

A Brief Glossary of Algebraic Structures

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Many mathematical texts define algebraic structures recursively, e.g.

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In a *ring* $(S, +, \cdot)$, $(S, +)$ is an *Abelian group*, (S, \cdot) is a *semigroup*, and $a \cdot (b + c) = a \cdot b + a \cdot c$, $(b + c) \cdot a = b \cdot a + c \cdot a$. (\cdot is distributive)

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A *group* $(S, *)$ is a *monoid* such that each element x has a unique inverse x^{-1} where $x^{-1} * x = x * x^{-1} = e$.

A *monoid* $(S, *)$ is a *semigroup* with unity e such that $e * x = x * e = x$ for all x and where e is unique.

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Such “explanations” do little to enlighten newcomers to the field. So let us present the facts and connections in the most straightforward manner.

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- matrix calculus ($M_{n \times n}, +, \cdot$) (Matrizenrechnung)
- switching algebra ($\{0,1\}, \vee, \wedge$) (Schaltalgebra)

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- ▶ a set of elements S
- ▶ one or more operations (denoted \boxplus, \boxdot, \dots)
- ▶ The nature of the operations involved determines which axioms are satisfied and which are not.
- ▶ Depending on the the axioms found to hold, each algebraic structure has been given a name by the mathematics community.

Axioms I

Consider a set of elements S and a first binary operation \boxplus (zweistellig)

1. Closure wrt \boxplus :
if a and b are in S then $a \boxplus b$ is also in S .
2. Associative law wrt \boxplus :
 $(a \boxplus b) \boxplus c = a \boxplus (b \boxplus c)$.
3. Identity element wrt \boxplus (“zero”): (Neutralelement der Addition)
There is a unique element e such that
 $a \boxplus e = e \boxplus a = a$ for any a .
4. Inverse element wrt \boxplus :
For every a in S there is an inverse $-a$ such that
 $a \boxplus -a = -a \boxplus a = e$.
5. Commutative law wrt \boxplus :
 $a \boxplus b = b \boxplus a$.

Axioms II

Consider a second binary operation \square that takes precedence over \boxplus

6. Closure wrt \square :
if a and b are in S then $a \square b$ is also in S .
7. Associative law wrt \square :
 $(a \square b) \square c = a \square (b \square c)$.
8. Identity element wrt \square ("unity"): (Neutralelement der Multiplikation)
There is a unique element i such that
 $a \square i = i \square a = a$ for any a .
9. Inverse element wrt \square :
For every a in S there is an inverse a^{-1} such that
 $a \square a^{-1} = a^{-1} \square a = i$, the only exception is e
for which no inverse exists.
10. Commutative law wrt \square :
 $a \square b = b \square a$.

Axioms III

... continued

11. Distributive law of \square over \boxplus :

$$a \square (b \boxplus c) = a \square b \boxplus a \square c \text{ (left-distributive) and} \\ (a \boxplus b) \square c = a \square c \boxplus b \square c \text{ (right-distributive).}$$

12. Distributive law of \boxplus over \square :

$$a \boxplus b \square c = (a \boxplus b) \square (a \boxplus c) \text{ (left-distributive) and} \\ a \square b \boxplus c = (a \boxplus c) \square (b \boxplus c) \text{ (right-distributive).}$$

13. Complement:

For every a in S there is a complement \bar{a} such that

$$a \boxplus \bar{a} = i \text{ and}$$

$$a \square \bar{a} = e.$$

Algebraic structures at a glance

Name of algebraic structure	Operations	Axioms satisfied												
		1	2	3	4	5	6	7	8	9	10	11	12	13
Set (Menge)														
Semigroup (Halbgruppe)	\boxplus	1	2											
Monoid	\boxplus	1	2	3										
Group (Gruppe)	\boxplus	1	2	3	4									
Abelian or commutative group	\boxplus	1	2	3	4	5								
Abelian semigroup	\boxplus	1	2			5								
Abelian monoid	\boxplus	1	2	3		5								
Ring	\boxplus, \boxtimes	1	2	3	4	5	6	7				11		
Ring with unity	\boxplus, \boxtimes	1	2	3	4	5	6	7	8			11		
Division ring aka skew field	\boxplus, \boxtimes	1	2	3	4	5	6	7	8	9		11		
Field (Körper)	\boxplus, \boxtimes	1	2	3	4	5	6	7	8	9	10	11		
Commutative ring	\boxplus, \boxtimes	1	2	3	4	5	6	7			10	11		
Commutative ring with unity	\boxplus, \boxtimes	1	2	3	4	5	6	7	8		10	11		
Semiring (Halbring)	\boxplus, \boxtimes	1	2			5	6	7				11		
Commutative semiring	\boxplus, \boxtimes	1	2			5	6	7			10	11		
Boolean algebra	\boxplus, \boxtimes	1	2	3		5	6	7	8		10	11	12	13

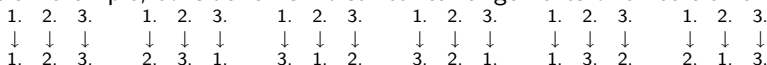
Examples of algebraic structures with one operation

- The set S_{DNA} of all possible DNA sequences of non-zero length with characters from $\{A,T,C,G\}$ together with the binary operation of string concatenation \smile forms an infinite **semigroup** (S_{DNA}, \smile) .

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- ▶ All permutations of a given number of elements form a **group** when combined with binary composition of such permutations as sole operation.

As an example, consider all six distinct rearrangements of three elements:



With S_3 denoting this set of permutations and \circ binary composition, (S_3, \circ) is a finite group.

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↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
1.	2.	3.	2.	3.	1.	3.	1.	2.	3.	2.	1.	1.	3.	2.	2.	1.	3.

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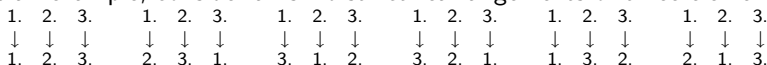
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- ▶ When supplemented with 0, the above structure turns into an **Abelian monoid** $(\mathbb{N}, +)$.
- ▶ $(\mathbb{Z}, +)$ is an infinite **Abelian group**.

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- ▶ All square matrices $M_{n \times n}$ with coefficients taken from a field together with matrix addition and matrix multiplication form an infinite **ring with unity** (multiplication is not commutative, inverse exists only if $\|M_{n \times n}\| \neq 0$).

Examples of algebraic structures with two operations II

- Any subset of integers $S = \{0, 1, \dots, p-1\}$ forms a **finite field** together with addition mod p and multiplication mod p iff p is a prime number. The term for this is **Galois field** $\text{GF}(p)$.

$\text{GF}(5) (\{0, 1, 2, 3, 4\}, + \bmod 5, \cdot \bmod 5)$ is the Galois field for $p = 5$.

Addition and multiplication tables are as follows

$\boxplus = + \bmod 5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\boxdot = \cdot \bmod 5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
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- As opposed to this, $(\{0, 1, \dots, m-1\}, + \text{ mod } m, \cdot \text{ mod } m)$ where m is not prime merely forms a finite commutative ring with unity as 0 is not the only element that lacks a multiplicative inverse. In the occurrence of $(\{0, 1, 2, 3, 4, 5, 6, 7, 8\}, + \text{ mod } 9, \cdot \text{ mod } 9)$, this also applies to 3 and 6.

Examples of algebraic structures with two operations III

- Cardinalities of finite fields are not confined to prime numbers.
A so-called **extension field** $\text{GF}(p^n)$ can be defined for any power p^n provided $2 \leq n \in \mathbb{N}^+$.

All polynomials $P(x)$ of degree $0, 1, \dots, n-1$ with coefficients from $\text{GF}(p)$ constitute the set of elements. The first operation is addition mod $M(x)$ and the second one multiplication mod $M(x)$ where $M(x)$ is an irreducible polynomial¹ of degree n with coefficients from $\text{GF}(p)$.

$\text{GF}(3^2)$, for instance, consists of the nine elements $\{0, 1, 2, x, x+1, x+2, 2x, 2x+1, 2x+2\}$, the operations are $+\text{mod}(x^2+1)$ and $\cdot\text{mod}(x^2+1)$ with $M(x) = x^2+1$ being an irreducible polynomial.

¹Cannot be expressed as a product of non-trivial polynomials of lower degree.

Examples of algebraic structures with two operations IV

- The factors of 30 together with operations least common multiple (lcm) and greatest common divisor (gcd) constitute a **Boolean algebra** of eight elements ($\{1,2,3,5,6,10,15,30\}$, lcm, gcd).
Taking the complement \bar{a} is tantamount to computing $\frac{30}{a}$.

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Let $\Omega = \{a,b,c\}$. The set of all sets that can be composed from those elements $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ is called power set $\mathfrak{P}(\Omega)$. $(\mathfrak{P}(\Omega), \cup, \cap)$ forms another Boolean algebra where the empty set \emptyset and the universal set Ω act as identity elements e and i respectively. Each element $x \in \mathfrak{P}(\Omega)$ has a complement $\bar{x} = \Omega - x$.

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- The well-known **switching algebra** $(\{0,1\}, \vee, \wedge)$ is a Boolean algebra with just two elements. The complement of an element is its logic inverse.

Examples of algebraic structures with two operations V

- The class of **semirings** is very broad and encompasses:

Constituent	S	\boxplus	\boxdot
the commutative semiring of natural numbers	\mathbb{N}	$+$	\cdot
the commutative ring with unity of integers	\mathbb{Z}	$+$	\cdot
the “ordinary” fields	$\mathbb{Q}, \mathbb{R}, \text{ or } \mathbb{C}$	$+$	\cdot
all Galois fields, e.g.	$\{0,1\}$	\oplus	\wedge
all other fields, e.g.	$\frac{P(x)}{Q(x)}$	$+$	\cdot
the switching algebra	$\{0,1\}$	\vee	\wedge
other finite Boolean algebras, e.g.	$\{1,2,3,4,6,12\}$	lcm	gcd
all other Boolean algebras, e.g.	$\mathfrak{P}(\Omega)$	\cup	\cap
the path algebras, e.g.	$\{0,1\}$	\max	\min
	$\mathbb{R} \cup \{\infty\}$	\min	$+$
	$\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$	\max	\cdot

Note

Many computational problems can be formulated in a semiring.

Engineering applications

Some of the not-so-common algebraic structures prove extremely helpful for studying real-world phenomena.

- ▶ AC circuits ← field of complex numbers.
- ▶ Digital circuits ← switching algebra.
- ▶ Error correction coding ← Galois fields.
- ▶ Cryptography ← all sorts of algebraic structures.
- ▶ Certain optimization problems ← path algebras.
- ▶ ...