma x}{(\Gamma a)\cosh\Gamma a}\right)$$ (structure displacement at the surface) (8.320) (23)

<u>Solution</u>:

Get ε_a' from Eq. (10) and substitute Eq. (1) for τ to obtain

$$\varepsilon_a'=\frac{1}{E_at_a}\tau=\frac{1}{\cancel{E_a}\cancel{t_a}}\frac{\cancel{t_a}}{\cancel{a}}\frac{\psi}{\alpha+\psi}\cancel{E_a}\varepsilon_{ISA}\left(\Gamma\cancel{a}\frac{\sinh\Gamma x}{\cosh\Gamma a}\right)=\frac{\psi}{\alpha+\psi}\varepsilon_{ISA}\left(\Gamma\frac{\sinh\Gamma x}{\cosh\Gamma a}\right) \tag{24}$$

Integrate ε_a' to obtain ε_a, i.e.,

$$\begin{aligned}\varepsilon_a(x)&=\int\varepsilon_a'(x)\,dx+C=\frac{\psi}{\alpha+\psi}\varepsilon_{ISA}\frac{\Gamma}{\cosh\Gamma a}\int\sinh\Gamma x\,dx+C\\&=\frac{\psi}{\alpha+\psi}\varepsilon_{ISA}\frac{\cancel{\Gamma}}{\cosh\Gamma a}\frac{\cosh\Gamma x}{\cancel{\Gamma}}+C\\&=\frac{\psi}{\alpha+\psi}\varepsilon_{ISA}\frac{\cosh\Gamma x}{\cosh\Gamma a}+C\end{aligned} \tag{25}$$

The constant C is found from the boundary condition. Recall Eq. (15) at $x=a$, i.e.,

$$\varepsilon_a(a)=\varepsilon_{ISA} \tag{26}$$

Substitution of Eq. (26) into Eq. (25) yields

$$\varepsilon_a(a)=\frac{\psi}{\alpha+\psi}\varepsilon_{ISA}\frac{\cosh\Gamma a}{\cosh\Gamma a}+C=\frac{\psi}{\alpha+\psi}\varepsilon_{ISA}+C=\varepsilon_{ISA} \tag{27}$$

Upon solution, Eq. (27) gives

$$C=\left(1-\frac{\psi}{\alpha+\psi}\right)\varepsilon_{ISA}=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA} \tag{28}$$

Substitution of Eq.(28) into Eq. (25) yields the required Eq. (307), i.e.,

$$\varepsilon_a(x)=\frac{\psi}{\alpha+\psi}\varepsilon_{ISA}\frac{\cosh\Gamma x}{\cosh\Gamma a}+\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(1+\frac{\psi}{\alpha}\frac{\cosh\Gamma x}{\cosh\Gamma a}\right) \tag{29}$$

Next, substitute Eq. (29) into Eq. (7) to obtain σ_a

$$\sigma_a = E_a(\varepsilon_a - \varepsilon_{ISA}) = E_a\left[\frac{\alpha}{\alpha+\psi}\left(1+\frac{\psi}{\alpha}\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)-1\right]\varepsilon_{ISA}$$

$$= E_a\left[\left(\frac{\alpha}{\alpha+\psi}-1\right)+\frac{\psi}{\alpha+\psi}\frac{\cosh\Gamma x}{\cosh\Gamma a}\right]\varepsilon_{ISA} = E_a\left(\frac{\cancel{\alpha}-\cancel{\alpha}-\psi}{\alpha+\psi}+\frac{\psi}{\alpha+\psi}\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)\varepsilon_{ISA} \tag{30}$$

$$= E_a\left(\frac{-\psi}{\alpha+\psi}+\frac{\psi}{\alpha+\psi}\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)\varepsilon_{ISA} = -\frac{\psi}{\alpha+\psi}E_a\varepsilon_{ISA}\left(1-\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)$$

The end part of Eq. (30) has the required form of Eq. (8.308).

Next, substitute Eq. (29) into Eq. (4) and integrate the result to get u_a, i.e.,

$$\frac{du_a}{dx} = \varepsilon_a = \frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(1+\frac{\psi}{\alpha}\frac{\cosh\Gamma x}{\cosh\Gamma a}\right) \quad (31)$$

Upon integration, Eq. (31) yields

$$u_a(x) = \int\frac{du_a}{dx}dx + C = \frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\int\left(1+\frac{\psi}{\alpha}\frac{1}{\cosh\Gamma a}\cosh\Gamma x\right)dx + C$$

$$= \frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(x+\frac{\psi}{\alpha}\frac{1}{\cosh\Gamma a}\frac{\sinh\Gamma x}{\Gamma}\right)+C \tag{32}$$

The constant C is found from the symmetry condition $u_a(x)\big|_{x=0} = 0$, i.e.,

$$u_a(0) = \frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(x+\frac{\psi}{\alpha}\frac{1}{\cosh\Gamma a}\frac{\sinh\Gamma x}{\Gamma}\right)\Bigg|_{x=0} + C = 0 + C = 0 \tag{33}$$

Solving Eq. (33) for C yields $C = 0$; hence, Eq. (32) becomes

$$u_a(x) = \frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(x+\frac{\psi}{\alpha}\frac{1}{\cosh\Gamma a}\frac{\sinh\Gamma x}{\Gamma}\right)$$

$$= \frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}a\left(\frac{x}{a}+\frac{\psi}{\alpha}\frac{\sinh\Gamma x}{\Gamma a\cosh\Gamma a}\right) \tag{34}$$

The end part of Eq. (34) has the required form of Eq. (8.309).

Next, Get ε' from Eq. (11) and substitute Eq. (1) for τ to obtain

$$\varepsilon' = -\frac{\alpha}{Et}\tau = -\frac{\alpha}{Et}\frac{t_a}{a}\frac{\psi}{\alpha+\psi}E_a\varepsilon_{ISA}\left(\Gamma a\frac{\sinh\Gamma x}{\cosh\Gamma a}\right) = -\alpha\frac{\psi}{\alpha+\psi}\frac{E_a t_a}{Et}\frac{\varepsilon_{ISA}}{\cancel{a}}\left(\Gamma\cancel{a}\frac{\sinh\Gamma x}{\cosh\Gamma a}\right)$$

$$= -\alpha\frac{\cancel{\psi}}{\alpha+\psi}\frac{1}{\cancel{\psi}}\varepsilon_{ISA}\left(\Gamma\frac{\sinh\Gamma x}{\cosh\Gamma a}\right) = -\alpha\frac{\cancel{\psi}}{\alpha+\psi}\frac{1}{\cancel{\psi}}\varepsilon_{ISA}\left(\Gamma\frac{\sinh\Gamma x}{\cosh\Gamma a}\right) \tag{35}$$

$$= -\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(\Gamma\frac{\sinh\Gamma x}{\cosh\Gamma a}\right)$$

Integrate ε' to get ε, i.e.,

$$\varepsilon(x) = \int\varepsilon'(x)\,dx + C = -\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(\Gamma\frac{\int\sinh\Gamma x\,dx}{\cosh\Gamma a}\right)+C = -\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\frac{\cosh\Gamma x}{\cosh\Gamma a}+C \tag{36}$$

The constant C is found from the boundary condition. Recall Eq. (15) at $x=a$, i.e.,

$$\varepsilon\left(a\right)=0 \tag{37}$$

Substitution of Eq. (36) into Eq. (37) gives

$$\varepsilon(a)=-\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left.\frac{\cosh\Gamma x}{\cosh\Gamma a}\right|_{x=a}+C=-\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}+C=0 \tag{38}$$

Solving Eq. (38) for C yields

$$C=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA} \tag{39}$$

Hence, Eq. (36) becomes

$$\varepsilon(x)=-\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\frac{\cosh\Gamma x}{\cosh\Gamma a}+\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(1-\frac{\cosh\Gamma x}{\cosh\Gamma a}\right) \tag{40}$$

The end part of Eq. (40) has the required form of Eq. (8.311).

Next, substitute Eq. (40) into Eq. (9) to obtain σ, i.e.,

$$\sigma=E\varepsilon=\frac{\alpha}{\alpha+\psi}E\varepsilon_{ISA}\left(1-\frac{\cosh\Gamma x}{\cosh\Gamma a}\right) \tag{41}$$

Eq. (41) has the required form of Eq. (8.312).

Finally, substitute Eq. (40) into Eq. (6) and integrate the result to get u, i.e.,

$$\frac{du}{dx}=\varepsilon(x)=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(1-\frac{\cosh\Gamma x}{\cosh\Gamma a}\right) \tag{42}$$

Upon integration, Eq. (42) becomes

$$u(x)=\int\frac{du}{dx}dx+C=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\int\left(1-\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)dx+C=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left(x-\frac{\sinh\Gamma x}{\Gamma\cosh\Gamma a}\right)+C \tag{43}$$

The constant C is found from the symmetry condition $\left.u(x)\right|_{x=0}=0$, i.e.,

$$u(0)=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}\left.\left(x-\frac{\sinh\Gamma x}{\Gamma\cosh\Gamma a}\right)\right|_{x=0}+C=0+C=0 \tag{44}$$

Solving Eq. (44) for C yields $C=0$; hence, Eq. (43) becomes, upon rearrangement,

$$u(x)=\frac{\alpha}{\alpha+\psi}\varepsilon_{ISA}a\left(\frac{x}{a}-\frac{\sinh\Gamma x}{\Gamma a\cosh\Gamma a}\right) \tag{45}$$

Eq. (45) has the required form of Eq. (8.313).

Problem 4: Prove the integral in Eqs. (8.344), (8.345), i.e.

$$\int_{-a}^{a}\left(1-\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)^2 dx = l_a I(\Gamma a) \qquad (8.344)\ (1)$$

$$I(\Gamma a) = 1-\frac{3}{2}\frac{\sinh\Gamma a}{\Gamma a\cosh\Gamma a}+\frac{1}{2}\frac{1}{(\cosh\Gamma a)^2} \qquad \text{(bond efficiency)} \qquad (8.345)\ (2)$$

Solution

Expand Eq. (1)

$$\begin{aligned}\int_{-a}^{a}\left(1-\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)^2 dx &= \int_{-a}^{a}\left[1-2\frac{\cosh\Gamma x}{\cosh\Gamma a}+\left(\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)^2\right]dx \\ &= \left[x-\frac{2}{\Gamma}\frac{\sinh\Gamma x}{\cosh\Gamma a}+\frac{\int(\cosh\Gamma x)^2\,dx}{(\cosh\Gamma a)^2}\right]_{-a}^{a}\end{aligned} \qquad (3)$$

The integral $\int(\cosh\Gamma x)^2\,dx$ is evaluated as

$$\int(\cosh\Gamma x)^2\,dx = \frac{1}{4\Gamma}\sinh 2\Gamma x+\frac{x}{2}+C \qquad \text{(Wikipedia)} \qquad (4)$$

After apply the limits of integration, Eq. (4) becomes

$$\begin{aligned}\int_{-a}^{a}(\cosh\Gamma x)^2\,dx &= \left(\frac{1}{4\Gamma}\sinh 2\Gamma x+\frac{x}{2}+C\right)\Bigg|_{-a}^{a} = \frac{1}{4\Gamma}2\sinh 2\Gamma a+\frac{2a}{2} \\ &= \frac{1}{2\Gamma}\sinh 2\Gamma a + a\end{aligned} \qquad (5)$$

Substitution of Eq. (5) into Eq. (3) yields

$$\begin{aligned}\left[x-\frac{2}{\Gamma}\frac{\sinh\Gamma x}{\cosh\Gamma a}+\frac{\int(\cosh\Gamma x)^2\,dx}{(\cosh\Gamma a)^2}\right]_{-a}^{a} &= \left[2a-\frac{2}{\Gamma}\frac{2\sinh\Gamma a}{\cosh\Gamma a}+\frac{\frac{1}{2\Gamma}\sinh 2\Gamma a+a}{(\cosh\Gamma a)^2}\right] \\ &= 2\left[a-\frac{2}{\Gamma}\frac{\sinh\Gamma a}{\cosh\Gamma a}+\frac{\sinh 2\Gamma a}{4\Gamma(\cosh\Gamma a)^2}+\frac{a}{2(\cosh\Gamma a)^2}\right] \\ &= 2\left[a-\frac{2}{\Gamma}\frac{\sinh\Gamma a}{\cosh\Gamma a}+\frac{\sinh 2\Gamma a}{4\Gamma(\cosh\Gamma a)^2}+\frac{a}{2(\cosh\Gamma a)^2}\right]\end{aligned} \qquad (6)$$

Further simplification of Eq. (6) gives

$$2\left[a-\frac{2}{\Gamma}\frac{\sinh\Gamma a}{\cosh\Gamma a}+\frac{\sinh\Gamma a}{2\Gamma\cosh\Gamma a}+\frac{a}{2(\cosh\Gamma a)^2}\right]=2\left[a-\frac{2}{3\Gamma}\frac{\sinh\Gamma a}{\cosh\Gamma a}+\frac{a}{2(\cosh\Gamma a)^2}\right]$$
$$=2a\left(1-\frac{3}{2}\frac{\sinh\Gamma a}{\Gamma a\cosh\Gamma a}+\frac{1}{2}\frac{1}{(\cosh\Gamma a)^2}\right)=l_a\left(1-\frac{3}{2}\frac{\sinh\Gamma a}{\Gamma a\cosh\Gamma a}+\frac{1}{2}\frac{1}{(\cosh\Gamma a)^2}\right)=l_a\, I(\Gamma a) \tag{7}$$

Comparison of Eqs. (1), (2), (7) indicates that

$$\int_{-a}^{a}\left(1-\frac{\cosh\Gamma x}{\cosh\Gamma a}\right)^2 dx=l_a\, I(\Gamma a) \tag{8}$$

QED

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Problem 5: Prove that the integral in Eq. (8.355) gives the function $I(\Gamma a)$ of Eq. (8.345)

$$\int_{-a}^{+a}\left(\Gamma a\frac{\sinh\Gamma x}{\cosh\Gamma a}\right)\left(\frac{x}{a}-\frac{\sinh\Gamma x}{(\Gamma a)\cosh\Gamma a}\right)dx=l_a\, I(\Gamma a) \qquad (8.355)\ (1)$$

$$I(\Gamma a)=1-\frac{3}{2}\frac{\sinh\Gamma a}{\Gamma a\cosh\Gamma a}+\frac{1}{2}\frac{1}{(\cosh\Gamma a)^2} \qquad \text{(bond efficiency)} \qquad (8.345)\ (2)$$

<u>Solution</u>

Expand the integral in Eq. (1) to get

$$\int_{-a}^{a}\left(\Gamma a\frac{\sinh\Gamma x}{\cosh\Gamma a}\right)\left(\frac{x}{a}-\frac{\sinh\Gamma x}{(\Gamma a)\cosh\Gamma a}\right)dx=\frac{\Gamma a}{\cosh\Gamma a}\int_{-a}^{a}\left(\frac{x}{a}\sinh\Gamma x-\frac{(\sinh\Gamma x)^2}{(\Gamma a)\cosh\Gamma a}\right)dx \tag{3}$$

Evaluate the integral in Eq. (3), i.e.,

$$\int_{-a}^{a}\left(\frac{x}{a}\sinh\Gamma x-\frac{(\sinh\Gamma x)^2}{(\Gamma a)\cosh\Gamma a}\right)dx \tag{4}$$

First, do the first term in the integral Eq. (4)

$$\int_{-a}^{a} x\sinh\Gamma x\,dx=\left.\frac{x\cosh\Gamma x}{\Gamma}\right|_{-a}^{+a}-\frac{1}{\Gamma}\int_{-a}^{a}\cosh\Gamma x\,dx=\frac{2a}{\Gamma}\cosh\Gamma a-\frac{2}{\Gamma^2}\sinh\Gamma a \tag{5}$$

Next, recall the primitive

$$\int(\sinh\Gamma x)^2\,dx=\frac{1}{4\Gamma}\sinh 2\Gamma x-\frac{x}{2}+C \qquad \text{(Wikipedia)} \tag{6}$$

Thus,

$$\int_{-a}^{a}(\sinh\Gamma x)^2\,dx=\left.\left(\frac{1}{4\Gamma}\sinh 2\Gamma x-\frac{x}{2}+C\right)\right|_{-a}^{+a}=\frac{2}{4\Gamma}\sinh 2\Gamma a-\frac{2a}{2}=\frac{1}{2\Gamma}\sinh 2\Gamma a-a \tag{7}$$

Substitute Eqs. (5), (7) into Eq. (4) to get

$$\int_{-a}^{a}\left(\frac{x}{a}\sinh\Gamma x-\frac{(\sinh\Gamma x)^2}{(\Gamma a)\cosh\Gamma a}\right)dx$$

$$=\frac{1}{a}\left(\frac{2a}{\Gamma}\cosh\Gamma a-\frac{2}{\Gamma^2}\sinh\Gamma a\right)-\frac{1}{(\Gamma a)\cosh\Gamma a}\left(\frac{1}{2\Gamma}\sinh 2\Gamma a-a\right) \qquad (8)$$

Substitute Eq. (8) into Eq. (3) to obtain

$$\int_{-a}^{a}\left(\Gamma a\frac{\sinh\Gamma x}{\cosh\Gamma a}\right)\left(\frac{x}{a}-\frac{\sinh\Gamma x}{(\Gamma a)\cosh\Gamma a}\right)dx$$

$$=\frac{\Gamma a}{\cosh\Gamma a}\left[\frac{1}{a}\left(\frac{2a}{\Gamma}\cosh\Gamma a-\frac{2}{\Gamma^2}\sinh\Gamma a\right)-\frac{1}{(\Gamma a)\cosh\Gamma a}\left(\frac{1}{2\Gamma}\sinh 2\Gamma a-a\right)\right]$$

$$=\left[2a-\frac{2a}{\Gamma a}\frac{\sinh\Gamma a}{\cosh\Gamma a}-\frac{2\sinh\Gamma a\ \cancel{\cosh\Gamma a}}{2\Gamma(\cosh\Gamma a)^{\cancel{2}}}+\frac{a}{(\cosh\Gamma a)^2}\right] \qquad (9)$$

$$=\left[2a-\frac{2a}{\Gamma a}\frac{\sinh\Gamma a}{\cosh\Gamma a}-\frac{2a}{2\Gamma a}\frac{\sinh\Gamma a}{\cosh\Gamma a}+\frac{2a}{2(\cosh\Gamma a)^2}\right]=2a\left[1-\frac{1}{\Gamma a}\frac{3}{2}\frac{\sinh\Gamma a}{\cosh\Gamma a}+\frac{1}{2(\cosh\Gamma a)^2}\right]$$

Since $2a=l_a$, Eq. (9) becomes

$$\int_{-a}^{a}\left(\Gamma a\frac{\sinh\Gamma x}{\cosh\Gamma a}\right)\left(\frac{x}{a}-\frac{\sinh\Gamma x}{(\Gamma a)\cosh\Gamma a}\right)dx=l_a\left(1-\frac{3}{2}\frac{1}{\Gamma a}\frac{\sinh\Gamma a}{\cosh\Gamma a}+\frac{1}{2}\frac{1}{(\cosh\Gamma a)^2}\right)=l_a\,I(\Gamma a) \qquad (10)$$

QED