A Brief Glossary of Algebraic Structures

Prof. Hubert Kaeslin
Microelectronics Design Center
ETH Zürich

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Many mathematical texts define algebraic structures recursively, e.g.

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An Abelian group $(S, *)$ is a group such that $*$ is commutative.

A group $(S, *)$ is a monoid such that each element x has a unique inverse x^{-1} where $x^{-1} * x = x * x^{-1} = e$.

A monoid $(S, *)$ is a semigroup with unity e such that $e * x = x * e = x$ for all x and where e is unique.

In a semigroup $(S, *)$, $*$ is associative.

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> *In a semigroup* $(S, *)$*,* $*$ *is associative.*

Such "explanations" do little to enlighten newcomers to the field. So let us present the facts and connections in the most straightforward manner.

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- matrix calculus ($\mathrm{M}_{n\mathrm{x}n}$, $+$, $\cdot$) (Matrizenrechnung)
- switching algebra ($\{0,1\}$, $\vee$, $\wedge$) (Schaltalgebra)

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- The nature of the operations involved determines which axioms are satisfied and which are not.
- Depending on the the axioms found to hold, each algebraic structure has been given a name by the mathematics community.

Axioms I

Consider a set of elements S and a first binary operation $\boxplus$ (zweistellig)

1. Closure wrt $\boxplus$:
   if a and b are in S then $a \boxplus b$ is also in S.
2. Associative law wrt $\boxplus$:
   $(a \boxplus b) \boxplus c = a \boxplus (b \boxplus c)$.
3. Identity element wrt $\boxplus$ ("zero"): (Neutralelement der Addition)
   There is a unique element e such that
   $a \boxplus e = e \boxplus a = a$ for any a.
4. Inverse element wrt $\boxplus$:
   For every a in S there is an inverse $-a$ such that
   $a \boxplus -a = -a \boxplus a = e$.
5. Commutative law wrt $\boxplus$:
   $a \boxplus b = b \boxplus a$.

Axioms II

Consider a second binary operation $\boxdot$ that takes precedence over $\boxplus$

6. Closure wrt $\boxdot$:
   if a and b are in S then $a \boxdot b$ is also in S.
7. Associative law wrt $\boxdot$:
   $(a \boxdot b) \boxdot c = a \boxdot (b \boxdot c)$.
8. Identity element wrt $\boxdot$ ("unity"): (Neutralelement der Multiplikation)
   There is a unique element i such that
   $a \boxdot i = i \boxdot a = a$ for any a.
9. Inverse element wrt $\boxdot$:
   For every a in S there is an inverse a^{-1} such that
   $a \boxdot a^{-1} = a^{-1} \boxdot a = i$, the only exception is e
   for which no inverse exists.
10. Commutative law wrt $\boxdot$:
    $a \boxdot b = b \boxdot a$.

Axioms III

... continued

11. Distributive law of $\boxdot$ over $\boxplus$:
    $a \boxdot (b \boxplus c) = a \boxdot b \boxplus a \boxdot c$ (left-distributive) and
    $(a \boxplus b) \boxdot c = a \boxdot c \boxplus b \boxdot c$ (right-distributive).

12. Distributive law of $\boxplus$ over $\boxdot$:
    $a \boxplus b \boxdot c = (a \boxplus b) \boxdot (a \boxplus c)$ (left-distributive) and
    $a \boxdot b \boxplus c = (a \boxplus c) \boxdot (b \boxplus c)$ (right-distributive).

13. Complement:
    For every a in S there is a complement $\overline{a}$ such that
    $a \boxplus \overline{a} = i$ and
    $a \boxdot \overline{a} = e$.

Algebraic structures at a glance

Name of algebraic structure	Opera-tions	Axioms satisfied												
		1	2	3	4	5	6	7	8	9	10	11	12	13
Set (Menge)														
Semigroup (Halbgruppe)	⊞	1	2											
Monoid	⊞	1	2	3										
Group (Gruppe)	⊞	1	2	3	4									
Abelian or commutative group	⊞	1	2	3	4	5								
Abelian semigroup	⊞	1	2			5								
Abelian monoid	⊞	1	2	3		5								
Ring	⊞⊡	1	2	3	4	5	6	7				11		
Ring with unity	⊞⊡	1	2	3	4	5	6	7	8			11		
Division ring aka skew field	⊞⊡	1	2	3	4	5	6	7	8	9		11		
Field (Körper)	⊞⊡	1	2	3	4	5	6	7	8	9	10	11		
Commutative ring	⊞⊡	1	2	3	4	5	6	7			10	11		
Commutative ring with unity	⊞⊡	1	2	3	4	5	6	7	8		10	11		
Semiring (Halbring)	⊞⊡	1	2			5	6	7				11		
Commutative semiring	⊞⊡	1	2			5	6	7			10	11		
Boolean algebra	⊞⊡	1	2	3		5	6	7	8		10	11	12	13

Examples of algebraic structures with one operation

- The set S_{DNA} of all possible DNA sequences of non-zero length with characters from {A,T,C,G} together with the binary operation of string concatenation $\smile$ forms an infinite semigroup (S_{DNA}, $\smile$).

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- All permutations of a given number of elements form a group when combined with binary composition of such permutations as sole operation.

  As an example, consider all six distinct rearrangements of three elements:

$$\begin{array}{ccc} 1. & 2. & 3. \\ \downarrow & \downarrow & \downarrow \\ 1. & 2. & 3. \end{array} \quad \begin{array}{ccc} 1. & 2. & 3. \\ \downarrow & \downarrow & \downarrow \\ 2. & 3. & 1. \end{array} \quad \begin{array}{ccc} 1. & 2. & 3. \\ \downarrow & \downarrow & \downarrow \\ 3. & 1. & 2. \end{array} \quad \begin{array}{ccc} 1. & 2. & 3. \\ \downarrow & \downarrow & \downarrow \\ 3. & 2. & 1. \end{array} \quad \begin{array}{ccc} 1. & 2. & 3. \\ \downarrow & \downarrow & \downarrow \\ 1. & 3. & 2. \end{array} \quad \begin{array}{ccc} 1. & 2. & 3. \\ \downarrow & \downarrow & \downarrow \\ 2. & 1. & 3. \end{array}$$

  With S_3 denoting this set of permutations and $\circ$ binary composition, $(S_3, \circ)$ is a finite group.

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- $(\mathbb{Z}, +)$ is an infinite Abelian group.

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  The set of all quotients of two polynomials $P(x)$ and $Q(x)$ with real-valued coefficients together with addition and multiplication makes up for yet another infinite field $(\frac{P(x)}{Q(x)}, +, \cdot)$.
- All square matrices M_{nxn} with coefficients taken from a field together with matrix addition and matrix multiplication form an infinite ring with unity (muliplication is not commutative, inverse exists only if $||\mathrm{M}_{nxn}|| \neq 0$).

Examples of algebraic structures with two operations II

- Any subset of integers $S = \{0, 1, ..., p-1\}$ forms a finite field together with addition mod p and multiplication mod p iff p is a prime number. The term for this is Galois field GF(p).

  GF(5) ($\{0,1,2,3,4\}$, $+ \bmod 5$, $\cdot \bmod 5$) is the Galois field for $p = 5$. Addition and multiplication tables are as follows

$\boxplus = + \bmod 5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\boxdot = \cdot \bmod 5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
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- As opposed to this, ($\{0,1,...,m-1\}$, $+ \bmod m$, $\cdot \bmod m$) where m is not prime merely forms a finite commutative ring with unity as 0 is not the only element that lacks a multiplicative inverse. In the occurrence of ($\{0,1,2,3,4,5,6,7,8\}$, $+ \bmod 9$, $\cdot \bmod 9$), this also applies to 3 and 6.

Examples of algebraic structures with two operations III

- Cardinalities of finite fields are not confined to prime numbers.
  A so-called extension field GF(p^n) can be defined for any power p^n provided $2 \leq n \in \mathbb{N}^+$.

  All polynomials $P(x)$ of degree $0, 1, ..., n-1$ with coefficients from GF(p) constitute the set of elements. The first operation is addition mod $M(x)$ and the second one multiplication mod $M(x)$ where $M(x)$ is an irreducible polynomial[1] of degree n with coefficients from GF(p).

  GF(3^2), for instance, consists of the nine elements
  $\{0\ ,\ 1\ ,\ 2\ ,\ x\ ,\ x+1\ ,\ x+2\ ,\ 2x\ ,\ 2x+1\ ,\ 2x+2\}$,
  the operations are $+ \bmod (x^2+1)$ and $\cdot \bmod (x^2+1)$
  with $M(x) = x^2 + 1$ being an irreducible polynomial.

[1] Cannot be expressed as a product of non-trivial polynomials of lower degree.

Examples of algebraic structures with two operations IV

- The factors of 30 together with operations least common multiple (lcm) and greatest common divisor $(\gcd)$ constitute a Boolean algebra of eight elements $(\{1,2,3,5,6,10,15,30\}, \mathrm{lcm}, \gcd)$.
  Taking the complement $\overline{a}$ is tantamount to computing $\frac{30}{a}$.

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  Let $\Omega = \{a,b,c\}$. The set of all sets that can be composed from those elements $\{\,\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\,\}$ is called power set $\mathfrak{P}(\Omega)$. $(\mathfrak{P}(\Omega), \cup, \cap)$ forms another Boolean algebra where the empty set $\emptyset$ and the universal set Ω act as identity elements e and i respectively. Each element $x \in \mathfrak{P}(\Omega)$ has a complement $\overline{x} = \Omega - x$.

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- The well-known switching algebra ($\{0,1\}$, $\vee$, $\wedge$) is a Boolean algebra with just two elements. The complement of an element is its logic inverse.

Examples of algebraic structures with two operations V

- The class of semirings is very broad and encompasses:

Constituent	S	$\boxplus$	$\boxdot$
the commutative semiring of natural numbers	$\mathbb{N}$	$+$	$\cdot$
the commutative ring with unity of integers	$\mathbb{Z}$	$+$	$\cdot$
the "ordinary" fields	$\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$	$+$	$\cdot$
all Galois fields, e.g.	$\{0,1\}$	$\oplus$	$\wedge$
all other fields, e.g.	$\frac{P(x)}{Q(x)}$	$+$	$\cdot$
the switching algebra	$\{0,1\}$	$\vee$	$\wedge$
other finite Boolean algebras, e.g.	$\{1,2,3,4,6,12\}$	lcm	gcd
all other Boolean algebras, e.g.	$\mathfrak{P}(\Omega)$	$\cup$	$\cap$
the path algebras, e.g.	$\{0,1\}$	$\max$	$\min$
	$\mathbb{R} \cup \{\infty\}$	$\min$	$+$
	$\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$	$\max$	$\cdot$

Note

Many computational problems can be formulated in a semiring.

Engineering applications

Some of the not-so-common algebraic structures prove extremely helpful for studying real-world phenomena.

- AC circuits ← field of complex numbers.
- Digital circuits ← switching algebra.
- Error correction coding ← Galois fields.
- Cryptography ← all sorts of algebraic structures.
- Certain optimization problems ← path algebras.
- ...