

Chapter 3

3.19 Apparent molecular weight of mixture = $\frac{\text{Total mass of mixture}}{\text{Number of kilomoles in mixture}}$
 1 kmol of CO₂ = (12 + 32) = 44 kg
 1 kmol of N₂ = 28 kg

100 cm³ of mixture will contain 95 cm³ of CO₂ and 5 cm³ of N₂

If V is volume occupied by 1 kmol of any gas:

$$\text{Number of kmols of CO}_2 \text{ in } 100 \text{ cm}^3 \text{ of mixture} = \frac{95}{V}$$

and,

$$\text{Number of kmol of N}_2 \text{ in } 100 \text{ cm}^3 \text{ of mixture} = \frac{5}{V}$$

$$\therefore \text{ Apparent molecular wt of mixture (M}_a) = \frac{\left(\frac{95}{V} \times 44\right) + \left(\frac{5}{V} \times 28\right)}{\frac{95}{V} + \frac{5}{V}}$$

$$\underline{\underline{\mathbf{M}_a = 43.2}}$$

$$\text{Gas constant for 1 kg of mixture} = \frac{R^*}{M_a} = \frac{8.3143 \times 10^3}{43.2}$$

$$\underline{\underline{= 192.46 \text{ J deg}^{-1} \text{ kg}^{-1}}}$$

3.20 If water vapor comprises 1% of the volume of the air (i.e., if it accounts for 1% of the molecules in air), what is the virtual temperature correction?

Answer $T_v - T \simeq 1 \text{ K}$ (or 1°C)

Solution: The apparent molecular weight of air that contains 1% by volume (i.e., by number of molecules) of water vapor is, $M_{\text{moist}} = (\text{Apparent Molecular Weight of dry air}) \times (\text{Fraction of Dry Air}) + (\text{Molecular Weight of Water Vapor}) \times (\text{Fraction of Water Vapor})$

$$= (28.97 \times 0.99) + (18 \times 0.01)$$

$$= 28.86$$

For moist air:

$$p = R_d \rho_{\text{moist}} T_v$$

Also, for moist air we could write:

$$p = R_{\text{moist}} \rho_{\text{moist}} T$$

$$\therefore \frac{T_v}{T} = \frac{R_{\text{moist}}}{R_d} = \frac{M_d}{M_{\text{moist}}}$$

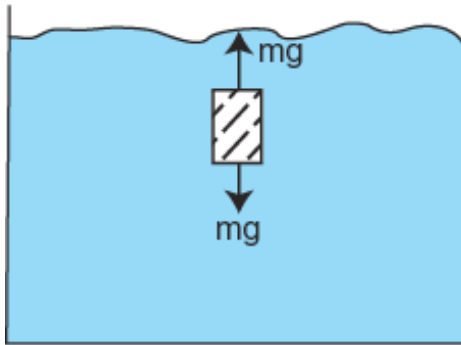
(Since R 's are inversely proportional to molecular weights.)

$$\begin{aligned}\therefore \frac{T_v}{T} &= \frac{28.97}{28.86} \\ &= 1.0038 \\ \therefore T_v &= T = 0.0038 T\end{aligned}$$

If we take $T = 288 \text{ K}$

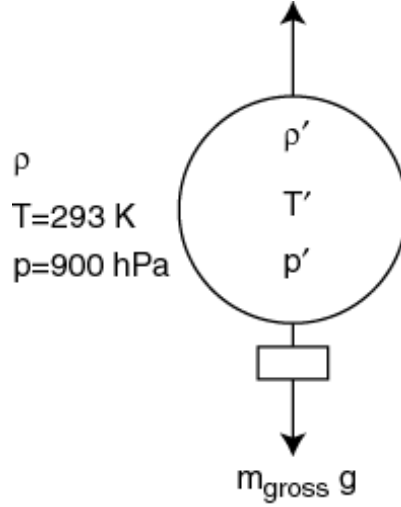
$$\underline{T_v - T = (0.0038) = 288 = 1.09^\circ\text{C} \simeq 1^\circ\text{C}}$$

- 3.21** Archimedes buoyancy principle asserts that an object placed in a fluid (liquid or gas) will be lighter by an amount equal to the weight of the fluid it displaces. Provide a proof of this principle. [Hint: Consider the vertical forces that act on a stationary element of fluid prior to the element being displaced by an object.]



Consider an element of the fluid with mass m . The downward force on this element is mg . Since the element of fluid is stationary (before it is displaced) the net upward force acting on it due to the surrounding fluid must be mg . If this element is displaced by an object, the surrounding fluid will exert the same net upward force on the object as it did on the element (namely, mg). Therefore, the object will be lighter a mass, that is, by the mass of the fluid displaced.

3.22



$\rho(\text{volume of balloon})g - \rho'(\text{volume of balloon})g = (\rho - \rho')(\text{volume of balloon})g.$

For hydrostatic equilibrium of balloon

$$m_{\text{gross}} = (\rho - \rho')(\text{volume of balloon}) \quad (1)$$

Also, $p=R_d\rho T$ and, since $p = p'$,

$$\rho T = \rho' T'$$

or

$$\rho' = \rho \frac{T}{T'} \quad (2)$$

From (1) and (2)

$$m_{\text{gross}} = \rho \left(1 - \frac{T}{T'}\right) (\text{volume of balloon}) \quad (3)$$

Now,

Density of air at 273 K and 1000 hPa = 1.275 kg m⁻³

and density varies at $\frac{p}{T}$

\therefore Density of air at 293 K and 900 hPa is

$$\begin{aligned} \rho &= (1.275) \frac{900}{1000} \frac{273}{293} \text{ kg m}^{-3} \\ &= 1.069 \text{ kg m}^{-3} \end{aligned} \quad (4)$$

From (3) and (4)

$$m_{\text{gross}} = 1.069 \left(1 - \frac{293}{T'}\right) (\text{volume of balloon})$$

or,

$$600 = 1.069 \left(1 - \frac{293}{T'}\right) 3000$$

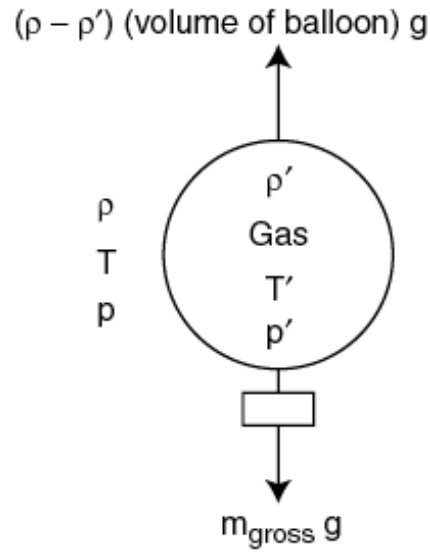
$$\therefore T' = \frac{293}{1 - \frac{600}{3000 \times 1.069}} = \frac{293}{1 - 0.1871}$$

$$= 360.43 \text{ K}$$

or,

$$\underline{T' = 87.43^\circ \text{C}}$$

3.23



For no vertical acceleration of balloon

$$m_{\text{gross}} = (\rho - \rho') (\text{volume of balloon})$$

Since m_{gross} is same for the two balloons,

$$(\rho - \rho') (\text{volume of balloon}) = \text{constant}$$

Therefore,

$$\begin{aligned} & \rho (\text{volume of helium balloon}) - (\text{density of helium}) (\text{volume of helium balloon}) \\ &= \rho (\text{volume of hot air balloon}) - \\ & \quad (\text{density of hot air}) (\text{volume of hot air balloon}) \end{aligned} \quad (1)$$

Also,

$$p = R_{\text{unit mass}} \rho T$$

and,

$$p = p'$$

therefore,

$$\rho \propto \frac{1}{R_{\text{unit mass}} T} \propto \frac{(\text{MW})_{\text{helium}}}{T} \quad (2)$$

Also,

$$T_{\text{helium}} = T_{\text{ambient}} = T$$

From (1) and (2)

$$\begin{aligned} & \frac{(\text{MW})_{\text{air}}}{T} (\text{volume of helium balloon}) - \frac{(\text{MW})_{\text{helium}}}{T} (\text{volume of helium balloon}) \\ &= \frac{(\text{MW})_{\text{air}}}{T} (\text{volume of hot air balloon}) - \frac{(\text{MW})_{\text{air}}}{T_{\text{hot air}}} (\text{volume of hot air balloon}) \end{aligned}$$

or,

$$\begin{aligned} & (\text{volume of balloon}) \frac{(\text{MW})_{\text{air}} - (\text{MW})_{\text{helium}}}{T} \\ &= (\text{volume of hot air balloon}) (\text{MW})_{\text{air}} \left(\frac{1}{T} - \frac{1}{T_{\text{hot air}}} \right) \end{aligned}$$

or,

$$\begin{aligned} & (\text{volume of balloon}) \frac{(\text{MW})_{\text{air}} - (\text{MW})_{\text{helium}}}{(\text{MW})_{\text{air}}} \\ &= (\text{volume of hot air balloon}) \left(\frac{T_{\text{hot air}} - T}{T_{\text{hot air}}} \right) \end{aligned}$$

Therefore,

$$\begin{aligned} & (\text{volume of helium balloon}) \frac{28.97 - 2}{28.97} = (\text{volume of hot air balloon}) \frac{363 - 273}{363} \\ \therefore \text{ volume of hot air balloon} &= (\text{volume of helium balloon}) \left(\frac{26.97}{28.99} \right) \frac{363}{100} \\ &= 1000 \left(\frac{26.97}{28.99} \right) \frac{363}{100} \\ &= \underline{\underline{3377 \text{ m}^3}} \end{aligned}$$

3.24 From (3.29)

$$\begin{aligned} p_1 &= p_2 \exp \frac{(Z_2 - Z_1)}{\bar{H}} \\ &\simeq p_2 \left(1 + \frac{(Z_2 - Z_1)}{\bar{H}} \right) \end{aligned}$$

$$\begin{aligned}\therefore \bar{T}_v &= 270 \frac{\ln\left(\frac{101}{20}\right)}{\ln\left(\frac{94}{20}\right)} \\ &= 282.38^\circ\text{K}\end{aligned}$$

$$\begin{aligned}\therefore \text{Average temperature difference between center} \\ \text{of the hurricane and its surroundings} \\ &= (282.38 - 270) \text{ K} \\ &\underline{\simeq 12 \text{ K} = 12^\circ\text{C}}\end{aligned}$$

$$\mathbf{3.27} \quad \left. \begin{array}{l} z = ? \quad 500 \text{ hPa} \\ T_v = 0^\circ\text{C} \quad 1000 \text{ hPa} \end{array} \right\}$$

From the hypsometric equation:

For 1000 – 500 hPa:

$$z(500 \text{ hPa}) - z(1000 \text{ hPa}) = \frac{R_d \bar{T}_v}{g} \ln\left(\frac{p_{1000}}{p_{500}}\right) = 29.26 \times \bar{T}_v (\ln 2)$$

$$\begin{aligned}\therefore z(500 \text{ hPa}) - z(1000 \text{ hPa}) &= 29.26 \times 273 \times 0.693 \\ &= 5536.8 \text{ m}\end{aligned}$$

For 1020 – 1000 hPa layer:

$$\begin{aligned}z(1000 \text{ hPa}) - z(1020 \text{ hPa}) &= \frac{R_d \bar{T}_v}{g} \ln\left(\frac{p_{1020}}{p_{1000}}\right) = 29.26 \times 288 \times \ln \frac{1020}{1000} \\ &= 29.26 \times 288 \times 0.02 \\ &= 167 \text{ m}\end{aligned}$$

(Note: We are not given \bar{T}_v between 1000 and 1020 hPa, however, because the height difference between these two pressure levels is small, \bar{T}_v will not differ greatly from the temperature at 1020 hPa, namely, 15°C or 288 K.)

Now,

$$\begin{aligned}z(1000 \text{ hPa}) - z(\text{sea level}) &= (167 - 50) \text{ m} \\ &= 117 \text{ m}\end{aligned}$$

Therefore,

$$\begin{aligned}\text{Height of 500 hPa level above sea level} &= 5537 + 117 = 5654 \text{ m} \\ &= \underline{\mathbf{5.654 \text{ m}}}\end{aligned}$$

(Note: Answer given in 1st Ed. of book is based on taking $R_d/g = 29.3$ instead of the more accurate value of 29.26 used above.)

3.28 $\frac{\text{mass} = m}{5 \times 10^6 \text{ J}}$ 500 hPa

$$\frac{\text{Area} = 1 \text{ m}^2}{1000 \text{ hPa}}$$

$$\begin{aligned} \text{Difference in pressure between the two layers} &= 500 \text{ hPa} \\ &= 5 \times 10^4 \text{ Pa} \end{aligned}$$

Therefore,

$$\begin{aligned} mg &= 5 \times 10^4 \\ m &= \frac{5 \times 10^4}{9.81} = 0.51 \times 10^4 \text{ kg} \end{aligned}$$

If ΔT is change in temperature of layer due to heating:

$$\begin{aligned} (0.51 \times 10^4) c_p \Delta T &= 5 \times 10^6 \\ \therefore \Delta T &= \frac{5 \times 10^6}{(0.51 \times 10^4)(1004)} \end{aligned}$$

$$\therefore \underline{\Delta T = 0.976^\circ\text{C} = \mathbf{0.98^\circ\text{C}}}$$

Also,

$$\begin{aligned} Z_{500} - Z_{100} &= \frac{R_d \bar{T}_v}{g_o} \ln(p_1/p_2) \\ \Delta(Z_{500} - Z_{100}) &= \frac{R_d}{g_o} \ln\left(\frac{1000}{500}\right) \Delta \bar{T}_v \\ &= \frac{287}{9.81} (\ln 2)(0.976) \end{aligned}$$

$$\therefore \underline{\Delta(Z_{500} - Z_{100}) = \mathbf{19.8 \text{ m} = 20 \text{ m}}}$$

3.29 From eqn. (3.29) in text:

$$\begin{aligned} Z_2 - Z_1 = \Delta Z &= \frac{R_d \bar{T}_v}{g_o} \ln \frac{p_1}{p_2} \\ \therefore \frac{\partial(\Delta Z)}{\partial \bar{T}_v} &= \frac{R_d}{g_o} \ln \frac{p_1}{p_2} \end{aligned}$$

or,

$$\begin{aligned}
\partial \bar{T}_v &= \frac{\partial(\Delta Z)}{\frac{R_d}{g_o} \ln \frac{p_1}{p_2}} \\
&= \frac{180}{\frac{287}{9.81} \ln \frac{1000}{500}} \\
&= \frac{180}{20.279} \\
&= \underline{\underline{8.9^\circ\text{C}}}
\end{aligned}$$

(Note: Since lapse rate does not change, change in surface temperature is equal to change in mean value of \bar{T}_v .)

- 3.30** Derive a relationship for the height of a given pressure surface (p) in terms of the pressure p_o and temperature T_o at sea level assuming that the temperature decreases uniformly with height at a rate Γ K km⁻¹.

$$\text{Answer } z = \frac{T_o}{T} \left[1 - \left(\frac{p}{p_o} \right)^{\frac{R\Gamma}{g}} \right]$$

Solution: Let the height of the pressure surface be z ; then its temperature T is given by

$$T = T_o - \Gamma z \quad (1)$$

combining the hydrostatic equation (3.17) with the ideal gas equation (3.2) yields

$$\frac{dp}{p} = -\frac{g}{RT} dz \quad (2)$$

From (1) and (2)

$$\frac{dp}{p} = -\frac{g}{R(T_o - \Gamma z)} dz$$

Integrating this equation between pressure levels p_o and p and corresponding heights 0 and z and neglecting the variation of g with z , we obtain

$$\int_{p_o}^p \frac{dp}{p} = -\frac{g}{R} \int_0^z \frac{dz}{(T_o - \Gamma z)}$$

or

$$\ln \frac{p}{p_o} = \frac{g}{R\Gamma} \ln \left(\frac{T_o - \Gamma z}{T_o} \right)$$

Therefore,

$$z = \frac{T_o}{\Gamma} \left[1 - \left(\frac{p}{p_o} \right)^{R\Gamma/g} \right] \quad (3)$$

This equation forms the basis for the calibration of aircraft altimeters. An altimeter is simply an aneroid barometer that measure ambient air pressure p . However, the scale of the altimeter is expressed at the height z of the aircraft, where z is related to p by (3) with values of T_o , p_o and Γ appropriate to the U.S. Standard Atmosphere, namely, $T_o = 288$ K, $p_o = 1013.25$ hPa, and $\Gamma = 6.50$ K km⁻¹.

3.31 Consider a hiker who gets to 1 km without any change in pressure due to synoptic conditions. Then, altimeter reads correct height of 1 km. Now, if pressure falls by 8 hPa, the altimeter (which is calibrated assuming that 1 hPa decrease in pressure corresponds to 8 m increase in height) will read 1000 m + (8 × 8) m = **1064 m**.

3.32 Work done by a system in changing its volume from v_1 to v_2 is given by:

$$\text{work} = \int_{v_1}^{v_2} p dv \quad (1)$$

The gas equation for 2 kg of dry air can be written as

$$pv = (2R_d)T \quad (2)$$

From (1) and (2):

$$\text{Work done on a system} = - \int_{v_1}^{v_2} \frac{2R_d T}{v} dv$$

For an isothermal transformation:

$$\text{Work done on a system} = -2R_d T \ln \frac{v_2}{v_1}$$

Since $v_2 = \frac{v_1}{10}$ and $T = 288$ K,

$$\begin{aligned} \text{Work done on a system} &= -2(287)288 \ln \frac{1}{10} \\ &= 2(287)(288)(2.3026) \\ &= 3.806 \times 10^5 \text{ J} \simeq \mathbf{3.8 \times 10^5 \text{ J}} \end{aligned}$$

3.33 (a)

$$\begin{aligned} dq &= du + dw \quad (\text{1st Law for unit mass}) \\ &= du + pd\alpha \\ &= \left[\left(\frac{\partial u}{\partial T} \right)_{\alpha} dT + \left(\frac{\partial u}{\partial \alpha} \right)_{T} d\alpha \right] + pd\alpha \end{aligned}$$

$$dq = c_v dT + p d\alpha \quad (i)$$

for an ideal gas, since $\left(\frac{\partial u}{\partial \alpha}\right)_T = 0$.

Also, $p\alpha = RT$

$$\therefore p d\alpha + \alpha dp = R dT \quad (ii)$$

From (i) and (ii) for an adiabatic transformation ($dq = 0$)

$$c_v(p d\alpha + \alpha dp) + R p d\alpha = 0$$

Also for an ideal gas:

$$R = c_p - c_v$$

$$\therefore c_v p d\alpha + c_v \alpha dp - (c_p - c_v) p d\alpha = 0$$

$$\text{or} \quad \frac{dp}{p} + \gamma \frac{d\alpha}{\alpha} = 0 \quad \text{where } c_p/c_v = \gamma$$

Integrating,

$$\ln p + \gamma \ln \alpha = \text{constant}$$

Hence,

$$\underline{p\alpha^\gamma = \text{constant}}$$

or, for volume v ,

$$\underline{pv^\gamma = \text{constant}}$$

(b) For the isothermal transformation:

$$\begin{aligned} p_1 v_1 &= p_2 v_2 \\ \therefore (1000)(7.5) &= p_2(2.5) \\ \therefore p_2 &= 3000 \text{ hPa} \end{aligned}$$

For the adiabatic transformation

$$\begin{cases} p_2 v_2^\gamma = p_3 v_3^\gamma \\ \frac{p_2 v_2}{T_2} = \frac{p_3 v_3}{T_3} \end{cases}$$

$$\begin{aligned} \therefore T_3 &= T_2 \left(\frac{v_2}{v_3}\right)^{\gamma-1} \\ &= 290 \left(\frac{2.5}{7.5}\right)^{\gamma-1} = 290 \left(\frac{2.5}{7.5}\right)^{0.4} \\ \therefore T_3 &= \mathbf{186.8^\circ K \text{ or } -86.2^\circ C} \end{aligned}$$

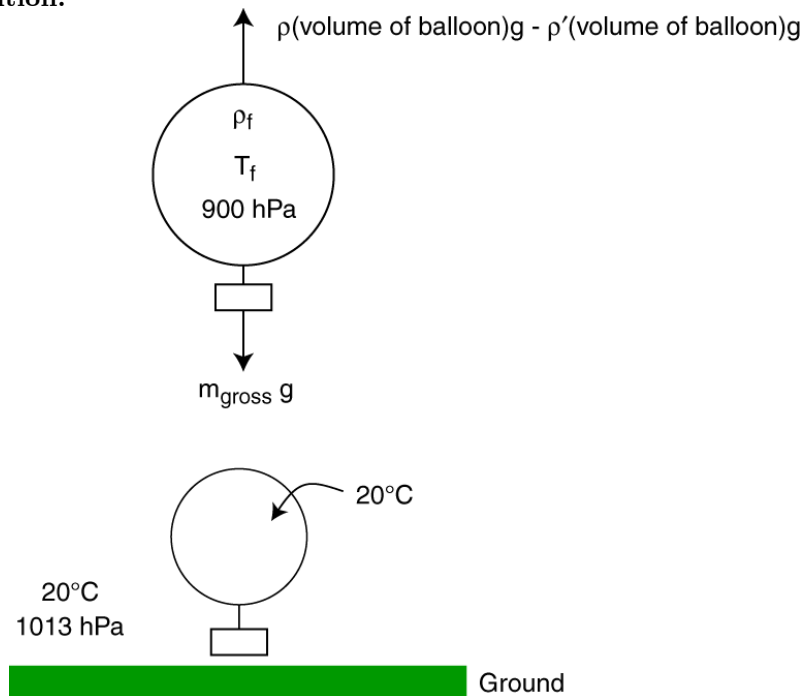
From

$$\begin{aligned}
 p_2 v_2^\gamma &= p_3 v_3^\gamma \\
 p_3 &= p_2 \left(\frac{v_2}{v_3} \right)^\gamma \\
 &= 3000 \left(\frac{2.5}{7.5} \right)^\gamma \\
 &= 3000 (0.333)^{1.4} \\
 \mathbf{p_3} &= \mathbf{643 \text{ hPa}}
 \end{aligned}$$

3.34 If the balloon in Exercise 3.22 is filled with air at the ambient temperature of 20°C at ground level where the pressure is 1013 hPa, estimate how much fuel will need to be burned to lift the balloon to its cruising altitude of 900 hPa. Assume that the balloon is perfectly insulated, and that the fuel releases energy at a rate of $5 \times 10^7 \text{ J kg}^{-1}$.

Answer 5.23 kg

Solution:



For the air in balloon at ground level:

$$p_i = 1013 \text{ hPa}$$

$$T_i = 20^\circ\text{C} = 293 \text{ K}$$

For the air in balloon at 900 hPa

$$p_f = 900 \text{ hPa}$$

$$T_f = 87.43^\circ\text{C} \text{ (from solution to Exercise 3.22)}$$

$$= 360.43 \text{ K}$$

Suppose the air in the balloon goes from its initial to its final state in two steps:

- (1) Heat is added at constant pressure at ground level until the temperature reaches T'_f .
- (2) Air in the balloon expands adiabatically as p drops from 1013 to 900 hPa, while T drops from T'_f to T_f .

For step (2):

$$pv^\gamma = \text{constant and } \frac{pv}{T} = \text{constant}$$

$$\therefore p^{1-\gamma} T^\gamma = \text{constant}$$

$$\therefore p_i^{1-\gamma} (T'_f)^\gamma = p_f^{1-\gamma} (T_f)^\gamma$$

$$\therefore T'_f = \left(\frac{p_f}{p_i}\right)^{\frac{1-\gamma}{\gamma}} T_f$$

$$= \left(\frac{p_f}{p_i}\right)^{\frac{1-1.4}{1.4}} T_f$$

$$= \left(\frac{p_i}{p_f}\right)^{0.286} (360.43)$$

$$= \left(\frac{1013}{900}\right)^{0.286} (360.43)$$

$$= (1.034) (360.43) = 372.83 \text{ K}$$

Heat required for heating of air = (mass of air) $(c_p) (\Delta T)$ Joules

$$\therefore \left(\begin{array}{l} \text{mass of fuel} \\ \text{needed in kg} \end{array}\right) (5 \times 7 \text{ J kg}^{-1}) = (\text{mass of air}) (c_p) \Delta T$$

$$\begin{aligned}
\text{Mass of air in balloon (in kg)} &= \left(\begin{array}{c} \text{volume of air} \\ \text{in balloon} \\ \text{at 900 hPa} \end{array} \right) \left(\begin{array}{c} \text{density of air} \\ \text{at 900 hPa} \\ \text{and 360.43 K} \end{array} \right) \\
&= (3000 \text{ m}^3) \left(\frac{p}{R_d T} \right) \\
&= (3000 \text{ m}^3) \left(\frac{900 \times 10^2}{287 \times 360.43} \right) \\
&= 2610.12 \text{ kg}
\end{aligned}$$

$$\begin{aligned}
\therefore \text{Mass of fuel needed (in kg)} &= (2610.12 \text{ kg}) (1004 \text{ J K}^{-1} \text{ kg}^{-1}) \frac{(372.83 - 273 \text{ K})}{5 \times 10^7 \text{ J kg}^{-1}} \\
&= \frac{(2610.12)(1004)(99.83)}{5 \times 10^7} \\
&= 52322110.5 \times 10^{-7} \text{ kg} \\
&= \underline{\underline{5.23 \text{ kg}}}
\end{aligned}$$

3.35

$$\begin{aligned}
H_2 - H_1 &= 3(L_f) + 3 \int_{273}^{373} c_{pw} dT \\
&= 3(3.34) \times 10^5 + 3 \int_{273}^{373} (4183.9 + 0.125 T) dT \\
&= (10.02 \times 10^5) + 3 \left[4183.9 T + \frac{0.125 T^2}{2} \right]_{273}^{373} \\
&= (10.02 \times 10^5) + 3 \left[(4183.9 \times 313) + \left(\frac{0.125 (313)^2}{2} \right) \right. \\
&\quad \left. - (4183.9 \times 273) - \left(\frac{0.125 (273)^2}{2} \right) \right] \\
&= (10.02 \times 10^5) + 3(168821) \\
&= (10.02 \times 10^5) + (5.06 \times 10^5) \\
&= \underline{\underline{15.1 \times 10^6 \text{ J}}}
\end{aligned}$$

3.36 We have to prove that for an adiabatic change

$$T_1 \left(\frac{p_o}{p_1} \right)^{R/c_p} = T_2 \left(\frac{p_o}{p_2} \right)^{R/c_p}$$

That is, to prove

$$\frac{T_1}{p_1^{R/c_p}} = \frac{T_2}{p_2^{R/c_p}} \quad (1)$$

We have for an adiabatic transformation

$$\begin{cases} p_1 v_1^\gamma = p_2 v_2^\gamma \\ \frac{p_1 v_1}{T_1} = \frac{p_2 v_2}{T_2} \end{cases}$$

Hence,

$$\begin{aligned} \frac{v_1}{v_2} &= \frac{p_2}{T_2} \frac{T_1}{p_1} \\ \therefore \frac{p_1}{p_2} &= \left(\frac{T_2 p_1}{p_2 T_1} \right)^\gamma \end{aligned}$$

or

$$\left(\frac{p_1}{p_2} \right)^{1-\gamma} = \left(\frac{T_2}{T_1} \right)^\gamma$$

or,

$$\left(\frac{p_1}{p_2} \right)^{\frac{1-\gamma}{\gamma}} = \frac{T_2}{T_1}$$

But,

$$\frac{1-\gamma}{\gamma} = \frac{1-c_p/c_v}{c_p/c_v} = \frac{c_v-c_p}{c_p} = -\frac{R}{c_p}$$

Hence,

$$\left(\frac{p_1}{p_2} \right)^{-R/c_p} = \frac{T_2}{T_1}$$

or

$$\frac{\mathbf{T}_1}{\mathbf{P}_1^{R/c_p}} = \frac{\mathbf{T}_2}{\mathbf{P}_2^{R/c_p}}, \text{ which is}$$

From (3.54)

$$\theta = T \left(\frac{p_o}{p} \right)^{R/c_p} \quad (2)$$

Hence, for T_1 and p_1 ,

$$\theta_1 = T_1 \left(\frac{p_o}{p_1} \right)^{R/c_p} \quad (3)$$

and for T_2 and p_2 ,

$$\theta_2 = T_2 \left(\frac{p_o}{p_2} \right)^{R/c_p} \quad (4)$$

From (1), (3) and (4),

$$\underline{\theta_1 = \theta_2}$$

3.37 *Accurate answer (by calculation):*

For an adiabatic transformation

$$pv^\gamma = \text{constant}$$

But

$$\frac{pv}{T} = \text{constant}$$

Therefore,

$$p \left(\frac{T}{p} \right)^\gamma = \text{constant}$$

$$\therefore p^{1-\gamma} T^\gamma = \text{constant} \quad (1)$$

$$\begin{aligned} p_1 &= 200 \text{ hPa} \\ T_1 &= -60^\circ\text{C} = 213 \text{ K} \\ p_2 &= 1000 \text{ hPa} \\ T_2 &= \theta \end{aligned}$$

\therefore From (1),

$$\begin{aligned} \theta &= \left(\frac{200}{1000} \right)^{\frac{1-\gamma}{\gamma}} T_1 \\ &= \left(\frac{1}{5} \right)^{\frac{1-\gamma}{\gamma}} 213 = \left(\frac{1}{5} \right)^{\frac{1-c_p/c_v}{c_p/c_v}} 213 = \left(\frac{1}{5} \right)^{R_d/c_p} 213 \quad (2) \end{aligned}$$

From c_p and c_v values given in 1st Ed. of Wallace & Hobbs we get $R_d/c_p = 0.286$, therefore

$$\begin{aligned} \theta &= (0.2)^{-0.286} (213) \\ &= 337.46 \text{ K} \\ &= \underline{\underline{64.5^\circ\text{C}}} \end{aligned}$$

The T versus $\ln p$ chart uses $c_p/c_v = 1.40449$ or $R_d/c_p = 0.288$. If this value is used in (2), we get $\theta = 65.59 \simeq \mathbf{66^\circ\text{C}}$ (i.e., same as from T versus $\ln p$ chart on the book web site)

3.38 (a) Macroscopic kinetic energy of an air parcel with mass m is

$$\begin{aligned}
 K_m &= \frac{1}{2} m c_s^2 = \frac{1}{2} m \left[(\gamma R_d T)^{1/2} \right]^2 \\
 &= \frac{1}{2} m \frac{c_p}{c_v} (c_p - c_v) T \\
 &= \frac{1}{2} m c_p T \left(\frac{c_p}{c_v} - 1 \right) \\
 &= \frac{1}{2} H (\gamma - 1) \\
 \mathbf{K_m} &= \frac{1}{2} \mathbf{H (0.40)} = \mathbf{0.20 H}
 \end{aligned}$$

(b)

$$\begin{aligned}
 K_m &= 0.20 H \\
 \therefore dK_m &= 0.20 dH \\
 &= 0.20(m c_p dT)
 \end{aligned} \tag{1}$$

$$\begin{aligned}
 \text{Also, } K_m &= \frac{1}{2} m c_s^2 \\
 \therefore dK_m &= m c_s dc_s
 \end{aligned} \tag{2}$$

From (1) and (2),

$$\begin{aligned}
 m c_s dc_s &= 0.20 m c_p dT \\
 \therefore \frac{dc_s}{c_s} &= \frac{0.20 c_p dT}{c_s^2} \\
 \therefore \frac{dc_s}{c_s dT} &= \frac{0.20 c_p}{c_s^2} = \frac{0.20 c_p}{\gamma R_d T} \\
 \frac{dc_s}{c_s dT} &= \frac{0.20 c_v}{R_d T}
 \end{aligned}$$

3.39

$$\begin{aligned}
 c_{\text{person}} M_{\text{person}} \Delta T &= m_{\text{evap}} L_{\text{evap}} \\
 \therefore \frac{m_{\text{evap}}}{M_{\text{person}}} &= c_{\text{person}} \frac{\Delta T}{L_{\text{evap}}} \\
 &= \frac{(4.2 \times 10^3) 5}{(2.5 \times 10^6)} \\
 &= 0.0084 \\
 &= \mathbf{0.84\%}
 \end{aligned}$$

- 3.40** Twenty liters of air at 20°C and a relative humidity of 60% are compressed isothermally to a volume of 4 liters. Calculate the mass of water condensed. The saturation vapor pressure of water at 20°C is 23 hPa. (Density of air at 0°C and 1000 hPa is 1.28 kg m⁻³.)

Answer 0.14 g

Solution: We must find mass vapor in the air before and after the compression.

Mass of water vapor in air initially

Initially, the partial pressure of the vapor

$$\begin{aligned} &= \frac{60}{100} \text{ (SVP of water at 20°C)} \\ &= \frac{60}{100} \times (23.371 \text{ hPa}) \\ &= 14.02 \text{ hPa} \end{aligned}$$

Since $p = R\rho T$

$$\rho \propto p/T$$

Therefore, for air:

$$\begin{aligned} \frac{\rho_{20^\circ\text{C}=293 \text{ K and } 14.02 \text{ hPa}}}{\rho_{0^\circ\text{C}=273 \text{ K and } 1000 \text{ hPa}}} &= \frac{14.02/293}{1000/273} \\ &= \frac{14.02}{1000} \frac{273}{293} \end{aligned}$$

Therefore,

$$\rho_{20^\circ\text{C and } 14.02 \text{ hPa}} = \frac{14.02}{1000} \frac{273}{293} (1.2754) \text{ kg m}^{-3}$$

or,

$$\rho_{20^\circ\text{C and } 14.02 \text{ hPa}} = 0.01666 \text{ kg m}^{-3}$$

Provided the water vapor and air are at the same temperature and pressure (i.e., 14.02 hPa and 20°C in this case).

$$\begin{aligned} \text{Density of water vapor} &= \frac{18.016}{28.97} \text{ (density of air)} \\ &= 0.62 \text{ (density of air)} \end{aligned}$$

Therefore, the mass (in kg) of water vapor that occupies 20 liters under these conditions is

$$\begin{aligned} &(0.62) \times \text{(density of air at 20°C and 14.02 hPa)} \times \\ &\quad \text{(volume of 20 liters in m}^3\text{)} \\ &= \frac{5}{8} (0.01666) (2,0000 \times 10^{-6}) \\ &= 2.0825 \times 10^{-4} \text{ kg} \end{aligned} \tag{1}$$

Mass of water vapor in air after compression

To saturate air at 20°C it would have to be compressed until the vapor pressure is raised from 60% to 100% of the saturation value. From Boyle's Law, this means that it becomes saturated when its volume is reduced to 60% of initial value. This compression is *exceeded* when volume is reduced from 20 to 4 liters. Therefore, after compression the air is saturated and its partial pressure is equal to the saturation vapor pressure of water at 20°C = 23.37 hPa.

Proceeding as before,

For air:

$$\frac{\rho_{293 \text{ K and } 23.371 \text{ hPa}}}{\rho_{273 \text{ K and } 1000 \text{ hPa}}} = \frac{23.371/293}{1000/273}$$

Therefore,

$$\rho_{293 \text{ K and } 23.371 \text{ hPa}} = \frac{23.371}{1000} \frac{273}{293} (\rho_{273 \text{ K and } 1000 \text{ hPa}})$$

Hence, density of air at 293 K and 23.371 hPa is:

$$\therefore \rho_{293 \text{ K and } 23.371 \text{ hPa}} = \frac{23.371}{1000} \times \frac{273}{293} \times (1.2754) \text{ kg m}^{-3}$$

Therefore, mass of water vapor that occupies 4 liters when air is saturated is

$$\begin{aligned} & \left(\frac{5}{8} \times \rho_{293 \text{ K and } 23.371 \text{ hPa}} \right) \times (4,000 \times 10^{-6}) \text{ kg} \\ &= \left(\frac{5}{8} \times \frac{23.371}{1000} \times \frac{273}{293} \times 1.2754 \right) \times (4,000 \times 10^{-6}) \\ &= \underline{6.943 \times 10^{-5} \text{ kg}} \end{aligned}$$

$$\begin{aligned} \text{Therefore, mass of water condensed} &= (\text{mass of water vapor in air initially}) - \\ & \quad (\text{mass of water vapor in air after compression}) \\ &= (2.0825 \times 10^{-4}) - (6.943 \times 10^{-5}) \text{ kg} \\ &= 0.139 \text{ grams} = \underline{\mathbf{0.14 \text{ grams}}} \end{aligned}$$

An alternative solution:

$$\text{Relative humidity} = \frac{e}{e_s} 100$$

Therefore, for initial state:

$$60 = \frac{e}{23.37} 100$$

or,

$$e = 14.022 \text{ hPa} = \underline{1402.2 \text{ Pa}}$$

Let us now assume system is compressed isothermally and consider (fictionally) a fixed mass of water vapor (i.e., no condensation), then for vapor:

$$(1402.2)20 = p_2 4$$

$$\text{Therefore, } \underline{p_2 = 7011.0 \text{ Pa} = 70.11 \text{ hPa}}$$

This exceeds the SVP at 20°C, which is 23.37 hPa, by 46.74 hPa. Therefore, water must condense to bring vapor pressure to saturation at 20°C (= 23.37 hPa). Mass fraction of water that condenses is

$$\frac{70.11 - 23.37}{70.11} = 0.67$$

But, mass of water vapor in air before condensation is (from first solution given above)

$$= 2.0825 \times 10^{-4} \text{ kg}$$

$$\begin{aligned} \text{Therefore, mass of water condensed} &= 0.67 \times (2.0828 \times 10^{-4}) \text{ kg} \\ &= 1.39 \times 10^{-4} \text{ kg} \\ &= \underline{\mathbf{0.14 \text{ grams}}} \end{aligned}$$

3.41

$$\text{Specific humidity (q)} \equiv \frac{m_v}{m_v + m_d} = \frac{m_v/m_d}{1 + m_v/m_d} = \frac{r}{1+r}$$

where $r = \text{mixing ratio} \equiv m_v/m_d$

$$\text{If } q = 0.0196$$

$$0.0196(1+r) = r$$

$$\therefore \underline{r = 0.02 \text{ kg/kg} = 20 \text{ g/kg}}$$

$$\begin{aligned} \text{Also, } T_v &= T(1 + 0.61r) \\ &= 303(1 + 0.61 \times 0.02) \\ &= 306.697 \text{ K} \end{aligned}$$

$$\therefore \underline{\mathbf{T_v = 33.7^\circ \text{C}}}$$

$$p = R_d \rho_{\text{moist}} T_v \text{ for moist air}$$

$$p = 1014 \text{ hPa} = 1014 \times 10^2 \text{ Pa}$$

$$\therefore 1.014 \times 10^5 = (298)(\rho_{\text{moist}})(306.7)$$

$$\therefore \rho_{\text{moist}} = \frac{1.014 \times 10^5}{287 \times 306.7} \text{ kg m}^{-3}$$

$$\underline{\rho_{\text{moist}} = \mathbf{1.15 \text{ kg m}^{-3}}}$$

3.42

$$\begin{aligned}
 e &= (\text{kilomol fraction of water vapor}) p \\
 &= \frac{m_v/M_w}{\frac{m_a}{M_a} + \frac{m_v}{M_w}} = \frac{w}{\varepsilon + w}
 \end{aligned}$$

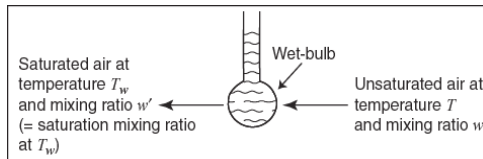
where,

$$\begin{aligned}
 \varepsilon &\equiv \frac{M_w}{M_a} = \frac{R_a}{R_w} = \frac{287}{461} = 0.623 \\
 w &= 1.80 \times 10^{-3} \text{ kg kg}^{-1} \\
 p &= 975 \text{ hPa}
 \end{aligned}$$

$$\begin{aligned}
 \therefore e &= \frac{1.80 \times 10^{-3}}{0.623 + 1.80 \times 10^{-3}} 975 \\
 \mathbf{e} &= \mathbf{2.81 \text{ hPa}}
 \end{aligned}$$

$$\begin{aligned}
 T_v &\simeq T (1 + 0.61 w) \\
 &= 288 (1 + 0.61 \times 1.80 \times 10^{-3}) \\
 &= 288 (1.001) \\
 &= 288.3^\circ\text{K} \\
 \mathbf{T_v} &= \mathbf{15.3^\circ\text{C}}
 \end{aligned}$$

3.43



Referring to above diagram, the heat released due to moist air being cooled from T to T_w is:

$$(T - T_w)(c_p + wc_{pv})$$

Heat required to evaporate water is:

$$L(w' - w)$$

$$\therefore (T - T_w)(c_p + wc_{pv}) = L(w' - w)$$

$$\begin{aligned}
\therefore w &= \frac{Lw' - (T - T_w) c_p}{c_{pv}(T - T_w) + L} \\
&= \frac{(2.25 \times 10^6) \times (8.7 \times 10^{-3}) - 6 \times 1004}{1952 \times 6 + 2.25 \times 10^6} \\
&= 5.99 \times 10^{-3} \text{ kg/kg} \\
&= \underline{\underline{5.99 \text{ g/kg} = 6.0 \text{ g/kg}}}
\end{aligned}$$

3.44

$$\begin{aligned}
du &= \left(\frac{du}{dT} \right)_v dT + \left(\frac{du}{dv} \right)_T dv \\
c_v &= \left(\frac{dq}{dT} \right)_v = \left(\frac{du}{dT} \right)_v \\
\therefore du &= c_v dT + \left(\frac{du}{dv} \right)_T dv. \text{ But for an ideal gas } \left(\frac{du}{dv} \right)_T = 0, \text{ therefore,} \\
du &= c_v dT. \text{ Therefore,}
\end{aligned}$$

$$\begin{aligned}
du &= mc_v dT \\
\therefore \frac{(du)_{\text{water}}}{(du)_{\text{air}}} &= \frac{[mc_v(T_2 - T_1)]_{\text{water}}}{[mc_v(T_2 - T_1)]_{\text{air}}} \\
&= \frac{4 \times 4218 \times (T_2 - T_1)}{1000 \times 717 \times (T_2 - T_1)} \\
&= 0.0235 \\
&= \underline{\underline{2.4\%}}
\end{aligned}$$

3.45 See skew $T - \ln p$ chart on the book web site.

Note that the moist adiabats on the skew $T - \ln p$ chart diverge with increasing height (or decreasing pressure). Hence, if the lapse rate follows a moist adiabat, a 1 K rise in temperature at the Earth's surface will be accompanied by a temperature rise of greater than 1 K in the upper troposphere.

To estimate how much larger the temperature rise in the upper troposphere would be, one need only estimate the rate of spreading of the moist adiabats. On the chart provided on the book web site the spacing between successive moist adiabats (in terms of temperature at the 1000 hPa level) is 5 K. To estimate the rate of spreading, identify the moist adiabat that passes through 25°C at the 1000 hPa level and the moist adiabats on either side of it. Follow these three contours up the the 250 hPa level and note the spacing between them (again in terms of temperature on a specific pressure level). The spacing is about 12.5 K. Hence, the multiplication factor is around 12.5 K per 5 K, or 2.5 K per degree K at the 1000-hPa-level.

3.46 See on the book web site skew $T - \ln p$ chart.

(a)

$$\begin{aligned}\text{Mixing ratio} &= w_s \text{ at } T_d \\ &= \mathbf{5.1 \text{ g kg}^{-1}}\end{aligned}$$

$$\begin{aligned}RH &= \frac{w_s \text{ at } T_d}{w_s \text{ at } T} \\ &= \frac{5.1}{10.8} \\ &= \mathbf{47\%}\end{aligned}$$

Wet-bulb temperature: Found, using Norman's Rule, to be: **9.3°C**.

Potential temperature: Since air parcel is at 1000 hPa:

$$\theta = T = 15^\circ\text{C} = \mathbf{288 \text{ K}}$$

Wet bulb potential temperature (θ_w): Since air parcel is at 1000 hPa

$$\theta_w = T_w = \mathbf{9.3^\circ\text{C}}$$

(b) If the parcel rises to 900 hPa:

Mixing ratio remains the same as **5.1 g kg⁻¹**

Since parcel is below LCL:

$$\begin{aligned}\text{Relative humidity is } \frac{w_s \text{ at } T_d}{w_s \text{ at } T} &= \frac{5.1}{6.8} 100 \\ &= \mathbf{75\%}\end{aligned}$$

Wet-bulb temperature is obtained by applying

Normand's rule at 900 hPa to give **4.5°C**

Potential temperature remains **unchanged at 288 K**

Wet-bulb potential temperature is also **conserved at 9.3°C**

(c) If the parcel rises to 800 hPa, it is now lifted above its LCL, therefore air becomes saturated at LCL.

Mixing ratio is changed to **4.3°C**

Since air is saturated **RH is 100%**

Since air is saturated **T_w = T = -1.1°C**

Potential temperature (θ) from **chart is 290°K** (not conserved)

Wet-bulb potential temperature (θ_w) is **conserved at 9.3°C**

(d) LCL = **847 hPa**

3.47 Solutions are found as follows:

- (a) Locate the point (A) at $p = 1000$ hPa and $T = 25^\circ\text{C}$, and the point (b) at $p = 1000$ hPa and $T_w = 20^\circ\text{C}$.

Find the LCL of this air by locating the point (C) at which the θ line through A intersects the θ_e (or θ_w) line through B. C is found to be at 900 hPa.

Now take the w_s line through C and extrapolate it to 1000 hPa (D). Then, from Normand's Rule, the temperature of D, **namely 18°C , is the dewpoint of the air.**

- (b) and (c) By definition of θ_e this is the temperature that the air would obtain if it were expanded until all of the moisture in it condensed and fell out and the air were then compressed to 1000 hPa.

Run up the saturated adiabat through C until this saturated adiabat runs parallel to a dry adiabat. Then, at this point, all of moisture has condensed and fallen out. The θ value of this dry adiabat is, by definition of θ , the temperature the dry air parcel would have if compressed adiabatically to 1000 hPa. This **θ value is about 62°C** . (Note: By definition, this is the θ_e value of the air at A.)

- (d) Wet-bulb potential temperature (θ_w) corresponding to the saturated adiabats are given by the temperature (in $^\circ\text{C}$) along the 200 hPa line in the skew $T - \ln p$ chart. The saturated adiabat that passes through C, has a **θ_w value of about 20°C** .

3.48 See skew $T - \ln p$ chart on the book web site.

$$\mathbf{T_d} = 14^\circ\text{C}$$

$$\text{Temperature at 900 hPa} \simeq \mathbf{19.8^\circ\text{C}}$$

- 3.49** (a) Show that when a parcel of dry air at temperature T' moves adiabatically in ambient air with temperature T , the temperature lapse rate following the parcel is given by

$$-\frac{dT'}{dz} = \frac{T'}{T} \frac{g}{c_p}$$

- (b) Explain why the lapse rate of the air parcel in this case differs from the dry adiabatic lapse rate (g/c_p). [Hint: Start with eqn. (3.54) with $T = T'$. Take the natural logarithm of both sides of this equation and then differentiate with respect to height z .]

Solution: From (3.54) with $T = T'$ we have for the air parcel

$$\begin{aligned}\theta &= T' \left(\frac{p_o}{p} \right)^{R^*/c_p} \\ \therefore \ln \theta &= \ln T' + \frac{R^*}{c_p} (\ln p_o - \ln p)\end{aligned}$$

Differentiating,

$$\frac{1}{\theta} \frac{d\theta}{dz} = \frac{1}{T'} \frac{dT'}{dz} - \frac{R^*}{c_p} \frac{1}{p} \frac{dp}{dz} \quad (1)$$

But for the ambient air we have, from the hydrostatic equation,

$$\frac{dp}{dz} = -g\rho \quad (2)$$

From (1) and (2):

$$\frac{1}{\theta} \frac{d\theta}{dz} = \frac{1}{T'} \frac{dT'}{dz} - \frac{R^*}{c_p} \frac{1}{p} (-g\rho)$$

For an adiabatic process θ is conserved (i.e., $\frac{d\theta}{dz} = 0$). Therefore,

$$0 = \frac{1}{T'} \frac{dT'}{dz} + \frac{R^* g \rho}{p c_p}$$

or,

$$\frac{dT'}{dz} = - \frac{R^* \rho T' g}{p c_p} \quad (3)$$

But, for the ambient air,

$$p = R^* \rho T \quad (4)$$

From (3) and (4),

$$\frac{dT'}{dz} = - \frac{T' g}{T c_p}$$

The dry adiabatic lapse rate $\left(\Gamma_d = \frac{g}{c_p} \right)$ is determined under the assumption that the air parcel develop no macroscopic kinetic energy (i.e., that $T' = T$), so

$$\Gamma_d = - \frac{dT'}{dz} = \frac{g}{c_p}.$$

3.50 Derive an expression for the rate of change in temperature with height (Γ_s) of a parcel of air undergoing a saturated adiabatic process. Assume that $\rho L_v \left(\frac{\partial w_s}{\partial p} \right)_T$ is small compared to 1.

$$\text{Answer } \Gamma_s \simeq \frac{\Gamma_d}{1 + \frac{L_v}{c_p} \left(\frac{\partial w_s}{\partial T} \right)_p}$$

Solution: Substituting (3.20) into (3.51) yields

$$dq = c_p dT + g dz \quad (1)$$

If the saturation ratio of the air with respect to water is w_s , the quantity of heat dq released into (or absorbed from) a unit mass of dry air due to condensation (or evaporation) of liquid water is $-L_v dw_s$, when L_v is the latent heat of condensation. Therefore,

$$-L_v dw_s = c_p dT + g dz \quad (2)$$

If we neglect the small amounts of water vapor associated with a unit mass of dry air, which are also warmed or cooled) by the release (or absorption) of the latent heat, then c_p in (2) is the specific heat at constant pressure of dry air. Dividing both sides of (2) by $c_p dz$ and rearranging terms, we obtain

$$\begin{aligned} \frac{dT}{dz} &= -\frac{L_v}{c_p} \frac{dw_s}{dz} - \frac{g}{c_p} \\ &= -\frac{L_v}{c_p} \frac{dz}{dz} \left[\left(\frac{dw_s}{dp} \right)_T dp + \left(\frac{\partial w_s}{\partial T} \right)_p dT \right] - \frac{g}{c_p} \\ \therefore \frac{dT}{dz} \left[1 + \frac{L_v}{c_p} \left(\frac{\partial w_s}{\partial T} \right)_p \right] &= -\frac{g}{c_p} \left[1 + \frac{L_v}{g} \left(\frac{\partial w_s}{\partial p} \right)_T \frac{dp}{dz} \right] \end{aligned} \quad (3)$$

Or, using the hydrostatic equation on the last term on the right side of (3)

$$\Gamma_s \equiv -\frac{dT}{dz} = \frac{\frac{g}{c_p} \left[1 - \rho L_v \left(\frac{\partial w_s}{\partial p} \right)_T \right]}{1 + \frac{L_v}{c_p} \left(\frac{\partial w_s}{\partial T} \right)_p}$$

or

$$\Gamma_s \equiv -\frac{dT}{dz} = \Gamma_d \frac{\left[1 - \rho L_v \left(\frac{\partial w_s}{\partial p} \right)_T \right]}{\left[1 + \frac{L_v}{c_p} \left(\frac{\partial w_s}{\partial T} \right)_p \right]} \quad (4)$$

In Exercise (3.51) we show that

$$-\rho L_v \left(\frac{\partial w_s}{\partial p} \right)_T \simeq 0.12$$

Therefore, from (4)

$$\Gamma_s \equiv -\frac{\partial T}{dz} \simeq \frac{\Gamma_d}{1 + \frac{L_v}{c_p} \left(\frac{\partial w_s}{\partial T} \right)_p}$$

3.51 In deriving the expression for the saturated adiabatic lapse rate in the previous exercise, it is assumed that $\rho L_v (\partial w_s / \partial p)_T$ is small compared to 1. Estimate the magnitude of $\rho L_v (\partial w_s / \partial p)_T$. Show that this last expression is dimensionless. [Hint: Use the skew $T - \ln p$ chart on the book web site to estimate the magnitude of $(\partial w_s / \partial p)_T$ for a pressure change of, say, 1000 to 950 hPa at 0°C.]

Answer About -0.12

Solution: Estimation of magnitude of $\rho L_v \left(\frac{\partial w_s}{\partial p} \right)_T$

Take

$$\rho \simeq 1.275 \text{ kg m}^{-3}$$

$$L_v = 2.5 \times 10^6 \text{ J kg}^{-1}$$

Suppose pressure changes from 1000 to 950 hPa, so that $dp = -50 \text{ hPa} = -5000 \text{ Pa}$. Then, from skew $T - \ln p$ chart we find that:

$$\begin{aligned} dw_s &\simeq (4 - 3.75) = 0.25 \text{ g/kg} \\ &\simeq 0.25 \times 10^{-3} \text{ kg/kg} \end{aligned}$$

Hence,

$$\begin{aligned} \rho L_v \left(\frac{\partial w_s}{\partial p} \right)_T &\simeq (1.275 \text{ kg m}^{-3}) (2.5 \times 10^6 \text{ J kg}^{-1}) \left(\frac{0.25 \times 10^{-3} \text{ kg kg}^{-1}}{-5000 \text{ Pa}} \right) \\ &\simeq \mathbf{-0.12} \end{aligned}$$

The units of $\rho L_v \left(\frac{\partial w_s}{\partial p} \right)_T$ are

$$\begin{aligned} &(\text{kg m}^{-3}) (\text{J kg}^{-1}) (\text{kg kg}^{-1}) \left(\frac{1}{\text{Pa}} \right) \\ &= \left(\frac{\text{kg}^2 \text{ m}^{-3}}{\text{kg}^2 \text{ s}^{-2}} \right) \left(\frac{\text{kg}^2 \text{ m}^2 \text{ s}^{-2}}{\text{kg}^2 \text{ m}^2 \text{ s}^{-2}} \right) \left(\frac{1}{\text{kg}^2 \text{ m}^{-1} \text{ s}^{-2}} \right) \end{aligned}$$

which is dimensionless.

3.52 In deriving the expression (3.71) for equivalent potential temperature it was assumed that

$$\frac{L_v}{c_p T} dw_s \simeq d \left(\frac{L_v w_s}{c_p T} \right)$$

Justify this assumption. [Hint: Differentiate the right-hand side of the above expression and, assuming L_v/c_p is independent of temperature, show that the above approximation holds provided

$$\frac{dT}{T} \ll \frac{dw_s}{w_s}$$

Verify this inequality by noting the relative changes in T and w_s for small incremental displacements along saturated adiabats on a skew $T - \ln p$ chart.]

Solution: Show that:

$$\frac{L_v}{c_p T} dw_s \simeq d \left(\frac{L_v w_s}{c_p T} \right) \quad (1)$$

Differentiate RHS assuming L_v/c_p is a constant:

$$\frac{L_v}{c_p} \left[\frac{1}{T} dw_s - w_s \frac{dT}{T^2} \right] = \frac{L_v}{T c_p} \left[dw_s - w_s \frac{dT}{T} \right] = \frac{L_v dT}{T c_p} \left[\frac{dw_s}{dT} - \frac{w_s}{T} \right] \quad (2)$$

If $\frac{dT}{T} \ll \frac{dw_s}{w_s}$ (which can be verified from skew $T - \ln p$ chart) then,

$$\frac{dw_s}{dT} \gg \frac{w_s}{T} \quad (3)$$

\therefore From (2) and (3):

$$\underline{\text{R.H.s of (1)} = \frac{L_v}{T c_p} dw_s = \text{LHS of (1)}} \quad QED$$

3.53 Solution: See skew $T - \ln p$ chart on the book web site.

Answer (a) AB, unstable; BC, neutral; CD, neutral; DE, stable; EF, stable, FG, stable. (b) All layers are convectively unstable except CD, which is convectively neutral.

3.54 *Potential density* D is defined as the density that dry air would attain if it were transformed reversibly and adiabatically from its existing conditions to a standard pressure p_0 (usually 1000 hPa).

(a) If the density and pressure of a parcel of the air are ρ and p , respectively, show that

$$D = \rho \left(\frac{p_0}{p} \right)^{c_v/c_p}$$

where c_p and c_v are the specific heats of air at constant pressure and constant volume, respectively.

- (b) Calculate the potential density of a quantity of air at a pressure of 600 hPa and a temperature of -15°C .

Answer 1.17 kg m^{-3}

- (c) Show that

$$\frac{1}{D} \frac{dD}{dz} = -\frac{1}{T} (\Gamma_d - \Gamma)$$

where Γ_d is the dry adiabatic lapse rate, Γ the actual lapse rate of the atmosphere, and T the temperature at height z . [Hint: Take the natural logarithms of both sides of the expression given in (a) and then differentiate with respect to height z .]

- (d) Show that the criteria for stable, neutral, and unstable conditions in the atmosphere are that the potential density decreases with increasing height, is constant with height, and increases with increasing height, respectively. [Hint: Use the expression given in (c).]
- (e) Compare the criteria given in (d) with those for stable, neutral, and unstable conditions for a liquid.

Solution:

- (a) For a reversible, adiabatic transformation of an ideal gas

$$pV^\gamma = \text{constant}$$

where, $\gamma = c_p/c_v$. For a unit mass of a gas, $V = 1/\rho$, where ρ is the density of the gas. Therefore,

$$\left(\frac{p}{\rho^\gamma}\right) = \text{constant}$$

Hence, if the initial pressure and density of a gas are p and ρ , and the final pressure and density are p_o and D , and the gas undergoes an adiabatic transformation,

$$\frac{p}{\rho^\gamma} = \frac{p_o}{D^\gamma}$$

or,

$$D = \rho \left(\frac{p_o}{p}\right)^{1/\gamma} = \rho \left(\frac{p_o}{p}\right)^{c_v/c_p} \quad (3.122)$$

- (b) From the ideal gas equation for a unit mass of air

$$p = R_d \rho T \quad (3.123)$$

From (3.122) and (3.123)

$$D = \frac{p}{R_d T} \left(\frac{p_o}{p}\right)^{c_v/c_p} \quad (3.124)$$

For $p = 600 \text{ hPa} = 6 \times 10^4 \text{ Pa}$, $T = (273 - 15) \text{ K} = 258 \text{ K}$, $p_o = 1000 \text{ hPa} = 10^5 \text{ Pa}$, $R_d = 287 \text{ J deg}^{-1} \text{ kg}^{-1}$, $c_v = 717 \text{ J deg}^{-1} \text{ kg}^{-1}$ and $c_p = 10004 \text{ J deg}^{-1} \text{ kg}^{-1}$, we have from (3.124)

$$D = \frac{(6 \times 10^4)}{(287)(258)} \left(\frac{10^5}{6 \times 10^4} \right) \frac{717}{1004}$$

$$D = 1.17 \text{ kg m}^{-3}$$

(c) Taking logarithms of (3.122), we have

$$\ln D = \ln \rho + \frac{c_v}{c_p} \ln \left(\frac{p_o}{p} \right)$$

Differentiating with respect to height z ,

$$\frac{1}{D} \frac{dD}{dz} = \frac{1}{\rho} \frac{d\rho}{dz} - \frac{c_v}{c_p} \frac{1}{p} \frac{dp}{dz} \quad (3.125)$$

Also, for a unit mass of dry air considered as an ideal gas

$$p = R_d \rho T$$

Therefore,

$$\ln p = \ln R_d + \ln \rho + \ln T$$

and, differentiating this last expression with respect to height z ,

$$\frac{1}{p} \frac{dp}{dz} = \frac{1}{\rho} \frac{d\rho}{dz} + \frac{1}{T} \frac{dT}{dz} \quad (3.126)$$

From (3.125) and (3.126)

$$\begin{aligned} \frac{1}{D} \frac{dD}{dz} &= -\frac{1}{T} \frac{dT}{dz} + \frac{1}{p} \frac{dp}{dz} \left(1 - \frac{c_v}{c_p} \right) \\ &= -\frac{1}{T} \frac{dT}{dz} + \frac{R_d}{p c_p} \frac{dp}{dz} \end{aligned} \quad (3.127)$$

where we have used the relation $c_p - c_v = R_d$ (see eqn. (3.45) in text). From the hydrostatic equation and the ideal gas equation,

$$\frac{dp}{dz} = -g\rho = -g \frac{p}{R_d T} \quad (3.128)$$

From (3.127) and (3.128),

$$\frac{1}{D} \frac{dD}{dz} = -\frac{1}{T} \left(\frac{g}{c_p} + \frac{dT}{dz} \right) \quad (3.129)$$

But, the dry adiabatic lapse rate Γ_d is given by (see eqn. (3.53) in the text)

$$\Gamma_d = \frac{g}{c_p} \quad (3.130)$$

From (3.129) and (3.130)

$$\frac{1}{D} \frac{dD}{dz} = -\frac{1}{T} (\Gamma_d - \Gamma) \quad (3.131)$$

where, $\Gamma = -\frac{\partial T}{\partial z}$ is the lapse rate in the atmosphere.

(d) From Section 3.6.1 in the text, we have

$$\begin{aligned} \text{for a stable atmosphere:} & \quad \Gamma < \Gamma_d \\ \text{for an unstable atmosphere:} & \quad \Gamma > \Gamma_d \\ \text{for a neutral atmosphere:} & \quad \Gamma = \Gamma_d \end{aligned}$$

Hence, from (3.131),

$$\begin{aligned} \text{for a stable atmosphere:} & \quad \frac{dD}{dz} \text{ is negative} \\ & \quad (\text{i.e., } D \text{ decreases with increasing height}) \end{aligned}$$

$$\begin{aligned} \text{for an unstable atmosphere:} & \quad \frac{dD}{dz} \text{ is positive} \\ & \quad (\text{i.e., } D \text{ increases with increasing height}) \end{aligned}$$

$$\begin{aligned} \text{for a neutral atmosphere:} & \quad \frac{dD}{dz} = 0 \\ & \quad (\text{i.e., } D \text{ is constant with height}) \end{aligned}$$

3.55 A necessary condition for the formation of a mirage is that the density of the air increase with increasing height. Show that this condition is realized if the decrease of atmospheric temperature with height exceeds $3.5 \Gamma_d$, where Γ_d is the dry adiabatic lapse rate. [Hint: Take the natural logarithm of both sides of the expression for D given in Exercise 3.54a, then differentiate with respect to height z . Follow the same two steps for the gas equation in the form $p = \rho R_d T$. Combine the two expressions so derived with the hydrostatic equation to show that $\frac{1}{\rho} \frac{d\rho}{dz} = -\frac{1}{T} (dT/dz + g/R_d)$. Hence, proceed to the solution.]

Solution: From the solution to Exercise 3.54(a)

$$D = \rho \left(\frac{p_0}{p} \right)^{c_v/c_p}$$

Therefore,

$$\ln D = \ln \rho + \frac{c_v}{c_p} \ln \left(\frac{p_0}{p} \right)$$

and, differentiating with respect to height z ,

$$\frac{1}{D} \frac{dD}{dz} = \frac{1}{\rho} \frac{d\rho}{dz} - \frac{c_v}{c_p} \frac{1}{p} \frac{dp}{dz} \quad (3.132)$$

From the ideal gas equation for a unit mass of dry air,

$$p = R_d \rho T$$

therefore,

$$\ln p = \ln R_d + \ln \rho + \ln T$$

and, differentiating with respect to height z ,

$$\frac{1}{p} \frac{dp}{dz} = \frac{1}{\rho} \frac{d\rho}{dz} + \frac{1}{T} \frac{dT}{dz} \quad (3.133)$$

From (3.132) and (3.133),

$$\frac{1}{D} \frac{dD}{dz} = -\frac{1}{T} \frac{dT}{dz} + \frac{R_d}{p c_p} \frac{dp}{dz} \quad (3.134)$$

From the hydrostatic equation and the ideal gas equation,

$$\frac{dp}{dz} = -g\rho = -\frac{g p}{R_d T} \quad (3.135)$$

From (3.133) and (3.135),

$$\frac{1}{\rho} \frac{d\rho}{dz} = -\frac{1}{T} \left(\frac{dT}{dz} + \frac{g}{R_d} \right) \quad (3.136)$$

For a mirage to occur, $\frac{d\rho}{dz}$ must be positive. Therefore, from (3.136), for a mirage to occur

$$\frac{dT}{dz} + \frac{g}{R_d} < 0$$

or,

$$-\frac{dT}{dz} > \frac{g}{R_d} = \frac{c_p}{R_d} \frac{g}{c_p} = \frac{c_p}{R_d} \Gamma_d \quad (3.137)$$

where, in the last step, we have used eqn. (3.53) in the text, that is, $\frac{g}{c_p} = \Gamma_d$. Substituting $c_p = 1004 \text{ J deg}^{-1} \text{ kg}^{-1}$ and $R_d = 287 \text{ J deg}^{-1} \text{ kg}^{-1}$ into (3.137), we find that a necessary condition for a mirage to occur is that

$$-\frac{dT}{dz} > 3.5 \Gamma_d$$

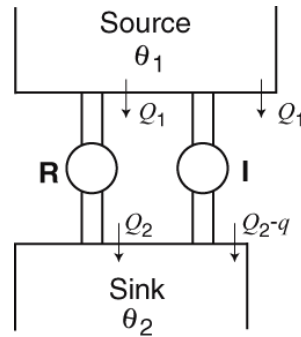
That is, the temperature of the air must decrease with height at more than $3.5 \Gamma_d = 3.5 (9.8^\circ\text{C km}^{-1}) = 34^\circ\text{C km}^{-1}$. Such a steep lapse rate generally occurs only over strongly heated surfaces, such as deserts and roads.

3.56 Assuming the truth of the Second Law of Thermodynamics, prove the following two statements (known as *Carnot's Theorems*):

- (a) No engine can be more efficient than a reversible engine working between the same limits of temperature. [Hint: The efficiency of any engine is given by eqn. (3.87); the distinction between a reversible (**R**) and an irreversible (**I**) engine is that **R** can be driven backward but **I** cannot. Consider a reversible and an irreversible engine working between the same limits of temperature. Suppose initially that **I** is more efficient than **R** and use **I** to drive **R** backwards. Show that this leads to a violation of the Second Law of Thermodynamics, and hence prove that **I** cannot be more efficient than **R**.]
- (b) All reversible engines working between the same limits of temperature have the same efficiency. [Hint: Proof is similar to that for part (a).]

Solution:

- (a)



To prove that no engine can be more efficient than a reversible engine working between the same limits of temperature consider a reversible (R) and irreversible (I) engine working between θ_1 and θ_2 . Assume I is more efficient than R . Then, if R takes heat Q_1 from source and yields heat Q_2 to sink. Therefore, if I takes Q_1 from source it must yield heat $Q_2 - q$ (q positive) to sink. Now let us use I to drive R backward. This will require I to do work $Q_1 - Q_2$ on R . But, in one cycle, I develops work $Q_1 - (Q_2 - q) = (Q_1 - Q_2) + q$. Hence, even when I is drawing R backwards, mechanical work q is still available. But, in one cycle of the combined system, heat $Q_2 - (Q_2 - q) = q$ is taken from colder body. This violates 2nd Law.

- (b) Take two reversible engines operating between θ_1 and θ_2 , and assume one engine is more efficient than the other. Then follow same procedure as in (a) above to show this would violate 2nd Law.

3.57 Lord Kelvin introduced the concept of *available energy*, which he defined as the maximum amount of heat that can be converted into work by using the coldest available body in a system as the sink for an ideal heat engine. By considering an ideal heat engine, that uses the coldest available body as a sink, show that the available energy of the universe is tending to zero and that

$$\text{loss of available energy} = T_0 (\text{increase in entropy})$$

where T_0 is the temperature of the coldest available body.

Solution: For an ideal reversible engine

$$\frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1}$$

$$\text{Work done in 1 cycle} = Q_1 - Q_2 = \frac{T_1 - T_2}{T_1} Q_1.$$

If engine operates with sink at $T_o (= T_2)$:

$$\text{Available energy} = \frac{T_1 - T_0}{T_1} Q_1$$

Let Q pass from T_1 to T_2 ($T_1 > T_2$) by, say, conduction or radiation. Then,

$$\begin{aligned} \text{Loss of available energy} &= \left(\frac{T_1 - T_0}{T_1} \right) Q - \left(\frac{T_2 - T_0}{T_2} \right) Q \\ &= QT_o \left(\frac{T_1 - T_2}{T_1 T_2} \right) \end{aligned}$$

Since $T_1 > T_2$, there is a loss of available energy for natural processes

$$\begin{aligned} \text{Loss of available energy} &= T_o \left(\frac{Q}{T_2} - \frac{Q}{T_1} \right) \\ &= T_o (\text{increase in entropy}) \end{aligned}$$

3.58

$$\begin{aligned} \eta &\equiv \frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1} \\ T_1 &= 373 \text{ K}, T_2 = 273 \text{ K} \\ \therefore \eta &= \frac{100}{373} \end{aligned}$$

$$\begin{aligned} \text{Work done in 1 cycle} &= Q_1 - Q_2 = Q_1 \eta \\ &= 20 \times \frac{100}{373} \\ &= 5.36 \text{ J} \end{aligned}$$

$$\text{Work done in 10 cycles} = 53.6 \text{ J}$$

$$\begin{aligned}
\text{Heat rejected to sink in 1 cycle} &= Q_2 \\
&= Q_1 (1 - \eta) \\
\text{Heat reflected to sink in 10 cycles} &= 10Q_1 (1 - \eta) \\
&= (10)(20) \left(1 - \frac{100}{373}\right) \\
&= \mathbf{146.4 \text{ J}}
\end{aligned}$$

3.59

$$\frac{Q_1 - Q_2}{Q_2} = \frac{T_1 - T_2}{T_2} = \frac{17}{273}$$

(Note this is not the same as efficiency η of engine $\equiv \frac{Q_1 - Q_2}{Q_1}$) For every Q_2 joules taken from water, $Q_1 - Q_2$ joules ($= Q_2 \frac{17}{273}$) of work has to be done by the motor that drive the refrigerator.

$$\begin{aligned}
\text{Heat required to freeze 20 kg of water} &= 20 \times L_m \\
&= 20 \times (3.34 \times 10^5) \text{ J} \\
&= 6.68 \times 10^6 \text{ J}
\end{aligned}$$

Hence, work done by motor to freeze 20 kg of water $= (6.68 \times 10^6) \frac{17}{273} \text{ J}$
 $= 4.16 \times 10^5 \text{ J}$. Since 1 watt $= 1 \text{ J s}^{-1}$ a 1 kW motor does 1000 J of work per second.

$$\begin{aligned}
\therefore \text{Time needed to do } 4.16 \times 10^5 \text{ J of work is } &\frac{4.16 \times 10^5}{1000} \text{ secs} = 416 \text{ secs} \\
&= \mathbf{6.93 \text{ mins}}
\end{aligned}$$

3.60 A Carnot engine operating in reverse (i.e., as an air conditioner) is used to cool a house. The indoor temperature of the house is maintained at T_i and the outdoor temperature is T_o ($T_o > T_i$). Because the walls of the house are not perfectly insulating, heat is transferred into the house at a constant rate given by

$$\left(\frac{dq}{dt}\right)_{\text{leakage}} = K(T_o - T_i)$$

where $K (> 0)$ is a constant.

- Derive an expression for the power (i.e., energy used per second) required to drive the Carnot engine in reverse in terms of T_o , T_i and K .
- During the afternoon, the outdoor temperature increases from 27 to 30°C. What percentage increase in power is required to drive the Carnot engine in reverse to maintain the interior temperature of the house at 21°C?

Answer (a) $K(T_o - T_i)^2/T_i$; (b) 125%

Solution:

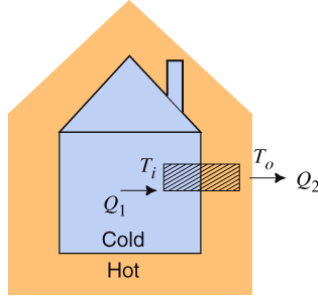


Figure 3.27

(a) See Figure 3.27. Let Q_1 and Q_2 be the heats that a Carnot engine running in reverse (i.e., serving as an air conditioner) takes in and rejects, respectively, in one cycle to keep a house at a temperature T_i that is below the outside temperature of T_o .

If T_i is to remain constant,

$$\left(\frac{dq}{dt}\right)_{\text{pumped out of house by air conditioner}} = \left(\frac{dq}{dt}\right)_{\text{heat leakage into house}}$$

Hence,

$$\frac{Q_1}{\Delta t} = K(T_o - T_i) \quad (3.138)$$

where, Δt is the time period for one cycle of the air conditioner. The power (i.e., work per unit time) needed to drive the air conditioner is

$$\begin{aligned} P &= \frac{\text{Work done to drive air conditioner per cycle}}{\Delta t} \\ &= \frac{Q_2 - Q_1}{\Delta t} \end{aligned} \quad (3.139)$$

Since the air conditioner is ideal

$$\frac{Q_2}{Q_1} = \frac{T_o}{T_i}$$

or

$$\frac{Q_2 - Q_1}{Q_1} = \frac{T_o - T_i}{T_i}$$

Therefore,

$$Q_2 - Q_1 = Q_1 \left(\frac{T_o - T_i}{T_i}\right) \quad (3.140)$$

From (3.139) and (3.140)

$$P = \frac{Q_1}{\Delta t} \left(\frac{T_o - T_i}{T_i} \right) \quad (3.141)$$

From (3.138) and (3.141)

$$P = K (T_o - T_i)^2 / T_i \quad (3.142)$$

- (b) For outside temperatures of T_{o1} and T_{o2} , the powers P_2 and P_1 , respectively, needed to drive the air conditioner to maintain the house at temperature T_i are, from (3.142) in (a) above,

$$\begin{aligned} \frac{P_2}{P_1} &= \frac{K (T_{o2} - T_i)^2 / T_i}{K (T_{o1} - T_i)^2 / T_i} \\ &= \frac{(T_{o2} - T_i)^2}{(T_{o1} - T_i)^2} \end{aligned}$$

For $T_i = 294$ K, $T_{o1} = 300$ K and $T_{o2} = 303$ K,

$$\frac{P_2}{P_1} = \frac{9^2}{6^2} = 2.25$$

Therefore, the increase in power needed to keep the house at 21°C or 294 K is **125%**.

3.61 Increase in entropy in warming ice from 263 to 273 K:

$$\begin{aligned} &= m_{\text{ice}} c_{\text{ice}} \int_{263}^{273} \frac{dT}{T} \\ &= m_{\text{ice}} c_{\text{ice}} \ln \frac{273}{263} \\ &= (0.002) (2106) \ln \frac{273}{263} \\ &= 4.212 (0.0373) \\ &= 0.1572 \text{ J deg}^{-1} \end{aligned}$$

Increase in entropy on melting ice at 273 K

$$\begin{aligned} &= \frac{(0.002) (3.34 \times 10^5)}{273} \text{ J deg}^{-1} \\ &= 2.447 \text{ J deg}^{-1} \end{aligned}$$

Increase in entropy on heating water from 273 K to 373 K

$$\begin{aligned} &= (0.002) (4218) \ln \frac{373}{273} \\ &= 2.633 \text{ J deg}^{-1} \end{aligned}$$

Increase in entropy on changing water to steam at 373 K:

$$\begin{aligned}
 &= \frac{(0.002)(2.25 \times 10^6)}{373} \\
 &= 12.06 \text{ J deg}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \text{Total increase in entropy} &= (0.1572 + 2.447 + 2.633 + 12.06) \\
 &= 17.297 \text{ J deg}^{-1} \\
 &= \mathbf{17.3 \text{ J deg}^{-1}}
 \end{aligned}$$

3.62 In general,

$$S_2 - S_1 = \int_1^2 \frac{dQ}{T}$$

For an ideal gas:

$$\begin{aligned}
 S_2 - S_1 &= \int_1^2 \frac{c_v dT}{T} + \int_1^2 \frac{pdv}{T} \\
 &= c_v \ln \frac{T_2}{T_1} + \int_1^2 R^* \frac{dv}{v} \\
 &= c_v \ln \frac{T_2}{T_1} + R^* \ln \frac{v_2}{v_1}
 \end{aligned}$$

Since,

$$\begin{aligned}
 \frac{p_1 v_1}{T_1} &= \frac{p_2 v_2}{T_2} \\
 S_2 - S_1 &= c_v \ln \frac{T_2}{T_1} + R^* \ln \left(\frac{p_1 T_2}{p_2 T_1} \right) \\
 &= c_v \ln \frac{T_2}{T_1} + R^* \ln \frac{p_1}{p_2} + R^* \ln \frac{T_2}{T_1}
 \end{aligned}$$

Since,

$$\begin{aligned}
 R^* &= c_p - c_v \\
 \therefore S_2 - S_1 &= c_p \ln \frac{T_2}{T_1} + R^* \ln \frac{p_1}{p_2}
 \end{aligned}$$

For a diatomic gas $c_p/c_v = \gamma = 1.4$. Also,

$$c_p = c_v + R^* = \frac{c_p}{\gamma} + R^*$$

or

$$\begin{aligned}
 c_p &= \frac{\gamma R^*}{\gamma - 1} \\
 &= (1.4 \times 8.3145) / 0.4 \text{ J deg}^{-1} \text{ mol}^{-1} \\
 &= 29.10 \text{ J deg}^{-1} \text{ mol}^{-1} \\
 \therefore S_2 - S_1 &= 29.10 \ln \left(\frac{373}{286} \right) + 8.3145 \ln \left(\frac{1}{2} \right) \\
 &= (29.10) (0.2656) + 8.3145 (-0.6931) \\
 &= 7.729 - 5.763 \\
 &= 1.966 \text{ J deg}^{-1} \\
 &= \mathbf{2.0 \text{ J deg}^{-1}}
 \end{aligned}$$

3.63 Show that the expression numbered (3.118) in the solution to Exercise 3.50 above can be written as

$$\Gamma_s = \Gamma_d \frac{(1 + w_s L_v / R_d T)}{(1 + w_s L_v^2 / c_p R_v T^2)}$$

Solution: Comparing expression (3.118) in Exercise 3.50 with the expression given in the statement of this exercise, we see that we have to prove (a) $-\rho L_v \left(\frac{dw_s}{dp} \right)_T = \frac{w_s L_v}{R_d T}$, and (b) $\frac{L_v}{c_p} \left(\frac{dw_s}{dT} \right)_p = \frac{w_s L_v^2}{c_p R_v T^2}$

(a) To show:

$$-\rho L_v \left(\frac{dw_s}{dp} \right)_T = \frac{w_s L_v}{R_d T}$$

From eqn. (3.63)

$$w_s \simeq 0.622 \frac{e_s}{p} \quad (3.143)$$

Also,

$$p = R_d \rho T \quad (3.144)$$

From (3.143) and (3.144),

$$\left(\frac{dw_s}{dp} \right)_T = \left[\frac{dw_s}{d(R_d \rho T)} \right]_T$$

or, since T is constant,

$$\left(\frac{dw_s}{dp} \right)_T = \frac{1}{R_d T} \frac{dw_s}{dp} \quad (3.145)$$

But,

$$w_s = \frac{\rho_{sv}}{\rho}$$

Therefore,

$$\frac{dw_s}{d\rho} = \rho_{sv} \frac{d}{d\rho} (\rho^{-1}) = -\frac{\rho_{sv}}{\rho^2} \quad (3.146)$$

From (3.145) and (3.146),

$$\left(\frac{dw_s}{dp} \right)_T = -\frac{1}{R_d T} \frac{\rho_{sv}}{\rho^2} \quad (3.147)$$

From (3.146) and (3.147),

$$\left(\frac{dw_s}{dp} \right)_T = -\frac{w_s}{R_d T \rho} \quad (3.148)$$

Therefore,

$$\underline{-\rho L_v \left(\frac{dw_s}{dp} \right)_T = L_v \frac{w_s}{R_d T}} \quad QED$$

(b) To show:

$$\frac{L_v}{c_p} \left(\frac{dw_s}{dT} \right)_p = \frac{w_s L_v^2}{c_p R_v T^2}$$

From (3.143) in (a) above:

$$\begin{aligned} \left(\frac{dw_s}{dT} \right)_p &= \frac{d}{dT} \left[\left(0.622 \frac{e_s}{p} \right) \right]_p \\ &= \frac{0.622}{p} \frac{de_s}{dT} \end{aligned} \quad (3.149)$$

Using the Clausius-Clapeyron eqn. (3.93) and the ideal gas equation for saturated water vapor,

$$\frac{de_s}{dT} = \frac{L_v}{T \alpha_2} = \frac{L_v}{T} \frac{L_v}{R_v T} = \frac{L_v e_s}{T^2 R_v} \quad (3.150)$$

From (3.149) and (3.150)

$$\left(\frac{dw_s}{dT} \right)_p = \frac{0.622}{p} \frac{L_v e_s}{T^2 R_v}$$

and using (3.143)

$$= \frac{w_s L_v}{T^2 R_v}$$

Therefore,

$$\frac{L_v}{c_p} \left(\frac{dw_s}{dT} \right)_p = \frac{w_s L_v^2}{c_p R_v T^2}$$

3.64 From (3.99)

$$\frac{\Delta T_B}{\Delta p_{\text{atmos}}} = \frac{T_B (\alpha_2 - \alpha_1)}{L_v}$$

with

$$\begin{aligned} dp_{\text{atmos}} &= (1013 - 600) \text{ hPa} \\ &= -413 \text{ hPa} \\ &= -4.13 \times 10^4 \text{ Pa} \\ T_B &= 373 \text{ K at } 1013 \text{ hPa} \\ L_v &= 2.25 \times 10^6 \text{ J K}^{-1} \\ \therefore dT_B &= \frac{-(373 \text{ K}) (1.66 - 0.001 \text{ m}^3 \text{ kg}^{-1}) (4.13 \times 10^4 \text{ Pa})}{2.25 \times 10^6 \text{ J K}^{-1}} \\ &= 11.30 \text{ deg} \end{aligned}$$

Hence, **new boiling point** = **88.7°C**

3.65 The change ΔT_M in the melting point of ice due to a change in pressure Δp is, by analogy with (3.99),

$$\Delta T_M = \frac{T_M (\alpha_w - \alpha_i)}{L_M} \Delta p$$

where α_w and α_i are the specific volumes of water and ice, respectively, and L_M is the latent heat of melting of ice. Substituting $T_M = 273 \text{ K}$, $(\alpha_w - \alpha_i) = (1.0010 - 1.0908) 10^{-3} \text{ m}^3 \text{ kg}^{-1} = -0.0898 \times 10^{-3} \text{ m}^3 \text{ kg}^{-1}$, $L_M = 3.34 \times 10^5 \text{ J kg}^{-1}$, and $\Delta p = 1 \text{ atm} = 1.013 \times 10^5 \text{ Pa}$ we obtain

$$\begin{aligned} \Delta T_M &= -\frac{273 \times (0.0898 \times 10^{-3})}{3.34 \times 10^5} (1.013 \times 10^5) \text{ }^\circ\text{C} \\ \therefore \Delta T_M &= -0.0074 \text{ }^\circ\text{C} \end{aligned}$$

Therefore, an increase in pressure of 1 atm decreases the melting point of ice by 0.0074 °C. (Ice is unusual in this respect; the specific volumes of the liquid forms of most materials is greater than the specific volumes of the solid forms. Consequently, the melting point of most materials *increases* with increasing pressure.)

3.66* By differentiating the enthalpy function, defined by eqn. (3.47), show that

$$\left(\frac{\partial p}{\partial T} \right)_s = \left(\frac{\partial s}{\partial \alpha} \right)_p$$

where s is entropy. Show that this relation is equivalent to the Clausius-Clapeyron equation.

[Hint:
$$dh = \left(\frac{\partial h}{\partial s} \right)_p ds + \left(\frac{\partial h}{\partial p} \right)_s dp$$

and, since h is a function of state,

$$\left. \frac{\partial}{\partial p} \left(\frac{\partial h}{\partial s} \right)_p = \frac{\partial}{\partial s} \left(\frac{\partial h}{\partial p} \right)_s \right]$$

Solution: From (3.47)

$$\begin{aligned} h &= u + p\alpha \\ \therefore dh &= du + p d\alpha + \alpha dp \\ &= (dq - p d\alpha) + p d\alpha + \alpha dp \\ dh &= T dS + \alpha dp \end{aligned} \quad (1)$$

Therefore

$$\begin{aligned} h &= h(s, p) \\ \therefore dh &= \left(\frac{\partial h}{\partial s} \right)_p ds + \left(\frac{\partial h}{\partial p} \right)_s dp \end{aligned} \quad (2)$$

From (1) and (2),

$$\left(\frac{\partial h}{\partial s} \right)_p = T \text{ and } \left(\frac{\partial h}{\partial p} \right)_s = \alpha \quad (3)$$

But,

$$\frac{\partial}{\partial p} \left(\frac{\partial h}{\partial s} \right)_p \equiv \frac{\partial}{\partial s} \left(\frac{\partial h}{\partial p} \right)_s \quad (4)$$

From (3) and (4),

$$\left(\frac{\partial T}{\partial p} \right)_s = \left(\frac{\partial \alpha}{\partial s} \right)_p$$

or,

$$\underline{\left(\frac{\partial p}{\partial T} \right)_s = \left(\frac{\partial s}{\partial \alpha} \right)_p} \quad (5)$$

Since $ds = \frac{dQ}{T} = \frac{L_v}{T}$ for a phase change from liquid to vapor at T , and if the vapor is saturated, so that $p = e_s$, and $d\alpha = \alpha_2 - \alpha_1$ (5) becomes

$$\left(\frac{\partial e_s}{\partial T} \right)_s = \frac{\mathbf{L}_v}{\mathbf{T}(\alpha_2 - \alpha_1)}$$

which is the *Clausius-Clapeyron equation*.