CHAPTER 9 PROBLEMS AND EXERCISES

Problem 1: Consider a linear PWAS transducer with length $l_a = 7\ \text{mm}$, width $b_a = 1.65\ \text{mm}$, thickness $t_a = 0.2\ \text{mm}$, and piezoelectric material properties as given in **Error! Reference source not found.**. Assume internal damping ratio η = 1% and electric loss factor δ = 1%.

(i) Calculate the complex compliance, $\bar{s}_{11}^E$, complex dielectric permittivity, $\bar{\varepsilon}_{33}^T$, complex coupling coefficient, $\bar{k}_{31}$, real and complex axial wave speeds in the piezoelectric material, c and $\bar{c}$, and real and complex electrical capacitances, C and $\bar{C}$.

(ii) For a 10-V, 100-kHz excitation, calculate the induced strain S_{ISA}, induced displacement u_{ISA}, and tip displacement $\hat{u}(a)$.

(iii) Calculate the natural resonance frequencies of the first, second, and third modes of antisymmetric axial vibration, in kHz. Plot on the same chart the first, second, and third modes of antisymmetric axial vibration.

(iv) Calculate the natural resonance frequencies of the first, second, and third modes of symmetric axial vibration, in kHz. Plot on the same chart the first, second, and third modes of symmetric axial vibration.

(v) Plot on separate charts the real part, the imaginary part, the amplitude, and the phase of the PWAS electromechanical admittance at 401 equally spaced points in the $10-1000\ \text{kHz}$ frequency range. Identify on this chart some of the resonance frequencies determined earlier. State the mode type (symmetric or antisymmetric) and mode number (e.g., f_2^A for the second antisymmetric mode).

(vi) Plot on separate charts the real part, the imaginary part, the amplitude, and the phase of the PWAS complex impedance at 401 equally spaced points in the 100-1000 kHz frequency range. Identify on this chart the frequencies at which the real part of the impedance peaks. Discuss how these frequencies are related, if at all, with the peaks of the PWAS admittance and with the mechanical resonances for in-plane vibration of the PWAS transducer.

<u>Solution</u>

(i) To calculate the complex compliance, $\bar{s}_{11}^E$ and complex dielectric permittivity, $\bar{\varepsilon}_{33}^T$, use Eq. (9.99), i.e.,

$$\bar{s}_{11}^E = s_{11}^E\left(1 - i\eta\right), \qquad \bar{\varepsilon}_{33}^T = \varepsilon_{33}^T\left(1 - i\delta\right) \tag{9.99) (1}$$

Upon calculation, we get

$$\bar{s}_{11}^E = \left(15.3 - i0.153\right)\ \text{Pa}^{-1}, \qquad \bar{\varepsilon}_{33}^T = \left(15.487 - i0.155\right)\ \text{nF/m} \tag{2}$$

To calculate the complex coupling coefficient, $\bar{k}_{31}$, use the formula

$$\bar{k}_{31}^2 = \frac{d_{31}^2}{\bar{s}_{11}^E\,\bar{\varepsilon}_{33}^T} \tag{3}$$

Upon calculation, one gets $\bar{k}_{31} = 0.3595 + i0.0036$.

The real wave speed in the piezoelectric material, c is calculated with the textbook Eq. (9.8), i.e.,

$$c = \sqrt{\frac{1}{\rho s_{11}^{E}}} \qquad (9.8)\ (4)$$

The complex wave speed is calculating with a similar formula, only that it uses the complex compliance, i.e.,

$$\bar{c} = \sqrt{\frac{1}{\rho \bar{s}_{11}^{E}}} \qquad (5)$$

Upon calculation, one gets $c = 2913\ \text{m/s}$, $\bar{c} = (2913 + i14.6)\ \text{m/s}$.

To calculate the real electrical capacitance, C, use the textbook Eq. (9.47), i.e.,

$$C = \varepsilon_{33}^{T} \frac{A}{t_a} \qquad (9.47)\ (6)$$

where $A = l_a b_a$. To calculate the complex $\bar{C}$ capacitance, use Eq. (6) but with the complex dielectric permittivity, $\bar{\varepsilon}_{33}^{T}$, i.e.,

$$\bar{C} = \bar{\varepsilon}_{33}^{T} \frac{A}{t_a} \qquad (7)$$

Upon calculation, one gets $C = 0.894\ \text{nF}$, $\bar{C} = (0.894 - i0.0089)\ \text{nF}$.

(ii) To calculate the induced strain S_{ISA}, recall Eq. (9.28) from textbook Chapter 9, i.e.,

$$S_{ISA} = d_{31}\hat{E}_3 \qquad \text{(piezoelectrically induced strain)} \qquad (9.28)\ (8)$$

where $\hat{E}_3 = -\hat{V}/t_a$. The piezoelectrically induced displacement u_{ISA} is calculated with Eq. (9.29) from textbook Chapter 9, i.e.,

$$u_{ISA} = 2S_{ISA}a = 2d_{31}\hat{E}_3 a \qquad \text{(piezoelectrically induced displacement)} \qquad (9.29)\ (9)$$

where $a = l_a/2$. The tip displacement $\hat{u}(a)$ is calculated with Eq. (9.36) from textbook Chapter 9, i.e.,

$$\hat{u}(\pm a) = \pm\tfrac{1}{2}\frac{u_{ISA}}{\gamma a}\frac{\sin\gamma a}{\cos\gamma a} = \pm\tfrac{1}{2}\frac{u_{ISA}}{\gamma a}\tan\gamma a \qquad \text{(tip displacement)} \qquad (9.36)\ (10)$$

where the wavenumber γ is given by Eq. (9.12), i.e., $\gamma = \omega/c$. Upon calculation, one gets $\hat{E}_3 = -0.05\ \text{kV/mm}$, $S_{ISA} = 8.75\ \mu\varepsilon$, $u_{ISA} = 61.2\ \text{nm}$, $\gamma = 0.216/\text{mm}$, $\hat{u}(a) = 38.2\ \text{nm}$.

(iii) To calculate the natural resonance frequencies of the first, second, and third modes of antisymmetric vibration, use the textbook Eq. (9.73), i.e.,

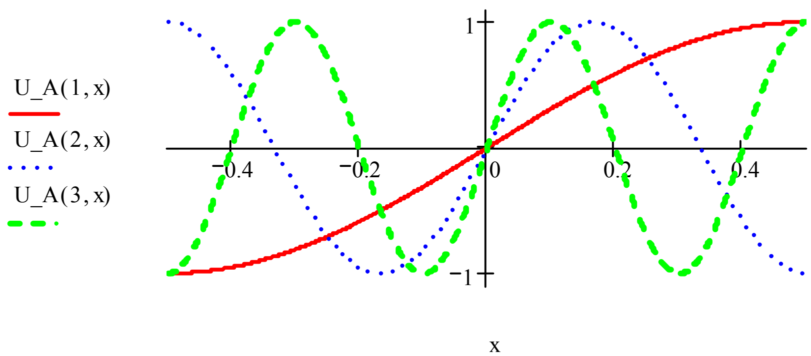
$$f_n^A = (2n-1)\frac{c}{2l}, \quad n = 1,2,3,... \tag{9.73) (11}$$

Upon calculation, one gets $f_1^A = 208$ kHz, $f_2^A = 624$ kHz, $f_3^A = 1041$ kHz.

To plot the first, second, and third modes of antisymmetric vibration, use textbook Eq. (9.76)

$$U_n^A \sin(2n-1)\pi\frac{x}{l}, \quad n = 1,2,3,... \tag{9.76) (12}$$

Upon calculation, one obtains the graphics below.



(iv) To calculate the natural resonance frequencies of the first, second, and third modes of symmetric vibration, we use the textbook Eq. (9.81), i.e.,

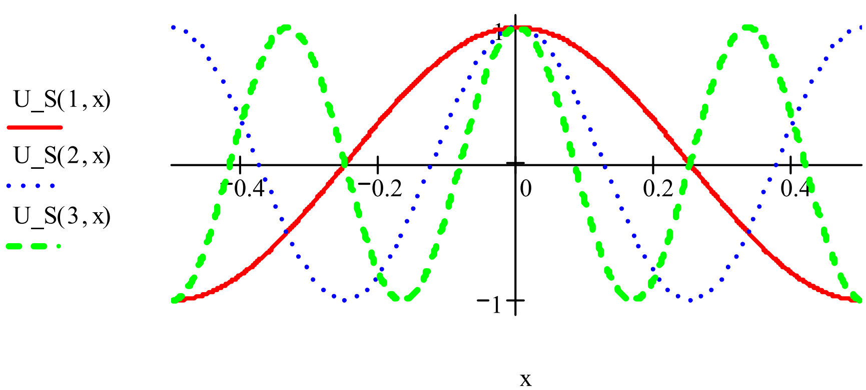
$$f_n^S = 2n\frac{c}{2l}, \quad n = 1,2,3,... \tag{9.81) (13}$$

Upon calculation, one gets $f_1^S = 416$ kHz, $f_2^S = 832$ kHz, $f_3^S = 1249$ kHz.

To plot the first, second, and third modes of antisymmetric vibration, use textbook Eq. (9.84)

$$U_n^S \cos 2n\pi\frac{x}{l}, \quad n = 1,2,3,... \tag{9.84) (14}$$

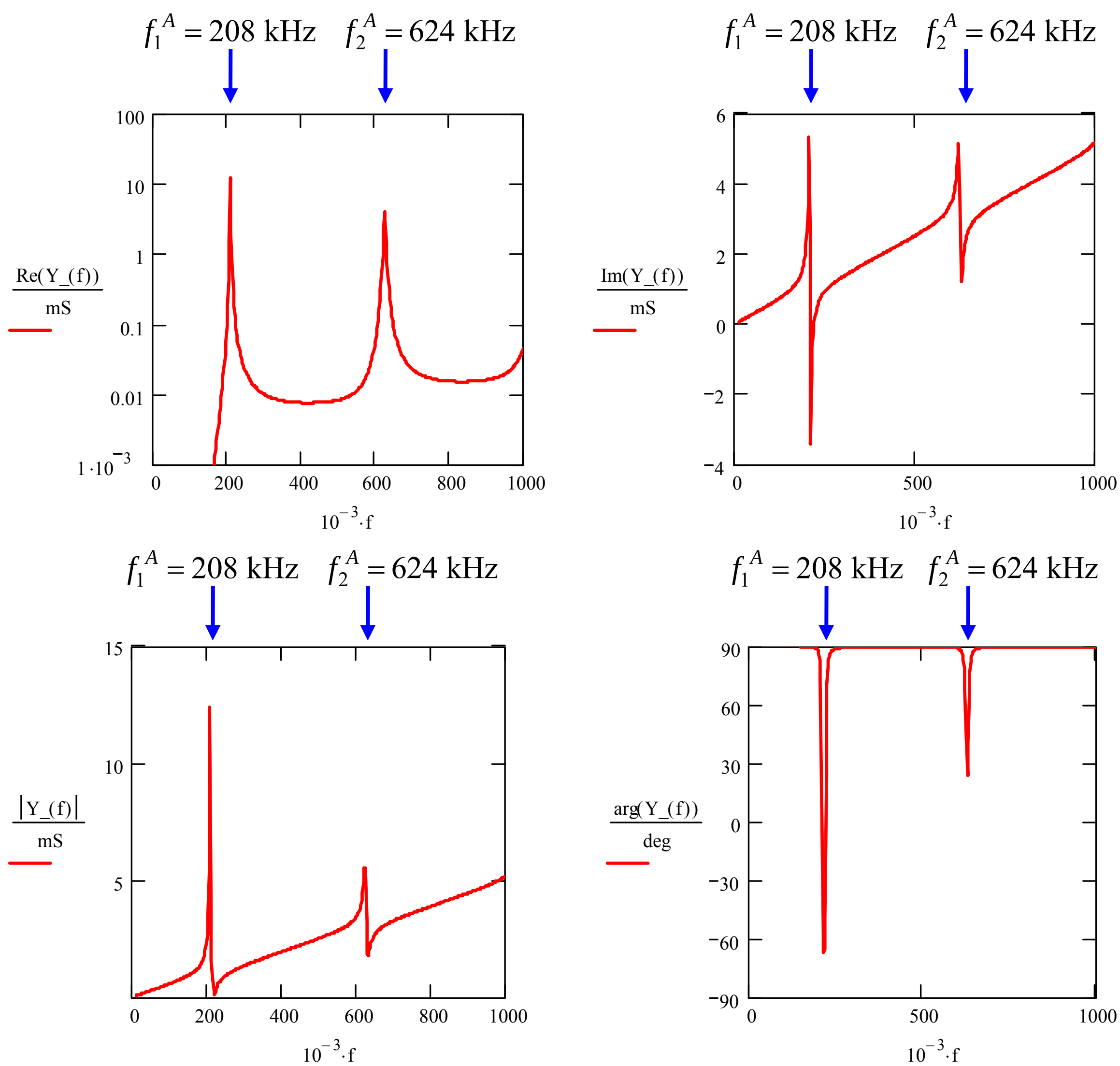
Upon calculation, one obtains the graphics below.



(v) To plot the real part, the imaginary part, the amplitude, and the phase of the PWAS electromechanical admittance we use the first part of the textbook Eq. (9.100), i.e.,

$$\bar{Y} = \mathrm{i}\omega\cdot\bar{C}\left[1-\bar{k}_{31}^2\left(1-\frac{1}{\bar{\varphi}\cot\bar{\varphi}}\right)\right] \qquad (9.100)\ (15)$$

where $\omega = 2\pi f$ and $\bar{\varphi} = \varphi\sqrt{1-\mathrm{i}\eta}$, $\phi = \frac{1}{2}\gamma l$ $\gamma = \omega / c$. Upon calculation, one gets the following plots.



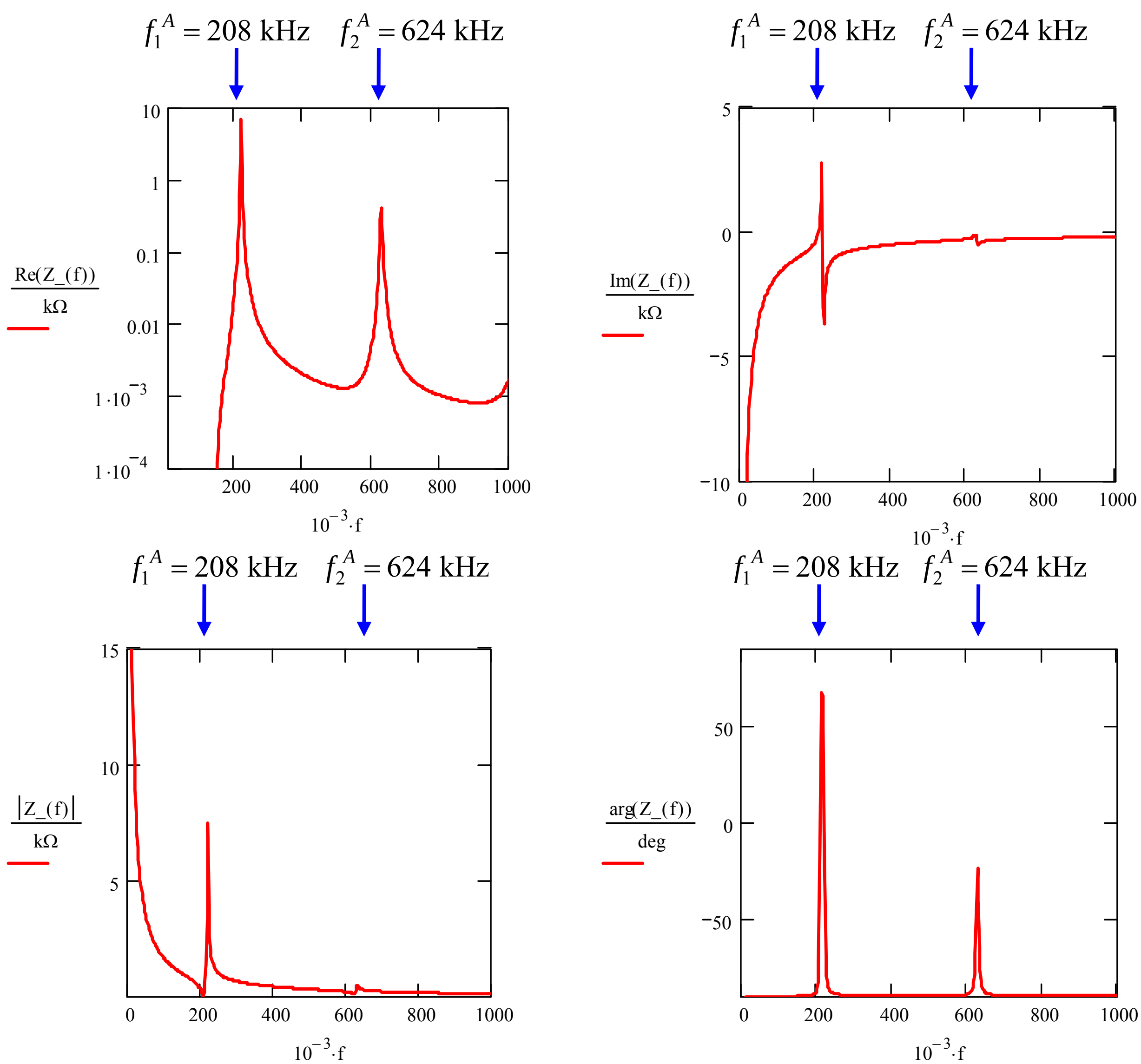
The chart of admittance vs. frequency shows peaks at $f_1^Y = 208$ kHz, $f_2^Y = 624$ kHz. These are antisymmetric frequencies $f_1^A = 208$ kHz, $f_2^A = 624$ kHz, at which the PWAS resonates. As the PWAS passes through these frequencies, its admittance goes through a maximum and takes up a lot of electrical power from the power supply and converts it into mechanical power needed to maintain the PWAS in a state of resonant vibration.

4

(vi) To plot the real part, the imaginary part, the amplitude, and the phase of the PWAS electromechanical admittance we use the second part of the textbook Eq. (9.100), i.e.,

$$\bar{Z} = \frac{1}{\mathrm{i}\omega\cdot\bar{C}}\left[1-\bar{k}_{31}^{2}\left(1-\frac{1}{\bar{\varphi}\cot\bar{\varphi}}\right)\right]^{-1} \qquad (9.100)\ (16)$$

where $\bar{\varphi} = \varphi\sqrt{1-\mathrm{i}\eta}$, $\phi = \frac{1}{2}\gamma l$ $\gamma = \omega / c$, and $\omega = 2\pi f$. Upon calculation, one gets the following plots.



5

Discussion

The peaks of the impedance plots represent anti-resonance frequencies. The textbook Table 9.5 illustrates this point by showing the difference between resonance frequencies (peaks of the admittance real part) and anti-resonance frequencies (peaks of the impedance real part). For example, the first peak of the admittance real part is given at $f_{Y_1} = 208\ \text{kHz}$ whereas the first peak of the impedance real part is given at $f_{Z_1} = 219\ \text{kHz}$. Hence, strictly speaking, the peaks of the impedance real part are not directly related to the resonance frequencies. However, numerically speaking, they are very close to the resonance frequencies, as illustrated by the numerical values given in the textbook Table 9.5. One notices that, as one goes up in the peak number, the numerical difference between these two frequencies diminishes. The textbook Table 9.5 also gives the actual ratio between the various values of f_Z and f_Y. One notices that this ratio shows a 5.7 % difference for the first peak, 0.7% difference for the second peak, 0.2% difference for the third peak, etc. In virtue of these observations, the peaks of the impedance real part, especially from the second peak onwards, can be considered reasonable approximations to the resonance frequencies.

PROBLEM 9.1 SOLUTION

Units $kHz := 1000\cdot Hz$ $pF := 10^{-12}\cdot F$ $nF := 10^{-9}\cdot F$ $\mu m := 10^{-6}\cdot m$ $\mu\varepsilon := 10^{-6}$ $MPa := 10^{6}\cdot Pa$ $GPa := 10^{9}\cdot Pa$

$mS := 10^{-3}\cdot S$ $k\Omega := 10^{3}\cdot \Omega$ $nm := 10^{-9}\cdot m$

Given:

$La := 7\cdot mm$ $ba := 1.65\cdot mm$ $ta := 0.2\cdot mm$ $a := \dfrac{La}{2}$ $a = 3.50mm$

$\varepsilon0 := 8.85\cdot 10^{-12}\cdot\dfrac{F}{m}$ $\rho := 7700\cdot\dfrac{kg}{m^3}$ $s11 := 15.3\cdot 10^{-12}\cdot\dfrac{1}{Pa}$ $d31 := -175\cdot 10^{-12}\cdot\dfrac{m}{V}$ $\nu := 0.35$ $\eta := 1\%$

$\varepsilon33 := 1750\cdot\varepsilon0$ $\varepsilon33 = 15.487\dfrac{nF}{m}$ $s33 := 17.3\cdot 10^{-12}\cdot\dfrac{1}{Pa}$ $d33 := 400\cdot 10^{-12}\cdot\dfrac{m}{V}$ $k33 := 0.72$ $\delta := 1\%$

Solution:

(i) Complex compliance $s11\_ := s11\cdot(1 - i\cdot\eta)$ $10^{12}\cdot s11\_ = 15.3000 - 0.1530i\dfrac{1}{Pa}$

Complex dielectric permittivity $\varepsilon33\_ := \varepsilon33\cdot(1 - i\cdot\delta)$ $\varepsilon33\_ = 15.487 - 0.155i\dfrac{nF}{m}$

Complex coupling coefficient $k31\_ := \sqrt{\dfrac{d31^2}{s11\_\cdot\varepsilon33\_}}$ $k31\_ = 0.3595 + 0.0036i$

Real sound speed in piezo material $c := \sqrt{\dfrac{1}{\rho\cdot s11}}$ $c = 2913\dfrac{m}{s}$

Complex sound speed $c\_ := \sqrt{\dfrac{1}{\rho\cdot s11\_}}$ $c\_ = 2913.3 + 14.6i\dfrac{m}{s}$

$A := ba\cdot La$ $A = 11.5500mm^2$

Real capacitance $C := \varepsilon33\cdot\dfrac{A}{ta}$ $C = 0.894nF$

Complex capacitance $C\_ := \varepsilon33\_\cdot\dfrac{A}{ta}$ $C\_ = 0.8944 - 0.0089i\,nF$

==

(ii) $VV := 10\cdot V$ $E3 := \dfrac{-VV}{ta}$ $E3 = -0.05\dfrac{kV}{mm}$

$f := 100\cdot kHz$ $\omega := 2\cdot\pi\cdot f$ $\gamma := \dfrac{\omega}{c}$ $\gamma = 0.216\dfrac{1}{mm}$

$S\_ISA := d31\cdot E3$ $S\_ISA = 8.75\mu\varepsilon$

$uISA := 2\cdot S\_ISA\cdot a$ $uISA = 61.25nm$

$ua := \dfrac{1}{2}\cdot\dfrac{uISA}{\gamma\cdot a}\cdot\tan(\gamma\cdot a)$ $ua = 38.16nm$

(iii) Antisymmetric vibrations

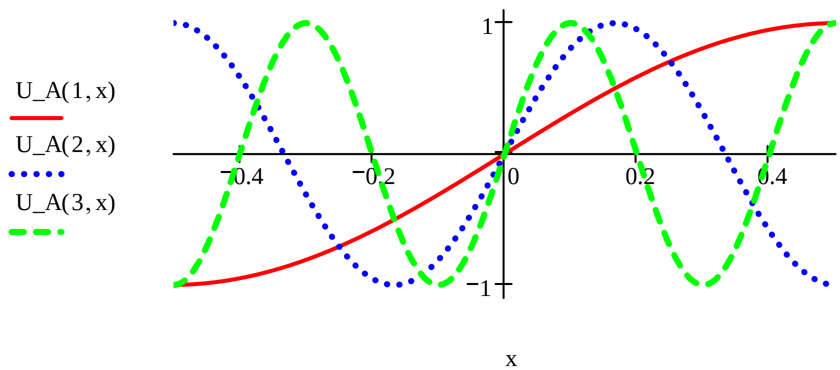
$$fA(n) := (2 \cdot n - 1) \cdot \frac{c}{2La}$$

$fA(1) = 208\,\text{kHz}$

$fA(2) = 624\,\text{kHz}$

$fA(3) = 1041\,\text{kHz}$

Antisymmetric modeshapes $U\_A(n, x) := \sin[(2 \cdot n - 1) \cdot \pi \cdot x]$



(iv) Symmetric vibrations

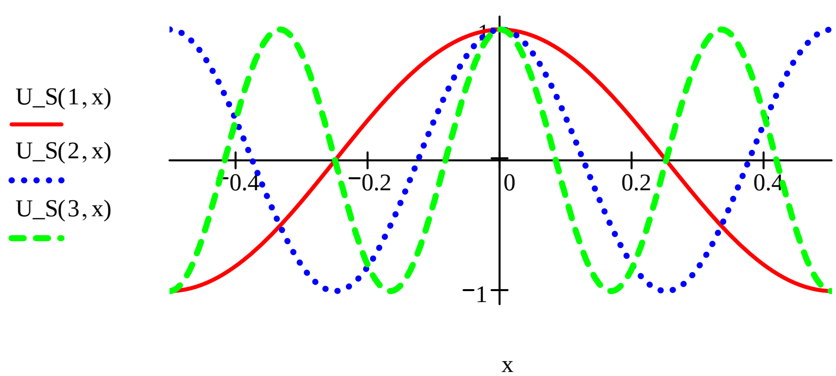
$$fS(n) := (2 \cdot n) \cdot \frac{c}{2La}$$

$fS(1) = 416\,\text{kHz}$

$fS(2) = 832\,\text{kHz}$

$fS(3) = 1249\,\text{kHz}$

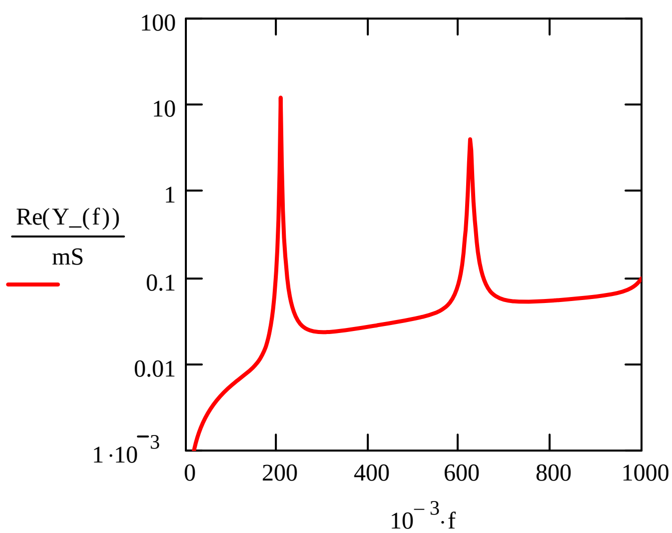
Symmetric modeshapes $U\_S(n, x) := \cos[(2 \cdot n) \cdot \pi \cdot x]$

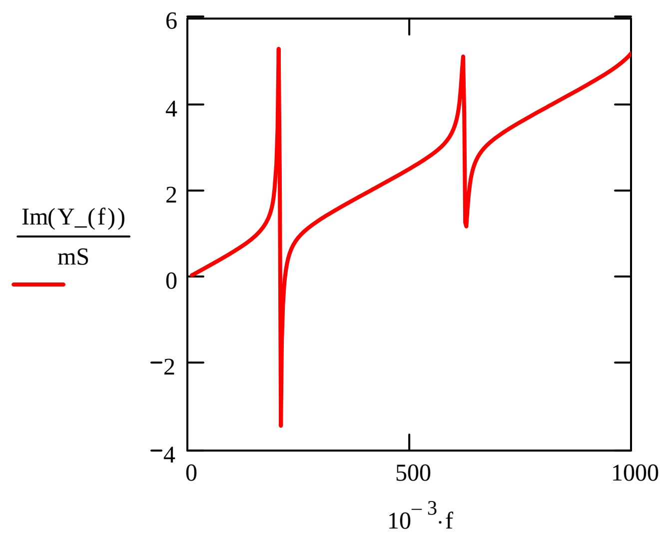


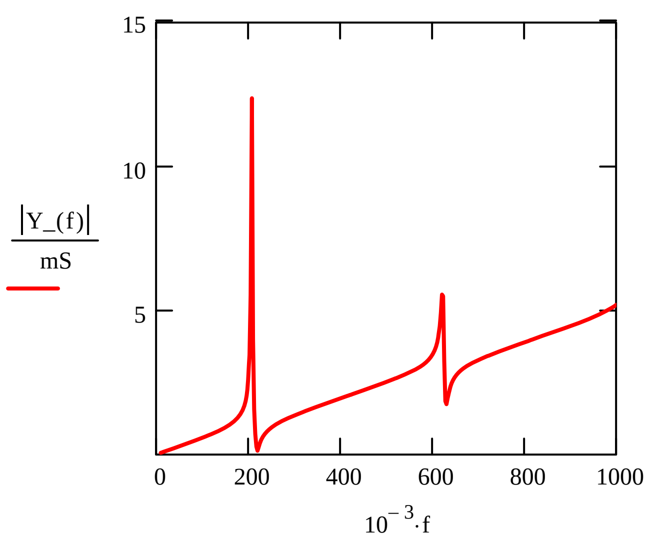
$f\_start := 10 \cdot 10^{3}$ $\quad f\_end := 1000 \cdot 10^{3}$ $\quad N := 401$ $\quad df := \dfrac{f\_end - f\_start}{N - 1}$ $\quad f := f\_start, f\_start + df \,..\, f\_end$

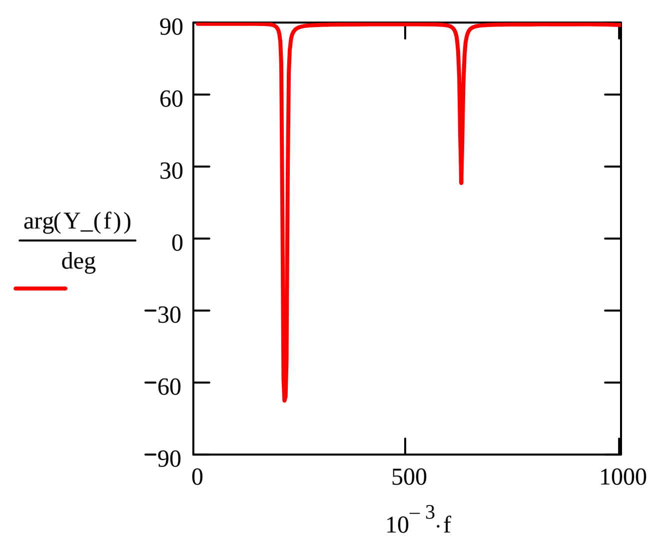
(v) Admittance $\quad \omega(f) := 2 \cdot \pi \cdot f$ $\quad \gamma\_(f) := \dfrac{\omega(f)}{c\_}$ $\quad \phi\_(f) := \dfrac{1}{2} \cdot \gamma\_(f) \cdot La$

$$Y\_(f) := i \cdot \omega(f) \cdot C\_ \cdot \left[1 - k31\_^{2} \cdot \left(1 - \frac{1}{\phi\_(f) \cdot \cot(\phi\_(f))}\right)\right]$$



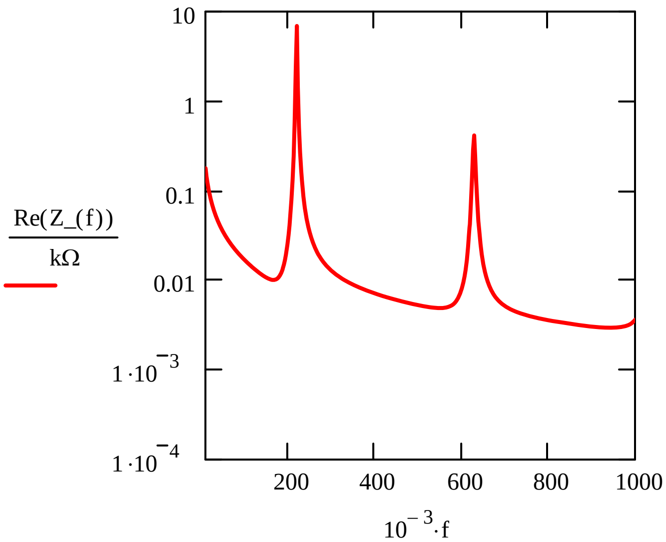


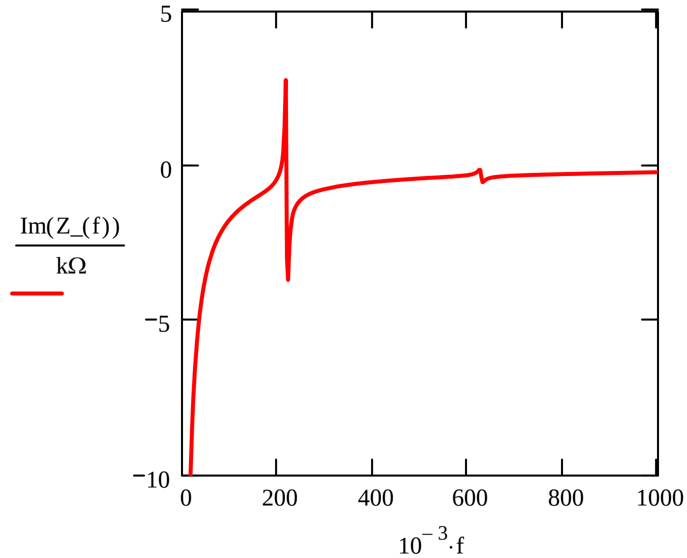


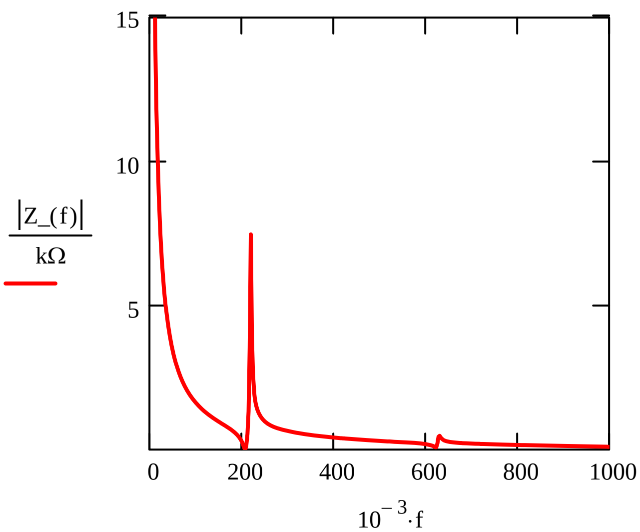


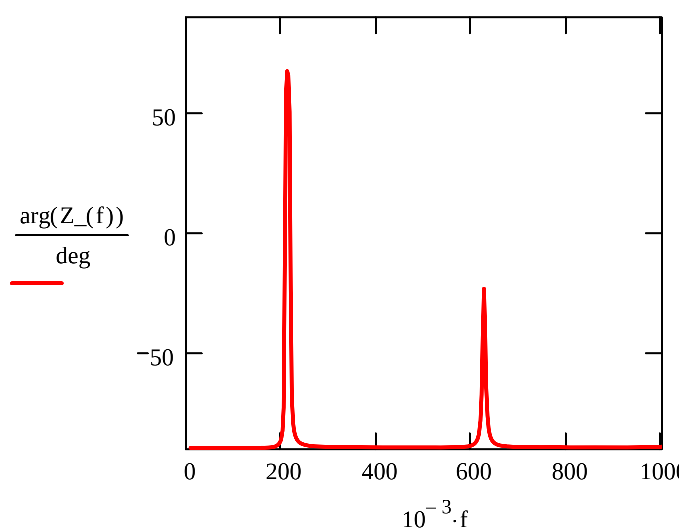
(vi) Impedance

$$Z\_(f) := \left[i \cdot \omega(f) \cdot C\_ \cdot \left[1 - k31\_^{2} \cdot \left(1 - \frac{1}{\phi\_(f) \cdot \cot(\phi\_(f))} \right) \right] \right]^{-1}$$









Problem 2: Consider a circular PWAS transducer with diameter $2a = 7$ mm, thickness $t_a = 0.2$ mm, and piezoelectric material properties as given in Table 7.1. Assume internal damping ratio $\eta = 1\%$ and electric loss factor $\delta = 1\%$.

(i) Calculate the complex compliance, $\bar{s}_{11}^E$, complex dielectric permittivity, $\bar{\varepsilon}_{33}^T$, complex planar coupling coefficient, $\bar{k}_p$, real and complex planar wave speeds in the piezoelectric material, c_p and $\bar{c}_p$, and real and complex electrical capacitances, C and $\bar{C}$.

(ii) For a 10-V, 100-kHz excitation, calculate the induced strain ε_{ISA}, induced displacement u_{ISA}, and edge displacement $u_r(a)$.

(iii) Calculate the natural resonance frequencies in kHz of the first, second, and third modes of vibration, under electric excitation. Plot on the same chart the first, second, and third modes of vibration.

(iv) Plot on separate charts the real part, the imaginary part, the amplitude, and the phase of the circular PWAS electromechanical admittance at 401 equally spaced points in the $10-1000$ kHz frequency range. Identify on this chart the resonance frequencies determined earlier.

(v) Plot on separate charts the real part, the imaginary part, the amplitude, and the phase of the PWAS complex impedance at 401 equally spaced points in the $10-1000$ kHz frequency range. Identify on this chart the frequencies at which the real part of the impedance peaks. Discuss how these frequencies are related, if at all, with the peaks of the PWAS admittance and with the mechanical resonances for in-plane vibration of the PWAS transducer.

Solution

(i) To calculate the complex compliance, $\bar{s}_{11}^E$ and complex dielectric permittivity, $\bar{\varepsilon}_{33}^T$, use Eq. (9.99), i.e.,

$$\bar{s}_{11}^E = s_{11}^E\left(1-i\eta\right), \qquad \bar{\varepsilon}_{33}^T = \varepsilon_{33}^T\left(1-i\delta\right) \tag{1}$$

Upon calculation, we get

$$\bar{s}_{11}^E = \left(15.3 - i0.153\right)\ \mathrm{Pa}^{-1}, \qquad \bar{\varepsilon}_{33}^T = \left(15.487 - i0.155\right)\ \mathrm{nF/m} \tag{2}$$

To calculate the complex planar coupling coefficient $\bar{k}_p$, use the formula in Chapter 9, Eq. (9.145), i.e.,

$$\bar{k}_p^2 = \frac{2}{(1-\nu)}\frac{d_{31}^2}{\bar{s}_{11}^E\,\bar{\varepsilon}_{33}^T} \tag{3}$$

Upon calculation, one gets $\bar{k}_p = 1.106 + i0.0111$.

The planar wave speed in the piezoelectric material, c_p is calculated with the textbook Eq. (9.117), i.e.,

$$c = \sqrt{\frac{1}{\rho s_{11}^E(1-\nu^2)}} \tag{4}$$

The complex wave speed is calculating with a similar formula, only that it uses the complex compliance, i.e.,

$$\bar{c} = \sqrt{\frac{1}{\rho \bar{s}_{11}^E(1-\nu^2)}} \tag{5}$$

Upon calculation, one gets $c = 3110$ m/s, $\bar{c} = \left(3110.1 + i15.5\right)$ m/s.

To calculate the real electrical capacitance, C, use the textbook Eq. (9.47), i.e.,

$$C = \varepsilon_{33}^T\frac{A}{t} \tag{6}$$

where $A = \pi a^2$. To calculate the complex $\bar{C}$ capacitance, use Eq. (6) but with the complex dielectric permittivity, $\bar{\varepsilon}_{33}^T$, i.e.,

$$\bar{C} = \bar{\varepsilon}_{33}^T\frac{A}{t} \tag{7}$$

Upon calculation, one gets $A = 38.5\ \mathrm{mm}^2$, $C = 2.98$ nF, $\bar{C} = \left(2.98 - i0.0298\right)$ nF.

(ii) To calculate the induced strain S_{ISA}, recall Eq. (9.131) from textbook Chapter 9, i.e.,

$$\varepsilon_{ISA} = d_{31}\hat{E}_3 \qquad \text{(piezoelectrically induced strain)} \qquad (9.131)\ (8)$$

where $\hat{E}_3 = -\hat{V}/t_a$. The piezoelectrically induced displacement u_{ISA} is calculated with Eq. (9.132) from textbook Chapter 9, i.e.,

$$u_{ISA} = d_{31}\hat{E}_3 a \qquad \text{(piezoelectrically induced displacement)} \qquad (9.132)\ (9)$$

where $a = l_a/2$. The tip displacement $\hat{u}(a)$ is calculated with Eq. (9.135) from textbook Chapter 9, i.e.,

$$u_r(a) = \frac{(1+\nu)J_1(\gamma a)}{(\gamma a)J_0(\gamma a)-(1-\nu)J_1(\gamma a)}u_{ISA} \qquad \text{(edge displacement)} \qquad (9.135)\ (10)$$

where the wavenumber is given by $\gamma = \omega/c_p$. Upon calculation, one gets $\hat{E}_3 = -0.05\ \text{kV/mm}$, $\varepsilon_{ISA} = 8.75\ \mu\varepsilon$, $u_{ISA} = 30.6\ \text{nm}$, $\gamma = 0.202/\text{mm}$, $\hat{u}(a) = 33.8\ \text{nm}$.

(iii) To calculate the natural resonance frequencies of the first, second, and third modes of vibration, use the textbook Eq. (9.171), i.e.,

$$f_j = \frac{1}{2\pi}\frac{c_p}{a}z_j, \qquad j = 1,2,3,... \qquad (9.171)\ (11)$$

where z_j are given by the textbook Eq. (9.170), i.e.,

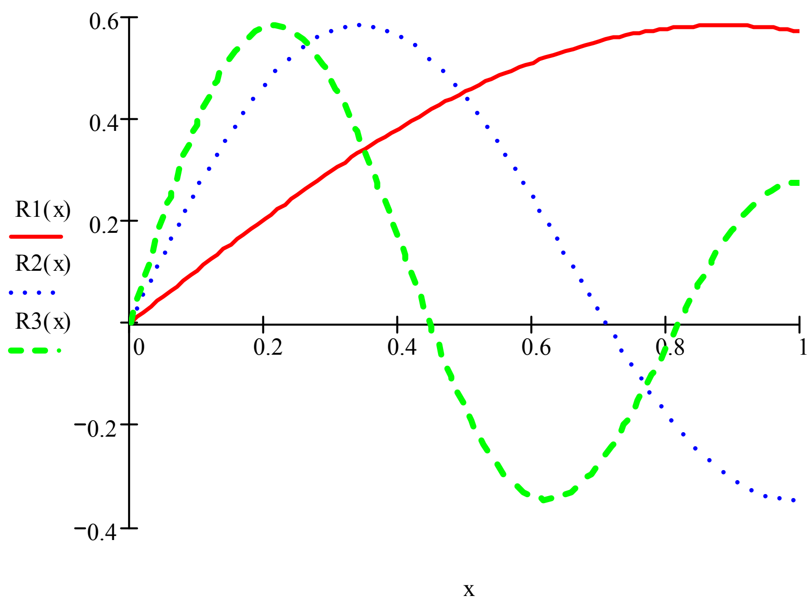
for $\nu = 0.35$ $\quad z =$ 2.079508; 5.398928; 8.577761; 11.736076 … (9.170) (12)

Upon calculation, one gets $f_1 = 294\ \text{kHz}$, $f_2 = 764\ \text{kHz}$, $f_3^A = 1213\ \text{kHz}$.

To plot the first, second, and third modes of vibration, use textbook Eq. (9.173)

$$R_j(r) = A_j\, J_1\left(z_j r/a\right), \qquad j = 1,2,3,... \qquad (9.173)\ (13)$$

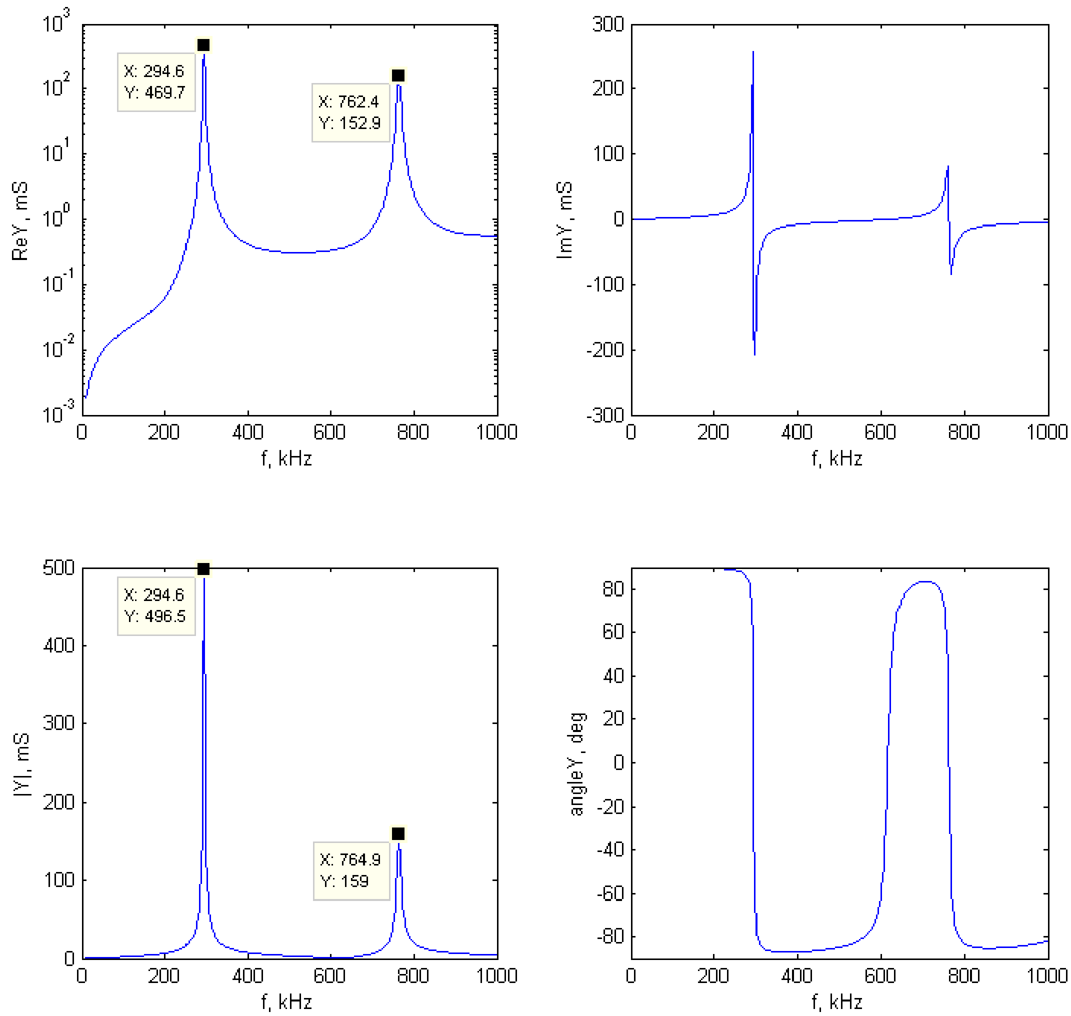
where, for convenience, the amplitudes A_j are taken equal to 1, i.e., $A_j = 1$. Upon calculation, one obtains the graphics below.



(iv) To plot the real part, the imaginary part, the amplitude, and the phase of the circular PWAS electromechanical admittance use the textbook Eq. (9.183), i.e.,

$$\bar{Y}(\omega) = i\omega\bar{C}\left[1-\bar{k}_p^2\left(1-\frac{(1+\nu)J_1(\bar{z})}{\bar{z}J_0(\bar{z})-(1-\nu)J_1(\bar{z})}\right)\right] \qquad (9.183)\ (14)$$

where $\omega = 2\pi f \quad \bar{z} = \dfrac{\omega a}{\bar{c}_p}$. Upon calculation, one gets the following plots.

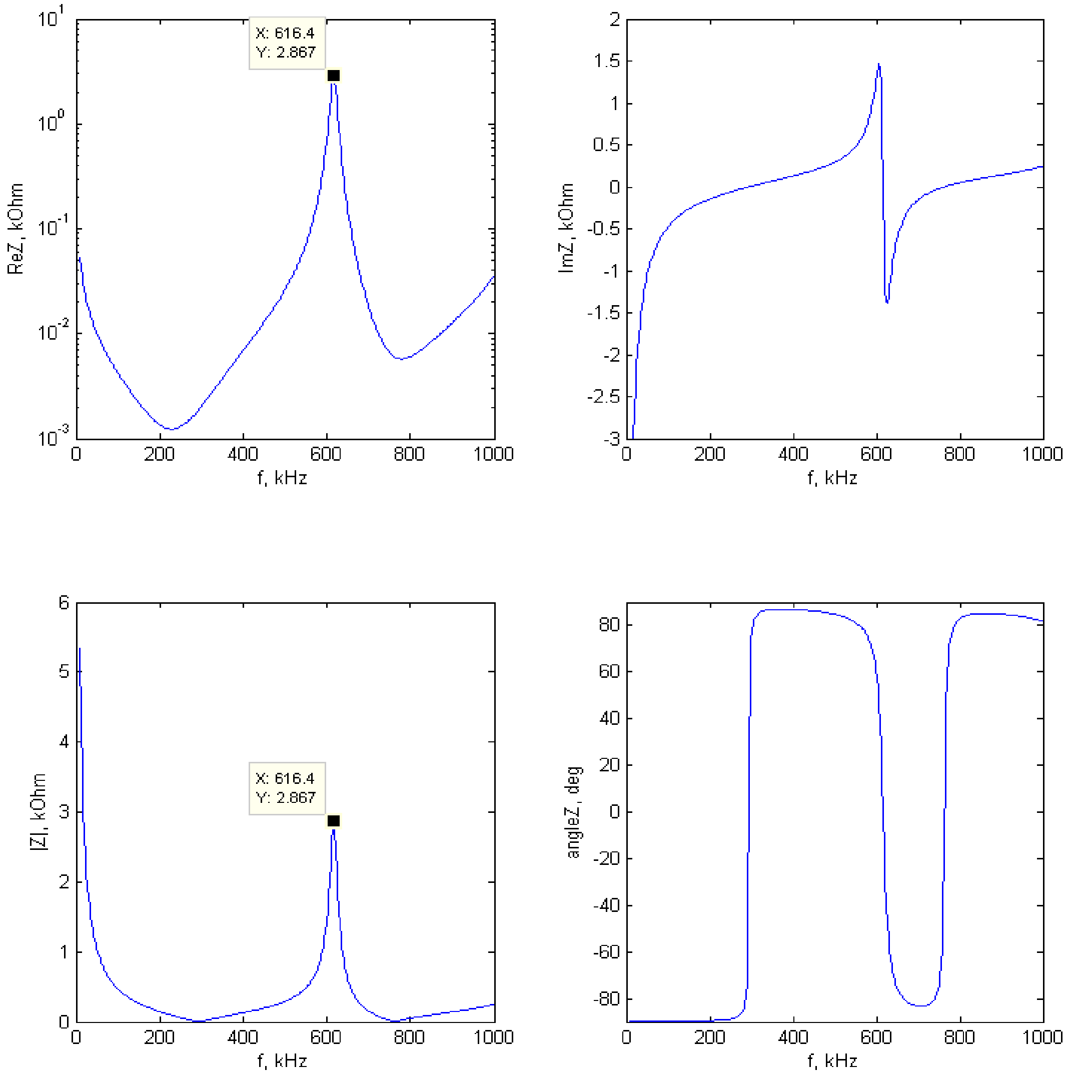


The chart of admittance vs. frequency shows peaks at $f_1^Y = 294.6$ kHz, $f_2^Y = 762.4$ kHz. These are frequencies $f_1 = 294$ kHz, $f_2 = 764$ kHz at which the PWAS resonates (the slight difference in numerical values are due to the limited number of points $N = 401$ taken in the numerical calculations and plot.) As the PWAS passes through these frequencies, its admittance goes through a maximum and takes up a lot of electrical power from the power supply and converts it into mechanical power needed to maintain the PWAS in a state of resonant vibration.

(v) To plot the real part, the imaginary part, the amplitude, and the phase of the circular PWAS electromechanical impedance use the textbook Eq. (9.184), i.e.,

$$\bar{Z}(\omega)=\frac{1}{i\omega\bar{C}}\left[1-\bar{k}_p^2\left(1-\frac{(1+\nu)J_1(\bar{z})}{\bar{z}J_0(\bar{z})-(1-\nu)J_1(\bar{z})}\right)\right]^{-1} \qquad (9.184)\ (15)$$

where $\omega=2\pi f \quad \bar{z}=\dfrac{\omega a}{\bar{c}_p}$. Upon calculation, one gets the following plots.



The frequency at which the impedance peaks is $f_1^Z=616.4$ kHz. This frequency corresponds to maximum impedance, hence minimum admittance. This frequency is not are related to frequencies at which the PWAS admittance peaks because it describes a opposite effect (minimum admittance, not maximum admittance). Neither is this frequency related to the mechanical resonances for in-plane vibration of the PWAS transducer; in fact this frequency is a frequency of anti-resonance, i.e., at which the motion of the PWAS transducer reaches a minimum.

PROBLEM 9.2 SOLUTION

$ORIGIN := 1$

Units $kHz := 1000 \cdot Hz$ $pF := 10^{-12} \cdot F$ $nF := 10^{-9} \cdot F$ $\mu m := 10^{-6} \cdot m$ $\mu\varepsilon := 10^{-6}$ $MPa := 10^{6} \cdot Pa$ $GPa := 10^{9} \cdot Pa$

$mS := 10^{-3} \cdot S$ $k\Omega := 10^{3} \cdot \Omega$ $nm := 10^{-9} \cdot m$

Given:

$a := \frac{7}{2} \cdot mm$ $ta := 0.2 \cdot mm$

$\varepsilon 0 := 8.85 \cdot 10^{-12} \cdot \frac{F}{m}$ $\rho := 7700 \cdot \frac{kg}{m^3}$ $s11 := 15.3 \cdot 10^{-12} \cdot \frac{1}{Pa}$ $d31 := -175 \cdot 10^{-12} \cdot \frac{m}{V}$ $\nu := 0.35$ $\eta := 1\%$

$\varepsilon 33 := 1750 \cdot \varepsilon 0$ $\varepsilon 33 = 15.487 \frac{nF}{m}$ $s33 := 17.3 \cdot 10^{-12} \cdot \frac{1}{Pa}$ $d33 := 400 \cdot 10^{-12} \cdot \frac{m}{V}$ $k33 := 0.72$ $\delta := 10\%$

Solution:

(i) Complex compliance $s11\_ := s11 \cdot (1 - i \cdot \eta)$ $10^{12} \cdot s11\_ = 15.3000 - 0.1530i \frac{1}{Pa}$

Complex dielectric permittivity $\varepsilon 33\_ := \varepsilon 33 \cdot (1 - i \cdot \delta)$ $\varepsilon 33\_ = 15.487 - 1.549i \frac{nF}{m}$

Complex coupling coefficient $kp\_ := \frac{2}{1 - \nu} \cdot \sqrt{\frac{d31^2}{s11\_ \cdot \varepsilon 33\_}}$ $kp\_ = 1.1017 + 0.0605i$

planar wave speed in piezo material $cp := \sqrt{\frac{1}{\rho \cdot s11 \cdot (1 - \nu^2)}}$ $cp = 3110 \frac{m}{s}$

Complex planar wave speed $cp\_ := \sqrt{\frac{1}{\rho \cdot s11\_ \cdot (1 - \nu^2)}}$ $cp\_ = 3110.1 + 15.5i \frac{m}{s}$

$A := \pi \cdot a^2$ $A = 38.5\, mm^2$

Real capacitance $C := \varepsilon 33 \cdot \frac{A}{ta}$ $C = 2.9801\, nF$

Complex capacitance $C\_ := \varepsilon 33\_ \cdot \frac{A}{ta}$ $C\_ = 2.9801 - 0.2980i\, nF$

==

(ii) $VV := 10 \cdot V$ $E3 := \frac{-VV}{ta}$ $E3 = -0.05\,\frac{kV}{mm}$

$\omega := 2 \cdot \pi \cdot 100 \cdot kHz$ $\gamma := \frac{\omega}{cp}$ $\gamma = 0.202\,\frac{1}{mm}$

$S\_ISA := d31 \cdot E3$ $S\_ISA = 8.75\,\mu\varepsilon$

$uISA := S\_ISA \cdot a$

$uISA = 30.62\,nm$

$$ua := \frac{(1+\nu)\cdot J1(\gamma \cdot a)}{\gamma \cdot a \cdot J0(\gamma \cdot a) - (1-\nu)\cdot J1(\gamma \cdot a)} \cdot uISA$$

$ua = 33.82\,nm$

(iii) vibrations $j := 1..3$

$z_1 := 2.079508$ $z_2 := 5.398928$ $z_3 := 8.577761$ $fn_j := \frac{1}{2\cdot\pi}\cdot\frac{cp}{a}\cdot z_j$

fn =

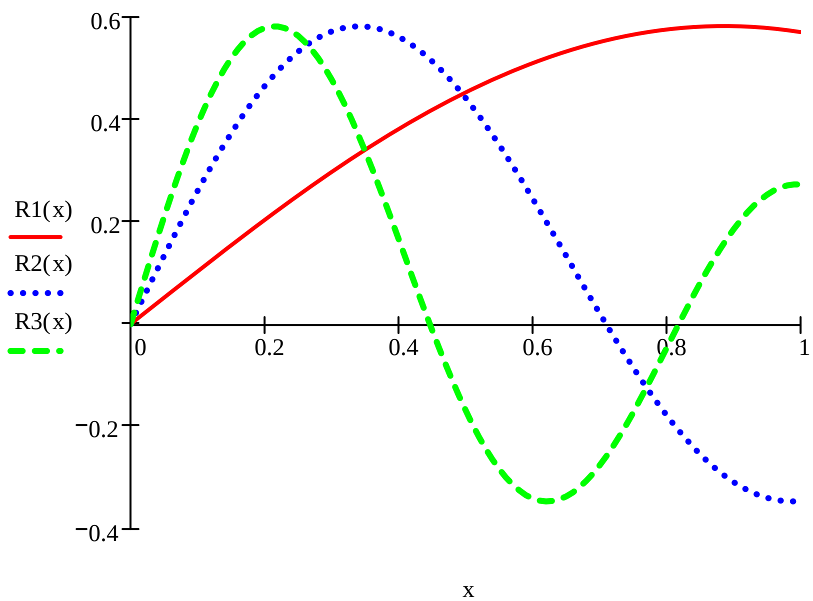
294
764
1213

kHz

modeshapes $R1(x) := J1(z_1\cdot x)$ $x := 0, 0.01..1$

$R2(x) := J1(z_2\cdot x)$

$R3(x) := J1(z_3\cdot x)$



(iv) and (v): the calculation of admittance Y and impedance Z is done in MATLAB because MathCad cannot handle Bessel functions with complex arguments

```
clc
% CIRCULAR RESONATOR ANALYSIS
% UNITS, CONSTANTS definition
kHz=1e3; mS=1e-3; kOhm=1e3; mm=1e-3; eps0=8.85*1e-12;

a=7/2*mm; ta=0.2*mm; A=pi*a^2;
e33=1750*eps0; rho=7700; s11=15.3*1e-12; d31=-175*1e-12;
nu=0.35; eta=1e-2; delta=1e-2;
s11_=s11*(1-1i*eta); e33_=e33*(1-1i*delta);
kp=2/(1-nu)*sqrt(d31^2/(s11*e33_));kp_=2/(1-nu)*sqrt(d31^2/(s11_*e33_));
cp=sqrt(1/(rho*s11*(1-nu^2))); cp_=sqrt(1/(rho*s11_*(1-nu^2)));
C_=e33_*A/ta;
N=401; fs=10*kHz; fe=1000*kHz;  df=(fe-fs)/(N-1); n=1:N;
for n=1:N; f(n)=fs+(n-1)*df; end
w=2*pi*f; Y0=1i*w*C_; z=w*a/cp_;
J0=besselj(0,z); J1=besselj(1,z);
num=(1+nu)*J1; den=z.*J0-(1-nu)*J1;
% admittance calculation and plot
Y=Y0.*(1-kp_^2*(1-num./den));
ReY=real(Y); ImY=imag(Y); absY=abs(Y); angleY=angle(Y)*180/pi;
figure(1)
subplot(2,2,1); semilogy(f/kHz,ReY/mS)
xlabel('f, kHz'); ylabel('ReY, mS');
subplot(2,2,2); plot(f/kHz,ImY/mS)
xlabel('f, kHz'); ylabel('ImY, mS'); % ylim([-0.3 0.3])
subplot(2,2,3); plot(f/kHz,absY/mS)
xlabel('f, kHz'); ylabel('|Y|, mS');
subplot(2,2,4); plot(f/kHz,angleY)
xlabel('f, kHz'); ylabel('angleY, deg'); ylim([-90 90])

%impedance calculation and plot
Z=1./Y;
ReZ=real(Z); ImZ=imag(Z); absZ=abs(Z); angleZ=angle(Z)*180/pi;
figure(2)
subplot(2,2,1); semilogy(f/kHz,ReZ/kOhm)
xlabel('f, kHz'); ylabel('ReZ, kOhm');
subplot(2,2,2); plot(f/kHz,ImZ/kOhm)
xlabel('f, kHz'); ylabel('ImZ, kOhm'); ylim([-3 2])
subplot(2,2,3); plot(f/kHz,absZ/kOhm)
xlabel('f, kHz'); ylabel('|Z|, kOhm');
subplot(2,2,4); plot(f/kHz,angleZ)
xlabel('f, kHz'); ylabel('angleZ, deg'); ylim([-90 90])
```

```
% I tried to code Eq. (5.62c) from Takuro Ikeda; but it gave numerical issues
% kp2=kp_^2; t1=kp2/(1-kp2); r1=z.*J0./J1; den1=r1-(1-nu); t2=(1+nu)./den;
% Y1=Y0.*t1.*t2
% ReY1=real(Y1); ImY1=imag(Y1); absY1=abs(Y1); angleY1=angle(Y1)*180/pi;
% figure(3)
% subplot(2,2,1); semilogy(f/kHz,ReY1/mS)
% xlabel('f, kHz'); ylabel('ReY1, mS');
% subplot(2,2,2); plot(f/kHz,ImY1)
% xlabel('f, kHz'); ylabel('ImY1, mS'); ylim([-0.3 0.3])
% subplot(2,2,3); plot(f/kHz,absY1)
% xlabel('f, kHz'); ylabel('|Y1|, mS');
% subplot(2,2,4); plot(f/kHz,angleY1)
% xlabel('f, kHz'); ylabel('angleY1, deg'); ylim([-90 90])
%
% these are some auxiliary plots I did to understand the numerical behavior
% subplot(2,2,2); plot(f/kHz,num./den); grid on;
% subplot(2,2,3); plot(f/kHz,1-num./den); grid on;
% subplot(2,2,4); plot(f/kHz,num); grid on;
```