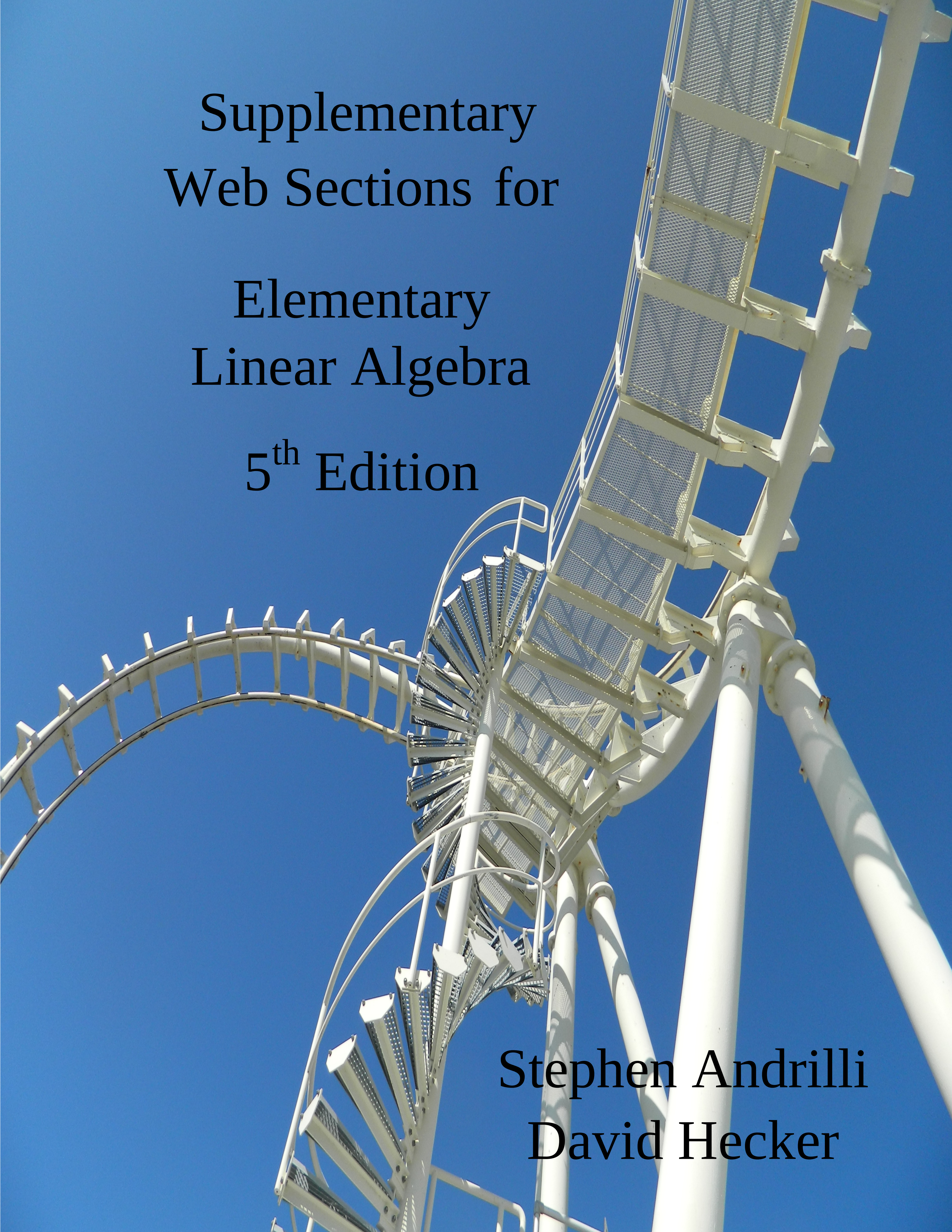

Supplementary Web Sections for

Elementary Linear Algebra

5th Edition

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Lines and Planes and the Cross Product in $\mathbb{R}^3$

Prerequisite: Section 1.2: The Dot Product

This section covers material which may already be familiar to you from analytic geometry. We will discuss analytic representations for lines and planes in $\mathbb{R}^3$. We will also introduce a new operation for vectors in $\mathbb{R}^3$, the cross product, and show its usefulness in geometric and physical calculations.

▶ Parametric Representation of a Line in $\mathbb{R}^3$

We begin by finding equations to describe a given line in $\mathbb{R}^3$. A line is determined uniquely once a point on the line as well as a direction for the line are known. Consider the following example.

EXAMPLE 1 We will find equations that represent the line passing through the origin $(0, 0, 0)$ in the direction of the vector $[1, -2, 7]$ (see Figure 1). Notice that a point is on the line if and only if it is the terminal point of a vector that starts at $(0, 0, 0)$ and is parallel to $[1, -2, 7]$. Every such vector is, of course, a scalar multiple of $[1, -2, 7]$, and hence has the form $t[1, -2, 7] = [t, -2t, 7t]$, for some real number t. Therefore, the points on the line are all of the form (x, y, z), where $x = t$, $y = -2t$, and $z = 7t$. Taken together, these three equations completely describe the points lying on the line. ■

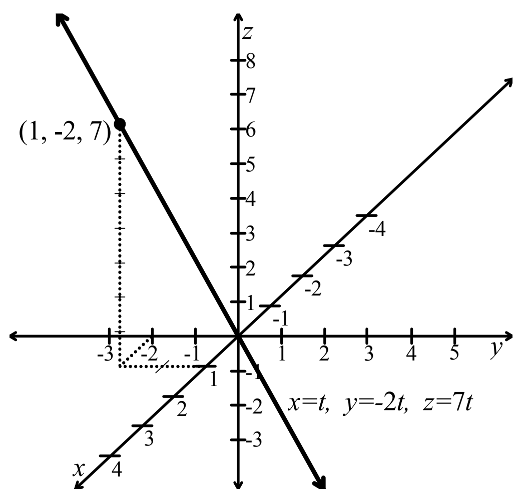

Figure 1 Line passing through the origin in the direction of $[1, -2, 7]$

The equations for the line in Example 1 are called **parametric equations**. The variable t in these equations is called the **parameter**. In general, to find parametric equations for the line passing through the point (x_0, y_0, z_0) in the direction of $\mathbf{v} = [a, b, c]$, we look for the terminal points of all vectors beginning at (x_0, y_0, z_0) that are parallel to $\mathbf{v}$ (see Figure 2).

Any vector parallel to $\mathbf{v}$ is of the form $[at, bt, ct]$, for some real number t, and since

$$[x_0, y_0, z_0] + [at, bt, ct] = [x_0 + at,\ y_0 + bt,\ z_0 + ct],$$

the terminal point of such a vector has the form $(x_0 + at, y_0 + bt, z_0 + ct)$. Therefore, we have proved the following theorem:

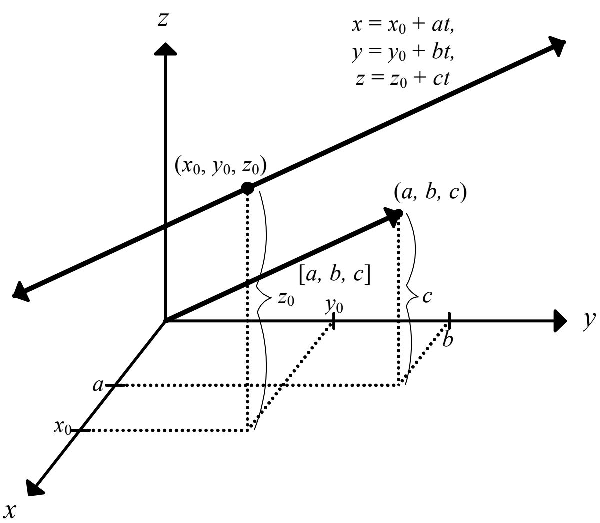

Figure 2 Line passing through (x_0, y_0, z_0) in the direction $[a, b, c]$

THEOREM 1
Parametric equations for the line l in $\mathbb{R}^3$ passing through (x_0, y_0, z_0) in the direction of $[a, b, c]$ are given by

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct,$$

where t represents a real parameter. That is, the points (x, y, z) in $\mathbb{R}^3$ which lie on l are precisely those which satisfy these equations for some real number t.

If we think of the parameter t as representing time (e.g., in seconds), and if we imagine an object starting at (x_0, y_0, z_0) at $t = 0$, traveling to new positions along the line l as the value of t changes, then the parametric equations for x, y, and z indicate the coordinates of the object at time t as it travels along l. Note that t can be negative (representing "past" time) as well as positive ("future" time).

We illustrate Theorem 1 with several examples.

EXAMPLE 2

We will find parametric equations for the line passing through $(-2, 7, 1)$ in the direction of the vector $[4, -3, 6]$, and then use these equations to find some other points on the line. By Theorem 1, the appropriate equations are:

$$x = -2 + 4t, \quad y = 7 - 3t, \quad z = 1 + 6t,$$

where $t \in \mathbb{R}$. Choosing arbitrary values for t in these equations will produce the coordinates of other points on the line. For example, letting $t = 1$ yields the point $(2, 4, 7)$. This is the terminal point of the vector $1[4, -3, 6]$ having initial point $(-2, 7, 1)$. Choosing $t = -2$ produces the point $(-10, 13, -11)$. This is the terminal point of the vector $-2[4, -3, 6]$ having initial point $(-2, 7, 1)$. ■

In the next example, we illustrate how to get the equation for a line when two points on the line are given. This example also shows that the parametric representation of a line is not unique.

EXAMPLE 3

We will calculate parametric equations for the line in $\mathbb{R}^3$ passing through $(7, 1, 1)$ and $(-3, 0, 5)$. In this case, we are not explicitly given the direction of the line. To find a vector in this direction, we take one of the points, say, $(-3, 0, 5)$, as the initial point, and the other, $(7, 1, 1)$, as the terminal point. This yields the direction vector $[7 - (-3), 1 - 0, 1 - 5] = [10, 1, -4]$. Then, using this vector together with the point $(7, 1, 1)$, we find that the parametric equations for the line are

$$x = 7 + 10t, \quad y = 1 + t, \quad z = 1 - 4t,$$

where $t \in \mathbb{R}$. Alternatively, notice that we could have used $(7, 1, 1)$ as the initial point and $(-3, 0, 5)$ as the terminal point in calculating the parametric equations. This choice gives us the direction vector $[-10, -1, 4]$ (why?), and we would then obtain the alternate parametric equations

$$x = -3 - 10s, \quad y = -s, \quad z = 5 + 4s,$$

where $s \in \mathbb{R}$. We used a different variable for the parameter in these last three equations to emphasize the fact that equal values of s and t do not correspond to the same point on the line. For example, $t = 0$ corresponds to the initial point $(7, 1, 1)$, while $s = 0$ produces $(-3, 0, 5)$. In order to produce $(-3, 0, 5)$ from the first set of parametric equations, we must use $t = -1$. ■

In the last example, notice that we also could have used any nonzero scalar multiple of $[10, 1, -4]$ as the direction vector. In particular, if we choose a unit vector in the direction of $[10, 1, -4]$ as the direction vector, the absolute value of the parameter t would represent the distance traveled along the line from the initial point.

In the next example, we consider two intersecting lines, and show how to find the point of intersection and the angle formed between the lines. Notice that whenever a pair of distinct lines intersects, there are at most two distinct angles formed, and these two angles are supplements of each other; that is, their angle measures sum to 180°. We define the **angle between two intersecting lines** as the minimum of these two angles (i.e., the angle that is $\leq \frac{\pi}{2}$ radians $= 90°$). We can find this angle by taking a vector in the direction of each line, calculating the angle between these vectors, and then taking the supplementary angle if necessary.

EXAMPLE 4 Let l_1 and l_2 be the lines with parametric equations

$$l_1\colon\ x = 8 - 5t,\ \ y = -3 + 2t,\ \ z = -7 + 7t, \qquad \text{where } t \in \mathbb{R},$$
$$\text{and} \quad l_2\colon\ x = 6 + 3s,\ \ y = -2 - s,\ \ z = 2 + 2s. \qquad \text{where } s \in \mathbb{R}.$$

First, let us determine if these lines intersect, and, if so, where. In order for l_1 and l_2 to intersect, we must find values for s and t such that all of the following equations are simultaneously satisfied:

$$\begin{cases} 8 - 5t &= 6 + 3s \\ -3 + 2t &= -2 - s \\ -7 + 7t &= 2 + 2s \end{cases}.$$

Solving for t in the first of these yields $t = -\frac{3}{5}s + \frac{2}{5}$. Substituting this into the second equation produces $-3+2(-\frac{3}{5}s+\frac{2}{5}) = -2-s$, which gives $s = -1$. Therefore, $t = 1$ (why?). We check that these values of s and t satisfy the third equation as well (they yield $0 = 0$), and therefore, the lines do intersect, and this occurs when $s = -1$ and $t = 1$. This intersection is at the point $(x, y, z) = (3, -1, 0)$ in $\mathbb{R}^3$ (why?).

Next, we determine the angle between these lines. To do this, we find a direction vector for each line, and then use the dot product to calculate the cosine of the angle θ between them. A vector in the direction of l_1 is $[-5, 2, 7]$ (because -5, 2, 7 are the coefficients of the parameters in the parametric equations for l_1) and a vector in the direction of l_2 is $[3, -1, 2]$ (because 3, -1, 2 are the coefficients of the parameters in the parametric equations for l_2). Therefore,

$$\cos\theta = \frac{[-5,2,7]\cdot[3,-1,2]}{\|[-5,2,7]\|\,\|[3,-1,2]\|} = \frac{-3}{\sqrt{78}\sqrt{14}} \approx -0.0908.$$

Therefore, $\theta \approx 95.2°$. Since this angle is greater than $90°$, the angle between the lines is the supplement of this, which is $\approx 84.8°$ (see Figure 3). ■

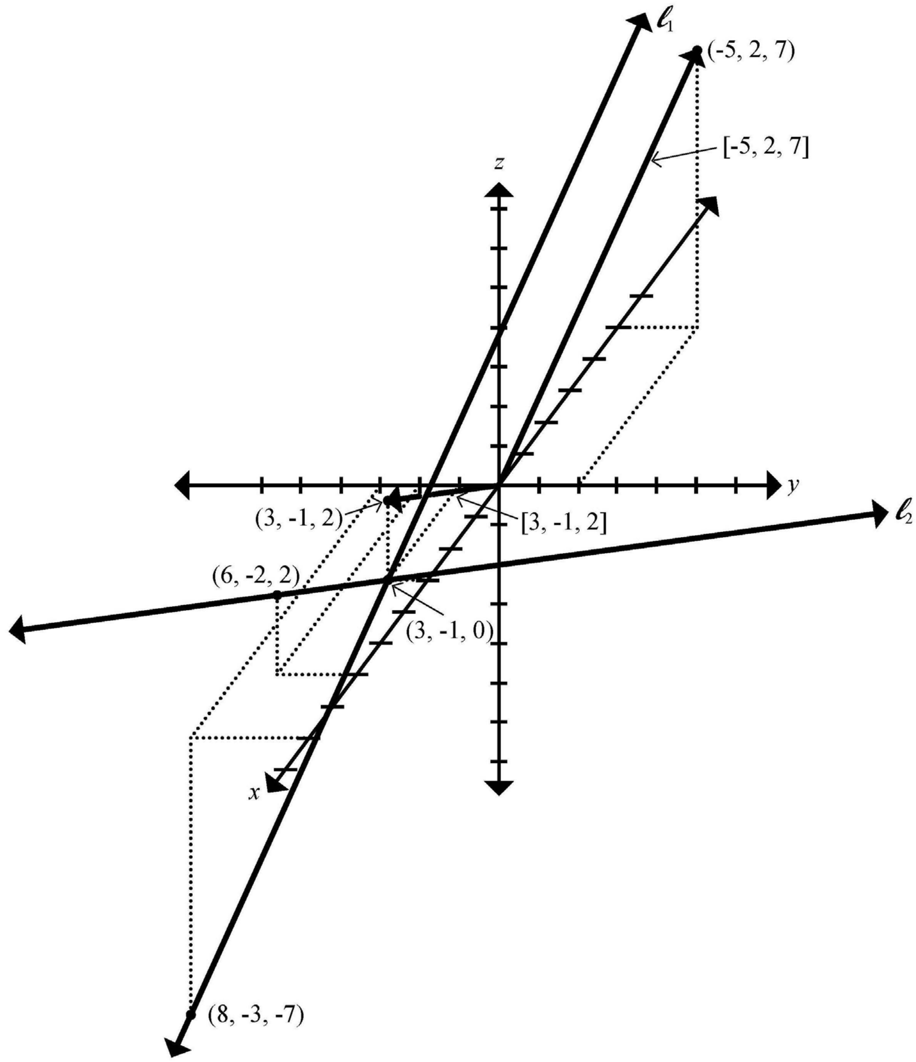

Figure 3 Lines l_1 and l_2 in Example 4

Notice in Example 4 that different parameters (t and s) were used to represent the lines l_1 and l_2. If, instead, we had used the same parameter t for both lines, finding the solution would not have been as straightforward. In particular, the equations for x would then be $x = 8 - 5t$ and $x = 6 + 3t$. Setting $8 - 5t = 6 + 3t$ would lead to $8t = 2$, or $t = \frac{1}{4}$, which does not agree with the value for t obtained in Example 4. This is because the true intersection point $(3, -1, 0)$ of the lines l_1 and l_2 occurs at *different* values of t on each line (at $t = 1$ for line l_1, and at $t = -1$ for line l_2). That is, different points on each line are produced by the same value of the parameter t. In effect, any value of the parameter t has a different "meaning" for each line. The moral here is that whenever we are working with two or more lines in the same problem, we must generally use a different parameter for each line.[1]

The next example illustrates a situation where two given lines have no point of intersection.

EXAMPLE 5 Consider the lines l_3 and l_4, whose parametric representations are

$$l_3\colon\ x = 5 - 3t,\ \ y = 4 + t,\ \ z = 3 - t, \qquad \text{where } t \in \mathbb{R},$$
$$\text{and} \quad l_4\colon\ x = 2 + 6s,\ \ y = -2s,\ \ z = 3 + 2s, \qquad \text{where } s \in \mathbb{R}.$$

Notice that l_3 is in a direction parallel to $[-3, 1, -1]$, and l_4 is in a direction parallel to $[6, -2, 2]$ (see Figure 4). Since $[6, -2, 2] = -2[-3, 1, -1]$, these two direction vectors are parallel. Therefore, l_3 and l_4 are either distinct parallel lines or different representations of the same line. To determine whether these lines are the same, we notice that $(5, 4, 3)$ lies on l_3. If $l_3 = l_4$, this point would also be on l_4, so there would be a real number s such that

$$5 = 2 + 6s, \quad 4 = -2s, \quad \text{and} \quad 3 = 3 + 2s.$$

The first of these equations yields $s = \frac{1}{2}$, which satisfies neither of the other equations. Therefore, $(5, 4, 3)$ does not lie on l_4, so l_3 and l_4 are distinct, parallel lines. ■

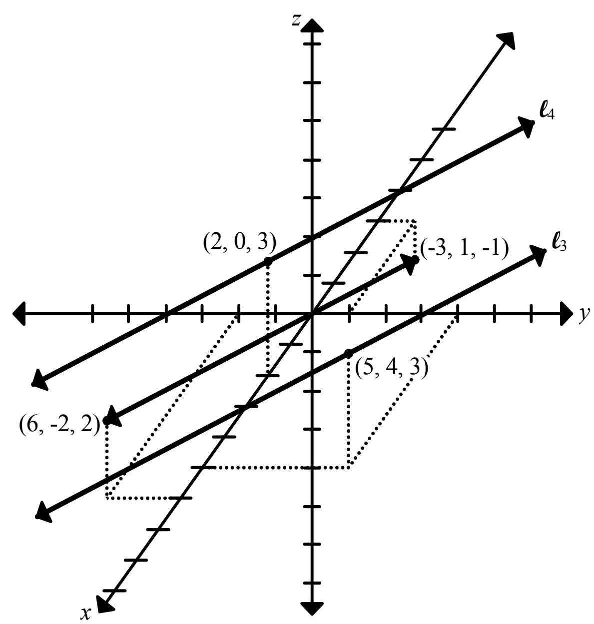

Figure 4 The lines l_3 and l_4 from Example 5, showing that $l_3 \parallel l_4$

In Example 5 the distinct lines l_3 and l_4 did not intersect because they are parallel. But in $\mathbb{R}^3$ (unlike $\mathbb{R}^2$), it is also possible for non-parallel lines to have no intersection at all. Such lines are called **skew lines**. For example, the lines

$$l_5\colon\ x = 2,\ \ y = t,\ \ z = 0,$$
$$\text{and} \quad l_6\colon\ x = t,\ \ y = 2,\ \ z = 1$$

have non-parallel direction vectors $[0, 1, 0]$ and $[1, 0, 0]$, respectively (why?), yet l_5 and l_6 never intersect since the z-coordinates of their points always differ.

[1] There are some applications where we might use the same parameter for both lines. For example, if the lines represent the paths that two objects are taking, and t represents time, then we can "synchonize the clocks" by using the same parameter for both lines. But this situation would be an exception to the general rule.

▶ Planes in $\mathbb{R}^3$

Unlike a line, a plane in $\mathbb{R}^3$ extends in many different directions. But all vectors *perpendicular* to a given plane are multiples of each other, a fact we will find useful when describing the plane. A vector perpendicular to a given plane is called a **normal vector** for that plane. In fact, a plane is completely determined once we know a point that lies in the plane and a normal vector for the plane.

Suppose (x_0, y_0, z_0) is a point lying in the plane $\mathcal{P}$, and suppose the vector $[a, b, c]$ is a normal vector for $\mathcal{P}$ (see Figure 5).

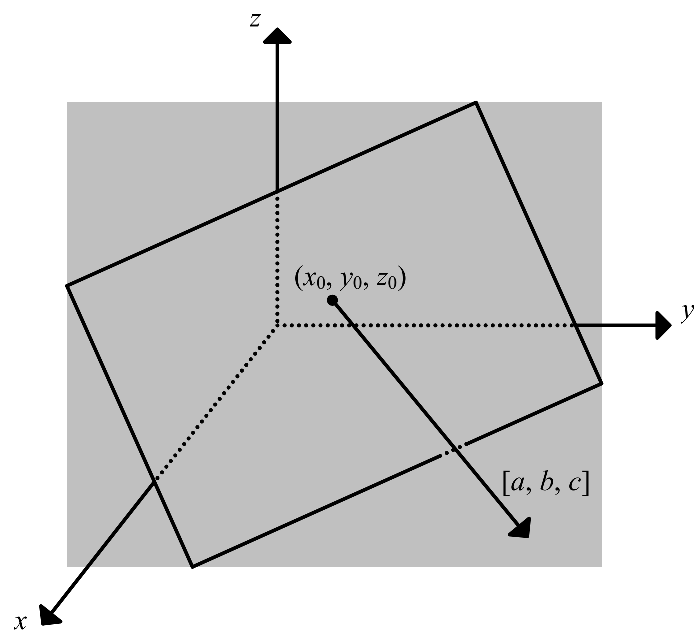

Figure 5 The plane in $\mathbb{R}^3$ perpendicular to $[a, b, c]$ and passing through (x_0, y_0, z_0)

Now, (x, y, z) is a point of $\mathcal{P}$ if and only if the vector $[x - x_0,\ y - y_0,\ z - z_0]$, with initial point at (x_0, y_0, z_0), lies entirely in the plane $\mathcal{P}$. Therefore, (x, y, z) lies in the plane precisely when this vector is perpendicular to the normal vector $[a, b, c]$; that is, if and only if

$$[a, b, c] \cdot [x - x_0, y - y_0, z - z_0] = 0, \quad \text{or,} \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Hence we have produced an equation for the plane. This result is stated in the following theorem.

THEOREM 2

Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ that passes through the point (x_0, y_0, z_0) having normal vector $[a, b, c]$. Then, $\mathcal{P}$ is precisely the set of points of the form (x, y, z) which satisfy the equation

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

EXAMPLE 6 The equation of the plane passing through $(x_0, y_0, z_0) = (1, -3, 1)$ and perpendicular to $[a, b, c] = [7, 1, 2]$ is

$$7x + y + 2z = 7 \cdot 1 + 1 \cdot (-3) + 2 \cdot 1, \text{ which is } 7x + y + 2z = 6.$$

■

The next example illustrates how to find a normal vector for a given plane.

EXAMPLE 7 Let $\mathcal{P}$ be the plane satisfying the equation $6x - 3y + 5z = 9$. Then, comparing this with the formula in Theorem 2, we find that $[6, -3, 5]$ is a normal vector for $\mathcal{P}$. A unit normal vector for $\mathcal{P}$ would be

$$\frac{[6,-3,5]}{\|[6,-3,5]\|} = \left[\frac{3\sqrt{70}}{35}, \frac{-3\sqrt{70}}{70}, \frac{\sqrt{70}}{14}\right] \approx [0.717, \ -0.359, \ 0.598].$$

■

Notice that if $[a, b, c]$ is a normal vector to a plane, then any nonzero scalar multiple of $[a, b, c]$ is also a normal vector to that plane as well.

When a pair of different planes intersects, there are at most two distinct angles formed, and these two angles are supplements of each other. These are, in fact, the same angles formed between normal vectors (in both directions) for the planes. Therefore, we define the **angle between two intersecting planes** as the minimum angle (i.e., the angle that is $\leq 90° = \frac{\pi}{2}$ radians) between a normal vector for one plane and a normal vector for the other. Exercise 7 explores this concept further.

You may recall from high school geometry that three non-collinear points in $\mathbb{R}^3$ uniquely determine a plane. Given three non-collinear points in $\mathbb{R}^3$, we can use Theorem 2 to find an equation for the plane that they determine once we calculate a normal vector for the plane. The next topic, the cross product of vectors, will furnish us with a method for finding a normal vector for a given plane.

► The Cross Product

The cross product operation for vectors in $\mathbb{R}^3$ takes two vectors and produces a third vector. This differs in two important ways from the dot product of vectors, which we discussed in Section 1.2. First, the cross product of two vectors will be another *vector*, not a scalar, as with the dot product. Second, the cross product is defined only for vectors in $\mathbb{R}^3$, while the dot product is defined in $\mathbb{R}^n$ for any positive integer n.

DEFINITION

Let $\mathbf{x} = [x_1, x_2, x_3]$ and $\mathbf{y} = [y_1, y_2, y_3]$ be two vectors in $\mathbb{R}^3$. Then, the **cross product** of $\mathbf{x}$ and $\mathbf{y}$, written $\mathbf{x} \times \mathbf{y}$, is the 3-vector

$$[(x_2y_3 - x_3y_2), \ (x_3y_1 - x_1y_3), \ (x_1y_2 - x_2y_1)].$$

The formula for the cross product is not difficult to remember. Notice that each coordinate of $\mathbf{x} \times \mathbf{y}$ is of the form $x_a y_b - x_b y_a$. The subscripts a and b are chosen so that in the ith coordinate of $\mathbf{x} \times \mathbf{y}$ neither a nor b equals i. For example, in the first coordinate of $\mathbf{x} \times \mathbf{y}$, only 2 and 3 are used as subscripts. Also, notice that in the first coordinate of $\mathbf{x} \times \mathbf{y}$, the x_2y_3 term is positive. In the second coordinate, the x_3y_1 term is positive. The subscripts of the positive term are always placed in the order in which they appear next to each other as the circle in Figure 6(a) is traversed in a *clockwise* direction. (When using this circle, be careful to write the x-factor of each term before the y-factor.) Another easy way to remember the cross product formula involves using the determinant of a 3×3 matrix, which is covered in Chapter 3. (See Exercise 8 of Section 3.1.)

EXAMPLE 8 We calculate several cross products:

(*a*) $[1, -3, -2] \times [2, -1, 0] = [((-3)\cdot 0 - (-2)\cdot(-1)), ((-2)\cdot 2 - 1\cdot 0), (1\cdot(-1) - (-3)\cdot 2)]$
$= [-2, -4, 5].$

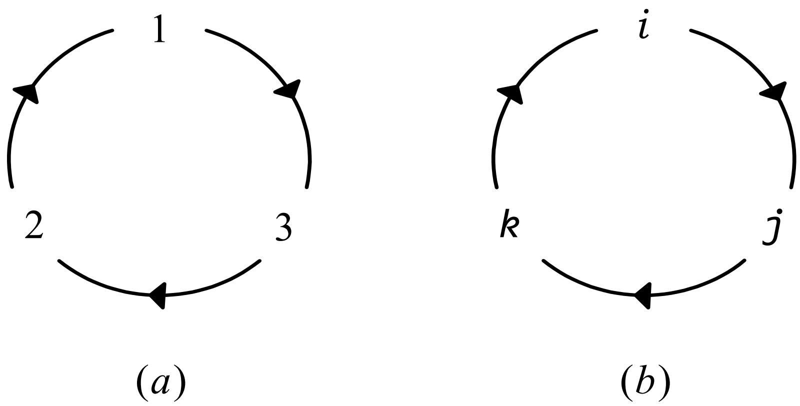

Figure 6 (a) Orders for subscripts in the cross product; (b) Orders for $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the cross product

$$
\begin{aligned}
(b)\quad \mathbf{i}\times\mathbf{j} &= [1,0,0]\times[0,1,0] = [(0\cdot 0-0\cdot 1),(0\cdot 0-1\cdot 0),(1\cdot 1-0\cdot 0)]\\
&= [0,0,1] = \mathbf{k}.\\
(c)\quad \mathbf{j}\times\mathbf{k} &= [0,1,0]\times[0,0,1] = [(1\cdot 1-0\cdot 0),(0\cdot 0-0\cdot 1),(0\cdot 0-1\cdot 0)]\\
&= [1,0,0] = \mathbf{i}.\\
(d)\quad \mathbf{k}\times\mathbf{i} &= [0,0,1]\times[1,0,0] = [(0\cdot 0-1\cdot 0),(1\cdot 1-0\cdot 0),(0\cdot 0-0\cdot 1)]\\
&= [0,1,0] = \mathbf{j}.\\
(e)\quad \mathbf{j}\times\mathbf{i} &= [0,1,0]\times[1,0,0] = [(1\cdot 0-0\cdot 0),(0\cdot 1-0\cdot 0),(0\cdot 0-1\cdot 1)]\\
&= [0,0,-1] = -\mathbf{k}.
\end{aligned}
$$
■

In the last example, we worked out four of the calculations in the following list, and the others are easily verified.

$$
\begin{aligned}
\mathbf{i}\times\mathbf{j} &= +\mathbf{k} & \mathbf{j}\times\mathbf{i} &= -\mathbf{k}\\
\mathbf{j}\times\mathbf{k} &= +\mathbf{i} & \mathbf{k}\times\mathbf{j} &= -\mathbf{i}\\
\mathbf{k}\times\mathbf{i} &= +\mathbf{j} & \mathbf{i}\times\mathbf{k} &= -\mathbf{j}.
\end{aligned}
$$

Notice that three of these cross products have "−" signs attached to the results. You should check that the result has a "+" sign attached if and only if the vectors to be crossed are adjacent when the circle in Figure 6(b) is traversed in a *clockwise* direction.

The calculations involving $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ above point out that the cross product is *not commutative*. In fact, part (1) of the next theorem shows that the cross product $\mathbf{x}\times\mathbf{y}$ is **anti-commutative** because $\mathbf{x}\times\mathbf{y} = -(\mathbf{y}\times\mathbf{x})$. That is, $\mathbf{x}\times\mathbf{y}$ is precisely the *reverse* vector of $\mathbf{y}\times\mathbf{x}$, since they have the same length, but $\mathbf{x}\times\mathbf{y}$ has the *opposite* direction as $\mathbf{y}\times\mathbf{x}$. Also note that the cross product is *not associative*: that is, in general, $(\mathbf{x}\times\mathbf{y})\times\mathbf{z} \neq \mathbf{x}\times(\mathbf{y}\times\mathbf{z})$. (See Exercise 11.)

▶ Basic Properties of the Cross Product

The next theorem lists many of the fundamental properties of the cross product operation.

THEOREM 3
(**Basic Properties of the Cross Product**)
Let $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ be vectors in $\mathbb{R}^3$, and let a be any real number. Then,

(1) $\mathbf{x} \times \mathbf{y} = -(\mathbf{y} \times \mathbf{x})$ (**Anti-Commutative Property**)

(2) $(a\mathbf{x}) \times \mathbf{y} = \mathbf{x} \times (a\mathbf{y}) = a(\mathbf{x} \times \mathbf{y})$ (**Scalar Associative Law**)

(3) $\mathbf{x} \times \mathbf{0} = \mathbf{0} \times \mathbf{x} = \mathbf{0}$ (**Zero Property**)

(4) $\mathbf{x} \times \mathbf{x} = \mathbf{0}$ (**Cancellation Property**)

(5) $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z})$ (**Distributive Law of Cross Product over Addition**)

(6) $\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\mathbf{x} \times \mathbf{y}) = 0$ (**Orthogonality**)

(7) $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$ (**Exchange of Cross and Dot Products)**

Part (6) of Theorem 3 gives us one of the most important properties of the cross product, since it asserts that if $\mathbf{x} \times \mathbf{y}$ is nonzero, then $\mathbf{x} \times \mathbf{y}$ is *perpendicular* (or, *orthogonal*) to both $\mathbf{x}$ and $\mathbf{y}$, since the dot product of $\mathbf{x} \times \mathbf{y}$ with each of $\mathbf{x}$ and $\mathbf{y}$ is zero. As we will see below, it is this property which makes the cross product useful in finding a normal vector for a plane. Part (7) is particularly amazing (and perhaps unexpected), showing that the roles of the cross product and dot product can be reversed if the order of the vectors involved does not change!

Before proving Theorem 3, we illustrate how some of the cross product properties can be used to simplify computations.

EXAMPLE 9 Notice that

$$
\begin{aligned}
& [2,2,-4] \times [-1,-1,2] \\
= \; & ((-2)[-1,-1,2]) \times [-1,-1,2] \\
= \; & (-2)([-1,-1,2] \times [-1,-1,2]) && \text{by part (2) of Theorem 3} \\
= \; & (-2)[0,0,0] && \text{by part (4) of Theorem 3} \\
= \; & [0,0,0].
\end{aligned}
$$

Of course, a brute-force calculation of the cross product yields

$$[2,2,-4] \times [-1,-1,2] \;=\; [(2\cdot 2-(-4)\cdot(-1),\; ((-4)\cdot(-1)-2\cdot 2),\; (2\cdot(-1)-2\cdot(-1))],$$

which equals $[0,0,0]$, as expected. ■

Proof **Proof of Theorem 3 (Abridged):** The proofs of all parts of Theorem 3 are done by brute force computation. It is enough to simplify each expression in each equation until there is a single vector or number on both sides of the equation. At that point, it will be obvious that both sides are equal. We prove part (1) and half of part (6), and ask you to do the remaining parts in Exercise 9. Throughout the proof, we assume that $\mathbf{x} = [x_1, x_2, x_3]$ and $\mathbf{y} = [y_1, y_2, y_3]$ are vectors in $\mathbb{R}^3$.

Proof of Part (1): Using the definition of the cross product, we see that

$$
\begin{aligned}
-(\mathbf{y} \times \mathbf{x}) &= -([(y_2x_3 - y_3x_2),\; (y_3x_1 - y_1x_3),\; (y_1x_2 - y_2x_1)] \\
&= [(-x_3y_2 + x_2y_3),\; (-x_1y_3 + x_3y_1),\; (-x_2y_1 + x_1y_2)],
\end{aligned}
$$

which is clearly equal to $\mathbf{x} \times \mathbf{y}$, completing the proof of part (1).

Proof of Part (6): We will compute $\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y})$ and show that the result is zero.

$$\begin{aligned}
&\text{Now, } \mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) \\
&= [x_1, x_2, x_3] \cdot ([(x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), (x_1y_2 - x_2y_1)]) \\
&= x_1(x_2y_3 - x_3y_2) + x_2(x_3y_1 - x_1y_3) + x_3(x_1y_2 - x_2y_1) \\
&= x_1x_2y_3 - x_1x_3y_2 + x_2x_3y_1 - x_2x_1y_3 + x_3x_1y_2 - x_3x_2y_1 \\
&= 0,
\end{aligned}$$

because the first and fourth, second and fifth, and third and sixth terms, respectively, cancel each other. **QED**

► Magnitude and Direction of the Cross Product

Next, we derive a formula for the magnitude of the cross product of two vectors, and then discuss the direction of the cross product vector in more detail.

THEOREM 4
Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^3$. Then,

$$\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\| \sin\theta,$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

We will illustrate Theorem 4 with an example before we present its proof.

EXAMPLE 10 Consider the vectors $\mathbf{x} = [1, -4, -1]$ and $\mathbf{y} = [4, -1, -1]$, and let θ represent the angle between $\mathbf{x}$ and $\mathbf{y}$. Then,

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{9}{\sqrt{18}\sqrt{18}} = \tfrac{1}{2}.$$

Therefore, $\theta = \arccos(\frac{1}{2}) = \frac{\pi}{3}$ (or $60°$), and $\sin\theta = \frac{\sqrt{3}}{2}$. Hence,

$$\|\mathbf{x}\|\|\mathbf{y}\| \sin\theta = \sqrt{18}\sqrt{18}\,\frac{\sqrt{3}}{2} = 9\sqrt{3}.$$

However, a brute-force computation of $\|\mathbf{x} \times \mathbf{y}\|$ gives

$$\|\mathbf{x} \times \mathbf{y}\| = \|[1, -4, -1] \times [4, -1, -1]\| = \|[3, -3, 15]\| = \sqrt{243},$$

which also equals $9\sqrt{3}$. ■

Proof **Proof of Theorem 4:** Let θ be the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$. Since the angle between vectors is always between 0 and π, we have $\sin\theta \geq 0$. Therefore,

$$\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\| \sin\theta$$

is true if and only if

$$\|\mathbf{x} \times \mathbf{y}\|^2 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \sin^2\theta,$$

holds, since all terms involved are nonnegative. But,

$$\begin{aligned}
\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \sin^2\theta &= \|\mathbf{x}\|^2\|\mathbf{y}\|^2(1 - \cos^2\theta) \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \left(1 - \left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right)^2\right) \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2.
\end{aligned}$$

Hence, it is enough to show that

$$\|\mathbf{x}\times\mathbf{y}\|^2 \;=\; \|\mathbf{x}\|^2\|\mathbf{y}\|^2-(\mathbf{x}\cdot\mathbf{y})^2.$$

Letting $\mathbf{x}=[x_1,x_2,x_3]$, and $\mathbf{y}=[y_1,y_2,y_3]$, a lengthy computation shows that both sides are equal to

$$x_1^2y_2^2+x_1^2y_3^2+x_2^2y_1^2+x_2^2y_3^2+x_3^2y_1^2+x_3^2y_2^2-2x_1x_2y_1y_2-2x_1x_3y_1y_3-2x_2x_3y_2y_3,$$

and we are done. **QED**

The following corollary of Theorem 4 gives an alternate test for parallel vectors in $\mathbb{R}^3$, since it states that *two nonzero vectors are parallel if and only if their cross product is zero.* You are asked to prove this result in Exercise 14.

COROLLARY 5
Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^3$. Then, $\mathbf{x}\times\mathbf{y}=\mathbf{0}$ if and only if $\mathbf{y}=a\mathbf{x}$ for some real number a.

Suppose $\mathbf{x}$ and $\mathbf{y}$ are two nonzero vectors in $\mathbb{R}^3$. In Theorem 4 we have a formula for the *magnitude* of $\mathbf{x}\times\mathbf{y}$, and we now turn our attention to the *direction* of $\mathbf{x}\times\mathbf{y}$. If $\mathbf{x}$ and $\mathbf{y}$ are parallel, Corollary 5 shows us that $\mathbf{x}\times\mathbf{y}$ is the zero vector, and so it has no direction. If $\mathbf{x}$ and $\mathbf{y}$ are not parallel, then part (6) of Theorem 3 tells us that $\mathbf{x}\times\mathbf{y}$ is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$. If we use the same initial point for $\mathbf{x}$ and $\mathbf{y}$, then $\mathbf{x}\times\mathbf{y}$ must be perpendicular to the plane which $\mathbf{x}$ and $\mathbf{y}$ determine. However, in $\mathbb{R}^3$, there are two opposite directions that such a vector might take. (See Figure 7.)

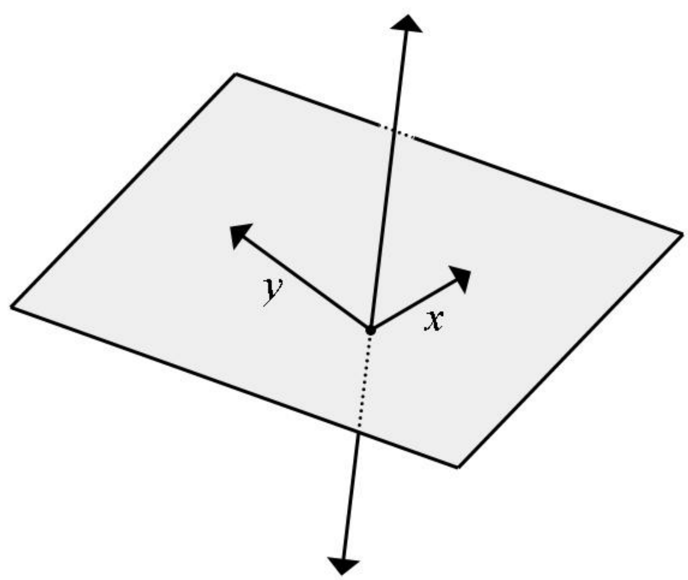

Figure 7 Two directions perpendicular to both $\mathbf{x}$ and $\mathbf{y}$

The **Right-Hand Rule** indicates the correct choice of direction for the cross product.

Right-Hand Rule If $\mathbf{x}$ and $\mathbf{y}$ are non-parallel nonzero vectors in $\mathbb{R}^3$, then the vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}\times\mathbf{y}$ (taken in that order) form a **right-handed system**.

An informal definition of a right-handed system can be given as follows: Suppose **x** and **y** are non-parallel nonzero vectors in $\mathbb{R}^3$, and **z** is any vector perpendicular to both **x** and **y**. Then **x**, **y**, and **z** (taken in that order) form a right-handed system if curling the fingers of your *right* hand from the vector **x** toward the vector **y** makes your thumb point in the direction of **z** (see Figure 8).[2] The Right-Hand Rule therefore states that $\mathbf{x} \times \mathbf{y}$ must point in the direction that makes the vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} \times \mathbf{y}$ (taken in that order) form a right-handed system.

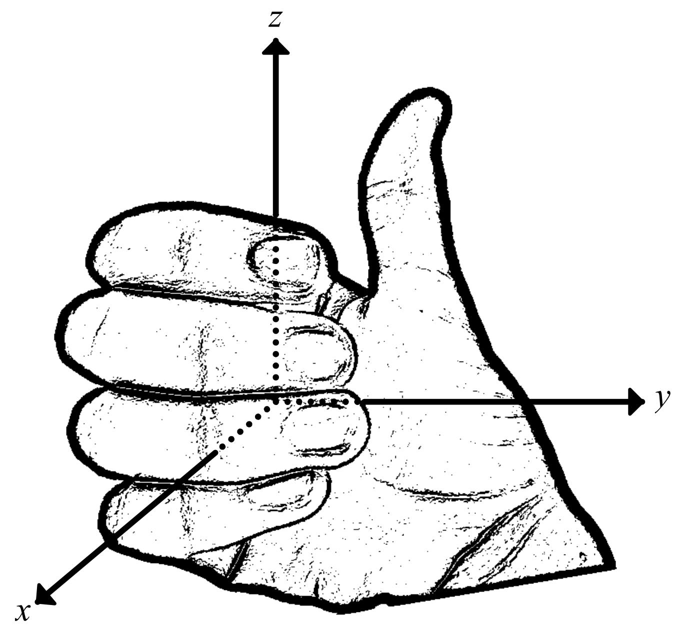

Figure 8 A right-handed system

We cannot give a proof of the Right-Hand Rule as a formal theorem since we have only informally defined what it means for a system to be right-handed. Our definition involves concepts we have not made mathematically precise here, such as "curling the fingers" from one vector to another (or, in the footnote, "looking down at the plane" from a particular side), which are beyond the scope of this section.

▸ Finding Equations for Lines and Planes Using the Cross Product

We now present several ways in which the cross product can be used to discover additional information about lines and planes.

Plane determined by three noncollinear points. As mentioned earlier, any three noncollinear points determine a unique plane. To find the equation of the plane, we first use the cross product to obtain the normal vector for the plane. We illustrate this method with an example.

EXAMPLE 11 Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ containing the points $P_1 = (1, -3, -2)$, $P_2 = (3, -4, -1)$, and $P_3 = (4, -1, -3)$. We will find an equation for $\mathcal{P}$. First, note that any vector which has both its initial and terminal points in $\mathcal{P}$ lies entirely in that plane.

[2] A slightly more formal way to define a right-handed system is as follows: Assume **x** and **y** are non-parallel nonzero vectors in $\mathbb{R}^3$ with the same initial point, and that **z** is perpendicular to both **x** and **y**. Then $\mathbf{x}, \mathbf{y}, \mathbf{z}$ (taken in that order) forms a right-handed system if, looking down at the plane formed by **x** and **y** *from the direction in which* **z** *points*, the angle from **x** to **y** in the *counterclockwise* direction has measure between 0 and π.

Therefore, the vectors $\mathbf{v} = [2, -1, 1]$ (initial point P_1, terminal point P_2) and $\mathbf{w} = [3, 2, -1]$ (initial point P_1, terminal point P_3) each lie entirely in $\mathcal{P}$. Hence, the vector $\mathbf{n} = \mathbf{v} \times \mathbf{w} = [2, -1, 1] \times [3, 2, -1] = [-1, 5, 7]$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. Using $\mathbf{n} = [-1, 5, 7]$ as a normal vector for $\mathcal{P}$ together with either P_1, P_2, or P_3 in Theorem 2 produces the equation $-x + 5y + 7z = -30$ for $\mathcal{P}$. (See Figure 9, where the vector $\mathbf{n}$ is drawn with P_1 as its initial point.) ■

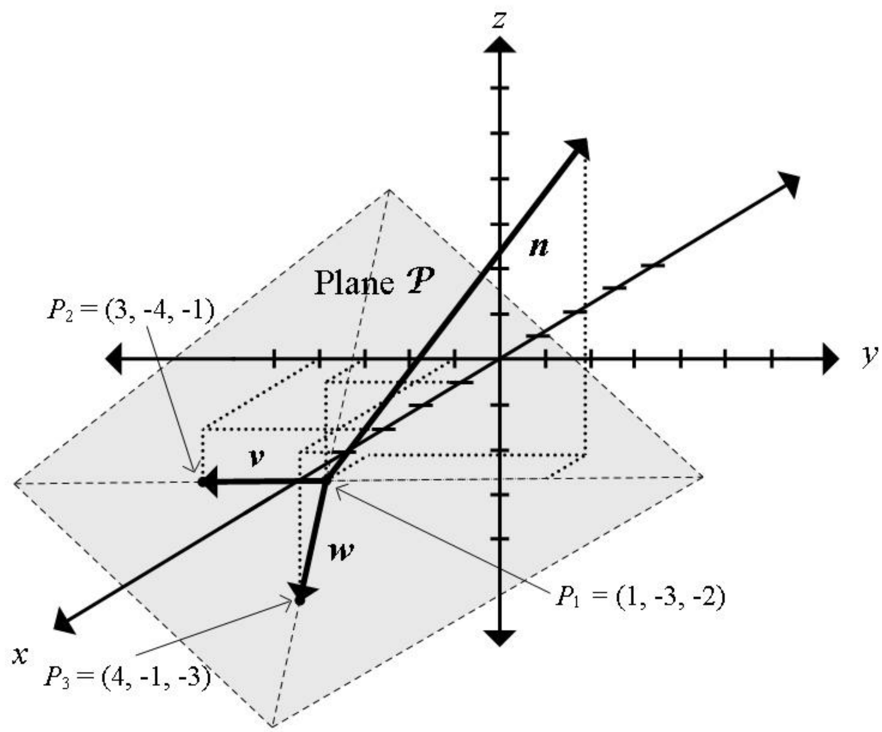

Figure 9 Normal vector $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ for the plane $\mathcal{P}$ in Example 11

Plane determined by two distinct, intersecting lines. Two distinct intersecting lines determine a unique plane. If we take the cross product of a vector in the direction of the first line with a vector in the direction of the other line, we will obtain a normal vector for that plane.

EXAMPLE 12 Consider the distinct, intersecting lines l_1 and l_2 from Example 4 above. We have

$$\begin{aligned} & l_1\colon\ x = 8 - 5t,\ \ y = -3 + 2t,\ \ z = -7 + 7t, \quad \text{where } t \in \mathbb{R}, \\ \text{and} \quad & l_2\colon\ x = 6 + 3s,\ \ y = -2 - s,\ \ z = 2 + 2s, \quad \text{where } s \in \mathbb{R}. \end{aligned}$$

A vector in the direction of l_1 is $[-5, 2, 7]$ (why?), and a vector in the direction of l_2 is $[3, -1, 2]$. Therefore, a vector normal to the plane determined by these lines is $[-5, 2, 7] \times [3, -1, 2] = [11, 31, -1]$. Now, in Example 4, we found $(3, -1, 0)$ to be an intersection point of l_1 and l_2. Using this point (although any point on either l_1 or l_2 could be used instead) together with the normal vector, we obtain $11x + 31y - z = 2$ as the equation of the plane (see Figure 10). ■

Line formed by two distinct, intersecting planes. Two non-parallel planes in $\mathbb{R}^3$ intersect along a line. To find parametric equations for this line, we must first find a vector in the direction of the line. Now, since the line lies in both planes, this direction vector will be parallel to both planes, and hence, this direction vector is

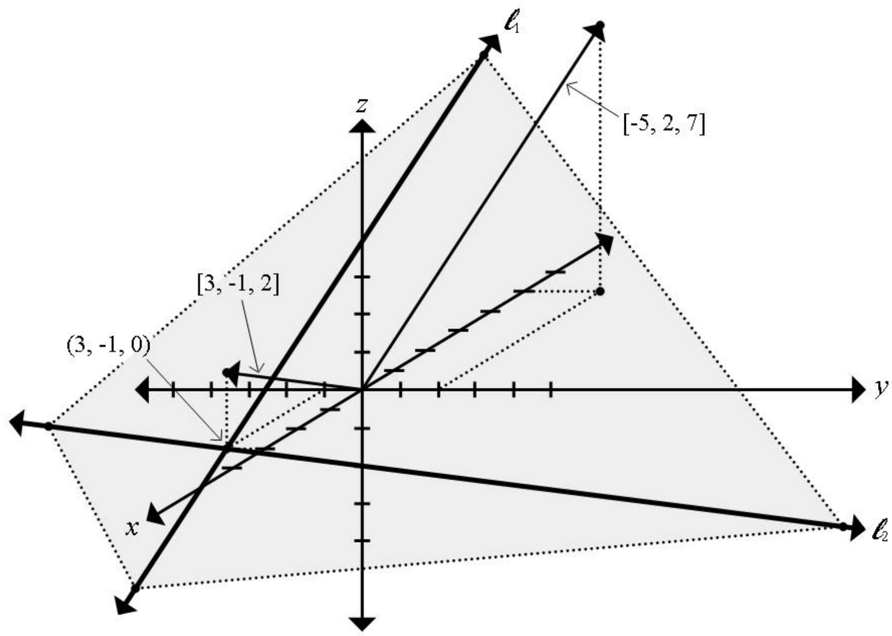

Figure 10 Plane determined by lines l_1 and l_2 in Examples 4 and 12

perpendicular to normal vectors for each of the planes. Thus, the cross product of these normal vectors gives a direction vector for the line.

EXAMPLE 13 Consider the planes $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^3$ satisfying the equations $3x - 2y + z = 2$ and $4x + y - 7z = -12$, respectively. Normal vectors for these planes are $\mathbf{v}_1 = [3, -2, 1]$ and $\mathbf{v}_2 = [4, 1, -7]$, respectively. Since the normal vectors are not parallel, the planes themselves will also not be parallel, and so they intersect along a line, l. A direction vector $\mathbf{w}$ for l is given by

$$\mathbf{w} = \mathbf{v}_1 \times \mathbf{v}_2 = [3, -2, 1] \times [4, 1, -7] = [13, 25, 11].$$

Next, we need to find a point on l. To do this, we choose an arbitrary value for z, say $z = 0$, and plug it into the equations for $\mathcal{P}$ and $\mathcal{Q}$, yielding

$$\begin{cases} 3x - 2y = 2 \\ 4x + y = -12 \end{cases}.$$

Solving the first of these for y produces $y = \frac{1}{2}(3x - 2)$. We plug this into the second equation to obtain $4x + \frac{1}{2}(3x - 2) = -12$, or $\frac{11}{2}x = -11$. This gives $x = -2$. Substituting -2 for x in $y = \frac{1}{2}(3x - 2)$ gives us $y = -4$. Hence, the point $(-2, -4, 0)$ satisfies the equations for both $\mathcal{P}$ and $\mathcal{Q}$, giving us a point on l.[3] Using this point together with the direction vector $\mathbf{w}$ above, we obtain the following parametric equations for l:

$$x = -2 + 13t, \quad y = -4 + 25t, \quad z = 11t.$$

■

[3] In this case, the choice $z = 0$ led to a point on the line l. However, choosing a particular value for one of the variables may sometimes lead to an inconsistent system in the other variables – for example, if the line l is perpendicular to one of the axes. In such a case, choose a particular value for a different variable instead.

▸ Calculating Shortest Distances using the Cross Product

Shortest distance from a point to a line. Let l be a line passing through $P_0 = (x_0, y_0, z_0)$ with direction vector $\mathbf{v} = [a, b, c]$, and let $P_1 = (x_1, y_1, z_1)$ be a point not on l. Let $\mathbf{w} = [(x_1 - x_0), (y_1 - y_0), (z_1 - z_0)]$ be the vector with initial point P_0 and terminal point P_1. Our goal is to calculate the shortest distance from P_1 to l. Using trigonometry (see Figure 11), we find that the desired distance equals $\|\mathbf{w}\| \sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.[4] Now, Theorem 4 tells us that $\|\mathbf{w}\| \sin\theta = \frac{\|\mathbf{v}\times\mathbf{w}\|}{\|\mathbf{v}\|}$, so we have the following result:

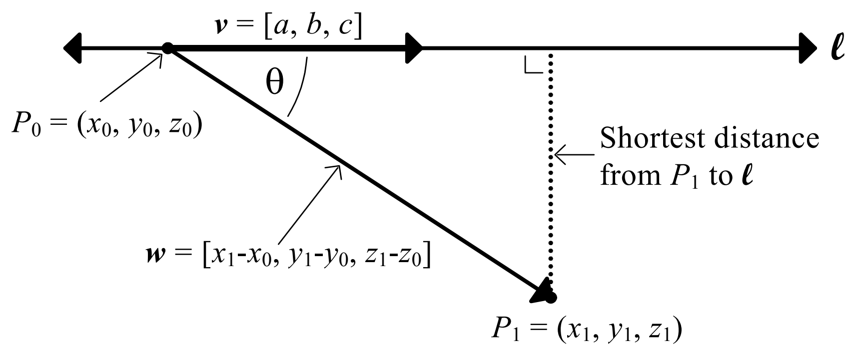

Figure 11 Shortest distance from a point (x_1, y_1, z_1) to the line through $P_0 = (x_0, y_0, z_0)$ with direction vector $\mathbf{v} = [a, b, c]$

> THEOREM 6
> Let l be the line through (x_0, y_0, z_0) with direction vector $[a, b, c]$, and let $P_1 = (x_1, y_1, z_1)$ be any point. Then the shortest distance from P_1 to l equals
> $$\frac{\|\,[a,b,c] \times [x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0]\|}{\sqrt{a^2 + b^2 + c^2}}.$$

EXAMPLE 14 Let l be the line with parametric equations

$$x = 3 - 3t, \quad y = 5, \quad z = 2t.$$

Notice that $P_1 = (x_1, y_1, z_1) = (1, -3, 2)$ is not on l. We will calculate the shortest distance from P_1 to l. Now, from the parametric equations for l, we see that one point on l is $P_0 = (x_0, y_0, z_0) = (3, 5, 0)$, and a direction vector for l is $[-3, 0, 2]$. Since $[x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0] = [-2, -8, 2]$, the shortest distance from P_1 to l is given by

$$\frac{\|[-3,0,2] \times [-2,-8,2]\|}{\sqrt{(-3)^2 + 0^2 + 2^2}} = \frac{\|[16, 2, 24]\|}{\sqrt{13}} = \frac{\sqrt{836}}{\sqrt{13}} \approx 8.02.$$

■

Shortest distance from a point to a plane. Let $\mathcal{P}$ be a plane with equation $ax + by + cz = d$, and let $P_1 = (x_1, y_1, z_1)$ be any point not in $\mathcal{P}$. We want to find the shortest distance from P_1 to $\mathcal{P}$. To do this, we must find the distance between P_1 and the nearest point to P_1 on $\mathcal{P}$. We take advantage of the fact that the vector

[4] The distance is still $\|\mathbf{w}\| \sin\theta$ even if $\mathbf{v}$ is pointing in the direction opposite to that shown in Figure 11, since supplementary angles produce the same value for the sine.

to P_1 from its nearest point in the plane is perpendicular to $\mathcal{P}$, and hence is parallel to $[a, b, c]$, a normal vector to $\mathcal{P}$.

To begin, we first choose an arbitrary point $P_0 = (x_0, y_0, z_0)$ in the plane, and find the vector $\mathbf{v}$ from P_0 to P_1. Then, the projection vector $\mathbf{p}$ of $\mathbf{v}$ onto the normal vector $[a, b, c]$ gives a vector to P_1 from its nearest point in the plane (see Figure 12). The length of $\mathbf{p}$ then gives the shortest distance between the plane $\mathcal{P}$ and P_1. In Exercise 22 you will be asked to show that these steps produce the formula given in the next theorem. (Note that this formula does not directly require the cross product assuming that a normal vector for the plane is known.)

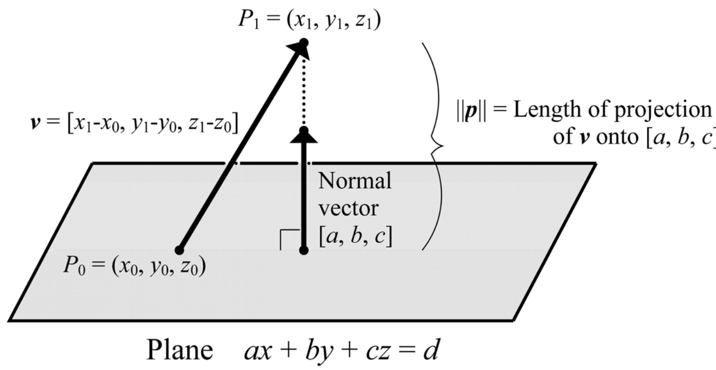

Figure 12 Shortest distance from the point (x_1, y_1, z_1) to the plane $ax + by + cz = d$

THEOREM 7
Let $ax + by + cz = d$ be the equation of a plane $\mathcal{P}$. Then, the shortest distance from any point $P_1 = (x_1, y_1, z_1)$ to $\mathcal{P}$ is given by

$$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

EXAMPLE 15 The shortest distance from the point $(2, 1, -3)$ to the plane $3x + y - z = 8$ is

$$\frac{|3 \cdot 2 + 1 \cdot 1 + (-1) \cdot (-3) - 8|}{\sqrt{3^2 + 1^2 + (-1)^2}} = \frac{2}{\sqrt{11}} = \frac{2\sqrt{11}}{11} \approx 0.603.$$

■

In Exercise 23, the cross product is used to give an analogous formula for the shortest distance between two (non-intersecting) lines. Exercise 24 discusses the shortest distance between two parallel planes.

▶ Applications of the Cross Product in Geometry and Physics

The cross product can also be used to find areas and volumes. A formula to determine the area of the unique triangle determined by three distinct points in $\mathbb{R}^3$ is presented in Exercise 25 using the cross product. (In Section 3.1 of the text we use other methods in linear algebra to determine various areas and volumes.)

There are many uses of the cross product in physics. For example, angular momentum, torque, and Lorentz force are all defined in terms of the cross product. The cross product is also used in Maxwell's Equations, which are the fundamental laws governing the behavior of electromagnetic fields. To conclude this section, we illustrate how the cross product is used to relate velocity and angular velocity.

Suppose an object travels in a circular path (orbit). Choose a fixed point P on the axis of rotation of the object, and let $\mathbf{r}$ represent the position vector from P (initial point) to the current location (terminal point) of the object along the orbit. Suppose $\mathbf{v}$ is the (regular) velocity vector of the object. Then $\mathbf{v}$ is perpendicular to $\mathbf{r}$; that is, $\mathbf{v}$ is tangent to the circle at that object. Notice that as the object proceeds along its orbit, for any given period of time, there is a corresponding central angle of the circle that is swept out. In physics, the **angular velocity** $\boldsymbol{\omega}$ of the object is represented by a vector whose *magnitude* is the amount of the central angle (in radians) of the circle that is traversed per second (or other appropriate unit of time), and whose *direction* is perpendicular to the plane of rotation (that is, parallel to the axis of rotation). Now, there are two opposite possibilities for the direction of $\boldsymbol{\omega}$, but in accordance with the Right-Hand Rule, *we determine the direction from which the orbital motion of the object appears counterclockwise, and then always choose positive angular velocity to be in that direction.* Notice also that since $\mathbf{v}$ lies in the plane of the circle, $\mathbf{v}$ is perpendicular to $\boldsymbol{\omega}$. Then, the following law of physics holds:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

This rule allows us to find the velocity of an object traveling in a circular motion if we know its angular velocity, as in the next example.

EXAMPLE 16 A small weight is attached to one end of a steel rod three meters long. The other end of the rod is secured at the origin of a coordinate system (see Figure 13). The rod pivots around the z-axis in a *clockwise* direction (as seen from above the z-axis), and makes one revolution every eight seconds. Hence the weight travels in a clockwise circular path about the z-axis. We will find the velocity vector and the speed of the weight at the point $(-1, 2, -2)$. We let $\mathbf{r}$ represent the vector from the origin (the rod's fixed point) to the weight; that is, $\mathbf{r} = [-1, 2, -2]$.

From the discussion above, the vector $\boldsymbol{\omega}$ describing the angular velocity for the rotation of the weight about the rod must point along the z-axis in the *negative* direction, since the z-axis is the axis of rotation, and the rotation is clockwise. (That is, from "below" the xy-plane, the rotation will appear to be counterclockwise.) Since one revolution takes eight seconds, we see that $\boldsymbol{\omega} = [0, 0, -\frac{\pi}{4}]$, measured in radians per second (why?). Thus, the velocity vector of the weight at the point $(-1, 2, -2)$ is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \left[0, 0, -\frac{\pi}{4}\right] \times [-1, 2, -2] = \left[-\frac{\pi}{2}, -\frac{\pi}{4}, 0\right].$$

The speed of the weight is $\|\mathbf{v}\| = \frac{\pi\sqrt{5}}{4} \approx 1.756$ m/sec. ■

EXAMPLE 17 The Earth revolves around the Sun in an orbit that is elliptical, but almost circular. The radius of this "circle" is approximately $92,900,000$ miles, and so the length of the Earth's orbit is about $584,000,000$ miles. Since the Earth completes one circuit in approximately $365\frac{1}{4}$ days, or 8766 hours,[5] the Earth's average speed is about $66,600$ miles per hour. For any given position of the Earth along its orbit, we can find its velocity vector $\mathbf{v}$, which will give us both the *magnitude and direction* for the Earth's movement at that position.

We set up a coordinate system with the Sun at the origin and the Earth's orbit in the xy-plane. The z-axis is chosen in the direction perpendicular to the plane of the orbit so that when we observe the orbital plane "from above," the Earth is traveling in a *counterclockwise* direction around the axis (see Figure 14). This means the angular velocity vector $\boldsymbol{\omega}$ for the revolution of the Earth about the Sun

[5]Except for the number of hours, we are generally rounding all values to three significant figures in this example.

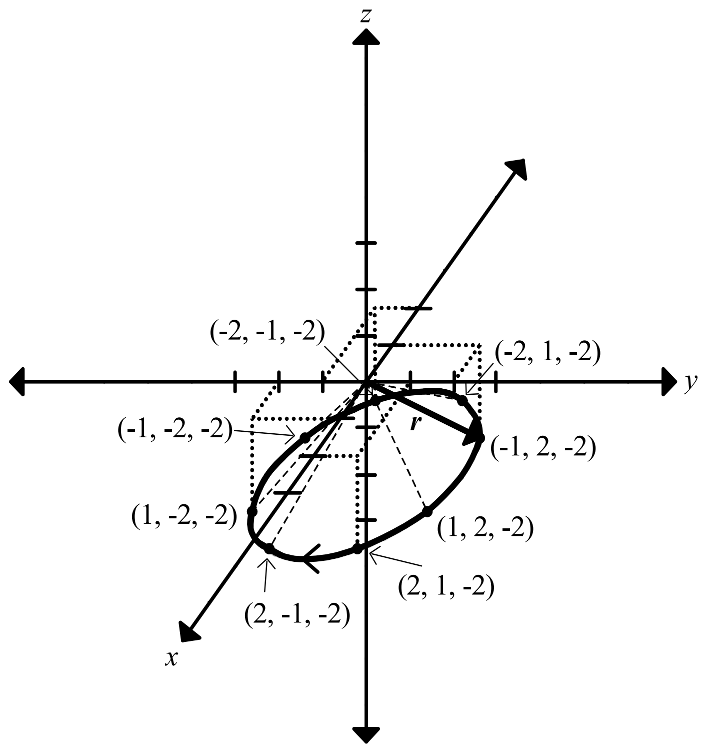

Figure 13 Path of weight attached to rod and revolving around z-axis

points in the direction of the *positive* z-axis. If we measure $\boldsymbol{\omega}$ in radians per hour, then $\boldsymbol{\omega} = [0, 0, \frac{2\pi}{8766}] = [0, 0, \frac{\pi}{4383}]$.

Suppose the current position of the Earth is given approximately by the vector $\mathbf{r} = [43600000,\ 82000000,\ 0]$, whose initial point is assumed to be at the origin. (Note that $\|\mathbf{r}\| \approx 92,900,000$.) Then the current velocity vector $\mathbf{v}$ of the Earth is

$$\begin{aligned}\mathbf{v} &= \left[0, 0, \frac{\pi}{4383}\right] \times [43600000,\ 82000000,\ 0] \\ &= \left[-\frac{82000000\pi}{4383},\ \frac{43600000\pi}{4383},\ 0\right],\end{aligned}$$

which is approximately $[-58800,\ 31300,\ 0]$. Notice that $\|\mathbf{v}\| \approx 66,600$ mi/hr, as expected. ■

▸ New Vocabulary

angle between two intersecting lines
angle between two intersecting planes
angular velocity
anti-commutative property for cross product
cancellation property for cross product
cross product of vectors
distributive law of cross product over addition
equation of a plane
exchange property of cross and dot product
normal vector to a plane
orthogonality property for cross product
parameter
parametric equations for a line

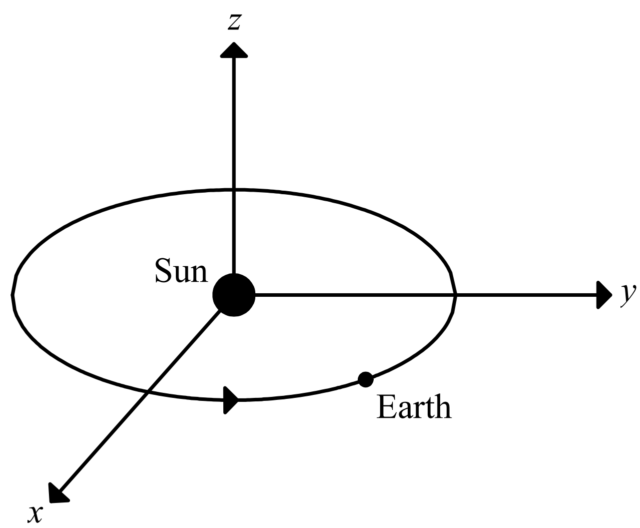

Figure 14 Orbit of the Earth

Right-Hand Rule
right-handed system
scalar associative law for cross product
skew lines
symmetric equations for a line (see Exercise 4)
zero property for cross product

▶ Highlights

- For a line l in $\mathbb{R}^3$ passing through (x_0, y_0, z_0) in the direction of $[a, b, c]$, the points (x, y, z) on l are precisely those which satisfy the parametric equations $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$, where t represents a real parameter.

- If l_1 and l_2 are two intersecting lines, and $\mathbf{v}$, $\mathbf{w}$ are vectors in the directions of l_1 and l_2, respectively, then the angle θ between l_1 and l_2 is the minimum angle (i.e., the angle θ that is $\leq 90° = \frac{\pi}{2}$ radians) between $\mathbf{v}$ and $\mathbf{w}$.

- For the plane $\mathcal{P}$ in $\mathbb{R}^3$ passing through the point (x_0, y_0, z_0) and having normal vector $[a, b, c]$, the points (x, y, z) on $\mathcal{P}$ are precisely those which satisfy the equation $ax + by + cz = ax_0 + by_0 + cz_0$.

- If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two intersecting planes, and $\mathbf{n}_1$, $\mathbf{n}_2$ are normal vectors for $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, then the angle θ between $\mathcal{P}_1$ and $\mathcal{P}_2$ is the minimum angle (i.e., the angle θ that is $\leq 90° = \frac{\pi}{2}$ radians) between $\mathbf{n}_1$ and $\mathbf{n}_2$.

- The cross product, $\mathbf{x} \times \mathbf{y}$, of vectors $\mathbf{x} = [x_1, x_2, x_3]$ and $\mathbf{y} = [y_1, y_2, y_3]$ in $\mathbb{R}^3$ is $[(x_2y_3 - x_3y_2),\ (x_3y_1 - x_1y_3),\ (x_1y_2 - x_2y_1)]$.

- Basic properties of the cross product for all vectors x, y, z in $\mathbb{R}^3$ and scalars a, include the following: anti-commutative: $\mathbf{x} \times \mathbf{y} = -(\mathbf{y} \times \mathbf{x})$; scalar associative: $(a\mathbf{x}) \times \mathbf{y} = \mathbf{x} \times (a\mathbf{y}) = a(\mathbf{x} \times \mathbf{y})$; zero: $\mathbf{x} \times \mathbf{0} = \mathbf{0} \times \mathbf{x} = \mathbf{0}$; cancellation: $\mathbf{x} \times \mathbf{x} = \mathbf{0}$; distributive: $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z})$; orthogonal: $\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{0}$; exchange: $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$.

- The magnitude of the cross product of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ is given by $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\| \sin\theta$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

- Two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ are parallel if and only if $\mathbf{x} \times \mathbf{y} = \mathbf{0}$.

- The direction of the cross product of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ is the direction determined by the Right-Hand Rule; that is, the direction that makes the vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} \times \mathbf{y}$ (taken in that order) form a right-handed system.

- If $\mathcal{P}$ is a plane containing three noncollinear points P_1, P_2, P_3, and $\mathbf{v}$ is the vector from P_1 to P_2, and $\mathbf{w}$ is the vector from P_1 to P_3, then the cross product $\mathbf{v} \times \mathbf{w}$ is a normal vector to $\mathcal{P}$.

- If l_1 and l_2 are two distinct, intersecting lines, having direction vectors $\mathbf{v}$ and $\mathbf{w}$, respectively, and $\mathcal{P}$ is a plane containing l_1 and l_2, then the cross product $\mathbf{v} \times \mathbf{w}$ is a normal vector to $\mathcal{P}$.

- If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two distinct, intersecting planes, having normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, and l is the line formed by the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$, then the cross product $\mathbf{n}_1 \times \mathbf{n}_2$ is a direction vector for l.

- The shortest distance from the point $P_1 = (x_1, y_1, z_1)$ to the line l through (x_0, y_0, z_0) with direction vector $[a, b, c]$ is given by
$$\frac{\|\,[a,b,c] \times [x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0]\,\|}{\sqrt{a^2 + b^2 + c^2}}.$$

- The shortest distance from the point $P_1 = (x_1, y_1, z_1)$ to the plane $ax + by + cz = d$ is given by
$$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

- The angular velocity $\boldsymbol{\omega}$ of an object traveling in a circle is a vector whose magnitude is the measure of the central angle (in radians) traversed per second, and whose direction is parallel to the axis of rotation so that the orbital motion of the object appears counterclockwise from that direction. If $\mathbf{v}$ is the velocity of the object, and $\mathbf{r}$ is the position vector of the object from the center of the circle, then $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

▶ EXERCISES

1. Find parametric equations for the line in $\mathbb{R}^3$ having the given properties:

★ **a)** passing through $(3, -1, 0)$, and in the direction $[0, 1, -4]$.

b) passing through $(-4, 2, 8)$, and in the direction $[-2, -3, 0]$.

★ **c)** passing through $(6, 2, 1)$ and $(4, -3, 7)$.

d) passing through $(4, -2, 9)$ and $(3, -2, 15)$.

★ **e)** passing through $(1, -5, -7)$, and parallel to the line with parametric equations $x = 5 - 2t$, $y = 7 + t$, $z = 9$.

f) passing through $(-2, 0, -5)$, and parallel to the line with parametric equations $x = -3t + 4$, $y = 4t - 2$, $z = -t - 1$.

2. Determine whether the two given lines intersect. If they do, find the point(s) of intersection.

★ **a)**
l_1: $x = -6 + 6t$, $y = 3 - 2t$, $z = 6 + t$
l_2: $x = 9 + 3s$, $y = 13 + 4s$, $z = 1 - 2s$

b)
l_1: $x = 8 + 2t$, $y = -5 - t$, $z = 10 - 4t$
l_2: $x = 7 - s$, $y = 1 + 6s$, $z = 5 - 5s$

★ **c)** l_1: $x = 4 - 6t$, $y = 5 + 9t$, $z = 6 - 3t$
l_2: $x = 8 - 4s$, $y = -1 + 6s$, $z = 9 - 2s$

d) l_1: $x = 5 + 3t$, $y = 12 - 2t$, $z = 5 + 7t$
l_2: $x = 3 - 5s$, $y = 11 + s$, $z = 6 - s$

★ **e)** l_1: $x = 4t - 7$, $y = 3t + 2$, $z = t - 5$
l_2: $x = 8s - 19$, $y = 6s - 7$, $z = 2s - 8$

f) l_1: $x = 3t - 2$, $y = t + 1$, $z = 2t - 7$
l_2: $x = 3s + 1$, $y = s + 2$, $z = 2s - 5$

3. Find the angle in each case between the pair of intersecting lines.

a) l_1: $x = 3t - 2$, $y = t + 2$, $z = 3t - 8$
l_2: $x = 4s - 7$, $y = 6s - 9$, $z = 7 - 6s$

★ **b)** l_1: $x = 9 - 3t$, $y = 1$, $z = 4t + 1$
l_2: $x = 3 - 3s$, $y = 5s - 4$, $z = 4s + 9$

c) l_1: $x = 1 - 7t$, $y = 7t$, $z = 5 - 8t$
l_2: $x = 5s - 4$, $y = 4s - 4$, $z = 16 - 11s$

d) l_1: $x = t + 1$, $y = -4t + 5$, $z = -t + 1$
l_2: $x = -5s + 1$, $y = 5s + 2$, $z = 2s + \frac{2}{5}$

4. **a)** Suppose that l is the line given in parametric form as: $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$. Show that if a, b, and c are nonzero, then l can be expressed in the form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

These equations, taken together, are known as the **symmetric equations** of the line l. (Hint: There are two parts to the proof: First, show that if a point lies on l, then it satisfies the symmetric equations for the line l. Then, show that if a point satisfies the symmetric equations for l, then that point lies on l.)

★ **b)** Use part (a) to state the symmetric equations for the line in part (c) of Exercise 1.

c) Use part (a) to state the symmetric equations for the line in part (f) of Exercise 1.

5. Find the equation for the given plane in $\mathbb{R}^3$.

★ **a)** The plane passing through $(1, 7, -2)$ having normal vector $[6, 1, 6]$.

b) The plane passing through $(0, -1, 1)$ having normal vector $[7, 3, -3]$.

c) The plane passing through $(-3, 5, 4)$ having normal vector $[9, 0, -2]$.

★ **d)** The xy-plane.

e) The yz-plane.

6. Find a unit normal vector for each of the following planes.

★ **a)** $2x - y + 2z = 7$

b) $x + 4y - 8z = -5$

c) $4x - 2y - z = -1$

7. In each case, calculate the angle between the two given intersecting planes:

★ **a)** $7x - 7y - 8z = 42$ and $4x + 5y - 11z = 23$.

b) $4x - 3y + 5z = 19$ and $4x - 3y = 48$.

c) $3x - 6y - 2z = 9$ and $8x + 5y - 3z = 72$.

8. Calculate each of the following.

★ **a)** $[1, 2, -1] \times [3, 7, 0]$

b) $[2, -1, 0] \times [1, -3, -2]$

c) $([1, 1, 0] \times [0, 1, -1]) \times [1, 2, 1]$

d) $[1, 1, 0] \times ([0, 1, -1] \times [1, 2, 1])$

e) $[3, -4, 1] \times [-6, 8, -2]$ (Think!)

f) $[3, 1, 2] \times [3, 1, 2]$

★ **g)** $[1, 2, -3] \cdot ([2, 0, -1] \times [-1, 2, 0])$

h) $[2, -5, -1] \cdot ([2, -5, -1] \times [-3, 4, 2])$

i) $\mathbf{i} \times (\mathbf{j} \times \mathbf{k})$

9. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^3$, and let a be any real number. Prove the following properties of the cross product stated in Theorem 3.

a) $(a\mathbf{x}) \times \mathbf{y} = \mathbf{x} \times (a\mathbf{y})$

b) $(a\mathbf{x}) \times \mathbf{y} = a(\mathbf{x} \times \mathbf{y})$

c) $\mathbf{x} \times \mathbf{0} = \mathbf{0}$

d) $\mathbf{x} \times \mathbf{x} = \mathbf{0}$

e) $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z})$

f) $\mathbf{y} \cdot (\mathbf{x} \times \mathbf{y}) = 0$

g) $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$

10. For vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^3$, explain why it is possible to calculate $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$, while the expression $(\mathbf{x} \cdot \mathbf{y}) \times \mathbf{z}$ does not make sense.

11. Find vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^3$ such that

$$(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} \neq \mathbf{x} \times (\mathbf{y} \times \mathbf{z}).$$

(This shows that there is no associative law for the cross product.)

12. Suppose that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$. Prove:

a) $(\mathbf{x} \times (\mathbf{y} \times \mathbf{z})) + (\mathbf{y} \times (\mathbf{z} \times \mathbf{x})) + (\mathbf{z} \times (\mathbf{x} \times \mathbf{y})) = \mathbf{0}$

b) $(\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{z} \times \mathbf{w}) = (\mathbf{x} \cdot \mathbf{z})(\mathbf{y} \cdot \mathbf{w}) - (\mathbf{x} \cdot \mathbf{w})(\mathbf{y} \cdot \mathbf{z})$

c) $\|\mathbf{x} \times \mathbf{y}\|^2 + (\mathbf{x} \cdot \mathbf{y})^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$

13. Let $\mathbf{x} = [5, -37, -8]$ and $\mathbf{y} = [-28, -25, 7]$.

a) Use the dot product to find the angle θ between $\mathbf{x}$ and $\mathbf{y}$.

b) Using your answer to part (a), verify that Theorem 4 holds for the given vectors $\mathbf{x}$ and $\mathbf{y}$.

14. The two parts of this exercise taken together prove Corollary 5. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^3$.

a) Prove that if $\mathbf{x} \times \mathbf{y} = \mathbf{0}$, then $\mathbf{x}$ and $\mathbf{y}$ are parallel. (Hint: Use Theorem 4.)

b) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are parallel, then $\mathbf{x} \times \mathbf{y} = \mathbf{0}$.

15. In each case, find the equation of the plane containing the given points.

★ **a)** $(1, 3, 0)$, $(-1, 4, 1)$, and $(3, 2, 2)$

b) $(-2, 7, 7)$, $(5, -2, -4)$, and $(-3, 4, 6)$

c) $(5, 2, 2)$, $(3, 2, -1)$, and $(8, 2, 9)$

d) $(-2, 5, 0)$, $(-3, -1, 4)$, and $(0, -2, 6)$

16. Explain what would happen if you attempted to use the technique illustrated in Example 11 to find the equation of a plane passing through three given points in the case where the three points are collinear. Provide an example with your explanation.

17. In each of parts (a) through (d) of Exercise 3 above, find the equation of the plane determined by the two given intersecting lines.

18. In each part, equations are given for a pair of planes in $\mathbb{R}^3$. Determine whether these planes intersect, and if they do, give parametric equations for the line of intersection.

a) $2x - y + z = 9$; $8x + 2y + z = 27$

b) $3x + 4y - z = 5$; $9x + 12y - 3z = 10$

★ **c)** $x + z = 6$; $y + z = 19$

d) $y = 16$; $x - 12z = 32$

19. Find the shortest distance from the given point $\mathcal{P}$ to the line l.

★ **a)** $P = (3, -1, 2)$; l: $x = 4 - 2t$, $y = t - 1$, $z = 5 + 2t$

b) $P = (0, 5, 4)$; l: $x = 2t - 1$, $y = 5 - 9t$, $z = 6t$

c) $P = (3, 1, -6)$; l: $x = 3 - 2t$, $y = -t$, $z = t - 8$

d) $P = (5, -2, 2)$; l: $x = -9t + 1$, $y = -6t - 5$, $z = -2t - 3$

20. Explain why the formula in Theorem 6 still gives the correct answer if the given point P_1 is actually on the line l.

21. Determine the shortest distance from the given point to the given plane.

★ **a)** Point: $(5, 2, 0)$; plane having equation: $2x - y - 2z = 12$

b) Point: $(3, 1, 1)$; plane having equation: $12x - 5z = -8$

★ **c)** Point: $(5, 0, -3)$; plane passing through $(3, 1, 5)$, $(1, -1, 2)$, and $(4, 3, 5)$

d) Point: $(2, 3, -1)$; plane passing through $(3, -1, -1)$, $(1, 1, 1)$, and $(5, 1, -2)$

22. Finish the argument in the text needed to prove that the formula for the shortest distance between a point and a plane in $\mathbb{R}^3$ given in Theorem 7 is correct.

23. (**Shortest distance between two non-parallel lines.**) If l_1 and l_2 are non-parallel lines, with direction vectors $[a_1, b_1, c_1]$ and $[a_2, b_2, c_2]$ respectively, and passing through (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, then the shortest distance from l_1 to l_2 equals

$$\frac{|(\mathbf{v} \cdot \mathbf{w})|}{\|\mathbf{v}\|},$$

where $\mathbf{v} = [a_1, b_1, c_1] \times [a_2, b_2, c_2]$, and $\mathbf{w} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1]$. (Since l_1 and l_2 are not parallel, $\mathbf{v} \neq \mathbf{0}$ by Exercise 14.) Using this formula, calculate the shortest distance between the two given lines in each case below.

★ **a)** l_1: $x = 5 - 2t$, $y = 3 + 2t$, $z = t - 1$
l_2: $x = 2$, $y = 5 + s$, $z = 4s$

b) l_1: $x = 3 - 2t$, $y = -4 + t$, $z = 1 + 3t$
l_2: $x = -2 + s$, $y = 3s - 1$, $z = -4s + 3$

★ **c)** l_1: $x = 3 - 7t$, $y = 1 + 4t$, $z = -8$
l_2: $x = -1 - s$, $y = 3s - 4$, $z = 1 - 3s$

24. (**Shortest distance between parallel planes.**) We can derive a formula for the distance between two parallel planes as follows: Any two parallel planes have the same normal vector, say, $[a, b, c]$. Then equations for the planes then have the form $ax + by + cz = d_1$ and $ax + by + cz = d_2$. The shortest distance between the planes is easily found by taking any point on the first plane, say (x_1, y_1, z_1) and then using Theorem 7 to find the shortest distance from that point to the second plane. This leads to the formula

$$\frac{|ax_1 + by_1 + cz_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

for the shortest distance between the planes. Use this formula to find the shortest distance between the following pairs of parallel planes:

★ **a)** $3x - y + 4z = 10$; $3x - y + 4z = 7$

b) $x - 2y + 5z = -3$; $x - 2y + 5z = 6$

★ **c)** $4x + 6y - 8z = 9$; $6x + 9y - 12z = -5$ (Hint: Use the same form for both planes.)

d) $10x - 25y + 20z = 4$; $8x - 20y + 16z = 11$ (Hint: Use the same form for both planes.)

25. (**Area of a triangle.**) Let (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) be three given points in $\mathbb{R}^3$. Then the area of the triangle determined by these points equals

$$\tfrac{1}{2}\|\, [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1] \times [x_3 - x_1,\ y_3 - y_1,\ z_3 - z_1]\, \|.$$

Using this formula, find the area of the triangle having the given vertices in each case:

★ **a)** $(2, -1, 0)$, $(3, 0, 1)$, and $(2, 2, 7)$.

b) $(1, 0, 2)$, $(2, 3, 4)$, and $(0, 1, 2)$.

26. Show that all of the points $P_1 = (2, 1, -3)$, $P_2 = (3, 1, -4)$, $P_3 = (5, 2, -5)$, $P_4 = (5, 3, -4)$ and $P_5 = (4, 3, -3)$ are in the same plane. Assuming that the figure $P_1P_2P_3P_4P_5$ having these points as vertices is a convex pentagon, find the area of that figure. (Hint: Break the figure into triangles and use Exercise 25.)

[Note: The five points here, taken in the given order, do, in fact, form a convex pentagon. This can be checked by examining each of the ten possible triangles $P_iP_jP_k$ (with $i < j < k$) that are formed using any three of these five vertices. Now, for each such triangle $P_iP_jP_k$, the cross product of the vector P_iP_j and the vector P_jP_k is certainly normal to the common plane containing all five points, but there are potentially two possible directions for each cross product. However, for this pentagonal figure, each of these ten cross products points in the same direction (they are all positive multiples of $[1, -1, 1]$). It can be shown that in such a case, none of the segments connecting a pair of vertices falls outside the figure. Thus, the given pentagon is convex.]

27. Prove that the formula given in Exercise 25 for the area of a triangle is correct. (**Hints:** Let P_1, P_2, and P_3 represent the three points, respectively. Consider the side P_1P_2 as the base of the triangle. The height is then the perpendicular distance from the point P_3 to the line containing P_1P_2. Use the formula in Theorem 6 for the shortest distance between a point and a line to get the height of the triangle.)

28. If the three given points in the formula in Exercise 25 are collinear, show that the area of the corresponding triangle is zero. (Hint: Notice that the vectors $[x_2 - x_1,\, y_2 - y_1,\, z_2 - z_1]$ and $[x_3 - x_1, y_3 - y_1,\, z_3 - z_1]$ are scalar multiples of each other when the given points are collinear.)

29. Suppose a weight is attached to one end of an inflexible spinning rod, whose other end is fixed at the origin (as in Example 16), so that the motion of the weight is circular. Find the velocity vector and speed of the weight at the given point in each case.

★ **a)** Point $(6, 3, -2)$ (measured in feet), where the rod makes one counter-clockwise revolution about the z-axis every 12 seconds

b) Point $(9, -12, 8)$ (measured in feet), where the rod makes one counter-clockwise revolution about the x-axis every 2 seconds

30. The planet Mars revolves around the Sun in an orbit that is elliptical, but almost circular. The radius of this "circle" is approximately $141,600,000$ miles. Mars completes one revolution about the Sun in approximately 1.88 years (assuming $365\frac{1}{4}$ days per year). Consider a coordinate system with the Sun at the origin and the orbit of Mars in the xy-plane. Assume the z-axis is chosen perpendicular to the plane of the orbit so that when when observed "from above," Mars is traveling in a *counterclockwise* direction around the axis. Calculate the angular velocity vector $\boldsymbol{\omega}$ for the revolution of Mars about the Sun, and then use $\boldsymbol{\omega}$ to find the velocity vector $\mathbf{v}$ and the speed $||\mathbf{v}||$ in radians per hour, if the current position of Mars is given (approximately) by the vector $\mathbf{r} = [85210000,\, 113100000,\, 0]$, whose initial point is assumed to be at the origin. (Note that $\|\mathbf{r}\| \approx 141,600,000$.)

31. Consider a coordinate system with the origin at the center of the Earth and z-axis running (in the positive direction) through the North Pole (see Figure 15). The Earth rotates around the z-axis once every 24 hours in the *counter-clockwise* direction as viewed from a point above the North Pole. The Earth's radius is approximately 6369 km. Let $(6369, 0, 0)$ represent the point where the Equator meets the Prime Meridian (which passes through Greenwich, England). Consider the point $(4246, 4246, 2123)$ (measured in km) on the Earth's surface. (This location is in the South Arabian peninsula.) Calculate the angular velocity $\boldsymbol{\omega}$ at this point, and then use $\boldsymbol{\omega}$ to find the velocity vector $\mathbf{v}$ and the speed $\|\mathbf{v}\|$ in km/sec at that point.

32. The latitude of a location on the Earth's surface is determined by drawing a line from that location to the center of the Earth and measuring the angle between that line and the plane of the Equator. It is typically measured in degrees rather than radians. For example, a point on the Equator is at $0°$ latitude, the North Pole is at $90°$ latitude, and a point halfway between them would have $45°$ latitude.

★ **a)** Given the latitude, θ, of a location on the surface of the Earth, use the coordinate system for the Earth and the information given in Exercise 31 to calculate the magnitude of the velocity at that point due to the Earth's rotation. (Hint: Show that the x- and y-coordinates of any point on the Earth's surface at latitude θ have the property $x^2 + y^2 = (6369 \cos\theta)^2$.)

b) Find the latitude of Philadelphia, PA (USA), and calculate the magnitude of the velocity due to the Earth's rotation there.

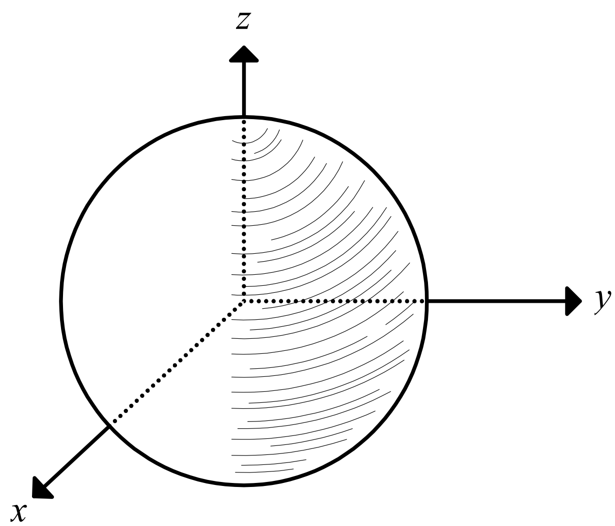

Figure 15 The Earth rotating about the z-axis

★ **33.** True or False:

a) There is a unique direction vector for any line in $\mathbb{R}^3$.

b) The angle between two distinct non-parallel lines in $\mathbb{R}^3$ is always defined.

c) The angle between two distinct non-parallel planes in $\mathbb{R}^3$ is always defined.

d) A normal vector to the plane $ax + by + c = d$ is $[-a, -b, -c]$.

e) For all vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathbb{R}^3$, $(\mathbf{x} + \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \times \mathbf{z}) + (\mathbf{y} \times \mathbf{z})$.

f) For all vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathbb{R}^3$, $\mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = (\mathbf{y} \times \mathbf{z}) \cdot \mathbf{x}$.

g) If $\mathbf{x}$ and $\mathbf{y}$ are parallel vectors in $\mathbb{R}^3$, then $\mathbf{x} \times \mathbf{y} \neq \mathbf{0}$.

h) If θ is the angle between two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$, then

$$\sin\theta = \|\mathbf{x} \times \mathbf{y}\| / (||\mathbf{x}|| \, ||\mathbf{y}||).$$

i) $\mathbf{k} \times \mathbf{i} = \mathbf{i} \times \mathbf{k}$.

j) The vectors $\mathbf{x} \times \mathbf{y}$, $\mathbf{x}$ and $\mathbf{y}$, taken in that order, form a right-handed system.

k) The shortest distance from a point P to a plane $\mathcal{P}$ can be found by taking any vector $\mathbf{v}$ from a point of $\mathcal{P}$ to P, and then calculating the length of the projection of $\mathbf{v}$ onto a normal vector for $\mathcal{P}$.

l) If an object travels in a circular path, its angular velocity is equal to the cross product of its velocity and its position vector from the center of the circle.

▶ Answers to Selected Exercises

(1) (a) $x = 3,\ y = -1 + t,\ z = -4t\ (t \in \mathbb{R})$
(c) $x = 6 - 2t,\ y = 2 - 5t,\ z = 1 + 6t\ (t \in \mathbb{R})$.
(Another valid form: $x = 4 + 2t,\ y = -3 + 5t,\ z = 7 - 6t\ (t \in \mathbb{R})$)
(e) $x = 1 - 2t,\ y = -5 + t,\ z = -7\ (t \in \mathbb{R})$

(2) (a) The lines intersect at a single point: $(0, 1, 7)$.
(c) The lines do not intersect.
(e) The lines are identical, so the intersection of the lines is the set of all points on (either) line; that is, the intersection consists of all points on the line $x = 4t - 7,\ y = 3t + 2,\ z = t - 5\ (t \in \mathbb{R})$.

(3) (b) $\theta = 45°$

(4) (b) $\frac{x-6}{-2} = \frac{y-2}{-5} = \frac{z-1}{6}$ (Another valid form: $\frac{x-4}{2} = \frac{y+3}{5} = \frac{z-7}{-6}$)

(5) (a) $6x + y + 6z = 1$ (d) $z = 0$

(6) (a) $\left[\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right]$

(7) (a) $\theta = 60°$

(8) (a) $[7, -3, 1]$ (g) -8

(15) (a) $x + 2y = 7$

(17) (b) $4x + 3z = 39$

(18) (c) $x = 6 - t,\ y = 19 - t,\ z = t\ (t \in \mathbb{R})$

(19) (a) $\frac{\sqrt{74}}{3} \approx 2.867$

(21) (a) $\frac{4}{3}$ (c) $\frac{31}{7} \approx 4.429$

(23) (a) $\frac{7\sqrt{13}}{39} \approx 0.647$
(c) 0 (The lines intersect at $(-4, 5, -8)$.)

(24) (a) $\frac{3\sqrt{26}}{26} \approx 0.588$ (c) $\frac{37\sqrt{29}}{174} \approx 1.145$

(25) (a) $\frac{\sqrt{74}}{2} \approx 4.301$

(29) (a) velocity $= [-\frac{\pi}{2}, \pi, 0]$ ft/sec; speed $= \frac{\pi\sqrt{5}}{2} \approx 3.512$ ft/sec

(32) (a) $\|\mathbf{v}\| = \frac{6369\pi}{43200} \cos\theta \approx 0.4632 \cos\theta$ km/sec

(33) (a) F (c) T (e) T (g) F (i) F (k) T
(b) F (d) T (f) T (h) T (j) T (l) F

Change of Variables and the Jacobian

Prerequisite: Section 3.1, Introduction to Determinants

In this section, we show how the determinant of a matrix is used to perform a change of variables in a double or triple integral. This technique generalizes to a change of variables in higher dimensions as well. Although the prerequisite for this section is listed as Section 3.1, we will also need the fact that $|\mathbf{A}| = |\mathbf{A}^T|$ from Section 3.3.

Substitution in One Variable

The following example serves to recall the method of integration by substitution from calculus:

EXAMPLE 1 To compute $\int_1^5 \sqrt{3x+1}\,dx$, we first make the substitution $u = 3x+1$. Then $du = 3\,dx$, and so

$$\begin{aligned}\int_1^5 \sqrt{3x+1}\,dx &= \tfrac{1}{3}\int_1^5 \sqrt{3x+1}\,(3\,dx) = \tfrac{1}{3}\int_4^{16} \sqrt{u}\,du \\ &= \tfrac{1}{3}\cdot\tfrac{2}{3}u^{\frac{3}{2}}\Big|_4^{16} = \tfrac{2}{9}(16^{\frac{3}{2}} - 4^{\frac{3}{2}}) = \tfrac{2}{9}(64-8) = \tfrac{112}{9}.\end{aligned}$$

Note the factor of 3 in $du = 3\,dx$. This indicates that the variable u covers 3 units of distance for each single unit of x. (It is as if u is measured in feet, while x is measured in yards.) Note that the length of the x-interval is only 4 units (from 1 to 5), while the length of the u-interval is 12 units (from 4 to 16). The factor of 3 in the du term compensates for this change. ■

In Example 1, the substitution variable u is a linear function of x, and so the change in units is constant throughout the given interval. In the next example, however, the substitution is non-linear.

EXAMPLE 2 Consider $\int_1^2 \frac{2x}{(x^2+1)^2}\,dx$. Let $u = x^2+1$. Then $du = 2x\,dx$. The integral is then calculated as

$$\int_1^2 \frac{2x}{(x^2+1)^2}\,dx = \int_2^5 \frac{du}{u^2} = -\frac{1}{u}\Big|_2^5 = -\tfrac{1}{5} - \left(-\tfrac{1}{2}\right) = \tfrac{3}{10}.$$

The factor $2x$ in $du = 2x\,dx$ indicates that the unit conversion from x to u is not constant. As the x-interval $[1,2]$ is stretched into the u-interval $[2,5]$, the stretching is done unevenly. For example, at $x = 1$, the scaling factor $2x = 2(1) = 2$, and so at this point, the length of a u-unit is 2 times smaller than the length of an x-unit. However, at $x = 1.5$, the scaling factor $2x = 2(1.5) = 3$, and so at this point, a u-unit is 3 times smaller than an x-unit.

In particular, the x-interval $[1.5, 1.51]$ (of length 0.01) is mapped to the u-interval $[3.25, 3.2801]$ (having length 0.0301). That is, the u-interval is approximately 3 times as long, because the scaling factor is 3 at $x = 1.5$. The error in using 3 as the scaling factor in this case is 0.0001, or $0.3\overline{3}\%$. As the length of the x-interval approaches 0, as it would in computing Riemann sums for integrals, the percent error in the scaling factor also approaches 0. ■

In general, since $\frac{du}{dx}$ is the rate of change of u with respect to x, its presence in the formula $du = \frac{du}{dx}\,dx$ keeps track of the amount of stretching involved in converting from x-coordinates to u-coordinates. Thus, $\frac{du}{dx}$ is the desired scaling factor for a change of variable in single-variable integration.

▶ Double Integrals

We now consider the analogous situation using two variables.

EXAMPLE 3 The area of the parallelogram P indicated in Figure 1 is given by the following double integral:

$$\text{Area} = \iint_P 1\, dx\, dy.$$

Converting this double integral into an iterated integral would be tedious. However, we can compute the area of P using Theorem 3.1. The vectors $\mathbf{w}_1 = [2, 1]$ and $\mathbf{w}_2 = [-1, 1]$ correspond to the sides of P, and so

$$\text{area of } P = \text{absolute value of } \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = |2 - (-1)| = 3.$$

We now examine the effect of a change of variables on the area. Since the sides of P are the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, we first create new variables u and v to satisfy the equation

$$[x, y] = u\mathbf{w}_1 + v\mathbf{w}_2 + [1, 1] = u[2, 1] + v[-1, 1] + [1, 1];$$

that is, $x = 2u - v + 1$, $y = u + v + 1$. Then, (x, y) vertices correspond to (u, v) vertices as follows:

(x, y)	(u, v)
$(1, 1)$	$(0, 0)$
$(0, 2)$	$(0, 1)$
$(3, 2)$	$(1, 0)$
$(2, 3)$	$(1, 1)$

Thus, in converting to the (u, v) coordinate system, the parallelogram P is mapped to the unit square S shown in Figure 2. Therefore, it follows that

$$\iint_S 1\, du\, dv = \text{area of } S = 1.$$

Since the parallelogram P does not have area 1, we must be missing a scaling factor of the type seen in the single variable case. Note that the scaling factor must be constant in this case, as in Example 1, because the change of coordinates involves only linear functions. Since the area of $P = 3(\text{area of } S)$, the scaling factor must be precisely 3. ■

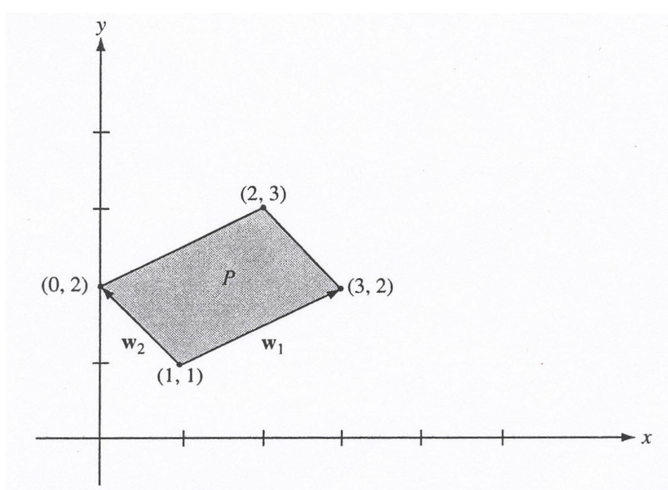

Figure 1: The parallelogram in the (x, y) system with vertices $(1, 1), (0, 2), (3, 2), (2, 3)$

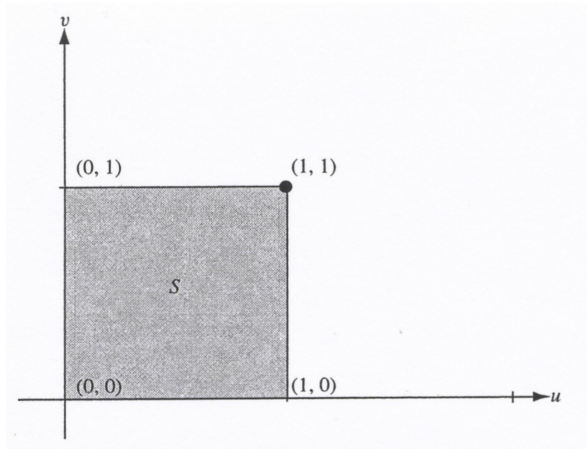

Figure 2: The square in the (u, v) system with vertices $(0,0), (0,1), (1,0), (1,1)$

Note in Example 3 that we can work backwards to compute the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ from the formulas for x and y as $\mathbf{w}_1 = \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right]$, and $\mathbf{w}_2 = \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right]$. This will work in general for all change of variable transformations. The idea behind this is that a unit rectangle in (u, v) coordinates is mapped to a region in (x, y) coordinates that is approximated by a parallelogram whose sides are $\mathbf{w}_1$ and $\mathbf{w}_2$, as in Figure 3. The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are tangent to the curved boundary of the actual image of the rectangle under the transformation. But differentiation, along with finding the tangent direction, also measures the rate of change, and so the lengths of $\mathbf{w}_1$ and $\mathbf{w}_2$ also represent the amount of stretching taking place in each of these directions. Hence, the scaling factor needed for the change of variable is the area of this approximating parallelogram, which, by Theorem 3.1, is the absolute value of $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$.

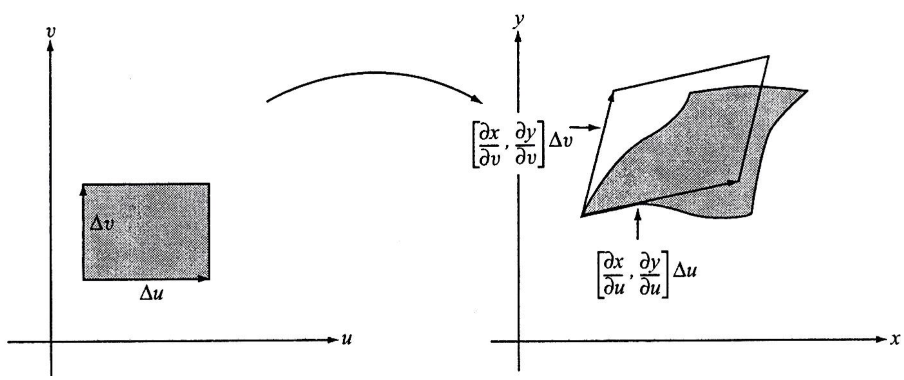

Figure 3: Converting a rectangle in (u, v) coordinates to an approximate parallelogram in (x, y) coordinates

In Section 3.3, it is proved that for any square matrix $\mathbf{A}$, $|\mathbf{A}| = |\mathbf{A}^T|$. Hence we could have also found the scaling factor as the absolute value of $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ instead. The matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

is called the **Jacobian matrix** of the change of coordinates function $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$.
We will refer to $|\mathbf{J}|$ as the **Jacobian determinant**. In general, the correct scaling factor to change an integral $\iint\limits_R f(x,y)\,dx\,dy$ over a region R into (u,v) coordinates is the absolute value of the Jacobian determinant, that is, $\left||\mathbf{J}|\right|$. Therefore, if S is the region in (u,v) coordinates that corresponds to R, then

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_S f(x(u,v),y(u,v))\,\left||\mathbf{J}|\right|\,du\,dv.$$

Just as in the one-variable case, the scaling factor can vary if the change of coordinates is nonlinear, as we will see shortly.

▶ Polar Coordinates

The polar coordinate system is frequently used to represent points in 2-dimensional space. In polar coordinates, each point $P = (x,y)$ in the plane is assigned a pair[6] of coordinates (r,θ), where r is the distance from the origin to P, and θ is the angle between the positive x-axis and the vector having initial point at the origin and terminal point P (see Figure 4). In all quadrants, the transformation from polar coordinates to standard (rectangular) coordinates is given by $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$. We can also convert from rectangular coordinates to polar coordinates using

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \qquad (\text{when } x \neq 0) \end{cases}.$$

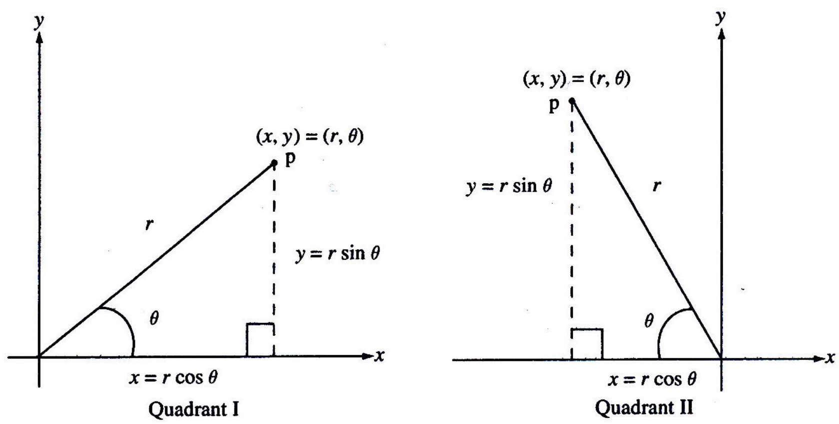

Figure 4: Relationship between standard coordinates and polar coordinates in Quadrants I and II

It is useful to express certain double integrals in polar coordinates if the region of integration (and/or the function involved) has radial or angular symmetry. In these instances, we need to compute the determinant of the Jacobian matrix in order to

[6] The assignment of polar coordinates to a given point (x,y) is not unique. For example, $(x,y) = \left(\sqrt{3},1\right)$ in rectangular coordinates can be represented as (r,θ) in polar coordinates as $\left(2,\frac{\pi}{6}\right)$, $\left(2,\frac{13\pi}{6}\right)$, or $\left(-2,\frac{7\pi}{6}\right)$. In general, $\left(\sqrt{3},1\right)$ can be expressed in polar coordinates as (r,θ), where $r = \pm\sqrt{(\sqrt{3})^2+1^2} = \pm 2$, and $\theta = \frac{\pi}{6} + k\pi$, where k is an even integer when r is positive, and k is an odd integer when r is negative.

include the proper scaling factor when we change coordinates. This determinant is

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

If we are careful to ensure that $r \geq 0$, the absolute value of $|\mathbf{J}|$ is also r, and so this is our scaling factor. Hence,

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(x(r,\theta), y(r,\theta))\,r\,dr\,d\theta,$$

where R^* is the region in the polar coordinate system corresponding to R. The next example illustrates this geometrically.

Example 4 Consider the square S in the (r,θ) (polar) coordinate system with left bottom corner at $(2, \frac{\pi}{6})$, width $\Delta r = 0.1$, and height $\Delta\theta = 0.1$. The image R of this square in the (x,y) system under the polar coordinate mapping $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ is shown in Figure 5.

Now, the square S has area $\Delta r \Delta\theta = (0.1)(0.1) = 0.01$, and thus the area of R is approximately equal to the product of the Jacobian determinant, $r = 2$, with the area of S. Hence, the area of $R \approx 2(0.01) = 0.02$.

To understand this approximation, recall that the columns of the Jacobian matrix represent vectors tangent at the corner point to the curved edges of R. When these vectors are scaled properly by multiplying by Δr and $\Delta\theta$, respectively, they represent the sides of a parallelogram (shown in Figure 6) whose area approximates the area of R. (In this particular case, the dot product of the columns is zero, and so the parallelogram is a rectangle.)

Finally, we compute the actual area of R for comparison purposes. The actual area of R is $\frac{\Delta\theta}{2\pi}$(the portion of the circle involved) times the differences of the areas of the circles of radii 2.1 and 2.0. Hence,

$$\text{area of } R = \frac{\Delta\theta}{2\pi}(\pi(2.1^2) - \pi(2^2)) = \frac{0.1}{2\pi}(\pi(0.41)) = \frac{0.041}{2} = 0.0205.$$

Thus, in this case, the scale factor obtained from the Jacobian induces an error of only 0.0005, or, 2.5%. Of course, in the actual integration, both $\Delta r \to 0$ and $\Delta\theta \to 0$, which makes the percent error approach 0 as well (although we do not prove this here). ∎

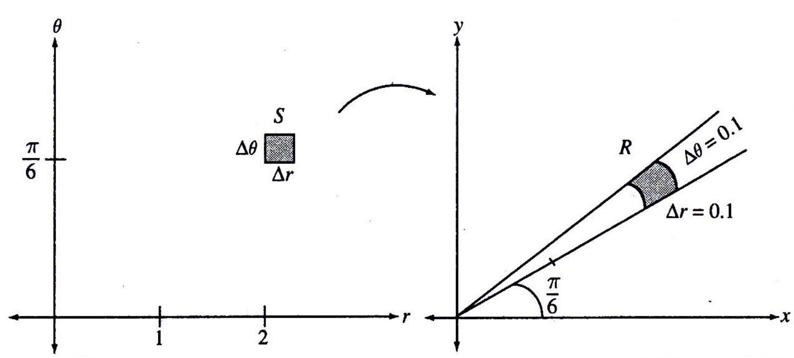

Figure 5: Image R of polar coordinate system square S in rectangular coordinates

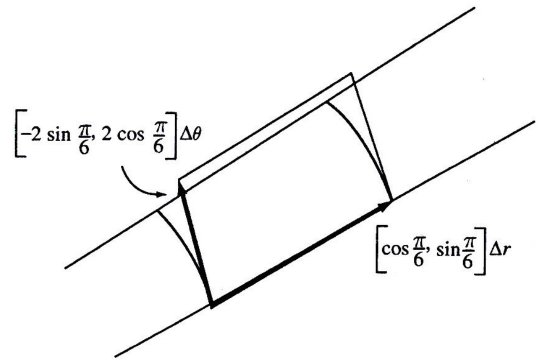

Figure 6: The parallelogram formed by the columns of the Jacobian at the point $(2, \frac{\pi}{6})$

EXAMPLE 5 Consider $\iint_R \sqrt{x^2+y^2}\,dx\,dy$ over the region R given by $0 \le r \le 1+\cos\theta$ in polar coordinates (see Figure 7). Now, $\sqrt{x^2+y^2} = r$, and so

$$\begin{aligned}
\iint_R \sqrt{x^2+y^2}\,dx\,dy &= \iint_R r\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^{1+\cos\theta} r^2\,dr\,d\theta \\
&= \int_0^{2\pi} \left(\frac{r^3}{3}\right)\Bigg|_0^{1+\cos\theta} d\theta = \tfrac{1}{3}\int_0^{2\pi} (1+\cos\theta)^3\ d\theta \\
&= \tfrac{1}{3}\int_0^{2\pi} (1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\ d\theta \\
&= \tfrac{1}{3}\int_0^{2\pi} (3\cos\theta+\cos^3\theta)\ d\theta + \tfrac{1}{3}\int_0^{2\pi}(1+3\cos^2\theta)\ d\theta.
\end{aligned}$$

An appeal to symmetry considerations (or a tedious computation) shows the first integral equals 0. Using a double-angle formula on the second integral, we obtain

$$\tfrac{1}{3}\int_0^{2\pi} \left(1+3\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)\right)\ d\theta = \left(\tfrac{5}{6}\theta+\tfrac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{5\pi}{3}.$$

■

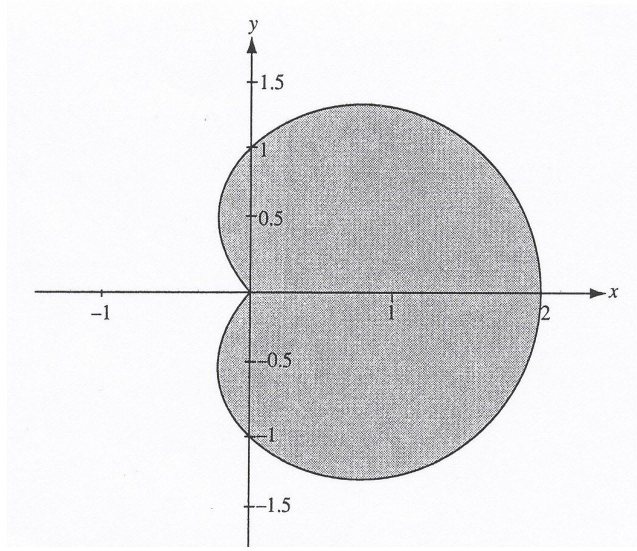

Figure 7: The region R in polar coordinates given by $0 \leq r \leq 1 + \cos\theta$

▶ Triple Integrals

The situation for change of variables in three dimensions is similar. When converting an integral in (x, y, z) coordinates to an integral in (u, v, w) coordinates, any rectangular solid based at the point (x, y, z) and having sides Δx, Δy, and Δz is mapped to a region approximated by a parallelepiped. The sides of this parallelepiped are the columns of the Jacobian matrix evaluated at (x, y, z) multiplied by Δx, Δy, and Δz, respectively. Thus, by Theorem 3.1, the absolute value of the Jacobian determinant

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

provides the correct scaling factor for converting from xyz-space to uvw-space. That is, $dx\, dy\, dz = \Big||\mathbf{J}|\Big|\, du\, dv\, dw$.

▶ Spherical Coordinates

One coordinate system frequently used in three dimensions is spherical coordinates. If $P = (x, y, z)$ is a point in the rectangular coordinate system and $\mathbf{v}$ is a vector from the origin to P, then P is assigned coordinates (ρ, ϕ, θ) in spherical coordinates, where $\rho = ||\mathbf{v}||$, ϕ is the angle between the vector $[0, 0, 1]$ and $\mathbf{v}$, and θ is the angle between the vector $[1, 0, 0]$ and the projection of $\mathbf{v}$ onto the xy-plane (see Figure 8). From elementary trigonometry, we find that

$$\begin{aligned} x &= \rho \sin\phi \cos\theta & \qquad \rho^2 &= x^2 + y^2 + z^2 \\ y &= \rho \sin\phi \sin\theta & \qquad \tan\theta &= \tfrac{y}{x}, \text{ when } x \neq 0 \\ z &= \rho \cos\phi & \qquad \cos\phi &= \tfrac{z}{\sqrt{x^2+y^2+z^2}}. \end{aligned}$$

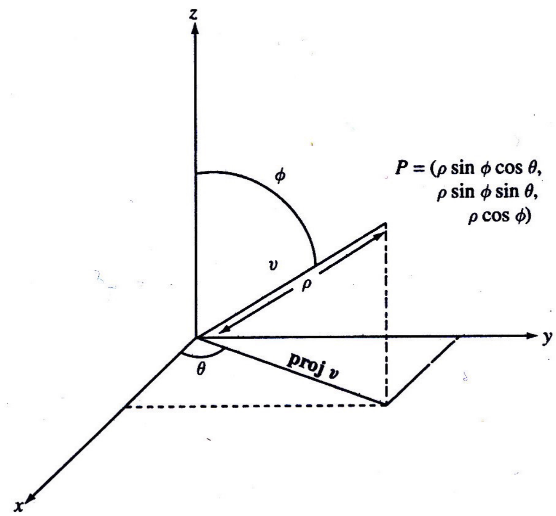

Figure 8: Spherical coordinates for $P = (x, y, z)$

Hence,

$$
\begin{aligned}
|\mathbf{J}| &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} - (-\rho\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&= \cos\phi(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta) \\
&\qquad + \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta) \\
&= \rho^2\cos^2\phi\sin\phi(\cos^2\theta + \sin^2\theta) + \rho^2\sin^3\phi(\cos^2\theta + \sin^2\theta) \\
&= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi \\
&= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) \\
&= \rho^2\sin\phi.
\end{aligned}
$$

Since $0 \le \phi \le \pi$ in spherical coordinates, the quantity $\rho^2\sin\phi$ is always nonnegative. Hence, when converting an integral from xyz-coordinates to $\rho\phi\theta$-coordinates, we have

$$dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Example 6 We find the volume of the region R bounded below by the upper half of the cone $z^2 = x^2 + y^2$ and bounded above by the sphere $x^2 + y^2 + z^2 = 8$ (see Figure 9). Now,

$$\text{volume of } R = \iiint\limits_R 1\,dx\,dy\,dz.$$

Converting to spherical coordinates, we have

$$\text{volume of } R = \iiint\limits_R \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Since the radius of the sphere is $\sqrt{8}$, ρ ranges from 0 to $\sqrt{8}$. The sides of the cone are at a 45° angle from the z-axis, and so ϕ ranges from 0 to $\frac{\pi}{4}$. Hence, changing to an iterated integral, we obtain

$$\begin{aligned}
\text{volume of } R &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{8}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left(\frac{\rho^3}{3}\sin\phi\right)\Big|_0^{\sqrt{8}} \, d\phi \, d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{8\sqrt{8}}{3}\sin\phi \, d\phi \, d\theta \\
&= -\frac{8\sqrt{8}}{3}\int_0^{2\pi} (\cos\phi)\Big|_0^{\frac{\pi}{4}} \, d\theta \\
&= -\frac{8\sqrt{8}}{3}\int_0^{2\pi} \left(\frac{\sqrt{2}}{2} - 1\right) d\theta \\
&= -\frac{8\sqrt{8}}{3}\left(\frac{\sqrt{2}}{2} - 1\right)(2\pi) \\
&= \frac{32\pi}{3}(\sqrt{2} - 1).
\end{aligned}$$

■

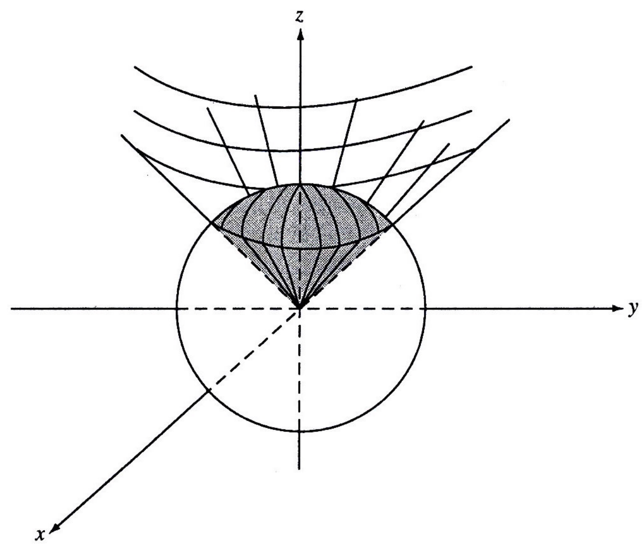

Figure 9: Region R bounded below by the upper half of the cone $z^2 = x^2 + y^2$ and bounded above by the sphere $x^2 + y^2 + z^2 = 8$

▸ Cylindrical Coordinates

Another frequently used three-dimensional coordinate system is cylindrical coordinates, (r, θ, z), in which the r and θ variables provide a polar coordinate system in the xy-plane, and z is unchanged from rectangular coordinates. Thus,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \ . \\ z &= z \end{aligned}$$

In Exercise 3, you are asked to show that the Jacobian determinant for a transformation from rectangular to cylindrical coordinates is r, and hence

$$dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

Higher Dimensions

The method we have shown for changing variables in double and triple integrals also works in general for multiple integrals in $\mathbb{R}^n$. In particular, to change from $x_1 x_2 \ldots x_n$-coordinates to $u_1 u_2 \ldots u_n$-coordinates, we must calculate the absolute value of the determinant of the Jacobian matrix,

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \cdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix},$$

and then we have

$$dx_1\, dx_2 \ldots dx_n = \Big||\mathbf{J}|\Big|\, du_1\, du_2 \ldots du_n.$$

New Vocabulary

cylindrical coordinates
Jacobian determinant
Jacobian matrix
polar coordinates
spherical coordinates

Highlights

- For the change of coordinates function $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$, the Jacobian matrix is $\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$, and its determinant, $|\mathbf{J}|$, is called the Jacobian determinant.

- If f is a function of variables x and y, R is a region in (x,y) coordinates, and S is the corresponding region in (u,v) coordinates, then

$$\iint_R f(x,y)\, dx\, dy \;=\; \iint_S f(x(u,v), y(u,v))\, \Big||\mathbf{J}|\Big|\, du\, dv.$$

  That is, the scaling factor involved when converting a double integral from (x,y) coordinates to (u,v) coordinates is the absolute value of the Jacobian determinant.

- When converting from (x,y) coordinates to (u,v) coordinates, we have $dx\, dy = \Big||\mathbf{J}|\Big|\, du\, dv$. In particular, in polar coordinates, where $x = r\cos\theta$ and $y = r\sin\theta$, we have $dx\, dy = r\, dr\, d\theta$.

- When converting an integral in (x,y,z) coordinates to an integral in (u,v,w) coordinates, the absolute value of the Jacobian determinant

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

  provides the correct scaling factor for converting from xyz-space to uvw-space. That is, $dx\, dy\, dz = \Big||\mathbf{J}|\Big|\, du\, dv\, dw$.

- In spherical coordinates, where $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, we have $dx\, dy\, dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

- In cylindrical coordinates, where $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, we have $dx\,dy\,dz = r\,dr\,d\theta\,dz$.
- When converting from $x_1x_2\ldots x_n$-coordinates to $u_1u_2\ldots u_n$-coordinates, the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \cdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{bmatrix},$$

and we have $dx_1\,dx_2\ldots dx_n = \left||\mathbf{J}|\right|\,du_1\,du_2\ldots du_n$.

▶ EXERCISES

1. For each change of variable formula, compute $dx\,dy$ in terms of $du\,dv$.

★ **a)** $x = u + v$, $y = u - v$

b) $x = u^2 + v^2$, $y = u^2 - v^2$

★ **c)** $x = u^2 - v^2$, $y = 2uv$

d) $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

★ **e)** $x = \frac{2u}{(u+1)^2+v^2}$, $y = \frac{1-(u^2+v^2)}{(u+1)^2+v^2}$

2. For each change of variable formula, compute $dx\,dy\,dz$ in terms of $du\,dv\,dw$.

★ **a)** $x = u + v$, $y = v + w$, $z = w + u$

b) $x = 3u + v + w$, $y = 3v + w$, $z = w$

★ **c)** $x = \frac{u}{u^2+v^2+w^2}$, $y = \frac{v}{u^2+v^2+w^2}$, $z = \frac{w}{u^2+v^2+w^2}$

d) $x = \frac{w}{u}$, $y = u$, $z = u\cos v$ (for $u > 0$)

3. Show that $|\mathbf{J}| = r$ for the change of variables from rectangular coordinates to cylindrical coordinates.

4. Compute each of the following integrals by changing to the indicated coordinate system:

★ **a)** $\iint\limits_R (x + y)\,dx\,dy$, where R is the region in the first quadrant between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$; polar coordinates

b) $\iint\limits_R 1\,dx\,dy$, where R is the region inside the innermost ring of the spiral $r = \theta$ in the first quadrant (see Figure 10); polar coordinates

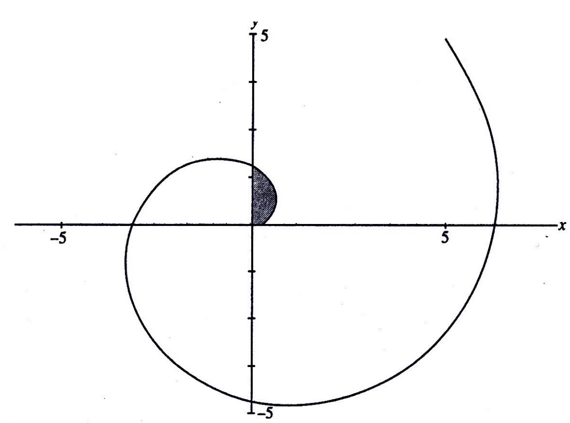

Figure 10: The spiral $r = \theta$

★ **c)** $\iiint\limits_R z\,dx\,dy\,dz$, where R is the half of the sphere of radius 1 centered at the origin which is above the xy-plane; spherical coordinates

d) $\iiint\limits_R \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz$, where R is the shell between the spheres of radii 2 and 3 centered at the origin; spherical coordinates

★ **e)** $\iiint\limits_R (x^2+y^2+z^2)\,dx\,dy\,dz$, where R is the region defined by $x^2+y^2 \le 4$ and $-3 \le z \le 5$; cylindrical coordinates

★ **5.** True or False:

a) A linear change of coordinates for an integration results in a constant scaling factor with respect to the associated integrals.

b) For the change of variables $u = y$, $v = x$, we have $du\,dv = 1\,dx\,dy$.

c) A rectangle in uv-coordinates with sides Δu and Δv is mapped by a change of coordinates to a region whose area is approximated by the area of the parallelogram with sides $\left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right]\Delta u$ and $\left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right]\Delta v$.

d) The scaling factor for a change of variables in integrals is always the determinant of the Jacobian matrix.

▶ Answers to Selected Exercises

(1) (a) $dx\ dy = 2\ du\ dv$

(c) $dx\ dy = 4(u^2 + v^2)\ du\ dv$

(e) $dx\ dy = \left(\frac{8|v|}{((u+1)^2+v^2)^3}\right)\ du\ dv$

(2) (a) $dx\ dy\ dz = 2\ du\ dv\ dw$

(c) $dx\ dy\ dz = \left(\frac{1}{(u^2+v^2+w^2)^3}\right)\ du\ dv\ dw$

(4) (a) $\frac{52}{3}$ (c) $\frac{\pi}{4}$ (e) $\frac{800\pi}{3}$

(5) (a) T (b) T (c) T (d) F

Function Spaces

Prerequisite: Section 4.7, Coordinatization

In this section, we apply the techniques of Chapter 4 to vector spaces whose elements are functions. The vector spaces $\mathcal{P}_n$ and $\mathcal{P}$ are familiar examples of such spaces. Other important examples are $C^0(\mathbb{R}) = \{$all continuous real-valued functions with domain $\mathbb{R}\}$ and $C^1(\mathbb{R}) = \{$all continuously differentiable real-valued functions with domain $\mathbb{R}\}$.

Linear Independence in Function Spaces

Proving that a finite subset S of a function space is linearly independent usually requires a modification of the strategy used in $\mathbb{R}^n$.

EXAMPLE 1 Consider the subset $S = \left\{x^3 - x,\, xe^{-x^2},\, \sin\left(\frac{\pi}{2}x\right)\right\}$ of $C^1(\mathbb{R})$. We will show that S is linearly independent using the definition of linear independence. Let a, b, and c be real numbers such that

$$a\left(x^3 - x\right) + b\left(xe^{-x^2}\right) + c\left(\sin\left(\frac{\pi}{2}x\right)\right) = 0$$

for every value of x. We must show that $a = b = c = 0$.

The above equation must be satisfied for every value of x. In particular, it is true for $x = 1$, $x = 2$, and $x = 3$. This yields the following system:

$$\begin{cases} (\text{Letting } x = 1 \Longrightarrow) & a(0) + b\left(\dfrac{1}{e}\right) + c\,(1) = 0 \\ (\text{Letting } x = 2 \Longrightarrow) & a(6) + b\left(\dfrac{2}{e^4}\right) + c\,(0) = 0 \\ (\text{Letting } x = 3 \Longrightarrow) & a(24) + b\left(\dfrac{3}{e^9}\right) + c\,(-1) = 0 \end{cases}.$$

Row reducing the matrix

$$\begin{array}{c} \begin{matrix} a & b & c & \end{matrix} \\ \left[\begin{array}{ccc|c} 0 & \frac{1}{e} & 1 & 0 \\ 6 & \frac{2}{e^4} & 0 & 0 \\ 24 & \frac{3}{e^9} & -1 & 0 \end{array}\right] \end{array} \quad \text{to} \quad \begin{array}{c} \begin{matrix} a & b & c & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

shows that the trivial solution $a = b = c = 0$ is the only solution to this homogeneous system. Hence, the set S is linearly independent by the definition of linear independence. ■

When proving linear independence using the technique of Example 1, we try to choose "nice" values of x to make computations easier. Even so, the use of a calculator or computer is often desirable when working with function spaces.

Other problems may occur because of the choice of x-values. Returning to Example 1, if instead we had plugged in $x = -1$, $x = 0$, and $x = 1$, we would have obtained the system

$$\begin{cases} (x = -1 \Longrightarrow) & a(0) + b\left(-\dfrac{1}{e}\right) + c\,(-1) = 0 \\ (x = \ \ 0 \Longrightarrow) & a(0) + b\,(0) + c\,(0) = 0 \\ (x = \ \ 1 \Longrightarrow) & a(0) + b\left(\dfrac{1}{e}\right) + c\,(1) = 0 \end{cases},$$

which has infinitely many nontrivial solutions. To prove linear independence, we must examine further values of x, generating more equations for the system, until the new system we obtain has only the trivial solution, as in Example 1.

Suppose, however, that after substituting many values for x and creating a huge homogeneous system, we still have nontrivial solutions. We cannot conclude that the set of functions is linearly dependent, although we may suspect that it is. In general, to *prove* that a set of functions $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is linearly dependent, we must find real numbers $a_1, \ldots, a_n$, not all zero, such that

$$a_1\mathbf{f}_1(x) + a_2\mathbf{f}_2(x) + \cdots + a_n\mathbf{f}_n(x) = 0$$

is a functional identity for every value of x, not just those we have tried.

EXAMPLE 2 Let $S = \{\sin 2x, \cos 2x, \sin^2 x, \cos^2 x\}$, a subset of $C^1(\mathbb{R})$. Suppose we attempt to show that S is linearly independent using the definition of linear independence. Let a, b, c, and d represent real numbers such that

$$a(\sin 2x) + b(\cos 2x) + c(\sin^2 x) + d(\cos^2 x) = 0.$$

Since we have four vectors in S, we substitute four different values for x into this equation to obtain the following system:

$$\begin{cases} (x = 0 \implies) & a(0) + b(1) + c(0) + d(1) = 0 \\ (x = \frac{\pi}{4} \implies) & a(1) + b(0) + c\left(\frac{1}{2}\right) + d\left(\frac{1}{2}\right) = 0 \\ (x = \frac{\pi}{2} \implies) & a(0) + b(-1) + c(1) + d(0) = 0 \\ (x = \frac{3\pi}{4} \implies) & a(-1) + b(0) + c\left(\frac{1}{2}\right) + d\left(\frac{1}{2}\right) = 0 \end{cases}.$$

Since the coefficient matrix for this homogeneous system row reduces to

$$\begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

there are nontrivial solutions to the system, such as $a = 0$, $b = -1$, $c = -1$, $d = 1$.

At this point, we cannot infer that S is linearly independent because we have nontrivial solutions. We also cannot conclude that S is linearly dependent because we have tested only a few values for x. We could try more values, such as $x = \frac{\pi}{6}$ and $x = \pi$, but we would still find that $a = 0$, $b = -1$, $c = -1$, $d = 1$ satisfies each equation we generate. This situation leads us to believe that the set S is linearly dependent. To be certain, we must check that the values $a = 0$, $b = -1$, $c = -1$, and $d = 1$ yield a functional identity when plugged into the original functional equation. Substituting these values yields

$$0(\sin 2x) + (-1)(\cos 2x) + (-1)(\sin^2 x) + (1)(\cos^2 x) = 0,$$

or $\cos 2x = \cos^2 x - \sin^2 x$, a well-known trigonometric identity. Thus, one vector in S can be expressed as a linear combination of the other vectors in S, and S is linearly dependent. ■

▸ New Vocabulary

$C^0(\mathbb{R})$ (continuous real-valued functions on $\mathbb{R}$)
$C^1(\mathbb{R})$ (real-valued functions on $\mathbb{R}$ having a continuous derivative)
function spaces
linearly dependent set (in a function space)
linearly independent set (in a function space)

▶ Highlights

- Function spaces are vector spaces whose elements are functions, such as $\mathcal{P}_n$, $\mathcal{P}$, $C^0(\mathbb{R})$, and $C^1(\mathbb{R})$.
- A set of functions $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ (in a function space) is linearly independent if there are n different values of x so that the resulting n equations of the form $a_1\mathbf{f}_1(x) + a_2\mathbf{f}_2(x) + \cdots + a_n\mathbf{f}_n(x) = 0$ form a system having only the trivial solution $a_1 = a_2 = \cdots = a_n = 0$.
- A set of functions $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ (in a function space) is linearly dependent if there are real numbers $a_1, a_2, \ldots, a_n$, not all zero, such that $a_1\mathbf{f}_1(x)+a_2\mathbf{f}_2(x)+\cdots + a_n\mathbf{f}_n(x) = 0$ for every value of x.

▶ EXERCISES

1. In each part of this exercise, determine whether the given subset S of $C^1(\mathbb{R})$ is linearly independent. If S is linearly independent, prove that it is. If S is linearly dependent, solve for a functional identity that expresses one function in S as a linear combination of the others.

★ **a)** $S = \left\{e^x, e^{2x}, e^{3x}\right\}$

b) $S = \{\sin x,\ \sin 2x,\ \sin 3x,\ \sin 4x\}$

★ **c)** $S = \left\{\dfrac{5x-1}{1+x^2}, \dfrac{3x+1}{2+x^2}, \dfrac{7x^3-3x^2+17x-5}{x^4+3x^2+2}\right\}$

d) $S = \{\sin x,\ \ \sin(x+1),\ \ \sin(x+2),\ \ \sin(x+3)\}$

2. Recall that a function $\mathbf{f}(x) \in C^0(\mathbb{R})$ is **even** if $\mathbf{f}(x) = \mathbf{f}(-x)$ for all $x \in \mathbb{R}$ and is **odd** if $\mathbf{f}(x) = -\mathbf{f}(-x)$ for all $x \in \mathbb{R}$. Suppose we want to prove that a finite subset S of $C^0(\mathbb{R})$ is linearly independent by the method of Example 1.

a) Suppose that every element of S is an odd function of x (as in Example 1). Explain why we would not want to substitute both 1 and -1 for x into the appropriate functional equation. Also explain why $x = 0$ would be a poor choice.

b) Suppose that every element of S is an even function. Would we want to substitute both 1 and -1 for x into the appropriate functional equation? Why? How about $x = 0$?

3. Let S be the subset $\{\cos(x+1), \cos(x+2), \cos(x+3)\}$ of $C^1(\mathbb{R})$.

a) Show that span(S) has $\{\cos x,\ \sin x\}$ for a basis. (Hint: The identity $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ is useful.)

b) Use part (a) to prove that S is linearly dependent.

4. For each given subset S of $C^1(\mathbb{R})$, find a subset B of S that is a basis for $\mathcal{V} =$ span(S).

★ **a)** $S = \{\sin 2x,\ \cos 2x,\ \sin^2 x,\ \cos^2 x,\ \sin x\cos x,\ 1\}$

b) $S = \{e^x,\ 1,\ e^{-x}\}$

★ **c)** $S = \{\sin(x+1), \cos(x+1), \sin(x+2), \cos(x+2)\}$

5. In each part of this exercise, let B represent an ordered basis for a subspace $\mathcal{V}$ of $C^1(\mathbb{R})$ and find $[\mathbf{v}]_B$ for the given $\mathbf{v} \in \mathcal{V}$.

★ **a)** $B = (e^x,\ e^{2x},\ e^{3x}),\ \ \mathbf{v} = 5e^x - 7e^{3x}$

b) $B = (\sin 2x,\ \cos 2x,\ \sin^2 x)$, $\mathbf{v} = 1$

★ **c)** $B = (\sin(x+1), \sin(x+2))$, $\mathbf{v} = \cos x$

★ **6.** True or False:

a) A subset $\{\mathbf{f}_1, \mathbf{f}_2\}$ of nonzero functions in $C^0(\mathbb{R})$ is linearly dependent if and only if $\mathbf{f}_1$ is a nonzero constant multiple of $\mathbf{f}_2$.

b) The set $\{x^2, x^3, x^4, x^5\}$ is a linearly independent subset of $C^1(\mathbb{R})$.

c) Let $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \in C^0(\mathbb{R})$. If plugging values for x into $a\mathbf{f}_1(x) + b\mathbf{f}_2(x) + c\mathbf{f}_3(x) = 0$ leads to $a = b = c = 0$, then $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ are linearly dependent.

d) Let $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \in C^0(\mathbb{R})$. If plugging 3 different values for x into $a\mathbf{f}_1(x) + b\mathbf{f}_2(x) + c\mathbf{f}_3(x) = 0$ does not allow us to conclude that $a = b = c = 0$, then $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ are linearly dependent.

▶ Answers to Selected Exercises

(1) (a) Linearly independent. To prove that it is, substitute the values $x = 0$, $x = 1$, $x = 2$, and follow the method of Example 1.

(c) Linearly dependent ($a = -2$, $b = 1$, $c = 1$)

(4) (a) $B = \{\sin(2x), \cos(2x), \sin^2 x\}$

(c) $B = \{\sin(x + 1), \cos(x + 1)\}$

(5) (a) $[\mathbf{v}]_B = [5, 0, -7]$

(c) $[\mathbf{v}]_B = [-\frac{\cos 2}{\sin 1}, \frac{\cos 1}{\sin 1}] \approx [0.4945, 0.6421]$. (If your answer is more complicated than this, compare numerical approximations.)

(6) (a) T (b) T (c) F (d) F

Max-Min Problems in $\mathbb{R}^n$ and the Hessian Matrix

Prerequisite: Section 6.3, Orthogonal Diagonalization

In this section, we study the problem of finding local maxima and minima for real-valued functions defined on $\mathbb{R}^n$. The method we describe is the higher-dimensional analogue to finding critical points and applying the second derivative test to functions defined on $\mathbb{R}$ studied in first-semester calculus.

▶ Taylor's Theorem in $\mathbb{R}^n$

Let $f \in C^2(\mathbb{R}^n)$, where $C^2(\mathbb{R}^n)$ is the set of real-valued functions defined on $\mathbb{R}^n$ having continuous second partial derivatives. The method for solving for local extreme points of f relies upon Taylor's Theorem with second degree remainder terms, which we state here without proof. (In the following theorem, an **open hypersphere** centered at $\mathbf{x}_0$ is a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{x}_0\| < r\}$ for some positive real number r.)

THEOREM 1

(**Taylor's Theorem** in $\mathbb{R}^n$) Let A be an open hypersphere centered at $\mathbf{x}_0 \in \mathbb{R}^n$, let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$, and let $t \in \mathbb{R}$ such that $\mathbf{x}_0 + t\mathbf{u} \in A$. Suppose $f\colon A \to \mathbb{R}$ has continuous second partial derivatives throughout A; that is, $f \in C^2(A)$. Then there is a c with $0 \le c \le t$ such that

$$\begin{aligned} f(\mathbf{x}_0 + t\mathbf{u}) \;=\; & f(\mathbf{x}_0) + \sum_{i=1}^{n} \left(\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}_0} \right)(tu_i) + \tfrac{1}{2}\sum_{i=1}^{n} \left(\left.\frac{\partial^2 f}{\partial x_i^2}\right|_{\mathbf{x}_0 + c\mathbf{u}} \right)(t^2 u_i^2) \\ & + \sum_{i=1}^{n}\sum_{j=i+1}^{n} \left(\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\mathbf{x}_0 + c\mathbf{u}} \right)(t^2 u_i u_j). \end{aligned}$$

Taylor's Theorem in $\mathbb{R}^n$ is derived from the familiar Taylor's Theorem in $\mathbb{R}$ by applying it to the function $g(t) = f(\mathbf{x}_0 + t\mathbf{u})$. In $\mathbb{R}^2$, the formula in Taylor's Theorem is

$$\begin{aligned} f(\mathbf{x}_0 + t\mathbf{u}) \;=\; & f(\mathbf{x}_0) + \left(\left.\frac{\partial f}{\partial x}\right|_{\mathbf{x}_0} \right)(tu_1) + \left(\left.\frac{\partial f}{\partial y}\right|_{\mathbf{x}_0} \right)(tu_2) \\ & + \tfrac{1}{2}\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{\mathbf{x}_0 + c\mathbf{u}} \right)(t^2 u_1^2) + \tfrac{1}{2}\left(\left.\frac{\partial^2 f}{\partial y^2}\right|_{\mathbf{x}_0 + c\mathbf{u}} \right)(t^2 u_2^2) \\ & + \left(\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{\mathbf{x}_0 + c\mathbf{u}} \right)(t^2 u_1 u_2). \end{aligned}$$

Recall that the **gradient** of f is defined by $\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right]$. If we let $\mathbf{v} = t\mathbf{u}$, then, in $\mathbb{R}^2$, $\mathbf{v} = [v_1, v_2] = [tu_1, tu_2]$, and so the sum

$$\left(\left.\frac{\partial f}{\partial x}\right|_{\mathbf{x}_0} \right)(tu_1) + \left(\left.\frac{\partial f}{\partial y}\right|_{\mathbf{x}_0} \right)(tu_2) \qquad \text{simplifies to} \qquad \left(\left.\nabla f\right|_{\mathbf{x}_0} \right) \cdot \mathbf{v}.$$

Also, since f has continuous second partial derivatives, we have $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. Therefore,

$$\begin{aligned}
&\tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}(t^2u_1^2) + \tfrac{1}{2}\frac{\partial^2 f}{\partial y^2}(t^2u_2^2) + \frac{\partial^2 f}{\partial x \partial y}(t^2u_1u_2) \\
&= \tfrac{1}{2}v_1\left(\frac{\partial^2 f}{\partial x^2}v_1 + \frac{\partial^2 f}{\partial x \partial y}v_2\right) + \tfrac{1}{2}v_2\left(\frac{\partial^2 f}{\partial y \partial x}v_1 + \frac{\partial^2 f}{\partial y^2}v_2\right) \\
&= \tfrac{1}{2}\mathbf{v}^T \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \mathbf{v},
\end{aligned}$$

where $\mathbf{v}$ is considered to be a column vector. The matrix

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

in this expression is called the **Hessian matrix** for f. Thus, in the $\mathbb{R}^2$ case, with $\mathbf{v} = t\mathbf{u}$, the formula in Taylor's Theorem can be written as

$$f(\mathbf{x}_0 + \mathbf{v}) = f(\mathbf{x}_0) + \left(\nabla f\Big|_{\mathbf{x}_0}\right) \cdot \mathbf{v} + \tfrac{1}{2}\mathbf{v}^T\left(\mathbf{H}\Big|_{\mathbf{x}_0 + k\mathbf{v}}\right)\mathbf{v},$$

for some k with $0 \le k \le 1$ (where $k = \frac{c}{t}$). While we have derived this result in $\mathbb{R}^2$, the same formula holds in $\mathbb{R}^n$, where the Hessian $\mathbf{H}$ is the matrix whose (i, j) entry is $\frac{\partial^2 f}{\partial x_i \partial x_j}$.

▸ Critical Points

If A is a subset of $\mathbb{R}^n$, then we say that $f: A \to \mathbb{R}$ has a **local maximum** at a point $\mathbf{x}_0 \in A$ if and only if there is an open neighborhood $\mathcal{U}$ of $\mathbf{x}_0$ such that $f(\mathbf{x}_0) \ge f(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{U}$. A **local minimum** for a function f is defined analogously.

THEOREM 2

Let A be an open hypersphere centered at $\mathbf{x}_0 \in \mathbb{R}^n$, and let $f: A \to \mathbb{R}$ have continuous first partial derivatives on A. If f has a local maximum or a local minimum at $\mathbf{x}_0$, then $\nabla f(\mathbf{x}_0) = \mathbf{0}$.

Proof If $\mathbf{x}_0$ is a local maximum, then $f(\mathbf{x}_0 + h\mathbf{e}_i) - f(\mathbf{x}_0) \le 0$ for small h. Then, $\lim_{h\to 0^+} \frac{f(\mathbf{x}_0 + h\mathbf{e}_i) - f(\mathbf{x}_0)}{h} \le 0$. Similarly, $\lim_{h \to 0^-} \frac{f(\mathbf{x}_0 + h\mathbf{e}_i) - f(\mathbf{x}_0)}{h} \ge 0$. Hence, for the limit to exist, we must have $\frac{\partial f}{\partial x_i}\Big|_{\mathbf{x}_0} = 0$. Since this is true for each i, $\nabla f\Big|_{\mathbf{x}_0} = \mathbf{0}$. A similar proof works for local minimums. **QED**

Points $\mathbf{x}_0$ at which $\nabla f(\mathbf{x}_0) = \mathbf{0}$ are called **critical points**.

EXAMPLE 1 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = 7x^2 + 6xy + 2x + 7y^2 - 22y + 23.$$

Then $\nabla f = [14x + 6y + 2,\ 6x + 14y - 22]$. We find critical points for f by solving $\nabla f = \mathbf{0}$. This is the linear system

$$\begin{cases} 14x + 6y + 2 = 0 \\ 6x + 14y - 22 = 0 \end{cases},$$

which has the unique solution $\mathbf{x}_0 = [-1, 2]$. Hence, by Theorem 2, $(-1, 2)$ is the only possible extreme point for f. (We will see later that $(-1, 2)$ is a local minimum.) ■

▶ Sufficient Conditions for Local Extreme Points

If $\mathbf{x}_0$ is a critical point for a function f, how can we determine whether $\mathbf{x}_0$ is a local maximum or a local minimum? For functions on $\mathbb{R}$, we have the second derivative test from calculus, which says that if $f''(\mathbf{x}_0) < 0$, then $\mathbf{x}_0$ is a local maximum, but if $f''(\mathbf{x}_0) > 0$, then $\mathbf{x}_0$ is a local minimum. We now derive a similar test in $\mathbb{R}^n$.

Consider the following formula from Taylor's Theorem:

$$f(\mathbf{x}_0 + \mathbf{v}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot \mathbf{v} + \tfrac{1}{2}\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0 + k\mathbf{v}} \right) \mathbf{v}.$$

At a critical point $\mathbf{x}_0$, $\nabla f(\mathbf{x}_0) = \mathbf{0}$, and so

$$f(\mathbf{x}_0 + \mathbf{v}) = f(\mathbf{x}_0) + \tfrac{1}{2}\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0 + k\mathbf{v}} \right) \mathbf{v}.$$

Hence, if $\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0 + k\mathbf{v}} \right) \mathbf{v}$ is positive for all small nonzero vectors $\mathbf{v}$, then f will have a local minimum at $\mathbf{x}_0$. (Similarly, if $\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0 + k\mathbf{v}} \right) \mathbf{v}$ is negative, f will have a local maximum.) But since we assume that f has continuous second partial derivatives, $\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0 + k\mathbf{v}} \right) \mathbf{v}$ is continuous in $\mathbf{v}$ and k, and will be positive for small $\mathbf{v}$ if $\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0} \right) \mathbf{v}$ is positive for all nonzero $\mathbf{v}$. Hence,

> THEOREM 3
> Given the conditions of Taylor's Theorem for a set A and for a function $f\colon A \to \mathbb{R}$, f has a local minimum at a critical point $\mathbf{x}_0$ if $\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0} \right) \mathbf{v} > 0$ for all nonzero vectors $\mathbf{v}$. Similarly, f has a local maximum at a critical point $\mathbf{x}_0$ if $\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0} \right) \mathbf{v} < 0$ for all nonzero vectors $\mathbf{v}$.

▶ Positive Definite Quadratic Forms

If $\mathbf{v}$ is a vector in $\mathbb{R}^n$, and $\mathbf{A}$ is an $n \times n$ matrix, the expression $\mathbf{v}^T\mathbf{A}\mathbf{v}$ is known as a **quadratic form**. (For more details on the general theory of quadratic forms, see Section 8.10.) A quadratic form such that $\mathbf{v}^T\mathbf{A}\mathbf{v} > 0$ for all nonzero vectors $\mathbf{v}$ is said to be **positive definite**. Similarly, a quadratic form such that $\mathbf{v}^T\mathbf{A}\mathbf{v} < 0$ for all nonzero vectors $\mathbf{v}$ is said to be **negative definite**.

Now, in particular, the expression $\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0} \right) \mathbf{v}$ in Theorem 3 is a quadratic form. Theorem 3 then says that if $\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0} \right) \mathbf{v}$ is a positive definite quadratic form at a critical point $\mathbf{x}_0$, then f has a local minimum at $\mathbf{x}_0$. Theorem 3 also says that if $\mathbf{v}^T \left(\mathbf{H}\Big|_{\mathbf{x}_0} \right) \mathbf{v}$ is a negative definite quadratic form at a critical point $\mathbf{x}_0$, then f has a local maximum at $\mathbf{x}_0$. Therefore, we need a method to determine whether a quadratic form of this type is positive definite or negative definite.

Now, the Hessian matrix $\left(\mathbf{H}\Big|_{\mathbf{x}_0} \right)$, which we will abbreviate as $\mathbf{H}$, is symmetric because $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ (since $f \in C^2(A)$). Hence, by Corollary 6.23, $\mathbf{H}$ can be orthogonally diagonalized. That is, there is an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}\mathbf{H}\mathbf{P}^T = \mathbf{D}$, a diagonal matrix, and so, $\mathbf{H} = \mathbf{P}^T\mathbf{D}\mathbf{P}$. Hence,

$$\mathbf{v}^T\mathbf{H}\mathbf{v} = \mathbf{v}^T\mathbf{P}^T\mathbf{D}\mathbf{P}\mathbf{v} = (\mathbf{P}\mathbf{v})^T\mathbf{D}(\mathbf{P}\mathbf{v}).$$

Letting $\mathbf{w} = \mathbf{P}\mathbf{v}$, we get $\mathbf{v}^T\mathbf{H}\mathbf{v} = \mathbf{w}^T\mathbf{D}\mathbf{w}$. But $\mathbf{P}$ is nonsingular, so as $\mathbf{v}$ ranges over all of $\mathbb{R}^n$, so does $\mathbf{w}$, and vice-versa. Thus, $\mathbf{v}^T\mathbf{H}\mathbf{v} > 0$ for all nonzero $\mathbf{v}$ if and only if $\mathbf{w}^T\mathbf{D}\mathbf{w} > 0$ for all nonzero $\mathbf{w}$. Now, $\mathbf{D}$ is diagonal, and so

$$\mathbf{w}^T\mathbf{D}\mathbf{w} = d_{11}w_1^2 + d_{22}w_2^2 + \cdots + d_{nn}w_n^2.$$

But the d_{ii}'s are the eigenvalues of $\mathbf{H}$. Thus, it follows that $\mathbf{w}^T\mathbf{D}\mathbf{w} > 0$ for *all* nonzero $\mathbf{w}$ if and only if all of these eigenvalues are positive. (Set $\mathbf{w} = \mathbf{e}_i$ for each i to prove the "only if" part of this statement.) Similarly, $\mathbf{w}^T\mathbf{D}\mathbf{w} < 0$ for *all* nonzero $\mathbf{w}$ if and only if all of these eigenvalues are negative. Hence,

THEOREM 4
A symmetric matrix $\mathbf{A}$ defines a positive definite quadratic form $\mathbf{v}^T\mathbf{A}\mathbf{v}$ if and only if all of the eigenvalues of $\mathbf{A}$ are positive. A symmetric matrix $\mathbf{A}$ defines a negative definite quadratic form $\mathbf{v}^T\mathbf{A}\mathbf{v}$ if and only if all of the eigenvalues of $\mathbf{A}$ are negative.

Hence, Theorem 3 can be restated as follows:

Given the conditions of Taylor's Theorem for a set A and a function $f\colon A \to \mathbb{R}$:
(1) if all of the eigenvalues of $\mathbf{H}$ are positive at a critical point $\mathbf{x}_0$, then f has a local minimum at $\mathbf{x}_0$, and
(2) if all of the eigenvalues of $\mathbf{H}$ are negative at a critical point $\mathbf{x}_0$, then f has a local maximum at $\mathbf{x}_0$.

EXAMPLE 2 Consider the function

$$f(x,y) = 7x^2 + 6xy + 2x + 7y^2 - 22y + 23.$$

In Example 1, we found that f has a critical point at $\mathbf{x}_0 = [-1, 2]$. Now, the Hessian matrix for f at $\mathbf{x}_0$ is

$$\mathbf{H} = \left.\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}\right|_{\mathbf{x}_0} = \begin{bmatrix} 14 & 6 \\ 6 & 14 \end{bmatrix}.$$

But $p_{\mathbf{H}}(x) = x^2 - 28x + 160$, which has roots $x = 8$ and $x = 20$. Thus, $\mathbf{H}$ has all eigenvalues positive, and hence, $\mathbf{v}^T\mathbf{H}\mathbf{v}$ is positive definite. Theorem 4 then tells us that $\mathbf{x}_0 = [-1, 2]$ is a local minimum for f. ■

▸ Local Maxima and Minima in $\mathbb{R}^2$

It can be shown (see Exercise 3) that a 2×2 symmetric matrix $\mathbf{A}$ defines a positive definite quadratic form ($\mathbf{v}^T\mathbf{A}\mathbf{v} > 0$ for all nonzero $\mathbf{v}$) if and only if $a_{11} > 0$ and $|\mathbf{A}| > 0$. Similarly, a 2×2 symmetric matrix defines a negative definite quadratic form ($\mathbf{v}^T\mathbf{A}\mathbf{v} < 0$ for all nonzero $\mathbf{v}$) if and only if $a_{11} < 0$ and $|\mathbf{A}| > 0$.

EXAMPLE 3 Suppose $f(x,y) = 2x^2 - 2x^2y^2 + 2y^2 + 24y - x^4 - y^4$. First, we look for critical points by solving the system

$$\begin{cases} \frac{\partial f}{\partial x} = 4x - 4xy^2 - 4x^3 = 4x(1 - (y^2 + x^2)) = 0 \\ \frac{\partial f}{\partial y} = -4x^2y + 4y + 24 - 4y^3 = -4y(x^2 + y^2) + 4y + 24 = 0 \end{cases}.$$

Now $\frac{\partial f}{\partial x} = 0$ yields $x = 0$ or $y^2 + x^2 = 1$. If $x = 0$, then $\frac{\partial f}{\partial y} = 0$ gives $4y + 24 - 4y^3 = 0$. The unique real solution to this equation is $y = 2$. Thus, $[0, 2]$ is a critical point.

If $x \neq 0$, then $y^2 + x^2 = 1$. From $\frac{\partial f}{\partial y} = 0$, we have $0 = -4y(1) + 4y + 24 = 24$, a contradiction, so there is no critical point when $x \neq 0$.

Next, we compute the Hessian matrix at the critical point $[0, 2]$.

$$\mathbf{H} = \left.\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}\right|_{[\mathbf{0},\mathbf{2}]}$$

$$= \left.\begin{bmatrix} 4 - 4y^2 - 12x^2 & -8xy \\ -8xy & -4x^2 + 4 - 12y^2 \end{bmatrix}\right|_{[\mathbf{0},\mathbf{2}]} = \begin{bmatrix} -12 & 0 \\ 0 & -44 \end{bmatrix}.$$

Since the $(1, 1)$ entry is negative and $|\mathbf{H}| > 0$, $\mathbf{H}$ defines a negative definite quadratic form and so f has a local maximum at $[0, 2]$. ■

▶ An Example in $\mathbb{R}^3$

Example 4 Consider the function

$$g(x, y, z) = 5x^2 + 2xz + 4xy + 10x + 3z^2 - 6yz - 6z + 5y^2 + 12y + 21.$$

We find the critical points by solving the system

$$\begin{cases} \frac{\partial g}{\partial x} = 10x + 2z + 4y + 10 = 0 \\ \frac{\partial g}{\partial y} = 4x - 6z + 10y + 12 = 0 \\ \frac{\partial g}{\partial z} = 2x + 6z - 6y - 6 = 0 \end{cases}.$$

Using row reduction to solve this linear system yields the unique critical point $[-9, 12, 16]$. The Hessian matrix at $[-9, 12, 16]$ is

$$\mathbf{H} = \left.\begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial x \partial z} \\ \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2} & \frac{\partial^2 g}{\partial y \partial z} \\ \frac{\partial^2 g}{\partial z \partial x} & \frac{\partial^2 g}{\partial z \partial y} & \frac{\partial^2 g}{\partial z^2} \end{bmatrix}\right|_{[-9,12,16]} = \begin{bmatrix} 10 & 4 & 2 \\ 4 & 10 & -6 \\ 2 & -6 & 6 \end{bmatrix}.$$

A lengthy computation produces $p_{\mathbf{H}}(x) = x^3 - 26x^2 + 164x - 8$. The roots of $p_{\mathbf{H}}(x)$ are approximately 0.04916, 10.6011, and 15.3497. Since all of these eigenvalues for $\mathbf{H}$ are positive, $[-9, 12, 16]$ is a local minimum for g. ■

▶ Failure of the Hessian Matrix Test

In calculus, we discovered that the second derivative test fails when the second derivative is zero at a critical point. A similar situation is true in $\mathbb{R}^n$. If the Hessian matrix at a critical point has 0 as an eigenvalue, and all other eigenvalues have the same sign, then the function f could have a local maximum, a local minimum, or neither at this critical point. Of course, if the Hessian matrix at a critical point has two eigenvalues with opposite signs, the critical point is not a local extreme point (why?). Exercise 2 illustrates these concepts.

▶ New Vocabulary

$C^2(\mathbb{R}^n)$ (functions from $\mathbb{R}^n$ to $\mathbb{R}$ having continuous second partial derivatives)
critical point (of a function)
gradient (of a function on $\mathbb{R}^n$)
Hessian matrix
local maximum (of a function on $\mathbb{R}^n$)
local minimum (of a function on $\mathbb{R}^n$)
negative definite quadratic form

open hypersphere (in $\mathbb{R}^n$)
positive definite quadratic form
Taylor's Theorem (in $\mathbb{R}^n$)

▸ Highlights

- The gradient of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is defined by $\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right]$.

- Let A be an open hypersphere about $\mathbf{x}_0$, and let f be a function on A with continuous partial derivatives. If f has a local maximum or minimum at $\mathbf{x}_0$, then $\nabla f(\mathbf{x}_0) = \mathbf{0}$.

- For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, its corresponding Hessian matrix $\mathbf{H}$ is the $n \times n$ matrix whose (i,j) entry is $\frac{\partial^2 f}{\partial x_i \partial x_j}$. In particular, for a function $f\colon \mathbb{R}^2 \to \mathbb{R}$, the Hessian matrix $\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$.

- Taylor's Theorem in $\mathbb{R}^n$: Let A be an open hypersphere in $\mathbb{R}^n$ centered at $\mathbf{x}_0$, let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$, and let $t \in \mathbb{R}$ such that $\mathbf{x}_0 + t\mathbf{u}$ is in A. Suppose $f\colon A \to \mathbb{R}$ has continuous second partial derivatives throughout A; that is, $f \in C^2(A)$. Then there is a c with $0 \le c \le t$ such that

$$\begin{aligned} f(\mathbf{x}_0 + t\mathbf{u}) &= f(\mathbf{x}_0) + \sum_{i=1}^{n} \left(\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}_0} \right)(tu_i) + \tfrac{1}{2}\sum_{i=1}^{n} \left(\left.\frac{\partial^2 f}{\partial x_i^2}\right|_{\mathbf{x}_0 + c\mathbf{u}} \right)(t^2 u_i^2) \\ &\quad + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\mathbf{x}_0 + c\mathbf{u}} \right)(t^2 u_i u_j). \end{aligned}$$

  In particular, we have

$$f(\mathbf{x}_0 + \mathbf{v}) = f(\mathbf{x}_0) + \left(\left.\nabla f\right|_{\mathbf{x}_0} \right) \cdot \mathbf{v} + \tfrac{1}{2}\mathbf{v}^T \left(\left.\mathbf{H}\right|_{\mathbf{x}_0 + k\mathbf{v}} \right) \mathbf{v},$$

  for some k with $0 \le k \le 1$.

- Let A be an open hypersphere centered at $\mathbf{x}_0 \in \mathbb{R}^n$. If $f\colon A \to \mathbb{R}$ has continuous second partial derivatives throughout A, then f has a local minimum at a critical point $\mathbf{x}_0$ if $\mathbf{v}^T \left(\left.\mathbf{H}\right|_{\mathbf{x}_0} \right) \mathbf{v} > 0$ for all nonzero vectors $\mathbf{v}$. Similarly, f has a local maximum at a critical point $\mathbf{x}_0$ if $\mathbf{v}^T \left(\left.\mathbf{H}\right|_{\mathbf{x}_0} \right) \mathbf{v} < 0$ for all nonzero vectors $\mathbf{v}$.

- A quadratic form is an expression of the form $\mathbf{v}^T\mathbf{A}\mathbf{v}$, where $\mathbf{v}$ is a vector in $\mathbb{R}^n$, and $\mathbf{A}$ is an $n \times n$ matrix. A positive [negative] definite quadratic form is one such that $\mathbf{v}^T\mathbf{A}\mathbf{v} > 0$ [$\mathbf{v}^T\mathbf{A}\mathbf{v} < 0$] for all nonzero vectors $\mathbf{v}$.

- For a function $f\colon \mathbb{R}^n \to \mathbb{R}$ having Hessian matrix $\mathbf{H}$, if $\mathbf{v}^T \left(\left.\mathbf{H}\right|_{\mathbf{x}_0} \right) \mathbf{v}$ is a positive [negative] definite quadratic form at a critical point $\mathbf{x}_0$, then f has a local minimum [maximum] at $\mathbf{x}_0$.

- A symmetric matrix $\mathbf{A}$ defines a positive [negative] definite quadratic form $\mathbf{v}^T\mathbf{A}\mathbf{v}$ if and only if all of the eigenvalues of $\mathbf{A}$ are positive [negative].

- If $f\colon \mathbb{R}^n \to \mathbb{R}$ has Hessian matrix $\mathbf{H}$, and all eigenvalues of $\mathbf{H}$ are positive [negative] at a critical point $\mathbf{x}_0$, then f has a local minimum [maximum] at $\mathbf{x}_0$.

- A 2×2 symmetric matrix $\mathbf{A}$ has a positive [negative] definite quadratic form $\mathbf{v}^T\mathbf{A}\mathbf{v}$ if and only if $a_{11} > 0$ [$a_{11} < 0$] and $|\mathbf{A}| > 0$.

▶ EXERCISES

1. In each part, solve for all critical points for the given function. Then, for each critical point, use the Hessian matrix to determine whether the critical point is a local maximum, a local minimum, or neither.

★ **a)** $f(x, y) = x^3 + x^2 + 2xy - 3x + y^2$

b) $f(x, y) = 6x^2 + 4xy + 3y^2 + 8x - 9y$

★ **c)** $f(x, y) = 2x^2 + 2xy + 2x + y^2 - 2y + 5$

d) $f(x, y) = x^3 + 3x^2y - x^2 + 3xy^2 + 2xy - 3x + y^3 - y^2 - 3y$
(Hint: To solve for critical points, first set $\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 0$.)

★ **e)** $f(x, y, z) = 2x^2 + 2xy + 2xz + y^4 + 4y^3z + 6y^2z^2 - y^2 + 4yz^3 - 4yz + z^4 - z^2$

2. The parts of this exercise illustrate cases in which the Hessian Matrix Test is inconclusive.

a) Show that $f(x, y) = (x - 2)^4 + (y - 3)^2$ has a local minimum at $[2, 3]$, but its Hessian matrix at $[2, 3]$ has 0 as an eigenvalue.

b) Show that $f(x, y) = -(x - 2)^4 + (y - 3)^2$ has a critical point at $[2, 3]$, its Hessian matrix at $[2, 3]$ has all nonnegative eigenvalues, but $[2, 3]$ is not a local extreme point for f.

c) Show that $f(x, y) = -(x+1)^4 - (y+2)^4$ has a local maximum at $[-1, -2]$, but its Hessian matrix at $[-1, -2]$ is $\mathbf{O}_2$ and thus has all of its eigenvalues equal to zero.

d) Show that $f(x, y, z) = (x - 1)^2 - (y - 2)^2 + (z - 3)^4$ does not have any local extreme points. Then verify that its Hessian matrix has eigenvalues of opposite sign at the function's only critical point.

3. The parts of this exercise prove necessary and sufficient conditions for a symmetric 2×2 matrix to represent a positive definite or negative definite quadratic form.

a) Prove that a symmetric 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ defines a positive definite quadratic form if and only if $a > 0$ and $|\mathbf{A}| > 0$. (Hint: Compute $p_{\mathbf{A}}(x)$ and show that both roots are positive if and only if $a > 0$ and $|\mathbf{A}| > 0$.)

b) Prove that a symmetric 2×2 matrix $\mathbf{A}$ defines a negative definite quadratic form if and only if $a_{11} < 0$ and $|\mathbf{A}| > 0$.

★ **4.** True or False:

a) If $f\colon \mathbb{R}^n \to \mathbb{R}$ has continuous second partial derivatives, then the Hessian matrix is symmetric.

b) Every symmetric matrix $\mathbf{A}$ defines either a positive definite or a negative definite quadratic form.

c) A Hessian matrix for a function with continuous second partial derivatives evaluated at any point is diagonalizable.

d) $\mathbf{v}^T \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \mathbf{v}$ is a positive definite quadratic form.

e) $\mathbf{v}^T \begin{bmatrix} 3 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 4 \end{bmatrix} \mathbf{v}$ is a positive definite quadratic form.

▶ Answers to Selected Exercises

(1) (a) Critical points: $(1, -1)$, $(-1, 1)$; local minimum at $(1, -1)$

(c) Critical point: $(-2, 3)$; local minimum at $(-2, 3)$

(e) Critical points: $(0, 0, 0)$, $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$; local minimums at $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

(4) (a) T (b) F (c) T (d) T (e) F

Jordan Canonical Form

Prerequisites: Section 5.6, Diagonalization of Linear Operators; Section 7.2, Complex Eigenvalues and Complex Eigenvectors

We have seen that not every $n \times n$ matrix is diagonalizable. This can cause difficulties in certain applications. In this section, we define what it means for a matrix to be in Jordan Canonical Form, and assert that every $n \times n$ matrix with complex entries is similar to a matrix of this type. For a diagonalizable complex matrix, its Jordan Canonical Form is merely a diagonal matrix to which it is similar. However, a nondiagonalizable complex matrix $\mathbf{A}$ is similar to a matrix in Jordan Canonical Form which is *almost* diagonal, but with some nonzero entries directly above the main diagonal. For many applications, this is helpful, thus easing our difficulty with nondiagonalizable matrices. We will also show how to put a matrix into Jordan Canonical Form.

Defining Jordan Blocks

Before defining Jordan Canonical Form, we must first discuss Jordan blocks, the basic components from which a matrix in Jordan Canonical Form is constructed.

DEFINITION

A $k \times k$ matrix $\mathbf{A}$ is a **Jordan block** associated with an eigenvalue λ if and only if $\mathbf{A}$ has
(1) every diagonal entry equal to λ,
(2) every entry immediately above the main diagonal equal to 1, and
(3) every other entry equal to zero.

EXAMPLE 1 The matrices

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} -2i & 1 \\ 0 & -2i \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad [4]$$

are Jordan blocks, while the matrices

$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 5i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$$

are not. ■

Since a $k \times k$ Jordan block $\mathbf{A}$ is upper triangular with the value λ on each main diagonal entry, we see that $p_{\mathbf{A}}(x) = (x - \lambda)^k$. Thus λ is the only eigenvalue of $\mathbf{A}$. It is easy to show that $\mathbf{e}_1 = [1, 0, ..., 0]$ spans the eigenspace for $\mathbf{A}$ corresponding to λ. (See Exercise 1(a).) It is also easy to see that $\mathbf{A}$ is a $k \times k$ Jordan block for λ if and only if both of the following conditions hold: $\mathbf{A}\mathbf{e}_1 = \lambda\mathbf{e}_1$, and for $2 \le i \le k$, $\mathbf{A}\mathbf{e}_i = \lambda\mathbf{e}_i + \mathbf{e}_{i-1}$. (This last condition can also be expressed as $(\mathbf{A} - \lambda\mathbf{I}_k)\mathbf{e}_i = \mathbf{e}_{i-1}$.) (See Exercise 1(b).) Thus, if $\mathbf{A}$ is a $k \times k$ Jordan block for λ, $(\mathbf{A} - \lambda\mathbf{I}_k)\mathbf{e}_1 = \mathbf{0}_k$, and it follows by induction that $(\mathbf{A} - \lambda\mathbf{I}_k)^i\mathbf{e}_i = \mathbf{0}_k$ for $1 \le i \le k$. It is then straightforward to show that if $\mathbf{A}$ is a $k \times k$ Jordan block for λ, then $(\mathbf{A} - \lambda\mathbf{I}_k)^k = \mathbf{O}_k$. (See Exercise 1(c).)

EXAMPLE 2 Consider the 3×3 Jordan block $\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ associated with $\lambda = 4$. Then,

$$\mathbf{A}\mathbf{e}_1 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = 4\mathbf{e}_1,$$

and so $(\mathbf{A} - 4\mathbf{I}_3)\mathbf{e}_1 = \mathbf{0}$. Similarly,

$$\mathbf{A}\mathbf{e}_2 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = 4\mathbf{e}_2 + \mathbf{e}_1,$$

and so $(\mathbf{A} - 4\mathbf{I}_3)\mathbf{e}_2 = \mathbf{e}_1$. Also,

$$\mathbf{A}\mathbf{e}_3 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = 4\mathbf{e}_3 + \mathbf{e}_2,$$

and so $(\mathbf{A} - 4\mathbf{I}_3)\mathbf{e}_3 = \mathbf{e}_2$. Thus $(\mathbf{A} - 4\mathbf{I}_3)^2\mathbf{e}_2 = (\mathbf{A} - 4\mathbf{I}_3)(\mathbf{A} - 4\mathbf{I}_3)\mathbf{e}_2 = (\mathbf{A} - 4\mathbf{I}_3)\mathbf{e}_1 = \mathbf{0}$, and $(\mathbf{A} - 4\mathbf{I}_3)^3\mathbf{e}_3 = (\mathbf{A} - 4\mathbf{I}_3)^2(\mathbf{A} - 4\mathbf{I}_3)\mathbf{e}_3 = (\mathbf{A} - 4\mathbf{I}_3)^2\mathbf{e}_2 = \mathbf{0}$. Hence, we have:

$$(\mathbf{A} - 4\mathbf{I}_3)\mathbf{e}_1 = \mathbf{0}, \ \ (\mathbf{A} - 4\mathbf{I}_3)^2\mathbf{e}_2 = \mathbf{0}, \ \ \text{and} \ \ (\mathbf{A} - 4\mathbf{I}_3)^3\mathbf{e}_3 = \mathbf{0}.$$

■

Now suppose that $\mathbf{B}$ is a $k \times k$ matrix similar to a Jordan block $\mathbf{A}$ with diagonal entry λ; that is, $\mathbf{B} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$ for some nonsingular matrix $\mathbf{P}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be the columns of $\mathbf{P}$. After some thought, you will see that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ behave with respect to $\mathbf{B}$ in the same way that $\mathbf{e}_1, \ldots, \mathbf{e}_k$ behave with respect to $\mathbf{A}$. In particular, $\mathbf{B}\mathbf{v}_1 = \lambda\mathbf{v}_1$, and $\mathbf{B}\mathbf{v}_i = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for $2 \le i \le k$. (See Exercise 2(a).)

The above process is reversible. That is, suppose $\mathbf{B}$ is a given $k \times k$ matrix. If we can find a linearly independent sequence of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ such that $\mathbf{B}\mathbf{v}_1 = \lambda\mathbf{v}_1$, and $\mathbf{B}\mathbf{v}_i = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for $2 \le i \le k$, and if $\mathbf{P}$ is the matrix whose i^{th} column is $\mathbf{v}_i$, then $\mathbf{P}$ is nonsingular, and $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ is the $k \times k$ Jordan block associated with the eigenvalue λ. (See Exercise 2(b).)

► Generalized Eigenvectors

We have seen that for a $k \times k$ matrix $\mathbf{B}$ similar to a Jordan block matrix $\mathbf{A}$ with eigenvalue λ, there is a nonsingular matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$. We have also seen that the columns of $\mathbf{P}$ form a sequence of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ such that $\mathbf{B}\mathbf{v}_1 = \lambda\mathbf{v}_1$, and $\mathbf{B}\mathbf{v}_i = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for $2 \le i \le k$. It is straightforward to show that $(\mathbf{B} - \lambda\mathbf{I}_k)^i\,\mathbf{v}_i = \mathbf{0}_k$ for $1 \le i \le k$. (See Exercise 3(a).) Since $\mathbf{P}$ is nonsingular, the $\mathbf{v}_i$'s are linearly independent, and so every vector in $\mathbb{C}^k$ is a linear combination of the $\mathbf{v}_i$'s. Hence it is easy to show that in this particular case, every vector $\mathbf{v}$ in $\mathbb{C}^k$ has the property that $(\mathbf{B} - \lambda\mathbf{I}_k)^i\,\mathbf{v} = \mathbf{0}_k$ for some i, where $1 \le i \le k$. (See Exercise 3(b).)

Unfortunately, not every $n \times n$ matrix is similar to a Jordan block matrix. However, we can find vectors in $\mathbb{C}^n$ that behave in similar ways to the vectors associated with Jordan blocks above. We begin with the following definition:

Definition

Let $\mathbf{A}$ be a square matrix and let λ be an eigenvalue for $\mathbf{A}$. Then a nonzero vector $\mathbf{v}$ is a **generalized eigenvector** for $\mathbf{A}$ corresponding to λ if and only if there is a positive integer k such that $(\mathbf{A} - \lambda\mathbf{I}_n)^k \mathbf{v} = \mathbf{0}$. The set

$$\{\mathbf{v} \in \mathbb{C}^n \mid (\mathbf{A} - \lambda\mathbf{I}_n)^k \mathbf{v} = \mathbf{0} \text{ for some positive integer } k\}$$

(which includes the zero vector) is called the **generalized eigenspace** for $\mathbf{A}$ corresponding to λ.

Thus, for a matrix similar to a $k \times k$ Jordan block (having eigenvalue λ), every nonzero vector in $\mathbb{C}^k$ is a generalized eigenvector. The generalized eigenspace in this case is $\mathbb{C}^k$.

When we studied diagonalization in Sections 3.4 and 5.6, we saw that to diagonalize a matrix, computing a complete set of fundamental eigenvectors was a crucial step. Similarly, we will see that in obtaining a Jordan Canonical Form matrix for a given matrix $\mathbf{B}$, it is vital to find sequences of special generalized eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ such that $\mathbf{B}\mathbf{v}_1 = \lambda\mathbf{v}_1$, and $\mathbf{B}\mathbf{v}_i = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for $2 \le i \le k$. Hence, we make the following definition:

Definition

Let $\mathbf{A}$ be a square matrix and let λ be an eigenvalue for $\mathbf{A}$. Then a sequence of nonzero vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ such that $\mathbf{A}\mathbf{v}_1 = \lambda\mathbf{v}_1$, and $\mathbf{A}\mathbf{v}_i = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for $2 \le i \le k$ is called a **fundamental sequence of generalized eigenvectors for A of length** k corresponding to λ.

Notice that the $\mathbf{v}_i$'s in this definition are, in fact, generalized eigenvectors for λ because we saw earlier that the given conditions on the $\mathbf{v}_i$'s imply that $(\mathbf{A} - \lambda\mathbf{I})^i \mathbf{v}_i = \mathbf{0}$, for $1 \le i \le k$.

For example, the sequence $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a fundamental sequence of generalized eigenvectors of length 3 for the Jordan block matrix $\mathbf{A}$ in Example 2 corresponding to the eigenvalue 4, since we found $\mathbf{A}\mathbf{e}_1 = 4\mathbf{e}_1$, $\mathbf{A}\mathbf{e}_2 = 4\mathbf{e}_2 + \mathbf{e}_1$, and $\mathbf{A}\mathbf{e}_3 = 4\mathbf{e}_3 + \mathbf{e}_2$. Such fundamental sequences of generalized eigenvectors are the key to finding a Jordan Canonical Form for a general square matrix.

▶ Defining Jordan Canonical Form

Not every $k \times k$ matrix $\mathbf{B}$ is similar to a Jordan block, since vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ with the desired properties $\mathbf{B}\mathbf{v}_1 = \lambda\mathbf{v}_1$, and $\mathbf{B}\mathbf{v}_i = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for $2 \le i \le k$ (for some λ) may not exist. Thus, we must go beyond the concept of a Jordan block to Jordan Canonical Form.

Definition

A square matrix $\mathbf{A}$ is in **Jordan Canonical Form** if and only if there exist Jordan blocks $\mathbf{A}_1, \ldots, \mathbf{A}_m$ such that

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_2 & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{A}_m \end{bmatrix}.$$

That is, a matrix in Jordan Canonical Form can be divided into blocks such that every block centered on the main diagonal is a Jordan block, and every other block is a zero matrix.

Example 3 The following matrices are all in Jordan Canonical Form. The Jordan blocks are bracketed to make them stand out.

$$\begin{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{bmatrix} 3 \end{bmatrix} \end{bmatrix}, \quad \begin{bmatrix} \begin{bmatrix} 4 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} 5 \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} 6-3i \end{bmatrix} \end{bmatrix}, \quad \begin{bmatrix} \begin{bmatrix} -1 \end{bmatrix} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \end{bmatrix},$$

$$\begin{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} & \begin{bmatrix} 7 \end{bmatrix} \end{bmatrix}, \quad \begin{bmatrix} \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix}.$$

■

Because every matrix in Jordan Canonical Form is upper triangular, its characteristic polynomial is easy to compute. In particular, if $\lambda_1, \ldots, \lambda_n$ are the entries along the main diagonal of an $n \times n$ Jordan Canonical Form matrix $\mathbf{A}$, then $p_{\mathbf{A}}(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$.

Example 4 Consider the 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

from Example 3. Then, $p_{\mathbf{A}}(x) = (x-2)^2 (x-3)$. Hence, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2$ and $\lambda_2 = 3$. Because $\mathbf{A}\mathbf{e}_1 = 2\mathbf{e}_1$ and $\mathbf{A}\mathbf{e}_2 = 2\mathbf{e}_2 + \mathbf{e}_1$, we see that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a fundamental sequence of generalized eigenvectors for $\mathbf{A}$ of length 2 corresponding to $\lambda_1 = 2$. Similarly, since $\mathbf{A}\mathbf{e}_3 = 3\mathbf{e}_3$, $\{\mathbf{e}_3\}$ is a fundamental sequence of generalized eigenvectors for $\mathbf{A}$ of length 1 corresponding to $\lambda_2 = 3$. ■

We can generalize the principles in Example 4 as follows: Suppose $\mathbf{A}$ is a matrix in Jordan Canonical Form, with $\mathbf{A}_i$ as the i^{th} Jordan block on the main diagonal of $\mathbf{A}$. Also suppose that $\mathbf{A}_i$ is a $k \times k$ matrix associated with the eigenvalue λ_i with its first row appearing on the l^{th} row of $\mathbf{A}$. Then it is easy to verify that $\mathbf{A}\mathbf{e}_l = \lambda_i \mathbf{e}_l$, and that $\mathbf{A}\mathbf{e}_{l+m} = \lambda_i \mathbf{e}_{l+m} + \mathbf{e}_{l+m-1}$ for $1 \le m \le k-1$. Essentially, $\mathbf{A}$ has $\{\mathbf{e}_l, \mathbf{e}_{l+1}, \ldots, \mathbf{e}_{l+k-1}\}$ as a fundamental sequence of generalized eigenvectors for $\mathbf{A}$ of length k corresponding to λ_i.

Analogously, if $\mathbf{B} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$, with $\mathbf{A}$ in Jordan Canonical Form, having blocks $\mathbf{A}_1$, ..., $\mathbf{A}_t$, with sizes k_1 through k_t, respectively, then the columns of $\mathbf{P}$ form a collection of fundamental sequences of generalized eigenvectors $\{\mathbf{v}_{11}, \mathbf{v}_{12}, \ldots, \mathbf{v}_{1k_1}\}$,

$\{\mathbf{v}_{21}, \mathbf{v}_{22}, \ldots, \mathbf{v}_{2k_2}\}, \ldots, \{\mathbf{v}_{t1}, \mathbf{v}_{t2}, \ldots, \mathbf{v}_{tk_t}\}$ for $\mathbf{B}$ of lengths $k_1, \ldots, k_t$, respectively, each for its appropriate eigenvalue.

EXAMPLE 5 Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 3 & -1 & 1 \\ 5 & -3 & 2 \\ -2 & 1 & -1 \end{bmatrix},$$

and let

$$\mathbf{B} = \mathbf{PAP}^{-1} = \begin{bmatrix} 4 & -4 & -7 \\ 3 & -5 & -13 \\ -1 & 3 & 8 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}$ (and $\mathbf{B}$) are clearly $\lambda_1 = 2$ and $\lambda_2 = 3$.

Now, by examining the two Jordan blocks of $\mathbf{A}$, we see that the columns of $\mathbf{P}$ form two fundamental sequences of generalized eigenvectors $\{\mathbf{v}_{11}, \mathbf{v}_{12}\}$ and $\{\mathbf{v}_{21}\}$, for λ_1 and λ_2 respectively, with $\mathbf{v}_{11} = [3, 5, -2]$, $\mathbf{v}_{12} = [-1, -3, 1]$, and $\mathbf{v}_{21} = [1, 2, -1]$. Notice that

$$\mathbf{Bv}_{11} = \begin{bmatrix} 4 & -4 & -7 \\ 3 & -5 & -13 \\ -1 & 3 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} = \lambda_1 \mathbf{v}_{11},$$

and that

$$\begin{aligned} \mathbf{Bv}_{12} &= \begin{bmatrix} 4 & -4 & -7 \\ 3 & -5 & -13 \\ -1 & 3 & 8 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\ &= 2 \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} = \lambda_1 \mathbf{v}_{12} + \mathbf{v}_{11}. \end{aligned}$$

Also,

$$\mathbf{Bv}_{21} = \begin{bmatrix} 4 & -4 & -7 \\ 3 & -5 & -13 \\ -1 & 3 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \lambda_2 \mathbf{v}_{21}.$$

We leave it for you to verify that $(\mathbf{B} - 2\mathbf{I}_3)\,\mathbf{v}_{11} = \mathbf{0}_3$, $(\mathbf{B} - 2\mathbf{I}_3)\,\mathbf{v}_{12} = \mathbf{v}_{11}$, $(\mathbf{B} - 2\mathbf{I}_3)^2\,\mathbf{v}_{12} = \mathbf{0}_3$, and $(\mathbf{B} - 3\mathbf{I}_3)\,\mathbf{v}_{21} = \mathbf{0}_3$ (see Exercise 7(a)). ■

▸ Every Complex Matrix is Similar to a Matrix in Jordan Canonical Form

When we were *diagonalizing* matrices in Section 5.6, our goal for a given matrix $\mathbf{C}$ was to find a basis of fundamental eigenvectors for $\mathbf{C}$. We found that the vectors of this basis become the columns of a matrix $\mathbf{P}$ so that $\mathbf{P}^{-1}\mathbf{CP}$ is diagonal. Now, to "Jordanize" a matrix $\mathbf{B}$, we try to find a basis consisting of fundamental sequences of *generalized* eigenvectors in order to build a matrix $\mathbf{P}$ so that $\mathbf{P}^{-1}\mathbf{BP}$ is in Jordan Canonical Form.

The main result in this section is:

> THEOREM 1
> Let $\mathbf{B}$ be an $n \times n$ matrix with complex entries. Then there exist $n \times n$ matrices $\mathbf{P}$ and $\mathbf{A}$ with $\mathbf{P}$ nonsingular and $\mathbf{A}$ in Jordan Canonical Form such that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$ (and hence, $\mathbf{B} = \mathbf{PAP}^{-1}$). Moreover, the matrix $\mathbf{A}$ is unique, except for the order in which the Jordan blocks appear along the main diagonal.

Theorem 1 asserts that every complex matrix is similar to a matrix in Jordan Canonical Form. We will not prove this theorem. However, the idea behind the proof is to show the existence of a fundamental sequence of generalized eigenvectors $\{\mathbf{v}_{i1}, \ldots, \mathbf{v}_{ik}\}$ associated with each eigenvalue λ_i of $\mathbf{B}$ such that $\mathbf{B}\mathbf{v}_{i1} = \lambda_i \mathbf{v}_{i1}$ and $\mathbf{B}\mathbf{v}_{ij} = \lambda_i \mathbf{v}_{ij} + \mathbf{v}_{i(j-1)}$ for $2 \le j \le k_i$. The vectors in these sequences constitute the columns of $\mathbf{P}$. Exercise 20 proves the uniqueness claim in the theorem, assuming the existence of Jordan Canonical Form (see the comment in Exercise 20(g)).

It is straightforward to show that if two matrices $\mathbf{C}$ and $\mathbf{D}$ have Jordan Canonical Form matrices with identical blocks (in any order), then $\mathbf{C}$ and $\mathbf{D}$ are similar. (See part (d) of Exercise 8, as well as Exercise 15 which extends this result.) It is also easy to prove the inverse of this statement using the uniqueness assertion of Theorem 1 together with Exercise 15; that is, two matrices $\mathbf{C}$ and $\mathbf{D}$ are not similar if there are Jordan Canonical Form matrices for $\mathbf{C}$ and $\mathbf{D}$ that have different Jordan blocks (in any order).

▶ Finding a Jordan Canonical Form

The general idea behind the method for finding a Jordan Canonical Form for a matrix is simple, but the details are often complicated to work out in practice. One useful result, which we state without proof, is that *the dimension of the generalized eigenspace for a given eigenvalue always equals the algebraic multiplicity of that eigenvalue*. We begin with a basic example.

EXAMPLE 6 Consider the matrix

$$\mathbf{B} = \begin{bmatrix} -12 & 13 & -40 \\ 17 & -17 & 55 \\ 9 & -9 & 29 \end{bmatrix}.$$

We find a Jordan Canonical Form for $\mathbf{B}$. You can quickly calculate that

$$p_{\mathbf{B}}(x) = x^3 - 3x - 2 = (x+1)^2(x-2).$$

Thus, the eigenvalues for $\mathbf{B}$ are $\lambda_1 = -1$ and $\lambda_2 = 2$. We must find the sizes of the Jordan blocks corresponding to these eigenvalues, and a sequence of generalized eigenvectors corresponding to each block. Now, the Cayley-Hamilton Theorem (Theorem 5.29 in Section 5.6) tells us that $p_{\mathbf{B}}(\mathbf{B}) = \mathbf{O}_3$. We factor $p_{\mathbf{B}}(\mathbf{B})$ to obtain $p_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B} + \mathbf{I}_3)^2 (\mathbf{B} - 2\mathbf{I}_3) = \mathbf{O}_3$.

We begin with the eigenvalue $\lambda_1 = -1$. Let

$$\mathbf{D} = (\mathbf{B} - 2\mathbf{I}_3) = \begin{bmatrix} -14 & 13 & -40 \\ 17 & -19 & 55 \\ 9 & -9 & 27 \end{bmatrix}$$

Then $(\mathbf{B} + \mathbf{I}_3)^2 \mathbf{D} = \mathbf{O}_3$.

Next, we search for the *smallest* positive integer k such that $(\mathbf{B} + \mathbf{I}_3)^k \mathbf{D} = \mathbf{O}_3$. Now,

$$(\mathbf{B} + \mathbf{I}_3)\mathbf{D} = \begin{bmatrix} 15 & -30 & 75 \\ -15 & 30 & -75 \\ -9 & 18 & -45 \end{bmatrix} \neq \mathbf{O}_3,$$

while, as we have seen, $(\mathbf{B} + \mathbf{I}_3)^2 \mathbf{D} = \mathbf{O}_3$. Hence, $k = 2$.

Notice that, given any vector $\mathbf{v}$, $(\mathbf{B} + \mathbf{I}_3)^2 [(\mathbf{B} - 2\mathbf{I}_3)\mathbf{v}] = \mathbf{0}$. Hence, if $(\mathbf{B} - 2\mathbf{I}_3)\mathbf{v}$ is nonzero, $(\mathbf{B} - 2\mathbf{I}_3)\mathbf{v}$ is a generalized eigenvector corresponding to $\lambda_1 = -1$. Since $(\mathbf{B} + \mathbf{I}_3)[(\mathbf{B} + \mathbf{I}_3)\mathbf{D}] = \mathbf{O}_3$, each column of $(\mathbf{B} + \mathbf{I}_3)\mathbf{D}$ is in the kernel of $(\mathbf{B} + \mathbf{I}_3)$, and is thus a generalized eigenvector for $\mathbf{B}$ corresponding to $\lambda_1 = -1$.

We choose a linearly independent subset of columns of $(\mathbf{B} + \mathbf{I}_3)\mathbf{D}$ that contains as many vectors as possible, in order to get as many linearly independent generalized eigenvectors as we can. Since all of the columns are multiples of the first, the first column alone suffices. Thus,

$$\mathbf{v}_{11} = [15, -15, -9].$$

A natural inclination here is to simplify $\mathbf{v}_{11}$ by multiplying every entry by $\frac{1}{3}$. However, we need to *be careful when making such adjustments*. We can only multiply a generalized eigenvector by a scalar if we *multiply every generalized eigenvector in the same fundamental sequence by the same scalar*. We must therefore postpone such considerations in this case until $\mathbf{v}_{12}$ is determined.

Next, we work backwards through the products of the form $(\mathbf{B}+\mathbf{I}_3)^{k-j}\,\mathbf{D}$ for j running from 2 up to k, choosing the same column in which we found the generalized eigenvector $\mathbf{v}_{11}$. Because $k = 2$, the only value we need to consider here is $j = 2$. Hence, we let

$$\mathbf{v}_{12} = [-14, 17, 9],$$

the first column of $(\mathbf{B}+\mathbf{I}_3)^{(2-2)}\,\mathbf{D} = \mathbf{D}$. Since the entries of $\mathbf{v}_{12}$ are not exactly divisible by 3, we do not simplify the entries of $\mathbf{v}_{11}$ and $\mathbf{v}_{12}$ here.

Now by construction, $(\mathbf{B}+\mathbf{I}_3)\,\mathbf{v}_{12} = \mathbf{v}_{11}$ and $(\mathbf{B}+\mathbf{I}_3)\,\mathbf{v}_{11} = \mathbf{0}$. Hence, $\mathbf{v}_{11}$ and $\mathbf{v}_{12}$ are generalized eigenvectors for λ_1. From our observation just before this example, the dimension of the generalized eigenspace for λ_1 equals the algebraic multiplicity of λ_1, which is 2. Thus we can stop our work for λ_1 here, since $\mathbf{v}_{11}$ and $\mathbf{v}_{12}$ form a basis for the generalized eigenspace for λ_1. We therefore have a fundamental sequence $\{\mathbf{v}_{11}, \mathbf{v}_{12}\}$ of generalized eigenvectors corresponding to a 2×2 Jordan block associated with $\lambda_1 = -1$ in a Jordan Canonical Form for $\mathbf{B}$.

To complete this example, we still must find a generalized eigenvector corresponding to $\lambda_2 = 2$.

Recall that $p_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B}-2\mathbf{I}_3)\,(\mathbf{B}+\mathbf{I}_3)^2 = \mathbf{O}_3$. Let

$$\mathbf{D} = (\mathbf{B}+\mathbf{I}_3)^2 = \begin{bmatrix} -18 & 9 & -45 \\ 36 & -18 & 90 \\ 18 & -9 & 45 \end{bmatrix}.$$

Then $(\mathbf{B}-2\mathbf{I}_3)\,\mathbf{D} = \mathbf{O}_3$.

Next, we search for the *smallest* positive integer k such that $(\mathbf{B}-2\mathbf{I}_3)^k\,\mathbf{D} = \mathbf{O}_3$. However, it is obvious here that $k = 1$.

Since $k-1 = 0$, $(\mathbf{B}-2\mathbf{I}_3)^{k-1}\,\mathbf{D} = \mathbf{D}$. Hence, each nonzero column of $\mathbf{D} = (\mathbf{B}+\mathbf{I}_3)^2$ is a generalized eigenvector for $\mathbf{B}$ corresponding to $\lambda_2 = 2$. In particular, the first column of $(\mathbf{B}+\mathbf{I}_3)^2$ serves nicely as a generalized eigenvector $\mathbf{v}_{21}$ for $\mathbf{B}$ corresponding to $\lambda_2 = 2$. No further work for λ_2 is needed here because $\lambda_2 = 2$ has algebraic multiplicity 1, and hence only one generalized eigenvector corresponding to λ_2 is sufficient. Since there are no other generalized eigenvectors for λ_2, we can simplify $\mathbf{v}_{21}$ by multiplying by $\frac{1}{18}$, yielding

$$\mathbf{v}_{21} = [-1, 2, 1].$$

Thus, $\{\mathbf{v}_{21}\}$ is a sequence of generalized eigenvectors corresponding to the 1×1 Jordan block associated with $\lambda_2 = 2$ in a Jordan Canonical Form for $\mathbf{B}$.

Finally, we now have an ordered basis $(\mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{21})$ of generalized eigenvectors for $\mathbf{B}$. Letting $\mathbf{P}$ be the matrix whose columns are these basis vectors, we find that

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{B}\mathbf{P} &= \tfrac{1}{9}\begin{bmatrix} -1 & 5 & -11 \\ -3 & 6 & -15 \\ 18 & -9 & 45 \end{bmatrix}\begin{bmatrix} -12 & 13 & -40 \\ 17 & -17 & 55 \\ 9 & -9 & 29 \end{bmatrix}\begin{bmatrix} 15 & -14 & -1 \\ -15 & 17 & 2 \\ -9 & 9 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \end{aligned}$$

which gives us a Jordan Canonical Form for $\mathbf{B}$. Note that we could have used the generalized eigenvectors in the order $\mathbf{v}_{21}, \mathbf{v}_{11}, \mathbf{v}_{12}$ instead, which would have resulted in the other possible Jordan Canonical Form for $\mathbf{B}$,

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

■

▶ Basic Method for Finding a Jordan Canonical Form

We summarize the basic method below. However, we will see that adjustments to this method must be made if the matrix has two or more Jordan blocks of different sizes corresponding to the same eigenvalue. Those adjustments are illustrated in Example 8.

Basic Method for Finding a Jordan Canonical Form for a Given Square Matrix (JORDAN FORM METHOD)
Suppose $\mathbf{B}$ is a square matrix with characteristic polynomial $p_{\mathbf{B}}(x) = (x-\lambda_1)^{m_1} \cdots (x-\lambda_l)^{m_l}$, where $\lambda_1, \ldots, \lambda_l$ are the distinct eigenvalues of $\mathbf{B}$.

Step A: For each eigenvalue λ_i in turn, perform the following:
Step A1: Let $\mathbf{D}$ be the matrix

$$(\mathbf{B}-\lambda_1\mathbf{I}_n)^{m_1} \cdots (\mathbf{B}-\lambda_{i-1}\mathbf{I}_n)^{m_{i-1}} (\mathbf{B}-\lambda_{i+1}\mathbf{I}_n)^{m_{i+1}} \cdots (\mathbf{B}-\lambda_l\mathbf{I}_n)^{m_l}.$$

That is, $\mathbf{D}$ is the product of the factors of $p_{\mathbf{B}}(\mathbf{B})$ with the factor $(\mathbf{B}-\lambda_i\mathbf{I}_n)^{m_i}$ missing.
Step A2: Find the smallest positive integer k such that $(\mathbf{B}-\lambda_i\mathbf{I}_n)^k \mathbf{D} = \mathbf{O}_n$. (The Cayley-Hamilton Theorem states that $p_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B}-\lambda_i\mathbf{I}_n)^{m_i} \mathbf{D} = \mathbf{O}_n$, and so $k \le m_i$.)
Step A3: Choose a set $\mathbf{v}_{11}, \ldots, \mathbf{v}_{s1}$ consisting of as many linearly independent columns of $(\mathbf{B}-\lambda_i\mathbf{I}_n)^{k-1}\mathbf{D}$ as possible (that is, so that no larger set of columns is linearly independent).
Step A4: For $1 \le t \le s$, and for each j in turn from 2 up to k, let $\mathbf{v}_{tj}$ be the column of $(\mathbf{B}-\lambda_i\mathbf{I}_n)^{k-j}\mathbf{D}$ corresponding to the column in which $\mathbf{v}_{t1}$ appeared in $(\mathbf{B}-\lambda_i\mathbf{I}_n)^{k-1}\mathbf{D}$.
Step A5: Assemble the s separate fundamental sequences of generalized eigenvectors, each of length k, consisting of $\{\mathbf{v}_{t1}, \ldots, \mathbf{v}_{tk}\}$.
Step A6: If the total number of generalized eigenvectors in all fundamental sequences corresponding to λ_i equals m_i (the algebraic multiplicity of λ_i), then the process for this eigenvalue is finished. Otherwise, perform the adjustment process for λ_i described later in this section.

Step B: Form the matrix $\mathbf{P}$ whose columns consist of all the fundamental sequences of generalized eigenvectors found in Step A (keeping the vectors in each sequence together and in order). Then $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ is a Jordan Canonical Form matrix similar to $\mathbf{B}$ with Jordan blocks along its main diagonal. Each block corresponds to a fundamental sequence of generalized eigenvectors for a particular eigenvalue.

The next example illustrates the method with a matrix having two Jordan blocks of the same size corresponding to the same eigenvalue.

EXAMPLE 7 Consider the 4×4 matrix

$$\mathbf{B} = \begin{bmatrix} -7 & -2 & 6 & 12 \\ 2 & -2 & -3 & -6 \\ 4 & 3 & -8 & -10 \\ -3 & -2 & 4 & 5 \end{bmatrix}.$$

We will follow the Jordan Form Method to find a Jordan Canonical Form matrix similar to $\mathbf{B}$.

Now,

$$p_{\mathbf{B}}(x) = x^4 + 12x^3 + 54x^2 + 108x + 81 = (x+3)^4.$$

Therefore, $\mathbf{B}$ has only one eigenvalue, $\lambda = -3$, having algebraic multiplicity 4. Thus, we must find 4 generalized eigenvectors for λ. Note that by the Cayley-Hamilton Theorem, $p_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B} + 3\mathbf{I}_4)^4 = (\mathbf{B} + 3\mathbf{I}_4)^4\mathbf{I}_4 = \mathbf{O}_4$.

Step A1: Let $\mathbf{D} = \mathbf{I}_4$.

Step A2: We search for the smallest positive integer k such that $(\mathbf{B}+3\mathbf{I}_4)^k\mathbf{D} = \mathbf{O}_4$. Computing successive powers of $(\mathbf{B} + 3\mathbf{I}_4)$, and multiplying by $\mathbf{D}$, yields

$$(\mathbf{B} + 3\mathbf{I}_4)\mathbf{D} = (\mathbf{B} + 3\mathbf{I}_4)\mathbf{I}_4 = \begin{bmatrix} -4 & -2 & 6 & 12 \\ 2 & 1 & -3 & -6 \\ 4 & 3 & -5 & -10 \\ -3 & -2 & 4 & 8 \end{bmatrix},$$

and $(\mathbf{B} + 3\mathbf{I}_4)^2\mathbf{D} = \mathbf{O}_4$. Thus, $k = 2$.

Step A3: Each column of $(\mathbf{B} + 3\mathbf{I}_4)\mathbf{D}$ is a generalized eigenvector for $\mathbf{B}$ corresponding to $\lambda = -3$. We choose a linearly independent subset of columns from $(\mathbf{B} + 3\mathbf{I}_4)\mathbf{D}$ that is as large as possible, in order to get as many linearly independent generalized eigenvectors as we can. Let $\mathbf{v}_{11} = 1^{st}$ column of $(\mathbf{B}+3\mathbf{I}_4)\mathbf{D}$. The second column of $(\mathbf{B} + 3\mathbf{I}_4)\mathbf{D}$ is not a scalar multiple of the first, so we let $\mathbf{v}_{21} = 2^{nd}$ column of $(\mathbf{B} + 3\mathbf{I}_4)\mathbf{D}$. However,

$$\begin{aligned} 3^{rd} \text{ column of } (\mathbf{B} + 3\mathbf{I}_4)\mathbf{D} &= -2\mathbf{v}_{11} + \mathbf{v}_{21}, \\ 4^{th} \text{ column of } (\mathbf{B} + 3\mathbf{I}_4)\mathbf{D} &= -4\mathbf{v}_{11} + 2\mathbf{v}_{21}, \end{aligned}$$

and so $\{\mathbf{v}_{11}, \mathbf{v}_{21}\}$ is a set containing as many linearly independent columns as possible. Thus, so far, we have found two generalized eigenvectors:

$$\mathbf{v}_{11} = [-4, 2, 4, -3] \quad \text{and} \quad \mathbf{v}_{21} = [-2, 1, 3, -2].$$

Step A4: Because $k = 2$, we only need consider $j = 2$. Now $(\mathbf{B} + 3\mathbf{I}_4)^{(2-2)}\mathbf{D} = \mathbf{D}$. Choose $\mathbf{v}_{12}$ and $\mathbf{v}_{22}$ as the first two columns of $\mathbf{D}$, respectively; that is, let

$$\mathbf{v}_{12} = [1, 0, 0, 0] \quad \text{and} \quad \mathbf{v}_{22} = [0, 1, 0, 0].$$

Thus, $(\mathbf{B} + 3\mathbf{I}_4)\mathbf{v}_{12} = \mathbf{v}_{11}$ and $(\mathbf{B} + 3\mathbf{I}_4)\mathbf{v}_{22} = \mathbf{v}_{21}$.

Step A5: This gives us *two* fundamental sequences of generalized eigenvectors; $\{\mathbf{v}_{11}, \mathbf{v}_{12}\}$ and $\{\mathbf{v}_{21}, \mathbf{v}_{22}\}$, each corresponding to a 2×2 Jordan block for $\lambda = -3$ in a Jordan Canonical Form for $\mathbf{B}$.

Step A6: Since $\lambda = -3$ has algebraic multiplicity 4, and we have 4 generalized eigenvectors, we are finished.

Step B: We now have $(\mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{21}, \mathbf{v}_{22})$, an ordered basis of generalized eigenvectors for $\mathbf{B}$. Letting $\mathbf{P}$ be the matrix whose columns are these basis vectors, we find that

$$\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \begin{bmatrix} 0 & 0 & -2 & -3 \\ 1 & 0 & -2 & -4 \\ 0 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -7 & -2 & 6 & 12 \\ 2 & -2 & -3 & -6 \\ 4 & 3 & -8 & -10 \\ -3 & -2 & 4 & 5 \end{bmatrix} \begin{bmatrix} -4 & 1 & -2 & 0 \\ 2 & 0 & 1 & 1 \\ 4 & 0 & 3 & 0 \\ -3 & 0 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix},$$

which is a Jordan Canonical Form for $\mathbf{B}$. ■

Example 8 below illustrates an "adjustment" process that must be used to find a Jordan Canonical Form when there are two (or more) Jordan blocks of different sizes for the same eigenvalue. The problem here is that when dealing with such an eigenvalue, the fundamental sequences corresponding to smaller Jordan blocks are hidden during the computation of longer fundamental sequences. Hence, after finding the longer sequences, an adjustment must be made to the given matrix to allow the shorter sequences to be found. The formal method for this adjustment process can be found directly after the example.

EXAMPLE 8 Let

$$\mathbf{B} = \begin{bmatrix} -9 & -2 & 4 & 8 & -18 \\ -38 & -1 & 20 & 28 & -78 \\ -58 & -9 & 25 & 56 & -131 \\ -89 & -12 & 38 & 89 & -210 \\ -43 & -6 & 18 & 44 & -103 \end{bmatrix}.$$

A lengthy calculation yields

$$p_{\mathbf{B}}(x) = x^5 - x^4 - 6x^3 + 14x^2 - 11x + 3 = (x-1)^4(x+3).$$

Therefore, the eigenvalues for $\mathbf{B}$ are $\lambda_1 = 1$ and $\lambda_2 = -3$.
Step A: We begin by finding generalized eigenvectors for λ_1.

Step A1: $p_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B} - \mathbf{I}_5)^4(\mathbf{B} + 3\mathbf{I}_5) = \mathbf{O}_5$. We let $\mathbf{D} = \mathbf{B} + 3\mathbf{I}_5$.

Step A2: We calculate as follows:

$$\mathbf{D} = \begin{bmatrix} -6 & -2 & 4 & 8 & -18 \\ -38 & 2 & 20 & 28 & -78 \\ -58 & -9 & 28 & 56 & -131 \\ -89 & -12 & 38 & 92 & -210 \\ -43 & -6 & 18 & 44 & -100 \end{bmatrix},$$

$$(\mathbf{B} - \mathbf{I}_5)\mathbf{D} = \begin{bmatrix} -34 & -8 & 12 & 32 & -68 \\ 6 & 24 & 28 & -96 & 140 \\ -53 & -4 & 30 & 16 & -58 \\ -16 & 16 & 32 & -64 & 80 \\ -2 & 8 & 12 & -32 & 44 \end{bmatrix},$$

$$(\mathbf{B} - \mathbf{I}_5)^2\mathbf{D} = \begin{bmatrix} 24 & 0 & -16 & 0 & 16 \\ -72 & 0 & 48 & 0 & -48 \\ 12 & 0 & -8 & 0 & 8 \\ -48 & 0 & 32 & 0 & -32 \\ -24 & 0 & 16 & 0 & -16 \end{bmatrix},$$

and $(\mathbf{B} - \mathbf{I}_5)^3\mathbf{D} = \mathbf{O}_5$. Hence $k = 3$.

Step A3: Each nonzero column of $(\mathbf{B} - \mathbf{I}_5)^2\mathbf{D}$ is a generalized eigenvector for

$\mathbf{B}$ corresponding to $\lambda_1 = 1$. We let

$$\mathbf{v}_{11} = [24, -72, 12, -48, -24],$$

the first column of $(\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D}$. (We do not divide by 12 here since not all of the entries of the vectors $\mathbf{v}_{12}$ and $\mathbf{v}_{13}$ calculated in the next step are divisible by 12.) Notice that the other columns of $(\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D}$ are scalar multiples of $\mathbf{v}_{11}$, so we have found as many linearly independent eigenvectors as possible at this stage.

Step A4: We let

$$\mathbf{v}_{12} = [-34, 6, -53, -16, -2] \quad \text{and} \quad \mathbf{v}_{13} = [-6, -38, -58, -89, -43],$$

the first columns of $(\mathbf{B} - \mathbf{I}_5) \mathbf{D}$, and $\mathbf{D}$, respectively. Hence, $(\mathbf{B} - \mathbf{I}_5) \mathbf{v}_{12} = \mathbf{v}_{11}$, and $(\mathbf{B} - \mathbf{I}_5)^2 \mathbf{v}_{13} = \mathbf{v}_{11}$. Thus we have our first fundamental sequence of generalized eigenvectors, $\{\mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{13}\}$, corresponding to a 3×3 Jordan block for $\lambda_1 = 1$.

Steps A5 and A6: We have one fundamental sequence of generalized eigenvectors consisting of a total of 3 vectors. But, since the algebraic multiplicity of λ_1 is 4, we must still find another generalized eigenvector. We do this by making an adjustment to the matrix $\mathbf{D}$.

Recall that each column of $(\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D}$ is a scalar multiple of $\mathbf{v}_{11}$. In fact,

$$\begin{aligned}
1^{st} \text{ column of } (\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D} &= 1\mathbf{v}_{11} = f_1 \mathbf{v}_{11}, \\
2^{nd} \text{ column of } (\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D} &= 0\mathbf{v}_{11} = f_2 \mathbf{v}_{11}, \\
3^{rd} \text{ column of } (\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D} &= -\tfrac{2}{3}\mathbf{v}_{11} = f_3 \mathbf{v}_{11}, \\
4^{th} \text{ column of } (\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D} &= 0\mathbf{v}_{11} = f_4 \mathbf{v}_{11}, \\
5^{th} \text{ column of } (\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D} &= \tfrac{2}{3}\mathbf{v}_{11} = f_5 \mathbf{v}_{11},
\end{aligned}$$

where the f_i's represent the respective coefficients of $\mathbf{v}_{11}$. We create a new matrix $\mathbf{F}$ whose i^{th} column is $f_i \mathbf{v}_{13}$. Then, since $\mathbf{v}_{13}$ is the first column of $\mathbf{D}$, $(\mathbf{B} - \mathbf{I}_5)^2 \mathbf{v}_{13} = \mathbf{v}_{11}$. Thus, for each i, $(\mathbf{B} - \mathbf{I}_5)^2 f_i \mathbf{v}_{13} = f_i \mathbf{v}_{11}$, and so $(\mathbf{B} - \mathbf{I}_5)^2 \mathbf{F} = (\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D}$. Let $\mathbf{D}_1 = \mathbf{D} - \mathbf{F}$. Clearly, $(\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D}_1 = \mathbf{O}_5$. We now revisit Steps A2 through A6 using the matrix $\mathbf{D}_1$ instead of $\mathbf{D}$. The purpose of this adjustment to the matrix $\mathbf{D}$ is to attempt to eliminate the effects of the fundamental sequence of length 3, thus unmasking shorter fundamental sequences.

Step A2: We have

$$\mathbf{D}_1 = \begin{bmatrix} 0 & -2 & 0 & 8 & -14 \\ 0 & 2 & -\frac{16}{3} & 28 & -\frac{158}{3} \\ 0 & -9 & -\frac{32}{3} & 56 & -\frac{277}{3} \\ 0 & -12 & -\frac{64}{3} & 92 & -\frac{452}{3} \\ 0 & -6 & -\frac{32}{3} & 44 & -\frac{214}{3} \end{bmatrix},$$

$$(\mathbf{B} - \mathbf{I}_5) \mathbf{D}_1 = \begin{bmatrix} 0 & -8 & -\frac{32}{3} & 32 & -\frac{136}{3} \\ 0 & 24 & 32 & -96 & 136 \\ 0 & -4 & -\frac{16}{3} & 16 & -\frac{68}{3} \\ 0 & 16 & \frac{64}{3} & -64 & \frac{272}{3} \\ 0 & 8 & \frac{32}{3} & -32 & \frac{136}{3} \end{bmatrix},$$

and $(\mathbf{B} - \mathbf{I}_5)^2 \mathbf{D}_1 = \mathbf{O}_5$. Hence $k = 2$.

Step A3: We look for new generalized eigenvectors among the columns of $(\mathbf{B} - \mathbf{I}_5) \mathbf{D}_1$. We must choose columns of $(\mathbf{B} - \mathbf{I}_5) \mathbf{D}_1$ that are not only linearly independent of each other, but also of our previously computed generalized eigenvectors. However, each column of $(\mathbf{B} - \mathbf{I}_5) \mathbf{D}_1$ is a scalar multiple of $\mathbf{v}_{11}$. In particular,

$$\begin{array}{rcrcl} 1^{st} \text{ column of } (\mathbf{B}-\mathbf{I}_5)\,\mathbf{D}_1 &=& 0\mathbf{v}_{11} &=& g_1\mathbf{v}_{11}, \\ 2^{nd} \text{ column of } (\mathbf{B}-\mathbf{I}_5)\,\mathbf{D}_1 &=& -\frac{1}{3}\mathbf{v}_{11} &=& g_2\mathbf{v}_{11}, \\ 3^{rd} \text{ column of } (\mathbf{B}-\mathbf{I}_5)\,\mathbf{D}_1 &=& -\frac{4}{9}\mathbf{v}_{11} &=& g_3\mathbf{v}_{11}, \\ 4^{th} \text{ column of } (\mathbf{B}-\mathbf{I}_5)\,\mathbf{D}_1 &=& \frac{4}{3}\mathbf{v}_{11} &=& g_4\mathbf{v}_{11}, \\ 5^{th} \text{ column of } (\mathbf{B}-\mathbf{I}_5)\,\mathbf{D}_1 &=& -\frac{17}{9}\mathbf{v}_{11} &=& g_5\mathbf{v}_{11}, \end{array}$$

where the g_i's represent the respective coefficients of $\mathbf{v}_{11}$.

Thus, $\mathbf{D}_1$ produced no new generalized eigenvectors for λ_1. Actually, this was to be expected, since we only needed one more generalized eigenvector, and if we found one at this stage, it would be part of a sequence of length $k = 2$. However, we still use $\mathbf{D}_1$ for our next adjustment. We create a new matrix $\mathbf{G}$ whose i^{th} column is $g_i\mathbf{v}_{12}$. Then, since $(\mathbf{B}-\mathbf{I}_5)\,\mathbf{v}_{12} = \mathbf{v}_{11}$, we see that $(\mathbf{B}-\mathbf{I}_5)\,\mathbf{G} = (\mathbf{B}-\mathbf{I}_5)\,\mathbf{D}_1$, implying that $(\mathbf{B}-\mathbf{I}_5)(\mathbf{D}_1-\mathbf{G}) = \mathbf{O}_5$. Let $\mathbf{D}_2 = \mathbf{D}_1 - \mathbf{G}$. We now revisit Steps A2 through A6 using the matrix $\mathbf{D}_2$.

Step A2:

$$\mathbf{D}_2 = \begin{bmatrix} 0 & -\frac{40}{3} & -\frac{136}{9} & \frac{160}{3} & -\frac{704}{9} \\ 0 & 4 & -\frac{8}{3} & 20 & -\frac{124}{3} \\ 0 & -\frac{80}{3} & -\frac{308}{9} & \frac{380}{3} & -\frac{1732}{9} \\ 0 & -\frac{52}{3} & -\frac{256}{9} & \frac{340}{3} & -\frac{1628}{9} \\ 0 & -\frac{20}{3} & -\frac{104}{9} & \frac{140}{3} & -\frac{676}{9} \end{bmatrix}$$

and $(\mathbf{B}-\mathbf{I}_5)\,\mathbf{D}_2 = \mathbf{O}_5$. Hence $k = 1$.

Step A3: Notice that the 2^{nd} column of $\mathbf{D}_2$ is not a scalar multiple of $\mathbf{v}_{11}$. Thus, we let $\mathbf{v}_{21} = -\frac{3}{4}\left(2^{nd} \text{ column of } \mathbf{D}_2\right)$, where we have multiplied by $-\frac{3}{4}$ to simplify the form of the vector. That is,

$$\mathbf{v}_{21} = [10, -3, 20, 13, 5].$$

Step A4: Because $k = 1$, the generalized eigenvector $\mathbf{v}_{21}$ for λ_1 corresponds to a 1×1 Jordan block. We do not need to find more vectors in this sequence.

Step A5: We now have the following two fundamental sequences of generalized eigenvectors: $\{\mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{13}\}$, and $\{\mathbf{v}_{21}\}$.

Step A6: Thus, we have now found 4 generalized eigenvectors for $\mathbf{B}$ corresponding to $\lambda_1 = 1$. Since the algebraic multiplicity of λ_1 is 4, we are finished with this eigenvalue. Note, by the way, that the remaining columns of $\mathbf{D}_2$ do not produce any more generalized eigenvectors independent of $\mathbf{v}_{11}$ and $\mathbf{v}_{21}$ since

$$\begin{array}{rcrcl} 3^{rd} \text{ column of } \mathbf{D}_2 &=& \frac{1}{9}\mathbf{v}_{11} &-& \frac{16}{9}\mathbf{v}_{21}, \\ 4^{th} \text{ column of } \mathbf{D}_2 &=& -\frac{5}{9}\mathbf{v}_{11} &+& \frac{20}{3}\mathbf{v}_{21}, \\ 5^{th} \text{ column of } \mathbf{D}_2 &=& \mathbf{v}_{11} &-& \frac{92}{9}\mathbf{v}_{21}. \end{array}$$

Finally, we need to find one generalized eigenvector for $\mathbf{B}$ corresponding to $\lambda_2 = -3$. We start Step A for this eigenvalue.

Step A1: Since $p_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B}+3\mathbf{I}_5)(\mathbf{B}-\mathbf{I}_5)^4 = \mathbf{O}_5$, each nonzero column of $(\mathbf{B}-\mathbf{I}_5)^4$ is an eigenvector for $\mathbf{B}$ corresponding to $\lambda_2 = -3$. Let $\mathbf{D} = (\mathbf{B}-\mathbf{I}_5)^4$.

Step A2: Now,

$$\mathbf{D} = (\mathbf{B}-\mathbf{I}_5)^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2816 & 512 & -1024 & -3328 & 7424 \\ 2816 & 512 & -1024 & -3328 & 7424 \\ 5632 & 1024 & -2048 & -6656 & 14848 \\ 2816 & 512 & -1024 & -3328 & 7424 \end{bmatrix}$$

and $(\mathbf{B} + 3\mathbf{I}_5)\,\mathbf{D} = \mathbf{O}_5$. Thus, $k = 1$.

Step A3: All columns of $\mathbf{D}$ are scalar multiples of the first column. Hence, we let $\mathbf{v}_{31} = \frac{1}{2816}\,(1^{st}$ column of $\mathbf{D})$; that is,

$$\mathbf{v}_{31} = [0, 1, 1, 2, 1].$$

Steps A4, A5, and A6: Since the algebraic multiplicity of λ_2 is 1, and we have found one generalized eigenvector, namely $\mathbf{v}_{31}$, we have finished Step A for λ_2. The fundamental sequence of generalized eigenvectors is $\{\mathbf{v}_{31}\}$, corresponding to a 1×1 Jordan block for $\lambda_2 = -3$.

Step B: We now have the ordered basis $(\mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{13}, \mathbf{v}_{21}, \mathbf{v}_{31})$ for $\mathbb{C}^5$ consisting of 3 sequences of generalized eigenvectors for $\mathbf{B}$. If we let $\mathbf{P}$ be the matrix whose columns are the vectors in this ordered basis, then we obtain the following Jordan Canonical Form for $\mathbf{B}$:

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

■

Here is a formal step to be added to the Method if Step A6 does not produce enough generalized eigenvectors:

Adjustment Process (if necessary) for Step A6 of the Method

Suppose we have already found the fundamental sequences $\{\mathbf{v}_{11}, \mathbf{v}_{12}, \ldots\}, \ldots, \{\mathbf{v}_{s1}, \mathbf{v}_{s2}, \ldots\}$ of generalized eigenvectors for $\mathbf{B}$ corresponding to λ_i. Also suppose $\mathbf{D}$ is the most recent matrix used in Step A1, and that k is the smallest positive integer such that $(\mathbf{B} - \lambda_i \mathbf{I}_n)^k\,\mathbf{D} = \mathbf{O}_n$.

For each q, express the q^{th} column of $(\mathbf{B} - \lambda_i \mathbf{I}_n)^{(k-1)}\,\mathbf{D}$ as a linear combination of $\mathbf{v}_{11}, \ldots, \mathbf{v}_{s1}$; that is, solve for $f_{1q}, \ldots, f_{sq}$ such that the q^{th} column of $(\mathbf{B} - \lambda_i \mathbf{I}_n)^{(k-1)}\,\mathbf{D}$ equals $f_{1q}\mathbf{v}_{11} + \cdots + f_{sq}\mathbf{v}_{s1}$.

Let $\mathbf{F}$ be the matrix whose q^{th} column equals $f_{1q}\mathbf{v}_{1k} + \cdots + f_{sq}\mathbf{v}_{sk}$.

Let $\mathbf{D}_1 = \mathbf{D} - \mathbf{F}$. By construction, we will have $(\mathbf{B} - \lambda_i \mathbf{I}_n)^{(k-1)}\,\mathbf{D}_1 = \mathbf{O}_n$.

Then follow Steps A2 through A6 of the Method using the new matrix $\mathbf{D}_1$ in place of $\mathbf{D}$.

▸ Conclusion

The Jordan Canonical Form for a matrix is important because every square matrix with complex entries has such a form. There are no "nonJordanizable" matrices. One application of the Jordan Canonical Form is to extend the method for solving homogeneous systems of linear differential equations discussed in Section 8.8 of the textbook so that the complete solution set can be found for every such system. The details of this appear in a section entitled "Solving First Order Systems of Linear Homogeneous Differential Equations," that is part of these additional companion "websections" for the textbook.

▸ New Vocabulary

fundamental sequence of generalized eigenvectors
generalized eigenspace
generalized eigenvector
Jordan block
Jordan Canonical Form

▶ Highlights

- A Jordan block for an eigenvalue λ is a square matrix with every main diagonal entry equal to λ, every entry immediately above the main diagonal equal to 1, and every other entry equal to 0.

- A square matrix is in Jordan Canonical Form if and only if it can be broken into blocks (submatrices) in such a way that the blocks along the main diagonal are Jordan blocks, and all other blocks are zero matrices. That is, a matrix $\mathbf{A}$ is in Jordan Canonical Form if

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_2 & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{A}_m \end{bmatrix},$$

  where each $\mathbf{A}_i$ is a Jordan block.

- If λ is an eigenvalue for a square matrix $\mathbf{B}$, then a nonzero vector $\mathbf{v}$ in $\mathbb{C}^n$ is a generalized eigenvector for $\mathbf{B}$ corresponding to λ if and only if there is a positive integer k such that $(\mathbf{B} - \lambda\mathbf{I})^k \mathbf{v} = \mathbf{0}$.

- If λ is an eigenvalue for a square matrix $\mathbf{B}$, the generalized eigenspace for $\mathbf{B}$ corresponding to λ is the set of all generalized eigenvectors for $\mathbf{B}$ corresponding to λ along with the zero vector.

- If λ is an eigenvalue for a square matrix $\mathbf{B}$, a sequence of nonzero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ for $\mathbf{B}$ such that $\mathbf{B}\mathbf{v}_1 = \lambda\mathbf{v}_1$, and $\mathbf{B}\mathbf{v}_i = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for $2 \le i \le k$, is called a fundamental sequence of generalized eigenvectors for $\mathbf{B}$ of length k corresponding to λ.

- Every square matrix with complex entries is similar to a matrix in Jordan Canonical Form.

- The Jordan Canonical Form for a given square matrix is unique except for the order in which the Jordan blocks appear on the main diagonal.

- A formal Method, together with examples in the text, illustrates how to use the Cayley-Hamilton Theorem (Theorem 5.29 in Section 5.6) to find fundamental sequences of generalized eigenvectors for a square matrix $\mathbf{B}$ corresponding to an eigenvalue λ of the form $\{\mathbf{v}_{i1}, \mathbf{v}_{i2}, \ldots, \mathbf{v}_{ik}\}$ such that $(\mathbf{B} - \lambda\mathbf{I})^j \mathbf{v}_{ij} = \mathbf{0}$ for $1 \le j \le k$. Each such fundamental sequence is associated with a $k \times k$ Jordan block corresponding to λ in a Jordan Canonical Form matrix for $\mathbf{B}$. The total number of vectors in all such fundamental sequences for λ equals the algebraic multiplicity of λ.

- If λ is an eigenvalue for a square matrix $\mathbf{B}$, and if the Jordan Canonical Form for $\mathbf{B}$ has two or more Jordan blocks of different sizes for λ, an "adjustment" process must be used (as summarized after Example 8) in order to find a complete set of fundamental sequences of generalized eigenvectors for λ.

▶ EXERCISES

1. Verify the following claims from the paragraph following Example 1:

a) Show that if $\mathbf{A}$ is a $k \times k$ Jordan block associated with eigenvalue λ, then $\mathbf{e}_1 = [1, 0, ..., 0]$ spans the eigenspace for $\mathbf{A}$ corresponding to λ.

b) Show that $\mathbf{A}$ is a $k \times k$ Jordan block associated with eigenvalue λ if and only if the following two conditions hold: $\mathbf{A}\mathbf{e}_1 = \lambda\mathbf{e}_1$, and for $2 \le i \le k$, $\mathbf{A}\mathbf{e}_i = \lambda\mathbf{e}_i + \mathbf{e}_{i-1}$.

★ **c)** Show that if $\mathbf{A}$ is a $k \times k$ Jordan block associated with eigenvalue λ, then $(\mathbf{A} - \lambda\mathbf{I}_k)^k = \mathbf{O}_k$.

2. Verify the following claims from the paragraphs after Example 2:

a) Suppose that $\mathbf{A}$ is a $k \times k$ Jordan block matrix with diagonal entry λ and $\mathbf{B} = \mathbf{PAP}^{-1}$ for some nonsingular matrix $\mathbf{P}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be the columns of $\mathbf{P}$. Show that $\mathbf{B}\mathbf{v}_1 = \lambda\mathbf{v}_1$, and $\mathbf{B}\mathbf{v}_i = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for $2 \le i \le k$.

b) Suppose $\mathbf{B}$ is a given $k \times k$ matrix, and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a linearly independent sequence of vectors such that $\mathbf{B}\mathbf{v}_1 = \lambda\mathbf{v}_1$, and $\mathbf{B}\mathbf{v}_i = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for $2 \le i \le k$. If $\mathbf{P}$ is the (nonsingular) matrix whose i^{th} column is $\mathbf{v}_i$, show that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$ is the $k \times k$ Jordan block associated with the eigenvalue λ.

3. Suppose $\mathbf{B}$ is a given $k \times k$ matrix, and that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a linearly independent sequence of vectors such that $\mathbf{B}\mathbf{v}_1 = \lambda\mathbf{v}_1$, and $\mathbf{B}\mathbf{v}_i = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for $2 \le i \le k$.

a) Show that $(\mathbf{B} - \lambda\mathbf{I}_k)^i\,\mathbf{v}_i = \mathbf{0}_k$ for $1 \le i \le k$.

b) Show that every vector $\mathbf{v}$ in $\mathbb{C}^k$ has the property $(\mathbf{B} - \lambda\mathbf{I}_k)^i\,\mathbf{v} = \mathbf{0}_k$ for some i, where $1 \le i \le k$.

4. In each part, list all possible matrices in Jordan Canonical Form having the given polynomial as its characteristic polynomial. Place brackets around each Jordan block in each matrix. Also, indicate which of the matrices in your list are similar to each other.

★ **a)** $(x-2)^2(x+1)$

b) $x^2 + x - 6$

★ **c)** $x^2 + 6x + 10$

★ **d)** $x^4 + 4x^3 + 4x^2$

e) x^4

★ **f)** $x^4 - 3x^2 - 4$

5. If you have a calculator or software package available to perform matrix computations, use it to trace through *all* of the computations in Example 8 in this section. Conclude these computations by verifying that $\mathbf{P}^{-1}\mathbf{BP} = \mathbf{A}$ for the given 5×5 matrix $\mathbf{B}$ and the computed matrices $\mathbf{P}$ and $\mathbf{A}$. (This exercise is too tedious without a calculator or appropriate software.)

6. In each part, find a matrix $\mathbf{A}$ in Jordan Canonical Form similar to the given matrix $\mathbf{B}$. Also, specify a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{BP} = \mathbf{A}$.

★ **a)** $\mathbf{B} = \begin{bmatrix} -9 & 5 & 8 \\ -4 & 3 & 4 \\ -8 & 4 & 7 \end{bmatrix}$

b) $\mathbf{B} = \begin{bmatrix} 6 & -2 & -3 \\ -14 & 14 & 13 \\ 23 & -19 & -19 \end{bmatrix}$

(Hint: $p_{\mathbf{B}}(x) = x^3 - x^2 - 8x + 12 = (x-2)^2(x+3)$)

c) $\mathbf{B} = \begin{bmatrix} 5 & -3 & -6 \\ 7 & -5 & -14 \\ -2 & 2 & 6 \end{bmatrix}$

★ **d)** $\mathbf{B} = \begin{bmatrix} 8 & 5 & 0 \\ -5 & -3 & -1 \\ 10 & 7 & 0 \end{bmatrix}$

e) $\mathbf{B} = \begin{bmatrix} -4+5i & 2-2i & 1-2i \\ -8+6i & 4-2i & 2-3i \\ 4i & -2i & -i \end{bmatrix}$ (Hint: $p_{\mathbf{B}}(x) = x^3 - 2ix^2 - x$)

f) $\mathbf{B} = \begin{bmatrix} -12 & 5 & -11 & -10 \\ -7 & 3 & -7 & -8 \\ 15 & -6 & 14 & 12 \\ -4 & 1 & -4 & -3 \end{bmatrix}$

(Hint: $p_{\mathbf{B}}(x) = x^4 - 2x^3 - 3x^2 + 4x + 4 = (x-2)^2(x+1)^2$)

★ **g)** $\mathbf{B} = \begin{bmatrix} -3 & -2 & -1 & 5 & 3 \\ -8 & 2 & 7 & -2 & 1 \\ 0 & -3 & -4 & 6 & 3 \\ -4 & 1 & 3 & -1 & 1 \\ 0 & -2 & -2 & 4 & 1 \end{bmatrix}$

(Hint: $p_{\mathbf{B}}(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = (x+1)^5$)

7. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{P}$, $\mathbf{v}_{11}$, $\mathbf{v}_{12}$, and $\mathbf{v}_{21}$ be the matrices and vectors given in Example 5.

a) Verify by direct computation that $(\mathbf{B} - 2\mathbf{I}_3)\,\mathbf{v}_{11} = \mathbf{0}_3$, $(\mathbf{B} - 2\mathbf{I}_3)\,\mathbf{v}_{12} = \mathbf{v}_{11}$, $(\mathbf{B} - 2\mathbf{I}_3)^2\,\mathbf{v}_{12} = \mathbf{0}_3$, and $(\mathbf{B} - 3\mathbf{I}_3)\,\mathbf{v}_{21} = \mathbf{0}_3$.

★ **b)** Use the technique illustrated in the examples to find a Jordan Canonical Form for $\mathbf{B}$, along with vectors $\mathbf{u}_{11}$, $\mathbf{u}_{12}$, and $\mathbf{u}_{21}$ that the method produces. (Note that $\mathbf{u}_{11}$, $\mathbf{u}_{12}$, and $\mathbf{u}_{21}$ could be different than $\mathbf{v}_{11}$, $\mathbf{v}_{12}$, and $\mathbf{v}_{21}$.) If $\mathbf{Q}$ is the matrix having columns $\mathbf{u}_{11}$, $\mathbf{u}_{12}$, and $\mathbf{u}_{21}$, verify that $\mathbf{B} = \mathbf{QAQ}^{-1}$.

c) Verify that $(\mathbf{B} - 2\mathbf{I}_3)\,\mathbf{u}_{11} = \mathbf{0}_3$ and $(\mathbf{B} - 2\mathbf{I}_3)\,\mathbf{u}_{12} = \mathbf{u}_{11}$.

d) Note that $\mathbf{u}_{11} = -\mathbf{v}_{11}$. Explain why the second equations in parts (a) and (c) together imply that the sum of $\mathbf{u}_{12}$ and $\mathbf{v}_{12}$ is either the zero vector or an eigenvector for $\mathbf{B}$ for $\lambda_1 = 2$. Verify this.

e) Verify that $\mathbf{u}_{21}$ differs from $\mathbf{v}_{21}$ by an eigenvector for $\mathbf{B}$ for $\lambda_2 = 3$.

8. This exercise proves the similarity of Jordan Canonical Form matrices having identical blocks.

a) Suppose R is a Type (III) row operation, and that $\mathbf{E} = R(\mathbf{I}_n)$. Prove that $\mathbf{E} = \mathbf{E}^{-1} = \mathbf{E}^T$. (Hint: Use Theorem 2.1 and Theorem 6.7.)

b) Let R and $\mathbf{E}$ be as given in part (a), and let $\mathbf{A}$ be an $n \times n$ matrix. Prove that $R(\mathbf{A}) = \mathbf{EA}$.

c) Let R, $\mathbf{E}$, and $\mathbf{A}$ be as given in part (b). If R swaps rows i and j of $\mathbf{A}$, show that $\mathbf{AE}^{-1}$ is the matrix obtained from $\mathbf{A}$ after swapping columns i and j.

d) Show that if two Jordan Canonical Form $n \times n$ matrices $\mathbf{C}$ and $\mathbf{D}$ have identical blocks, but in a different order, then $\mathbf{C}$ and $\mathbf{D}$ are similar. (Hint: We need to reorder both the rows and columns of one of the matrices, say, $\mathbf{C}$, to match those of the other. Consider a sequence $R_1, R_2, \ldots, R_k$ of Type (III) row operations that reorders the rows (only) of the Jordan blocks of $\mathbf{C}$ in such a way so that the rows of $R_k(\cdots(R_2(R_1(\mathbf{C})))\cdots)$ are re-positioned to match the corresponding rows of $\mathbf{D}$. For $1 \le i \le k$, let $\mathbf{E}_i = R_i(\mathbf{I}_n)$. Use the results of parts (a), (b), and (c) to show that $(\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1)\,\mathbf{C}\,(\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1)^{-1} = \mathbf{D}$.)

9. Show that if $\mathbf{A}$ is an $n \times n$ matrix with $p_{\mathbf{A}}(x) = (x - \lambda)^n$, then every nonzero vector in $\mathbb{C}^n$ is a generalized eigenvector for $\mathbf{A}$ corresponding to λ.

10. (The results in this exercise are needed as lemmas for several of the exercises below.) Let $\mathbf{A}$ be an $n \times n$ matrix.

★ **a)** If a and b are scalars, prove that $(\mathbf{A} + a\mathbf{I}_n)$ and $(\mathbf{A} + b\mathbf{I}_n)$ commute.

b) If a and b are scalars, and k and j are positive integers, prove that $(\mathbf{A} + a\mathbf{I}_n)^k$ and $(\mathbf{A} + b\mathbf{I}_n)^j$ commute.

★ **11.** Suppose that $\mathbf{A}$ is an $n \times n$ matrix and that λ is an eigenvalue for $\mathbf{A}$ having algebraic multiplicity α. Suppose that $\mathbf{B}$ is a matrix in Jordan Canonical Form that is similar to $\mathbf{A}$. Finally, suppose that $\mathbf{J}$ is a Jordan block in $\mathbf{B}$ corresponding to λ. Prove that $\mathbf{J}$ has size $m \times m$, for some $m \le \alpha$. (The result in this exercise is needed as a lemma in Exercise 19(b) below.)

12. Let $\mathbf{A}$ be an $n \times n$ matrix and let λ_1 and λ_2 be distinct eigenvalues for $\mathbf{A}$. Show that a nonzero vector $\mathbf{v}$ cannot be a generalized eigenvector for $\mathbf{A}$ corresponding to both λ_1 and λ_2. (Hint: Use the results proven in Exercise 10. Compare this result to Exercise 15 in Section 5.6.)

13. Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be generalized eigenvectors for $\mathbf{A}$ corresponding, respectively, to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of $\mathbf{A}$. Prove that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent. (Hint: First note that Exercise 12 implies that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct. Compare this result with Theorem 5.23 in Section 5.6.)

14. Let $\mathbf{A}$ be a $k \times k$ Jordan block associated with eigenvalue λ, and suppose that $\beta \ne \lambda$. Prove that $(\mathbf{A} - \beta\mathbf{I}_k)^j$ is nonsingular for every positive integer j.

15. Prove that two square matrices $\mathbf{A}$ and $\mathbf{B}$ are similar if and only if they are both similar to the same matrix $\mathbf{C}$ in Jordan Canonical Form. (Hint: Use Exercises 13(d) and 13(e) in Section 3.3.)

★ **16.** (The result in this exercise is needed for Exercise 20 below.) Let $\mathbf{A}$ be an $n \times n$ matrix, let $\mathbf{P}$ be a nonsingular $n \times n$ matrix, and let $q(x)$ be a polynomial in x. Prove that if $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, then $q(\mathbf{B}) = \mathbf{P}^{-1}q(\mathbf{A})\mathbf{P}$. (Note that this statement not only claims that $q(\mathbf{B})$ and $q(\mathbf{A})$ are similar, but also that the same matrix $\mathbf{P}$ can be used to exhibit the similarity.)

17. (The note at the end of part (b) of this exercise is needed in the solutions for Exercises 19 and 20.)

★ **a)** Suppose $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, where $\mathbf{A}_{11}$ is a $k \times k$ matrix, $\mathbf{A}_{12}$ is a $k \times m$ matrix, $\mathbf{A}_{21}$ is an $m \times k$ matrix, and $\mathbf{A}_{22}$ is an $m \times m$ matrix. Prove that $\mathbf{A}^2 = \begin{bmatrix} \mathbf{A}_{11}^2 + \mathbf{A}_{12}\mathbf{A}_{21} & \mathbf{A}_{11}\mathbf{A}_{12} + \mathbf{A}_{12}\mathbf{A}_{22} \\ \mathbf{A}_{21}\mathbf{A}_{11} + \mathbf{A}_{22}\mathbf{A}_{21} & \mathbf{A}_{21}\mathbf{A}_{12} + \mathbf{A}_{22}^2 \end{bmatrix}$.

b) Suppose the matrix $\mathbf{A}$ in part (a) has the form $\mathbf{A} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_2 \end{bmatrix}$, where $\mathbf{J}_1$ and $\mathbf{J}_2$ are Jordan blocks. Use the result in part (a) to show that $\mathbf{A}^2 = \begin{bmatrix} \mathbf{J}_1^2 & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_2^2 \end{bmatrix}$. (Note: We state the following useful generalization of part (b) here without proof: If $\mathbf{A} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_2 & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{J}_k \end{bmatrix}$ is a matrix in Jordan Canonical Form, where $\mathbf{J}_1, \ldots, \mathbf{J}_k$ are Jordan blocks, and $q(x)$ is a polynomial, then $q(\mathbf{A}) = \begin{bmatrix} q(\mathbf{J}_1) & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & q(\mathbf{J}_2) & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & q(\mathbf{J}_k) \end{bmatrix}$.)

18. Let $\mathbf{A}$ be an $n \times n$ matrix, and let λ be an eigenvalue for $\mathbf{A}$.

a) Prove that the generalized eigenspace for $\mathbf{A}$ corresponding to λ is the kernel of $(\mathbf{A} - \lambda\mathbf{I}_n)^j$, for all $j \geq p$, for some positive integer p. (Hint: First choose a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_t\}$ for the generalized eigenspace. Find a p such that $(\mathbf{A} - \lambda\mathbf{I}_n)^j \mathbf{v}_i = \mathbf{0}$, for all $j \geq p$ and for $1 \leq i \leq t$. This will prove that the generalized eigenspace is contained in the desired kernel.)

b) Use part (a) to show that the dimension of the generalized eigenspace for $\mathbf{A}$ corresponding to λ is $n - \text{rank}\left((\mathbf{A} - \lambda\mathbf{I}_n)^j\right)$ for all values of $j \geq p$. (Hint: Use part (2) of Theorem 5.9.)

19. Suppose $\mathbf{A}$ is an $n \times n$ matrix having eigenvalue λ with algebraic multiplicity α. Let $q(x) = p_{\mathbf{A}}(x)/(x - \lambda)^{\alpha} = (x - \lambda_1)^{\alpha_1} \cdots (x - \lambda_s)^{\alpha_s}$, where $\lambda_1, \ldots, \lambda_s$ are the remaining distinct eigenvalues of $\mathbf{A}$, having respective algebraic multiplicities $\alpha_1, \ldots, \alpha_s$.

a) Show that if $\mathbf{J}$ is a Jordan block of $\mathbf{A}$ corresponding to the eigenvalue λ, then $q(\mathbf{J})$ is nonsingular, and hence for each such $q(\mathbf{J})$, its columns are linearly independent. (Hint: Use Exercise 14 in this section and part (c) of Exercise 15 in Section 4.5.)

b) Show that if $\mathbf{J}$ is a Jordan block of $\mathbf{A}$ corresponding to an eigenvalue $\lambda_i \neq \lambda$, then $q(\mathbf{J}) = \mathbf{O}$. (Hint: Use Exercises 11, 1(c), and 10.)

c) If $\mathbf{A}$ is in Jordan Canonical Form, having m Jordan blocks, where the first l Jordan blocks $\mathbf{J}_1$ through $\mathbf{J}_l$ correspond to λ, and the remaining Jordan blocks $\mathbf{J}_{l+1}$ through $\mathbf{J}_m$ correspond to eigenvalues not equal to λ, show that

$$q(\mathbf{A}) = \begin{bmatrix} q(\mathbf{J}_1) & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & q(\mathbf{J}_2) & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & q(\mathbf{J}_l) & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} \end{bmatrix}.$$

(Hint: Use the generalization of Exercise 17(b) stated at the conclusion of that exercise.)

20. As in Exercise 19, suppose $\mathbf{A}$ is an $n \times n$ matrix having eigenvalue λ with algebraic multiplicity α. Let $q(x) = p_{\mathbf{A}}(x)/(x-\lambda)^{\alpha} = (x-\lambda_1)^{\alpha_1}\cdots(x-\lambda_s)^{\alpha_s}$, where $\lambda_1, \ldots, \lambda_s$ are the remaining distinct eigenvalues of $\mathbf{A}$, having respective algebraic multiplicities $\alpha_1, \ldots, \alpha_s$. For each integer $k \geq 0$, define

$$r_k(\mathbf{A}) = \operatorname{rank}\left((\mathbf{A} - \lambda\mathbf{I}_n)^k\, q(\mathbf{A}) \right).$$

a) Suppose $\mathbf{B}$ is similar to $\mathbf{A}$. Then $p_{\mathbf{A}}(x) = p_{\mathbf{B}}(x)$ (by Exercise 6 in Section 3.4), so $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues with the same multiplicities. Therefore, computing $q(x)$, as above, results in the same polynomial if we use the matrix $\mathbf{B}$ instead of the matrix $\mathbf{A}$. Thus,

$$r_k(\mathbf{B}) = \operatorname{rank}\left((\mathbf{B} - \lambda\mathbf{I}_n)^k\, q(\mathbf{B}) \right)$$

for each integer $k \geq 0$, using the same polynomial $q(x)$ as was used for $\mathbf{A}$. Show that $r_k(\mathbf{B}) = r_k(\mathbf{A})$. (Hint: Consider the polynomial $(x-\lambda)^k q(x)$ together with Exercise 16 above. Then use Exercise 16 in the Chapter Review Exercises for Chapter 2 of the textbook.)

b) If $\mathbf{A}$ is in Jordan Canonical Form, show that $r_0(\mathbf{A})$ equals the dimension of the generalized eigenspace for $\mathbf{A}$ corresponding to λ. (Hint: Show that both values equal α. Use part (c) of Exercise 19 as well as Exercise 18.)

c) Use parts (a) and (b) to show that for the general (not necessarily Jordan Canonical Form) matrix $\mathbf{A}$ (as given before part (a) above), the value of $r_0(\mathbf{A})$ equals the dimension of the generalized eigenspace for $\mathbf{A}$ corresponding to λ.

d) If $\mathbf{A}$ is in Jordan Canonical Form, prove that $r_0(\mathbf{A}) - r_1(\mathbf{A})$ gives the total number of Jordan blocks of $\mathbf{A}$ corresponding to λ. (Substantial hints are given in the Answers to Selected Exercises.) [The result in part (d) is also true for a general (not necessarily Jordan Canonical Form) matrix $\mathbf{A}$. This generalization can be proven in a manner similar to the solution of part (c).]

e) If $\mathbf{A}$ is in Jordan Canonical Form, prove that $r_0(\mathbf{A}) - r_1(\mathbf{A})$ equals the dimension of E_λ, the eigenspace for $\mathbf{A}$ corresponding to λ. (Substantial hints are given in the Answers to Selected Exercises.) [The result in part (e) is also true for a general (not necessarily Jordan Canonical Form) matrix $\mathbf{A}$. This generalization can be proven in a manner similar to the solution of part (c).]

f) If $\mathbf{A}$ is in Jordan Canonical Form, prove that $r_{k-1}(\mathbf{A}) - r_k(\mathbf{A})$ gives the total number of Jordan blocks of $\mathbf{A}$ corresponding to λ having size at least $k \times k$. (Substantial hints are given in the Answers to Selected Exercises.) [The result in part (f) is also true for a general (not necessarily Jordan Canonical Form) matrix $\mathbf{A}$. This generalization can be proven in a manner similar to the solution of part (c).]

★ **g)** For the general matrix $\mathbf{A}$ (as given before part (a) above), prove that the number of $k \times k$ Jordan blocks corresponding to λ in any Jordan Canonical Form for $\mathbf{A}$ is given by $r_{k-1}(\mathbf{A}) - 2r_k(\mathbf{A}) + r_{k+1}(\mathbf{A})$. [Note that this exercise can be used to prove the uniqueness assertion in Theorem 1, because it gives us a method of computing the number of Jordan blocks of a given size for each eigenvalue. However, the proofs in this exercise assume the existence of a Jordan Canonical Form matrix similar to a given matrix, and so this exercise can not be used to prove the existence claim in Theorem 1.]

h) Using a calculator or appropriate software to perform row reduction, verify the formula in part (g) for the 5×5 matrix in Example 8 for each of its eigenvalues, and for each appropriate value of k.

★ **21.** True or False:

a) Every square complex matrix is similar to a Jordan block matrix.

b) The dimension of the generalized eigenspace corresponding to an eigenvalue λ for a square matrix $\mathbf{A}$ equals the algebraic multiplicity of λ.

c) The dimension of the generalized eigenspace corresponding to an eigenvalue λ for a square matrix $\mathbf{A}$ equals the geometric multiplicity of λ.

d) If a Jordan Canonical Form for a matrix $\mathbf{A}$ has at least two Jordan blocks, then $\mathbf{A}$ has at least two different Jordan Canonical Forms.

e) If $\mathbf{A}$ is a square matrix, $\mathbf{P}$ and $\mathbf{Q}$ are nonsingular matrices, and $\mathbf{J}$ is in Jordan Canonical Form such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{J} = \mathbf{Q}^{-1}\mathbf{AQ}$, then $\mathbf{P} = \mathbf{Q}$.

f) If $\mathbf{A}$ is an $n \times n$ matrix, then every vector in $\mathbb{C}^n$ can be expressed as a linear combination of generalized eigenvectors for $\mathbf{A}$.

▶ Answers to Selected Exercises

(1) (c) By parts (a) and (b), $(\mathbf{A}-\lambda\mathbf{I}_k)\,\mathbf{e}_1 = \mathbf{0}_k$, and $(\mathbf{A}-\lambda\mathbf{I}_k)\,\mathbf{e}_i = \mathbf{e}_{i-1}$ for $2 \le i \le k$. We use induction to prove that $(\mathbf{A}-\lambda\mathbf{I}_k)^i\,\mathbf{e}_i = \mathbf{0}_k$ for all i, $1 \le i \le k$. The Base Step holds since we have already noted that $(\mathbf{A}-\lambda\mathbf{I}_k)\,\mathbf{e}_1 = \mathbf{0}_k$. For the Inductive Step, we assume that $(\mathbf{A}-\lambda\mathbf{I}_k)^{i-1}\,\mathbf{e}_{i-1} = \mathbf{0}_k$ and show that $(\mathbf{A}-\lambda\mathbf{I}_k)^i\,\mathbf{e}_i = \mathbf{0}_k$. But

$$(\mathbf{A}-\lambda\mathbf{I}_k)^i\,\mathbf{e}_i = (\mathbf{A}-\lambda\mathbf{I}_k)^{i-1}(\mathbf{A}-\lambda\mathbf{I}_k)\,\mathbf{e}_i = (\mathbf{A}-\lambda\mathbf{I}_k)^{i-1}\,\mathbf{e}_{i-1} = \mathbf{0}_k,$$

completing the induction proof.

Using $(\mathbf{A}-\lambda\mathbf{I}_k)^i\,\mathbf{e}_i = \mathbf{0}_k$, we see that

$$(\mathbf{A}-\lambda\mathbf{I}_k)^k\,\mathbf{e}_i = (\mathbf{A}-\lambda\mathbf{I}_k)^{k-i}(\mathbf{A}-\lambda\mathbf{I}_k)^i\,\mathbf{e}_i = (\mathbf{A}-\lambda\mathbf{I}_k)^{k-i}\,\mathbf{0}_k = \mathbf{0}_k$$

for all i, $1 \le i \le k$. Now, for $1 \le i \le k$, the i^{th} column of $(\mathbf{A}-\lambda\mathbf{I}_k)^k$ equals $(\mathbf{A}-\lambda\mathbf{I}_k)^k\,\mathbf{e}_i$. Therefore, we have shown that every column of $(\mathbf{A}-\lambda\mathbf{I}_k)^k$ is zero, and so $(\mathbf{A}-\lambda\mathbf{I}_k)^k = \mathbf{O}_k$.

(4) Note for all the parts below that two matrices in Jordan Canonical Form are similar to each other if and only if they contain the same Jordan blocks, rearranged in any order.

(a) $\begin{bmatrix} [2] & 0 & 0 \\ 0 & [2] & 0 \\ 0 & 0 & [-1] \end{bmatrix}, \begin{bmatrix} [2] & 0 & 0 \\ 0 & [-1] & 0 \\ 0 & 0 & [2] \end{bmatrix}, \begin{bmatrix} [-1] & 0 & 0 \\ 0 & [2] & 0 \\ 0 & 0 & [2] \end{bmatrix}$

$\begin{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & [-1] \end{bmatrix}, \begin{bmatrix} [-1] & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \end{bmatrix}$

The three matrices on the first line are all similar to each other. The two matrices on the second line are similar to each other, but not to those on the first line.

(c) Using the quadratic formula, $x^2+6x+10$ has roots $-3+i$ and $-3-i$, each having multiplicity 1;

$\begin{bmatrix} [-3+i] & 0 \\ 0 & [-3-i] \end{bmatrix}$ and $\begin{bmatrix} [-3-i] & 0 \\ 0 & [-3+i] \end{bmatrix}$,
which are similar to each other

(d) $x^4+4x^3+4x^2 = x^2(x+2)^2$;

$\begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix},$

which are similar to each other but to none of the others;

$\begin{bmatrix} [0] & 0 & \begin{matrix} 0 & 0 \end{matrix} \\ 0 & [0] & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} [0] & \begin{matrix} 0 & 0 \end{matrix} & 0 \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & \begin{matrix} 0 & 0 \end{matrix} & [0] \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} [0] & 0 \\ 0 & [0] \end{matrix} \end{bmatrix},$

which are similar to each other but to none of the others;

$\begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} [-2] & 0 \\ 0 & [-2] \end{matrix} \end{bmatrix}, \begin{bmatrix} [-2] & \begin{matrix} 0 & 0 \end{matrix} & 0 \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & \begin{matrix} 0 & 0 \end{matrix} & [-2] \end{bmatrix}, \begin{bmatrix} \begin{matrix} [-2] & 0 \\ 0 & [-2] \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix},$

which are similar to each other but to none of the others;

$$\begin{bmatrix} [-2] & 0 & 0 & 0 \\ 0 & [-2] & 0 & 0 \\ 0 & 0 & [0] & 0 \\ 0 & 0 & 0 & [0] \end{bmatrix}, \begin{bmatrix} [-2] & 0 & 0 & 0 \\ 0 & [0] & 0 & 0 \\ 0 & 0 & [-2] & 0 \\ 0 & 0 & 0 & [0] \end{bmatrix}, \begin{bmatrix} [-2] & 0 & 0 & 0 \\ 0 & [0] & 0 & 0 \\ 0 & 0 & [0] & 0 \\ 0 & 0 & 0 & [-2] \end{bmatrix},$$

$$\begin{bmatrix} [0] & 0 & 0 & 0 \\ 0 & [-2] & 0 & 0 \\ 0 & 0 & [-2] & 0 \\ 0 & 0 & 0 & [0] \end{bmatrix}, \begin{bmatrix} [0] & 0 & 0 & 0 \\ 0 & [-2] & 0 & 0 \\ 0 & 0 & [0] & 0 \\ 0 & 0 & 0 & [-2] \end{bmatrix}, \begin{bmatrix} [0] & 0 & 0 & 0 \\ 0 & [0] & 0 & 0 \\ 0 & 0 & [-2] & 0 \\ 0 & 0 & 0 & [-2] \end{bmatrix},$$

which are similar to each other but to none of the others

(f) $x^4 - 3x^2 - 4 = \left(x^2 - 4\right)\left(x^2 + 1\right) = (x-2)(x+2)(x-i)(x+i)$;
There are 24 possible Jordan Canonical Forms, all of which are similar to each other. Because each eigenvalue has algebraic multiplicity 1, all of the Jordan blocks have size 1×1. Hence, any Jordan Canonical Form matrix with these blocks is diagonal with the 4 eigenvalues 2, -2, i, and $-i$ on the main diagonal. The 24 possibilities result from all the possible orders in which these 4 eigenvalues can appear on the diagonal.

(6) One possible answer is given in each case.

(a) $\mathbf{A} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & [-1] \end{bmatrix}$; $\mathbf{P} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

(d) $\mathbf{A} = \begin{bmatrix} [3] & 0 & 0 \\ 0 & [1-i] & 0 \\ 0 & 0 & [1+i] \end{bmatrix}$; $\mathbf{P} = \begin{bmatrix} 1 & 2-i & 2+i \\ -1 & -3+i & -3-i \\ 1 & 1-2i & 1+2i \end{bmatrix}$.

(g) $\mathbf{A} = \begin{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} & [-1] \end{bmatrix}$; $\mathbf{P} = \begin{bmatrix} -2 & 1 & -2 & 0 & 1 \\ -8 & 0 & 3 & 1 & -2 \\ 0 & 0 & -3 & 0 & 2 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{bmatrix}$.

(7) (b) $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $\mathbf{P} = \begin{bmatrix} -3 & 1 & -1 \\ -5 & 3 & -2 \\ 2 & -1 & 1 \end{bmatrix}$. Hence, $\mathbf{u}_{11} = [-3, -5, 2]$, $\mathbf{u}_{12} = [1, 3, -1]$, and $\mathbf{u}_{21} = [-1, -2, 1]$.

(10) (a) Now,

$$\begin{aligned} (\mathbf{A} + a\mathbf{I}_n)(\mathbf{A} + b\mathbf{I}_n) &= (\mathbf{A} + a\mathbf{I}_n)\mathbf{A} + (\mathbf{A} + a\mathbf{I}_n)(b\mathbf{I}_n) \\ &= \mathbf{A}^2 + a\mathbf{I}_n\mathbf{A} + \mathbf{A}(b\mathbf{I}_n) + (a\mathbf{I}_n)(b\mathbf{I}_n) \\ &= \mathbf{A}^2 + a\mathbf{A} + b\mathbf{A} + ab\mathbf{I}_n \\ &= \mathbf{A}^2 + b\mathbf{A} + a\mathbf{A} + ba\mathbf{I}_n \\ &= \mathbf{A}^2 + b\mathbf{I}_n\mathbf{A} + \mathbf{A}(a\mathbf{I}_n) + (b\mathbf{I}_n)(a\mathbf{I}_n) \\ &= (\mathbf{A} + b\mathbf{I}_n)\mathbf{A} + (\mathbf{A} + b\mathbf{I}_n)(a\mathbf{I}_n) \\ &= (\mathbf{A} + b\mathbf{I}_n)(\mathbf{A} + a\mathbf{I}_n). \end{aligned}$$

(11) Use the fact that $p_{\mathbf{B}}(x) = p_{\mathbf{A}}(x)$ (Exercise 6 in Section 3.4) and that $\mathbf{B}$ is upper triangular with m of its main diagonal entries being main diagonal entries of $\mathbf{J}$. Further details can be found in the Student Solutions Manual for this section.

(16) Let k be the degree of the polynomial $q(x)$. Use induction on k. Note that $q(x) = a_{k+1}x^{k+1} + a_k x^k + \cdots + a_1 x + a_0 = \left(a_{k+1}x^k + a_k x^{k-1} + \cdots + a_1\right)x + a_0$. The details of the proof can be found in the Student Solutions Manual for this section.

(17) (a) Compute the (i, j) entry of $\mathbf{A}^2$ in four separate cases: $i \le k$, $j \le k$; $i > k$, $j \le k$; $i \le k$, $j > k$; $i > k$, $j > k$. For further details, see the Student Solutions Manual for this section.

(20) (d) Hints: Assume each Jordan block $\mathbf{J}_i$ has size $k_i \times k_i$. First consider the case where the first l Jordan blocks (that is, $\mathbf{J}_1, ..., \mathbf{J}_l$) of $\mathbf{A}$ represent all of the Jordan blocks corresponding to λ. Note that $q(\mathbf{J}_i) = \mathbf{O}$, for $i > l$ by Exercise 19. Consequently, by the block structure of $(\mathbf{A} - \lambda\mathbf{I}_n)\, q(\mathbf{A})$, we have

$$\operatorname{rank}\left((\mathbf{A} - \lambda\mathbf{I}_n)\, q(\mathbf{A})\right) = \sum_{i=1}^{l} \operatorname{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})\, q(\mathbf{J}_i)\right).$$

Also note that for $i \le l$,

$$\operatorname{rank}((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})\, q(\mathbf{J}_i)) = \operatorname{rank}((\mathbf{J}_i - \lambda\mathbf{I}_{k_i}))$$

by Exercise 16 in the Review Exercises of Chapter 2 since $q(\mathbf{J}_i)$ is nonsingular in that case. Show that for $i \le l$,

$$\operatorname{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})\, q(\mathbf{J}_i)\right) = k_i - 1.$$

Conclude that

$$r_1(\mathbf{A}) = \sum_{i=1}^{l} \operatorname{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})\, q(\mathbf{J}_i)\right) = \alpha - l.$$

(e) Hints: Assume each Jordan block $\mathbf{J}_i$ has size $k_i \times k_i$. First consider the case where the first l Jordan blocks (that is, $\mathbf{J}_1, ..., \mathbf{J}_l$) of $\mathbf{A}$ represent all of the Jordan blocks corresponding to λ. Note that when $i > l$, $\mathbf{J}_i - \lambda\mathbf{I}_{k_i}$ is nonsingular by Exercise 14, and so $\dim(\ker(\mathbf{J}_i - \lambda\mathbf{I}_{k_i})) = 0$. Consequently, by the block structure of $\mathbf{A} - \lambda\mathbf{I}_n$,

$$\dim(\ker(\mathbf{A} - \lambda\mathbf{I}_n)) = \sum_{i=1}^{l} \dim(\ker(\mathbf{J}_i - \lambda\mathbf{I}_{k_i})).$$

For $i \le l$, show that $\operatorname{rank}(\mathbf{J}_i - \lambda\mathbf{I}_{k_i}) = k_i - 1$. Use part (2) of Theorem 5.9 to conclude

$$\sum_{i=1}^{l} \dim(\ker(\mathbf{J}_i - \lambda\mathbf{I}_{k_i})) = l,$$

and use part (d).

(f) Hints: Assume each Jordan block $\mathbf{J}_i$ has size $k_i \times k_i$. First consider the case where the first l Jordan blocks (that is, $\mathbf{J}_1, ..., \mathbf{J}_l$) of $\mathbf{A}$ represent all of the Jordan blocks corresponding to λ. Note that for $i \le l$, $q(\mathbf{J}_i)$ is nonsingular (with rank k_i) by Exercise 19(a). Also note that, for $1 \le p \le k_i$,

$$\operatorname{rank}((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^p\, q(\mathbf{J}_i)) = \operatorname{rank}((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^p)$$

by Exercise 16 in the Review Exercises of Chapter 2. Consider the nature of the rows and columns of $\mathbf{J}_i - \lambda\mathbf{I}_{k_i}$, and use a proof by induction to conclude that, for $1 \le p < k_i$,

$$\operatorname{rank}((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^p) = k_i - p.$$

Also note that for $p \geq k_i$, $(\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^p$ is the zero matrix. In addition, for $i > l$, $(\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^p\, q(\mathbf{J}_i) = \mathbf{O}$ for all p, by Exercise 19(b). Also,

$$r_p(\mathbf{A}) = \text{rank}\left((\mathbf{A} - \lambda\mathbf{I}_n)^p\, q(\mathbf{A})\right) = \sum_{i=1}^{m} \text{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^p\, q(\mathbf{J}_i)\right)$$

by the block structure of $(\mathbf{A} - \lambda\mathbf{I}_n)^p\, q(\mathbf{A})$, which reduces to

$$\sum_{i=1}^{l} \text{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^p\right).$$

Thus,

$$r_{k-1}(\mathbf{A}) - r_k(\mathbf{A}) = \sum_{i=1}^{l} \left(\text{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^{k-1}\right) - \text{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^{k}\right)\right).$$

Finally, show that if $k_i \geq k$ and $1 \leq i \leq l$, then

$$\text{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^{k-1}\right) - \text{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^{k}\right) = 1,$$

but if $k_i < k$, then

$$\text{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^{k-1}\right) - \text{rank}\left((\mathbf{J}_i - \lambda\mathbf{I}_{k_i})^{k}\right) = 0.$$

Conclude that $r_{k-1}(\mathbf{A}) - r_k(\mathbf{A})$ equals the total number of Jordan blocks of $\mathbf{A}$ corresponding to λ having size at least $k \times k$.

(g) The number of Jordan blocks having size *exactly* $k \times k$ is the number having size at least $k \times k$ minus the number of size at least $(k+1) \times (k+1)$. By part (f), $r_{k-1}(\mathbf{A}) - r_k(\mathbf{A})$ gives the total number of Jordan blocks having size at least $k \times k$ corresponding to λ. Similarly, the number of Jordan blocks having size at least $(k+1) \times (k+1)$ is $r_k(\mathbf{A}) - r_{k+1}(\mathbf{A})$. Hence the number of Jordan blocks having size exactly $k \times k$ equals

$$(r_{k-1}(\mathbf{A}) - r_k(\mathbf{A})) - (r_k(\mathbf{A}) - r_{k+1}(\mathbf{A})) = r_{k-1}(\mathbf{A}) - 2r_k(\mathbf{A}) + r_{k+1}(\mathbf{A}).$$

(21) (a) F (b) T (c) F (d) F (e) F (f) T

Solving First Order Systems of Linear Homogeneous Differential Equations

Prerequisites: Web Section on Jordan Canonical Form; Section 8.8, Differential Equations

In Section 8.8 of the textbook, Theorem 8.9 gives the complete set of continuously differentiable solutions for the linear first order system of differential equations $\mathbf{F}'(t) = \mathbf{AF}(t)$ when the matrix $\mathbf{A}$ is diagonalizable. However, if the matrix $\mathbf{A}$ is not diagonalizable, then the technique described there fails to provide all of the solutions to the system. However, since every matrix $\mathbf{A}$ can be placed in Jordan Canonical Form, we now describe a technique that uses this form to completely solve the first-order system.

The concept behind the technique is based upon Lemma 8.8 in Section 8.8, which states that all continuously differentiable solutions to $f'(t) = af(t)$ are of the form $f(t) = be^{at}$ for some constant b. Thus, we might expect solutions of the system $\mathbf{F}'(t) = \mathbf{AF}(t)$ to be of the form $\mathbf{F}(t) = e^{\mathbf{A}t}\mathbf{c}$, for some constant vector $\mathbf{c}$, assuming that we can define the expression $e^{\mathbf{A}t}$ appropriately.

► Defining $e^{\mathbf{A}t}$

Suppose $\mathbf{A}$ is an $n \times n$ matrix, and let λ be an eigenvalue for $\mathbf{A}$ (possibly complex) with corresponding generalized eigenvector $\mathbf{v}$. Then there is a positive integer k such that $(\mathbf{A} - \lambda\mathbf{I}_n)^k \mathbf{v} = \mathbf{O}_n$.

Next, recall that e^t can be expressed as the power series

$$e^t = \sum_{j=0}^{\infty} \frac{t^j}{j!} = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots.$$

Assuming an analogous formula for $e^{\mathbf{A}t}$ would lead to

$$\begin{aligned} e^{\mathbf{A}t}\mathbf{v} &= e^{(\lambda\mathbf{I}_n + \mathbf{A} - \lambda\mathbf{I}_n)t}\mathbf{v} \\ &= e^{\lambda\mathbf{I}_n t}e^{(\mathbf{A} - \lambda\mathbf{I}_n)t}\mathbf{v} \\ &= e^{\lambda t}\left(\sum_{j=0}^{\infty} \frac{((\mathbf{A} - \lambda\mathbf{I}_n)\,t)^j}{j!}\right)\mathbf{v} \\ &= e^{\lambda t}\sum_{j=0}^{\infty} \frac{t^j}{j!}\left((\mathbf{A} - \lambda\mathbf{I}_n)^j\,\mathbf{v}\right). \end{aligned}$$

But, $(\mathbf{A} - \lambda\mathbf{I}_n)^j\,\mathbf{v} = \mathbf{0}$ for $j \geq k$, and so this would imply that

$$e^{\mathbf{A}t}\mathbf{v} = e^{\lambda t}\sum_{j=0}^{k-1} \frac{t^j}{j!}\,(\mathbf{A} - \lambda\mathbf{I}_n)^j\,\mathbf{v}.$$

This formula is useful because it only involves a *finite* sum of vectors rather than an infinite series, and therefore motivates the following definition.

Definition

Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalue λ and corresponding generalized eigenvector $\mathbf{v}$. Suppose k is the smallest positive integer such that $(\mathbf{A} - \lambda \mathbf{I}_n)^k \mathbf{v} = \mathbf{0}$. Then we define $e^{\mathbf{A}t}\mathbf{v}$ by the formula

$$e^{\mathbf{A}t}\mathbf{v} = e^{\lambda t} \sum_{j=0}^{k-1} \frac{t^j}{j!} (\mathbf{A} - \lambda \mathbf{I}_n)^j \mathbf{v}.$$

Example 1 Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 6 & -11 \\ -3 & -6 & 20 \\ -1 & -3 & 10 \end{bmatrix}.$$

In Exercise 1, you will be asked to show that $p_{\mathbf{A}}(x) = (x-3)^3$, and that $\mathbf{v} = [1, 0, 0]$ is a generalized eigenvector for $\mathbf{A}$ corresponding to $\lambda = 3$, with $(\mathbf{A} - 3\mathbf{I}_3)^3 \mathbf{v} = \mathbf{0}$. Then

$$\begin{aligned} e^{\mathbf{A}t}\mathbf{v} &= e^{3t} \sum_{j=0}^{2} \frac{t^j}{j!} (\mathbf{A} - 3\mathbf{I}_3)^j \mathbf{v} \\ &= e^{3t} \left(\mathbf{I}_3 \mathbf{v} + t (\mathbf{A} - 3\mathbf{I}_3) \mathbf{v} + \frac{t^2}{2} (\mathbf{A} - 3\mathbf{I}_3)^2 \mathbf{v} \right) \\ &= e^{3t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 & 6 & -11 \\ -3 & -9 & 20 \\ -1 & -3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} -3 & -9 & 21 \\ 1 & 3 & -7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \\ &= e^{3t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 + 2t - \frac{3}{2}t^2 \\ -3t + \frac{1}{2}t^2 \\ -t \end{bmatrix} e^{3t}. \end{aligned}$$

■

▸ Solving a Linear First-Order Homogeneous System

Returning to our discussion of the first-order system $\mathbf{F}'(t) = \mathbf{A}\mathbf{F}(t)$, we expect the solution set to be all vector-valued functions of the form $\mathbf{F}(t) = e^{\mathbf{A}t}\mathbf{c}$. Since we have now defined $e^{\mathbf{A}t}\mathbf{v}$ in the case in which $\mathbf{v}$ is a generalized eigenvector for $\mathbf{A}$, our strategy will be to find an ordered basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for $\mathbb{C}^n$ consisting of generalized eigenvectors for $\mathbf{A}$, and express the constant vector $\mathbf{c}$ as a linear combination of the vectors in this basis; that is, $\mathbf{c} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$. Then,

$$\mathbf{F}(t) = c_1 e^{\mathbf{A}t}\mathbf{v}_1 + \cdots + c_n e^{\mathbf{A}t}\mathbf{v}_n,$$

where $e^{\mathbf{A}t}\mathbf{v}_i$ is as defined above. The following theorem states that the set of all functions of this form gives the complete solution set of the first-order system.

THEOREM 1
Let $\mathbf{A}$ be an $n \times n$ matrix, and let $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis for $\mathbb{C}^n$ consisting of generalized eigenvectors for $\mathbf{A}$. Then the complete set of continuously differentiable solutions for the first-order linear homogeneous system of differential equations given by $\mathbf{F}'(t) = \mathbf{AF}(t)$ is the set of all functions of the form

$$\mathbf{F}(t) = c_1 e^{\mathbf{A}t}\mathbf{v}_1 + c_2 e^{\mathbf{A}t}\mathbf{v}_2 + \cdots + c_n e^{\mathbf{A}t}\mathbf{v}_n,$$

for $c_1, \ldots, c_n \in \mathbb{C}$.

In practice, the basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ in Theorem 1 is usually chosen so that if $\mathbf{P}$ is the matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then $\mathbf{P}^{-1}\mathbf{AP}$ is in Jordan Canonical Form. This is done for two reasons: first, because we have a process for finding such a basis of generalized eigenvectors (the Jordan Form Method[7]), and second, because the sequencing of the vectors makes it easier to compute $(\mathbf{A} - \lambda\mathbf{I}_n)^j\,\mathbf{v}_i$ as part of the calculation of $e^{\mathbf{A}t}\mathbf{v}_i$. This is demonstrated in Example 2 below.

In Exercise 8 you will be asked to show that every function of the form given in Theorem 1 is indeed a solution to the first-order system. We omit the proof that these are indeed *all* of the continuously differentiable solutions.

EXAMPLE 2 Consider the first-order system

$$\mathbf{F}'(t) = \begin{bmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \\ f_4'(t) \\ f_5'(t) \end{bmatrix} = \begin{bmatrix} -6 & 8 & 17 & -6 & 28 \\ -5 & 0 & 10 & -5 & 20 \\ 7 & -2 & -14 & 8 & -28 \\ -6 & -3 & 11 & -4 & 24 \\ -7 & 2 & 15 & -7 & 29 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \\ f_5(t) \end{bmatrix} = \mathbf{AF}(t).$$

To solve this system, we begin by putting the coefficient matrix $\mathbf{A}$ into Jordan Canonical Form using the Jordan Form Method.[8]

First, a lengthy calculation produces

$$\begin{aligned} p_{\mathbf{A}}(x) &= x^5 - 5x^4 + 14x^3 - 22x^2 + 17x - 5 \\ &= (x-1)^3\left(x^2 - 2x + 5\right) \\ &= (x-1)^3\,(x - (1+2i))\,(x - (1-2i)). \end{aligned}$$

Let $\lambda_1 = 1$, $\lambda_2 = 1 + 2i$, and $\lambda_3 = 1 - 2i$. Next, we follow Step A of the Jordan Form Method for each eigenvalue in turn to find generalized eigenvectors associated with these eigenvalues.

We begin with $\lambda_1 = 1$. We need to find three generalized eigenvectors for λ_1 since this is the algebraic multiplicity of λ_1.

Step A1: Since $p_{\mathbf{A}}(\mathbf{A}) = (\mathbf{A} - \mathbf{I}_5)^3\left(\mathbf{A}^2 - 2\mathbf{A} + 5\mathbf{I}_5\right)$, we let $\mathbf{D}_1 = \mathbf{A}^2 - 2\mathbf{A} + 5\mathbf{I}_5$. A short computation then yields

$$\mathbf{D}_1 = \begin{bmatrix} -28 & -24 & 60 & -28 & 128 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 8 & -4 & 4 & -16 \\ -4 & -4 & 8 & 0 & 16 \\ -10 & -11 & 19 & -9 & 44 \end{bmatrix}.$$

[7] See the web section on Jordan Canonical Form.
[8] Note that the Jordan Form Method calls the matrix "**B**," while we have named it "**A**."

Step A2: We search for the smallest value of k such that $(\mathbf{A} - \mathbf{I}_5)^k \mathbf{D}_1 = \mathbf{O}_5$. A few short computations yield

$$(\mathbf{A} - \mathbf{I}_5)\mathbf{D}_1 = \begin{bmatrix} 8 & 20 & -4 & 12 & -32 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & -12 & 12 & -4 & 32 \\ -8 & -12 & 12 & -4 & 32 \\ 4 & 8 & -4 & 4 & -16 \end{bmatrix},$$

$$(\mathbf{A} - \mathbf{I}_5)^2 \mathbf{D}_1 = \begin{bmatrix} -32 & -48 & 48 & -16 & 128 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & -12 & 12 & -4 & 32 \end{bmatrix},$$

and

$$(\mathbf{A} - \mathbf{I}_5)^3 \mathbf{D}_1 = \mathbf{O}_5.$$

Thus, $k = 3$.

Step A3: All columns of $(\mathbf{A} - \mathbf{I}_5)^2 \mathbf{D}_1$ are multiples of the first column. Hence, we choose the first column of $(\mathbf{A} - \mathbf{I}_5)^2 \mathbf{D}_1$ to be $\mathbf{v}_{11}$; that is, $\mathbf{v}_{11} = [-32, 0, 0, 0, -8]$.

Step A4: Let $\mathbf{v}_{12}$ be the first column of $(\mathbf{A} - \mathbf{I}_5)\mathbf{D}_1$, and let $\mathbf{v}_{13}$ be the first column of $\mathbf{D}_1$. To simplify matters, we divide all entries of these vectors by 4 to obtain

$$\begin{aligned} \mathbf{v}_{11} &= [-8, 0, 0, 0, -2], \\ \mathbf{v}_{12} &= [2, 0, -2, -2, 1], \\ \text{and } \mathbf{v}_{13} &= \left[-7, 0, 1, -1, -\tfrac{5}{2}\right]. \end{aligned}$$

Step A5: We have the single sequence $\{\mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{13}\}$ of generalized eigenvectors corresponding to $\lambda_1 = 1$. The following useful facts about $\mathbf{v}_{11}$, $\mathbf{v}_{12}$, and $\mathbf{v}_{13}$ will come in handy later:

$$(\mathbf{A} - \mathbf{I}_5)\mathbf{v}_{12} = \mathbf{v}_{11}, \quad (\mathbf{A} - \mathbf{I}_5)\mathbf{v}_{13} = \mathbf{v}_{12}, \quad \text{and} \quad (\mathbf{A} - \mathbf{I}_5)^2 \mathbf{v}_{13} = \mathbf{v}_{11}.$$

Step A6: Because we have 3 generalized eigenvectors for $\lambda_1 = 1$ and the algebraic multiplicity of λ_1 is 3, we do not need to find any more generalized eigenvectors corresponding to λ_1.

Next, we find the generalized eigenvectors for $\lambda_2 = 1 + 2i$. Since the algebraic multiplicity of λ_2 is 1, we only need one generalized eigenvector for λ_2. This will be an actual eigenvector.

Step A1: We let $\mathbf{D}_2 = (\mathbf{A} - (1 - 2i)\mathbf{I}_5)(\mathbf{A} - \mathbf{I}_5)^3$. A long but straightforward calculation yields

$$\mathbf{D}_2 = \begin{bmatrix} 96 + 72i & 48 - 24i & -192 - 144i & 96 + 72i & -384 - 288i \\ 40i & 16 + 8i & -80i & 40i & -160i \\ -16 - 72i & -32 - 8i & 32 + 144i & -16 - 72i & 64 + 288i \\ 16 + 32i & 16 & -32 - 64i & 16 + 32i & -64 - 128i \\ 32 + 64i & 32 & -64 - 128i & 32 + 64i & -128 - 256i \end{bmatrix}.$$

Step A2: Now $(\mathbf{A} - (1 + 2i)\mathbf{I}_5)\mathbf{D}_2 = p_{\mathbf{A}}(\mathbf{A}) = \mathbf{O}_5$ by the Cayley-Hamilton Theorem. Hence, $k = 1$.

Step A3: Because $k = 1$, finding an eigenvector for λ_2 amounts to choosing a nonzero column of $\mathbf{D}_2$. Since the second column of $\mathbf{D}_2$ has the simplest form, we choose that one. (Note that all of the other columns of $\mathbf{D}_2$ are scalar multiples of the second column.) We further simplify this choice by dividing each entry by 8 to get

$$\mathbf{v}_{21} = [6 - 3i,\ 2 + i,\ -4 - i,\ 2,\ 4].$$

Steps A4 – A6: These steps are unnecessary, since we have the only eigenvector we need for λ_2.

Finally, we must find an eigenvector for $\lambda_3 = 1 - 2i$. This computation is completely analogous to the one performed for λ_2. In Exercise 6 you are asked to verify that this process produces the vector

$$\mathbf{v}_{31} = [6 + 3i,\, 2 - i,\, -4 + i,\, 2,\, 4].$$

We now have an ordered basis $B = (\mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{13}, \mathbf{v}_{21}, \mathbf{v}_{31})$ of generalized eigenvectors for $\mathbf{A}$. By Theorem 1, every continuously differentiable solution of the system $\mathbf{F}'(t) = \mathbf{A}\mathbf{F}(t)$ has the form

$$\mathbf{F}(t) = c_1 e^{\mathbf{A}t}\mathbf{v}_{11} + c_2 e^{\mathbf{A}t}\mathbf{v}_{12} + c_3 e^{\mathbf{A}t}\mathbf{v}_{13} + c_4 e^{\mathbf{A}t}\mathbf{v}_{21} + c_5 e^{\mathbf{A}t}\mathbf{v}_{31}.$$

We need to work out each of the expressions in this sum. While the "$e^{\mathbf{A}t}\mathbf{v}$" portion of each expression has the form

$$e^{\lambda t}\sum_{j=0}^{k-1}\frac{t^j}{j!}\left(\mathbf{A} - \lambda\mathbf{I}_5\right)^j\mathbf{v},$$

(where $\mathbf{v}$ represents each of $\mathbf{v}_{11}$, $\mathbf{v}_{12}$, $\mathbf{v}_{13}$, $\mathbf{v}_{21}$, $\mathbf{v}_{31}$ in turn), these differ depending on the values of λ, k, and the particular $\mathbf{v}$ involved.

First, for $c_1 e^{\mathbf{A}t}\mathbf{v}_{11}$, we have $\lambda = 1$ and $k = 1$ (since $\left(\mathbf{A} - \mathbf{I}_5\right)^1\mathbf{v}_{11} = \mathbf{0}$). Thus,

$$\begin{aligned} c_1 e^{\mathbf{A}t}\mathbf{v}_{11} &= c_1 e^t \sum_{j=0}^{k-1}\frac{t^j}{j!}\left(\mathbf{A} - 1\mathbf{I}_5\right)^j\mathbf{v}_{11} \\ &= c_1 e^t\left(\frac{t^0}{0!}\left(\mathbf{A} - 1\mathbf{I}_5\right)^0\mathbf{v}_{11}\right) = c_1 e^t\left(\mathbf{v}_{11}\right). \end{aligned}$$

Next, for $c_2 e^{\mathbf{A}t}\mathbf{v}_{12}$, we have $\lambda = 1$ and $k = 2$ (since $\left(\mathbf{A} - \mathbf{I}_5\right)^2\mathbf{v}_{12} = \mathbf{0}$, but $\left(\mathbf{A} - \mathbf{I}_5\right)^1\mathbf{v}_{12} \neq \mathbf{0}$). Thus,

$$\begin{aligned} c_2 e^{\mathbf{A}t}\mathbf{v}_{12} &= c_2 e^t \sum_{j=0}^{k-1}\frac{t^j}{j!}\left(\mathbf{A} - 1\mathbf{I}_5\right)^j\mathbf{v}_{12} \\ &= c_2 e^t\left(\frac{t^0}{0!}\left(\mathbf{A} - 1\mathbf{I}_5\right)^0\mathbf{v}_{12} + \frac{t^1}{1!}\left(\mathbf{A} - 1\mathbf{I}_5\right)^1\mathbf{v}_{12}\right) \\ &= c_2 e^t\left(\mathbf{v}_{12} + t\left(\mathbf{A} - 1\mathbf{I}_5\right)\mathbf{v}_{12}\right) \\ &= c_2 e^t\left(\mathbf{v}_{12} + t\mathbf{v}_{11}\right) \qquad \text{since } \left(\mathbf{A} - 1\mathbf{I}_5\right)\mathbf{v}_{12} = \mathbf{v}_{11}. \end{aligned}$$

Next, for $c_3 e^{\mathbf{A}t}\mathbf{v}_{13}$, we have $\lambda = 1$ and $k = 3$ (since $\left(\mathbf{A} - \mathbf{I}_5\right)^3\mathbf{v}_{13} = \mathbf{0}$, but $\left(\mathbf{A} - \mathbf{I}_5\right)^2\mathbf{v}_{13} \neq \mathbf{0}$). Thus,

$$\begin{aligned} c_3 e^{\mathbf{A}t}\mathbf{v}_{13} &= c_3 e^t \sum_{j=0}^{k-1}\frac{t^j}{j!}\left(\mathbf{A} - 1\mathbf{I}_5\right)^j\mathbf{v}_{13} \\ &= c_3 e^t\left(\frac{t^0}{0!}\left(\mathbf{A} - 1\mathbf{I}_5\right)^0\mathbf{v}_{13} + \frac{t^1}{1!}\left(\mathbf{A} - 1\mathbf{I}_5\right)^1\mathbf{v}_{13} + \frac{t^2}{2!}\left(\mathbf{A} - 1\mathbf{I}_5\right)^2\mathbf{v}_{13}\right) \\ &= c_3 e^t\left(\mathbf{v}_{13} + t\left(\mathbf{A} - 1\mathbf{I}_5\right)\mathbf{v}_{13} + \frac{t^2}{2}\left(\mathbf{A} - 1\mathbf{I}_5\right)^2\mathbf{v}_{13}\right) \\ &= c_3 e^t\left(\mathbf{v}_{13} + t\mathbf{v}_{12} + \frac{t^2}{2}\mathbf{v}_{11}\right) \end{aligned}$$

since $\left(\mathbf{A} - 1\mathbf{I}_5\right)\mathbf{v}_{13} = \mathbf{v}_{12}$ and $\left(\mathbf{A} - 1\mathbf{I}_5\right)^2\mathbf{v}_{13} = \mathbf{v}_{11}$.

Next, for $c_4 e^{\mathbf{A}t}\mathbf{v}_{21}$, we have $\lambda = 1 + 2i$ and $k = 1$ (since $\left(\mathbf{A} - (1 + 2i)\mathbf{I}_5\right)^1\mathbf{v}_{21}$

$= \mathbf{0}$). Thus,

$$\begin{aligned} c_4 e^{\mathbf{A}t}\mathbf{v}_{21} &= c_4 e^{(1+2i)t} \sum_{j=0}^{k-1} \frac{t^j}{j!} \left(\mathbf{A} - (1+2i)\mathbf{I}_5\right)^j \mathbf{v}_{21} \\ &= c_4 e^{(1+2i)t} \left(\frac{t^0}{0!} \left(\mathbf{A} - (1+2i)\mathbf{I}_5\right)^0 \mathbf{v}_{21} \right) = c_4 e^{(1+2i)t} \left(\mathbf{v}_{21}\right). \end{aligned}$$

Finally, for $c_5 e^{\mathbf{A}t}\mathbf{v}_{31}$, we have $\lambda = 1 - 2i$ and $k = 1$ (since $\left(\mathbf{A} - (1-2i)\mathbf{I}_5\right)^1 \mathbf{v}_{31}$ $= \mathbf{0}$). Thus,

$$\begin{aligned} c_5 e^{\mathbf{A}t}\mathbf{v}_{31} &= c_5 e^{(1-2i)t} \sum_{j=0}^{k-1} \frac{t^j}{j!} \left(\mathbf{A} - (1+2i)\mathbf{I}_5\right)^j \mathbf{v}_{31} \\ &= c_5 e^{(1-2i)t} \left(\frac{t^0}{0!} \left(\mathbf{A} - (1-2i)\mathbf{I}_5\right)^0 \mathbf{v}_{31} \right) = c_5 e^{(1-2i)t} \left(\mathbf{v}_{31}\right). \end{aligned}$$

Notice by the way, that the values of "k" corresponding to each of these generalized eigenvectors is 1 for $\mathbf{v}_{11}$, $\mathbf{v}_{21}$, and $\mathbf{v}_{31}$, 2 for $\mathbf{v}_{12}$, and 3 for $\mathbf{v}_{13}$. (That is, if you have created and labelled the generalized eigenvectors as described in the Jordan Form Method, the value of k for a given generalized eigenvector is the second number in the subscript for that generalized eigenvector.)

Now, combining all of these expressions, we have:

$$\begin{aligned} \mathbf{F}(t) &= c_1 e^t \mathbf{v}_{11} + c_2 e^t \left(\mathbf{v}_{12} + t\mathbf{v}_{11}\right) + c_3 e^t \left(\mathbf{v}_{13} + t\mathbf{v}_{12} + \frac{t^2}{2}\mathbf{v}_{11}\right) \\ &\quad + c_4 e^{(1+2i)t}\mathbf{v}_{21} + c_5 e^{(1-2i)t}\mathbf{v}_{31}. \end{aligned}$$

Substituting in the values for the vectors $\mathbf{v}_{11}$, $\mathbf{v}_{12}$, $\mathbf{v}_{13}$, $\mathbf{v}_{21}$ and $\mathbf{v}_{31}$ produces the general solution for this system, as follows:

$$\begin{aligned} \mathbf{F}(t) &= c_1 e^t \begin{bmatrix} -8 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} + c_2 e^t \left(\begin{bmatrix} 2 \\ 0 \\ -2 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -8 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} \right) \\ &\quad + c_3 e^t \left(\begin{bmatrix} -7 \\ 0 \\ 1 \\ -1 \\ -\frac{5}{2} \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -2 \\ -2 \\ 1 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} -8 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} \right) \\ &\quad + c_4 e^{(1+2i)t} \begin{bmatrix} 6-3i \\ 2+i \\ -4-i \\ 2 \\ 4 \end{bmatrix} + c_5 e^{(1-2i)t} \begin{bmatrix} 6+3i \\ 2-i \\ -4+i \\ 2 \\ 4 \end{bmatrix} \end{aligned}$$

This simplifies to

$$\left(c_1 \begin{bmatrix} -8 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2-8t \\ 0 \\ -2 \\ -2 \\ 1-2t \end{bmatrix} + c_3 \begin{bmatrix} -7+2t-4t^2 \\ 0 \\ 1-2t \\ -1-2t \\ -\frac{5}{2}+t-t^2 \end{bmatrix} \right) e^t$$

$$+ c_4 \begin{bmatrix} 6-3i \\ 2+i \\ -4-i \\ 2 \\ 4 \end{bmatrix} e^{(1+2i)t} + c_5 \begin{bmatrix} 6+3i \\ 2-i \\ -4+i \\ 2 \\ 4 \end{bmatrix} e^{(1-2i)t},$$

or,

$$\begin{bmatrix} (-8c_1 + (2-8t)c_2 + (-7+2t-4t^2)c_3)e^t + (6-3i)c_4 e^{(1+2i)t} + (6+3i)c_5 e^{(1-2i)t} \\ (2+i)c_4 e^{(1+2i)t} + (2-i)c_5 e^{(1-2i)t} \\ (-2c_2 + (1-2t)c_3)e^t + (-4-i)c_4 e^{(1+2i)t} + (-4+i)c_5 e^{(1-2i)t} \\ (-2c_2 + (-1-2t)c_3)e^t + 2c_4 e^{(1+2i)t} + 2c_5 e^{(1-2i)t} \\ (-2c_1 + (1-2t)c_2 + (-\frac{5}{2}+t-t^2)c_3)e^t + 4c_4 e^{(1+2i)t} + 4c_5 e^{(1-2i)t} \end{bmatrix},$$

and so the solution set of the given first-order system consists of all vectors of this general form. ■

In Example 2, we began with a first-order system of differential equations involving only real numbers, and so you might be interested in finding only the *real-valued* solutions to the system. This is done in the following example.

EXAMPLE 3 We use the general complex solution obtained for the system in Example 2 to isolate its *real-valued* solutions.

Since $\lambda_1 = 1$ is real, we can replace the complex coefficients c_1, c_2, c_3 with real coefficients a_1, a_2, a_3, to obtain all of the *real-valued* terms in the solution related to λ_1. Thus, we only need to consider the terms related to $\lambda_2 = 1+2i$ and $\lambda_3 = 1-2i$ (those involving c_4 and c_5), which are

$$c_4 e^{(1+2i)t}\mathbf{v}_{21} + c_5 e^{(1-2i)t}\mathbf{v}_{31} = c_4 e^t e^{2it}\mathbf{v}_{21} + c_5 e^t e^{-2it}\mathbf{v}_{31}.$$

You are asked in Exercise 7 to use the facts that λ_2 and λ_3 are complex conjugates of each other, as are their corresponding generalized eigenvectors $\mathbf{v}_{21}$ and $\mathbf{v}_{31}$, along with the formulas

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta.$$

to show that the sum $c_4 e^t e^{2it}\mathbf{v}_{21} + c_5 e^t e^{-2it}\mathbf{v}_{31}$ can be expressed in the form

$$e^t\left(a_4\cos(2t) + a_5\sin(2t)\right)\mathbf{u} + e^t\left(-a_4\sin(2t) + a_5\cos(2t)\right)\mathbf{w},$$

where

$$\begin{aligned} \mathbf{u} &= \text{ real part of } \mathbf{v}_{21} = \frac{\mathbf{v}_{21}+\mathbf{v}_{31}}{2} = [6,\,2,\,-4,\,2,\,4]\,, \quad \text{and} \\ \mathbf{w} &= \text{ imaginary part of } \mathbf{v}_{21} = \frac{\mathbf{v}_{21}-\mathbf{v}_{31}}{2i} = [-3,\,1,\,-1,\,0,\,0]\,. \end{aligned}$$

Because all of the functions now involved (e^t, $\cos(2t)$, and $\sin(2t)$) are real-valued for real values of t, we restrict the coefficients a_4 and a_5 to being real so that we only get real-valued solutions.

Therefore, we have determined that all *real-valued* solutions to the given first-order system are of the form

$$\mathbf{F}(t) = a_1e^t\mathbf{v}_{11} + a_2e^t(\mathbf{v}_{12} + t\mathbf{v}_{11}) + a_3e^t\left(\mathbf{v}_{13} + t\mathbf{v}_{12} + \frac{t^2}{2}\mathbf{v}_{11}\right)$$
$$+ e^t(a_4\cos(2t) + a_5\sin(2t))\mathbf{u} + e^t(-a_4\sin(2t) + a_5\cos(2t))\mathbf{w},$$

where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$, and with $\mathbf{u}$ and $\mathbf{w}$ as described above. ■

▶ New Vocabulary

$e^{\mathbf{A}t}\mathbf{v}$

▶ Highlights

- If $\mathbf{A}$ is a square matrix, λ is an eigenvalue for $\mathbf{A}$ with corresponding generalized eigenvector $\mathbf{v}$, and if k is the smallest positive integer such that $(\mathbf{A} - \lambda\mathbf{I})^k\mathbf{v} = \mathbf{0}$, then $e^{\mathbf{A}t}\mathbf{v} = e^{\lambda t}\sum_{j=0}^{k-1}\frac{t^j}{j!}(\mathbf{A} - \lambda\mathbf{I}_n)^j\mathbf{v}$.
- If $\mathbf{A}$ is an $n \times n$ matrix, then the complete set of continuously differentiable solutions for the system of differential equations given by $\mathbf{F}'(t) = \mathbf{AF}(t)$ is the set of all functions of the form $\mathbf{F}(t) = c_1e^{\mathbf{A}t}\mathbf{v}_1 + c_2e^{\mathbf{A}t}\mathbf{v}_2 + \cdots + c_ne^{\mathbf{A}t}\mathbf{v}_n$, where $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is an ordered basis for $\mathbb{C}^n$ consisting of generalized eigenvectors for $\mathbf{A}$ and $c_1, \ldots, c_n \in \mathbb{C}$.
- An ordered basis for $\mathbb{C}^n$ consisting of generalized eigenvectors for a square matrix $\mathbf{A}$ can be computed using the Jordan Form Method found in the web section on Jordan Canonical Form.
- If a matrix $\mathbf{A}$ has all real entries, but has complex eigenvalues, the set of real continuously differentiable solutions for $\mathbf{F}'(t) = \mathbf{AF}(t)$ involves isolating the real part of the general complex solution, using the formulas $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$.

▶ EXERCISES

1. Let $\mathbf{A}$ be the 3×3 matrix from Example 1.

a) Verify that $p_{\mathbf{A}}(x) = (x-3)^3$.

b) Verify that $\mathbf{v}_{11} = [-3, 1, 0]$, $\mathbf{v}_{12} = [2, -3, -1]$, $\mathbf{v}_{13} = [1, 0, 0]$ is a sequence of generalized eigenvectors of $\mathbf{A}$ for $\lambda = 3$ such that $(\mathbf{A} - 3\mathbf{I}_3)\mathbf{v}_{13} = \mathbf{v}_{12}$, $(\mathbf{A} - 3\mathbf{I}_3)\mathbf{v}_{12} = \mathbf{v}_{11}$, and $(\mathbf{A} - 3\mathbf{I}_3)\mathbf{v}_{11} = \mathbf{0}$.

2. In each part, you are given an $n \times n$ matrix $\mathbf{A}$, an eigenvalue λ of $\mathbf{A}$, and a generalized eigenvector $\mathbf{v}$ for $\mathbf{A}$ corresponding to λ. Calculate $e^{\mathbf{A}t}\mathbf{v}$.

★ **a)** $\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ -5 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, $\lambda = 2$, $\mathbf{v} = [0, 1, 0]$

b) $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -2 & 6 & -11 & 6 \\ 0 & 3 & -5 & 3 \\ 0 & 1 & -2 & 2 \end{bmatrix}$, $\lambda = 1$, $\mathbf{v} = [-1, 1, 0, -1]$

★ **c)** $\mathbf{A} = \begin{bmatrix} 3i & 1 & -4i & 0 \\ 0 & 3i & 0 & -4i \\ 2i & 0 & -3i & 1 \\ 0 & 2i & 0 & -3i \end{bmatrix}$, $\lambda = i$, $\mathbf{v} = [0, 2, 0, 1]$

3. In each part, find the general solution for the first-order linear homogeneous system $\mathbf{F}'(t) = \mathbf{AF}(t)$ for the given matrix $\mathbf{A}$. (Note: Parts (a) through (g) of this exercise use the same matrices that appear in Exercise 6 in the web section on Jordan Canonical Form. If you have already done that exercise, you can use the answers you computed to help you solve this exercise.)

★ **a)** $\mathbf{A} = \begin{bmatrix} -9 & 5 & 8 \\ -4 & 3 & 4 \\ -8 & 4 & 7 \end{bmatrix}$

b) $\mathbf{A} = \begin{bmatrix} 6 & -2 & -3 \\ -14 & 14 & 13 \\ 23 & -19 & -19 \end{bmatrix}$

(Hint: $p_{\mathbf{A}}(x) = x^3 - x^2 - 8x + 12 = (x-2)^2(x+3)$)

c) $\mathbf{A} = \begin{bmatrix} 5 & -3 & -6 \\ 7 & -5 & -14 \\ -2 & 2 & 6 \end{bmatrix}$

★ **d)** $\mathbf{A} = \begin{bmatrix} 8 & 5 & 0 \\ -5 & -3 & -1 \\ 10 & 7 & 0 \end{bmatrix}$ (Note: Exercise 7(c) below asks for only the *real-valued* solutions for this system.)

e) $\mathbf{A} = \begin{bmatrix} -4+5i & 2-2i & 1-2i \\ -8+6i & 4-2i & 2-3i \\ 4i & -2i & -i \end{bmatrix}$ (Hint: $p_{\mathbf{A}}(x) = x^3 - 2ix^2 - x$)

f) $\mathbf{A} = \begin{bmatrix} -12 & 5 & -11 & -10 \\ -7 & 3 & -7 & -8 \\ 15 & -6 & 14 & 12 \\ -4 & 1 & -4 & -3 \end{bmatrix}$

(Hint: $p_{\mathbf{A}}(x) = x^4 - 2x^3 - 3x^2 + 4x + 4 = (x-2)^2(x+1)^2$)

★ **g)** $\mathbf{A} = \begin{bmatrix} -3 & -2 & -1 & 5 & 3 \\ -8 & 2 & 7 & -2 & 1 \\ 0 & -3 & -4 & 6 & 3 \\ -4 & 1 & 3 & -1 & 1 \\ 0 & -2 & -2 & 4 & 1 \end{bmatrix}$

(Hint: $p_{\mathbf{A}}(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = (x+1)^5$)

★ **h)** $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

4. Show that it is consistent with the definition of $e^{\mathbf{A}t}\mathbf{v}$ to *define* $e^{\mathbf{O}_n}$ to be $\mathbf{I}_n$. (Consider both cases: $t = 0$ and $\mathbf{A} = \mathbf{O}_n$. You can adopt here the standard convention in the context of infinite series that $0^0 = 1$.)

5. Let $\mathbf{A}$ be an $n \times n$ matrix having eigenvalue λ with corresponding generalized eigenvector $\mathbf{v}$. Let $\mathbf{B} = c\mathbf{A}$ for some $c \in \mathbb{C}$. Suppose that $\mathbf{F}(t) = e^{\mathbf{A}t}\mathbf{v}$. Show that $\mathbf{v}$ is a generalized eigenvector for $\mathbf{B}$ corresponding to the eigenvalue $c\lambda$, and that $\mathbf{G}(t) = e^{\mathbf{B}t}\mathbf{v} = \mathbf{F}(ct)$.

6. Let $\mathbf{F}'(t) = \mathbf{AF}(t)$ be the first-order system given in Example 2. Verify that

$$\mathbf{v}_{31} = [6+3i,\ 2-i,\ -4+i,\ 2,\ 4]$$

is an eigenvector for $\mathbf{A}$ corresponding to the eigenvalue $\lambda_3 = 1 - 2i$.

7. This exercise will substantiate the claim made in Example 3.

a) Use the formulas

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

to show that if $z = x + iy$, then

$$aze^{i\theta} + b\overline{z}e^{-i\theta} = x\left(c\,\cos\theta + d\,\sin\theta\right) + y\left(-c\,\sin\theta + d\,\cos\theta\right),$$

where $a, b, z \in \mathbb{C}$, $\overline{z}$ is the complex conjugate of z, $c = a + b$, and $d = i(a - b)$.

b) Use part (a) to show that if $t \in \mathbb{R}$ and

$$\begin{aligned} \mathbf{v}_{21} &= [6 + 3i,\ 2 - i,\ -4 + i,\ 2,\ 4] \\ \text{and } \mathbf{v}_{31} &= [6 + 3i,\ 2 - i,\ -4 + i,\ 2,\ 4], \end{aligned}$$

then the real-valued solutions corresponding to

$$c_4 e^{(1+2i)t}\mathbf{v}_{21} + c_5 e^{(1-2i)t}\mathbf{v}_{31}$$

in Example 3 can be expressed in the form

$$e^t\left(a_4\cos(2t) + a_5\sin(2t)\right)\mathbf{u} + e^t\left(-a_4\sin(2t) + a_5\cos(2t)\right)\mathbf{w},$$

where $a_4,\ a_5 \in \mathbb{R}$, $\mathbf{u}$ is the real part of $\mathbf{v}_{21}$ and $\mathbf{w}$ is the imaginary part of $\mathbf{v}_{21}$.

★ **c)** For the matrix $\mathbf{A} = \begin{bmatrix} 8 & 5 & 0 \\ -5 & -3 & -1 \\ 10 & 7 & 0 \end{bmatrix}$ from part (d) of Exercise 3, find all the *real-valued* solutions for the first-order linear homogeneous system $\mathbf{F}'(t) = \mathbf{AF}(t)$ by following a technique similar to that shown in Example 3, making use of the formula in part (a) above.

► **8.** Let $\mathbf{A}$ be an $n \times n$ matrix.

a) Let λ be an eigenvalue for $\mathbf{A}$ and let $\mathbf{v}$ be a corresponding generalized eigenvector. Show that $e^{\mathbf{A}t}\mathbf{v}$ is a solution to the first-order system $\mathbf{F}'(t) = \mathbf{AF}(t)$.

b) Show that any finite linear combination of solutions to the first-order system $\mathbf{F}'(t) = \mathbf{AF}(t)$ is also a solution for the system. (This, together with part (a), shows that all of the functions given in Theorem 1 are indeed solutions for the system $\mathbf{F}'(t) = \mathbf{AF}(t)$.)

★ **9.** True or False:

a) To compute $e^{\mathbf{A}t}\mathbf{w}$ for a square matrix $\mathbf{A}$ and a vector $\mathbf{w}$, one must first express $\mathbf{w}$ as a linear combination of generalized eigenvectors for $\mathbf{A}$.

b) If a square matrix $\mathbf{A}$ is diagonalizable, then the techniques in Section 8.8 of the textbook are sufficient to completely solve the system $\mathbf{F}'(t) = \mathbf{AF}(t)$, so the techniques introduced in this section are not necessary in that case.

c) If $\mathbf{A}$ is an $n \times n$ matrix, then the complete set of continuously differentiable solutions for the system of differential equations given by $\mathbf{F}'(t) = \mathbf{AF}(t)$ is the set of all functions of the form $\mathbf{F}(t) = ce^{\mathbf{A}t}\mathbf{v}$, where $\mathbf{v}$ is a generalized eigenvector for $\mathbf{A}$ and $c \in \mathbb{C}$.

d) If $\mathbf{A}$ is a square matrix whose only eigenvalue is 0, the the complete set of continuously differentiable solutions for the system of differential equations given by $\mathbf{F}'(t) = \mathbf{A}\mathbf{F}(t)$ consists only of polynomials.

e) The complete set of real-valued continuously differentiable solutions for the system

$$\mathbf{F}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{F}(t)$$

is the set of all functions of the form

$$\mathbf{F}(t) = a \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + b \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix},$$

for $a, b \in \mathbb{R}$.

▶ Answers to Selected Exercises

(2) (a) $\begin{bmatrix} 0 \\ 1+t \\ t \end{bmatrix} e^{2t}$ (c) $\begin{bmatrix} 2t \\ 2 \\ t \\ 1 \end{bmatrix} e^{it}$

(3) (a) $$\left(c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2+t \\ 1+2t \\ 2 \end{bmatrix} \right) e^t + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-t} = \begin{bmatrix} (c_1 + c_2(2+t))e^t + c_3 e^{-t} \\ (2c_1 + c_2(1+2t))e^t \\ 2c_2 e^t + c_3 e^{-t} \end{bmatrix}$$

(d) $$c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 2-i \\ -3+i \\ 1-2i \end{bmatrix} e^{(1-i)t} + c_3 \begin{bmatrix} 2+i \\ -3-i \\ 1+2i \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} c_1 e^{3t} + c_2(2-i)e^{(1-i)t} + c_3(2+i)e^{(1+i)t} \\ -c_1 e^{3t} + c_2(-3+i)e^{(1-i)t} + c_3(-3-i)e^{(1+i)t} \\ c_1 e^{3t} + c_2(1-2i)e^{(1-i)t} + c_3(1+2i)e^{(1+i)t} \end{bmatrix}$$

(g) $$\left(c_1 \begin{bmatrix} -2 \\ -8 \\ 0 \\ -4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1-2t \\ -8t \\ 0 \\ -4t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 3 \\ -3 \\ 1 \\ -2 \end{bmatrix} + c_4 \begin{bmatrix} -2t \\ 1+3t \\ -3t \\ t \\ -2t \end{bmatrix} + c_5 \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right) e^{-t} = \begin{bmatrix} (-2c_1 + c_2 - 2c_3 + c_5) + (-2c_2 - 2c_4)t \\ (-8c_1 + 3c_3 + c_4 - 2c_5) + (-8c_2 + 3c_4)t \\ (-3c_3 + 2c_5) - 3c_4 t \\ (-4c_1 + c_3) + (-4c_2 + c_4)t \\ -2c_3 - 2c_4 t \end{bmatrix} e^{-t}$$

(h) $$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \frac{1}{2}t^2 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 t + \frac{c_3}{2}t^2 \\ c_2 + c_3 t \\ c_3 \end{bmatrix}$$

(7) (c) $$a_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \left(a_5 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + a_4 \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right) e^t \sin t + \left(a_4 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + a_5 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right) e^t \cos t$$
$$= \begin{bmatrix} a_1 e^{3t} + (2a_5 - a_4)e^t \sin t + (2a_4 + a_5)e^t \cos t \\ -a_1 e^{3t} + (-3a_5 + a_4)e^t \sin t - (3a_4 + a_5)e^t \cos t \\ a_1 e^{3t} + (a_5 - 2a_4)e^t \sin t + (a_4 + 2a_5)e^t \cos t \end{bmatrix},$$
where $a_1, a_4, a_5 \in \mathbb{R}$

(9) (a) T (b) T (c) F (d) T (e) T

Isometries on Inner Product Spaces

Prerequisites: Section 5.5, Isomorphism; Section 6.3, Orthogonal Diagonalization; Section 7.4, Orthogonality in $\mathbb{C}^n$; Section 7.5, Inner Product Spaces

In this section, we investigate functions between inner product spaces that preserve the distances between vectors. We will pay special attention to such functions that are also linear transformations.

▶ Definition of an Isometry

As we will see in the formal definition below, a function on an inner product space that preserves the distances between vectors is called an **isometry**.

EXAMPLE 1 The translation function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f([x, y]) = [x, y] + [2, -5]$ is an isometry from $\mathbb{R}^2$ to $\mathbb{R}^2$. The function f merely moves every vector in the plane 2 units in the positive x-direction and 5 units in the negative y-direction. In effect, f is a mapping taking the entire plane to itself in which the distance between any pair of vectors remains unchanged. Such a mapping is often referred to as a **rigid motion of the plane**, in which the plane is imagined as an infinite, unbreakable, unbendable sheet of glass that is merely shifted to a new position. Along with translations, rigid motions of the plane also include rotations about a fixed point and reflections about a fixed line, as well as compositions of these mappings. (In fact, it can be shown that all isometries of the plane are actually rigid motions.)

The conjugation function $g\colon \mathbb{C} \to \mathbb{C}$ given by $g(z) = \overline{z}$ for every $z \in \mathbb{C}$ is also an isometry, since, if $z_1, z_2 \in \mathbb{C}$, then

$$\begin{aligned} |\overline{z_1} - \overline{z_2}| &= |\overline{z_1 - z_2}| && \text{by Theorem C.1 in Appendix C} \\ &= |z_1 - z_2|, \end{aligned}$$

because a complex number and its conjugate have the same absolute value. Hence, the distance from $\overline{z_1}$ to $\overline{z_2}$ equals the distance from z_1 to z_2. ■

Although both of the functions in Example 1 are isometries, the first one is not a (real) linear transformation, while the second is not a complex linear transformation.

In particular, the translation function on $\mathbb{R}^2$ in Example 1 is not a linear transformation because it does not fix the origin; that is, $f(\mathbf{0}) \neq \mathbf{0}$. Because of this, while f preserves the *distance* between vectors, it does not preserve the *lengths* of vectors; that is, in general, $\|f(\mathbf{v})\| \neq \|\mathbf{v}\|$. (This is because the *length* of a vector equals its *distance* from the zero vector. We will see this in more detail in the proof of Theorem 1, below.)

In this section, we are mostly interested in those isometries that *are* linear transformations. We will see in Theorem 1, below, that such isometries preserve both distances and lengths.

EXAMPLE 2 Consider the linear operator $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ which rotates every vector (having its initial point at the origin) counterclockwise 45° about the z-axis. From a geometric point of view, L is a **rigid motion of 3-space**, in which 3-dimensional space is imagined as an infinite unbreakable, unbendable solid that is merely rotated to a new position, while keeping the z-axis fixed. Thus, L is a obviously an isometry since it does not change the distance between any two vectors. This operator can be expressed as

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \\ z \end{bmatrix}.$$

■

In what follows, we will study the notion of isometry in more general inner product spaces, not just in $\mathbb{R}^n$ and $\mathbb{C}^n$. Therefore, we will often distinguish the particular inner product involved by using a subscript to indicate its corresponding inner product space. For example, in an inner product space $\mathcal{V}$, we would express the inner product of $\mathbf{v}_1$ and $\mathbf{v}_2$ as $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathcal{V}}$. Similarly, we express the norm of $\mathbf{v}$ in $\mathcal{V}$ as $\|\mathbf{v}\|_{\mathcal{V}}$. Throughout this section, when working with $\mathbb{R}^n$ or $\mathbb{C}^n$, unless stated otherwise, we assume that we are using the standard dot product as the inner product.

DEFINITION

Let $\mathcal{V}$ and $\mathcal{W}$ be real [complex] inner product spaces (either both real or both complex). Then a function $f\colon\ \mathcal{V} \to \mathcal{W}$ is an **isometry** if and only if $\|f(\mathbf{v}_1) - f(\mathbf{v}_2)\|_{\mathcal{W}} = \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}}$ for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$.

Notice that $\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}}$ is computed using the norm on $\mathcal{V}$, while $\|f(\mathbf{v}_1) - f(\mathbf{v}_2)\|_{\mathcal{W}}$ is computed with the norm of $\mathcal{W}$.

EXAMPLE 3

Consider the inner product space $\mathcal{P}_n$ with inner product defined as follows: if $\mathbf{p}_1 = a_n x^n + \cdots + a_1 x + a_0$ and $\mathbf{p}_2 = b_n x^n + \cdots + b_1 x + b_0$, then

$$\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = a_n b_n + \cdots + a_1 b_1 + a_0 b_0.$$

Also consider the inner product space $\mathbb{R}^{n+1}$ with the usual inner (dot) product. Suppose $L\colon\ \mathcal{P}_n \to \mathbb{R}^{n+1}$ is the linear transformation given by

$$L(c_n x^n + \cdots + c_1 x + c_0) = [c_n, \ldots, c_1, c_0].$$

Now L is obviously an isometry since

$$\begin{aligned} & \|(c_n x^n + \cdots + c_1 x + c_0) - (d_n x^n + \cdots + d_1 x + d_0)\|_{\mathcal{P}_n} \\ = \ & \|(c_n - d_n)x^n + \cdots + (c_1 - d_1)x + (c_0 - d_0)\|_{\mathcal{P}_n} \\ = \ & \sqrt{\langle (c_n - d_n)x^n + \cdots + (c_0 - d_0),\ (c_n - d_n)x^n + \cdots + (c_0 - d_0) \rangle} \\ = \ & \sqrt{(c_n - d_n)^2 + \cdots + (c_0 - d_0)^2}, \end{aligned}$$

while

$$\begin{aligned} & \|L(c_n x^n + \cdots + c_1 x + c_0) - L(d_n x^n + \cdots + d_1 x + d_0)\|_{\mathbb{R}^{n+1}} \\ = \ & \|[c_n, \ldots, c_1, c_0] - [d_n, \ldots, d_1, d_0]\|_{\mathbb{R}^{n+1}} \\ = \ & \|[(c_n - d_n), \ldots, (c_1 - d_1), (c_0 - d_0)]\|_{\mathbb{R}^{n+1}} \\ = \ & \sqrt{[(c_n - d_n), \ldots, (c_0 - d_0)] \cdot [(c_n - d_n), \ldots, (c_0 - d_0)]} \\ = \ & \sqrt{(c_n - d_n)^2 + \cdots + (c_0 - d_0)^2}. \end{aligned}$$

■

▶ Properties of Isometries

While linear transformations that are isometries preserve (by definition) the distances between vectors, the next theorem shows that these types of isometries also preserve the norms of vectors, as well as inner products and orthonormal sets.

THEOREM 1
Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation between inner product spaces. Then the following are equivalent:
(1) L is an isometry.
(2) $\|\mathbf{v}\|_{\mathcal{V}} = \|L(\mathbf{v})\|_{\mathcal{W}}$ for every $\mathbf{v} \in \mathcal{V}$.
(3) $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathcal{V}} = \langle L(\mathbf{v}_1), L(\mathbf{v}_2) \rangle_{\mathcal{W}}$ for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$.
(4) The image under L of every orthonormal set in $\mathcal{V}$ is an orthonormal set in $\mathcal{W}$.

Proof Assume L is a linear transformation between inner product spaces $\mathcal{V}$ and $\mathcal{W}$. We prove Theorem 1 by showing that (1) $\iff$ (2), (1) $\implies$ (3), (3) $\implies$ (4), and (4) $\implies$ (2).

(1) $\implies$ (2): Let $\mathbf{v} \in \mathcal{V}$. Then,

$$\begin{aligned} \|\mathbf{v}\|_{\mathcal{V}} &= \|\mathbf{v} - \mathbf{0}_{\mathcal{V}}\|_{\mathcal{V}} = \|L(\mathbf{v}) - L(\mathbf{0}_{\mathcal{V}})\|_{\mathcal{W}} && \text{because } L \text{ is an isometry} \\ &= \|L(\mathbf{v}) - \mathbf{0}_{\mathcal{W}}\|_{\mathcal{W}} && \text{by Theorem 5.1} \\ &= \|L(\mathbf{v})\|_{\mathcal{W}}. \end{aligned}$$

(2) $\implies$ (1): Suppose $\|\mathbf{v}\|_{\mathcal{V}} = \|L(\mathbf{v})\|_{\mathcal{W}}$ for all $\mathbf{v} \in \mathcal{V}$. But then, for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$, $\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}} = \|L(\mathbf{v}_1 - \mathbf{v}_2)\|_{\mathcal{W}} = \|L(\mathbf{v}_1) - L(\mathbf{v}_2)\|_{\mathcal{W}}$, because L is a linear transformation. Hence, L is an isometry.

This concludes the proof of (1) $\iff$ (2), and so (1) and (2) are equivalent.

(1) $\implies$ (3): We present a proof for real inner product spaces using the Polarization Identity from part (a) of Exercise 8 in Section 7.5. A completely analogous proof holds for complex inner product spaces using the complex Polarization Identity in part (b) of that exercise.

Suppose $\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}} = \|L(\mathbf{v}_1) - L(\mathbf{v}_2)\|_{\mathcal{W}}$ for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$. Then,

$$\begin{aligned} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathcal{V}} &= \tfrac{1}{4}\left(\|\mathbf{v}_1 + \mathbf{v}_2\|_{\mathcal{V}}^2 - \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}}^2\right) \\ & \qquad \text{using the Polarization Identity in } \mathcal{V} \\ &= \tfrac{1}{4}\left(\|\mathbf{v}_1 - (-\mathbf{v}_2)\|_{\mathcal{V}}^2 - \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}}^2\right) \\ &= \tfrac{1}{4}\left(\|L(\mathbf{v}_1) - L(-\mathbf{v}_2)\|_{\mathcal{W}}^2 - \|L(\mathbf{v}_1) - L(\mathbf{v}_2)\|_{\mathcal{W}}^2\right) \\ &= \tfrac{1}{4}\left(\|L(\mathbf{v}_1) + L(\mathbf{v}_2)\|_{\mathcal{W}}^2 - \|L(\mathbf{v}_1) - L(\mathbf{v}_2)\|_{\mathcal{W}}^2\right) \\ & \qquad \text{since } L \text{ is a linear transformation} \\ &= \langle L(\mathbf{v}_1), L(\mathbf{v}_2) \rangle_{\mathcal{W}} \quad \text{using the Polarization Identity in } \mathcal{W}. \end{aligned}$$

(3) $\implies$ (4): Suppose $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathcal{V}} = \langle L(\mathbf{v}_1), L(\mathbf{v}_2) \rangle_{\mathcal{W}}$ for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$. Let B be an orthonormal set in $\mathcal{V}$. Then $\|\mathbf{u}_i\|_{\mathcal{V}} = 1$ for all $\mathbf{u}_i \in B$, and $\langle \mathbf{u}_i, \mathbf{u}_j \rangle_{\mathcal{V}} = 0$ for all distinct $\mathbf{u}_i, \mathbf{u}_j \in B$. But then

$$\langle L(\mathbf{u}_i), L(\mathbf{u}_j) \rangle_{\mathcal{W}} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle_{\mathcal{V}} = 0,$$

for all distinct $\mathbf{u}_i, \mathbf{u}_j \in B$, and

$$\|L(\mathbf{u}_i)\|_{\mathcal{W}} = \sqrt{\langle L(\mathbf{u}_i), L(\mathbf{u}_i) \rangle_{\mathcal{W}}} = \sqrt{\langle \mathbf{u}_i, \mathbf{u}_i \rangle_{\mathcal{V}}} = \|\mathbf{u}_i\|_{\mathcal{V}} = 1$$

for all $\mathbf{u}_i \in B$. Also note that, for all distinct $\mathbf{u}_i, \mathbf{u}_j \in B$, $L(\mathbf{u}_i)$ and $L(\mathbf{u}_j)$ are distinct because

$$\begin{aligned}
\|L(\mathbf{u}_i) - L(\mathbf{u}_j)\|_{\mathcal{W}} &= \sqrt{\langle L(\mathbf{u}_i) - L(\mathbf{u}_j), L(\mathbf{u}_i) - L(\mathbf{u}_j)\rangle_{\mathcal{W}}} \\
&= \sqrt{\langle L(\mathbf{u}_i - \mathbf{u}_j), L(\mathbf{u}_i - \mathbf{u}_j)\rangle_{\mathcal{W}}} \\
&= \sqrt{\langle (\mathbf{u}_i - \mathbf{u}_j), (\mathbf{u}_i - \mathbf{u}_j)\rangle_{\mathcal{V}}} \\
&= \sqrt{\langle \mathbf{u}_i, \mathbf{u}_i\rangle_{\mathcal{V}} - \langle \mathbf{u}_i, \mathbf{u}_j\rangle_{\mathcal{V}} - \langle \mathbf{u}_j, \mathbf{u}_i\rangle_{\mathcal{V}} + \langle \mathbf{u}_j, \mathbf{u}_j\rangle_{\mathcal{V}}} \\
&= \sqrt{1 - 0 - 0 + 1} = \sqrt{2} \neq 0.
\end{aligned}$$

Hence, $L(B)$ is an orthonormal set.

(4) $\Longrightarrow$ (2): Suppose the image of every orthonormal set in $\mathcal{V}$ is an orthonormal set in $\mathcal{W}$. We must show that $\|L(\mathbf{v})\|_{\mathcal{W}} = \|\mathbf{v}\|_{\mathcal{V}}$ for all $\mathbf{v} \in \mathcal{V}$.

First, if $\mathbf{v} = \mathbf{0}_{\mathcal{V}}$, then $L(\mathbf{v}) = L(\mathbf{0}_{\mathcal{V}}) = \mathbf{0}_{\mathcal{W}}$, and so

$$\|L(\mathbf{v})\|_{\mathcal{W}} = \|\mathbf{0}_{\mathcal{W}}\|_{\mathcal{W}} = 0 = \|\mathbf{0}_{\mathcal{V}}\|_{\mathcal{V}} = \|\mathbf{v}\|_{\mathcal{V}}.$$

Next, suppose that $\mathbf{v} \neq \mathbf{0}_{\mathcal{V}}$. Now $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|_{\mathcal{V}}}$ is a unit vector, and so $\{\mathbf{u}\}$ is an orthonormal set in $\mathcal{V}$. Hence, $\{L(\mathbf{u})\}$ is an orthonormal set in $\mathcal{W}$. Therefore, $L(\mathbf{u})$ is a unit vector. Thus,

$$\begin{aligned}
1 &= \|L(\mathbf{u})\|_{\mathcal{W}} = \left\| L\left(\frac{\mathbf{v}}{\|\mathbf{v}\|_{\mathcal{V}}}\right)\right\|_{\mathcal{W}} \\
&= \left\| \left(\frac{1}{\|\mathbf{v}\|_{\mathcal{V}}}\right) L(\mathbf{v})\right\|_{\mathcal{W}} && \text{because } L \text{ is a linear transformation} \\
&= \frac{1}{\|\mathbf{v}\|_{\mathcal{V}}} \|L(\mathbf{v})\|_{\mathcal{W}} && \text{by Theorem 7.13.}
\end{aligned}$$

But then $\|\mathbf{v}\|_{\mathcal{V}} = \|L(\mathbf{v})\|_{\mathcal{W}}$. **QED**

Example 4 Consider again the linear operator L on $\mathbb{R}^3$ from Example 2, involving a counterclockwise rotation about the z-axis through an angle of 45°, along with an inner product on $\mathbb{R}^3$ given by the standard dot product. All of the properties of isometries given in Theorem 1 are satisfied for this linear transformation.

In particular, because the rotation L involves a rigid motion of space that fixes the origin, L preserves the lengths of vectors, which verifies that property (2) is satisfied.

Next, if B is any orthonormal set of vectors in $\mathbb{R}^3$, then the vectors in B are unit vectors, and because L preserves lengths, the vectors in $L(B)$ are also unit vectors. Moreover, since any two vectors in B are orthogonal to each other, their images in $L(B)$ after rotation must also be orthogonal to each other. Thus, $L(B)$ is also an orthonormal set, verifying that property (4) is satisfied.

However, verifying property (3), preservation of the inner product, requires some algebraic computation:

$$\begin{aligned}
L([x_1,y_1,z_1])\cdot L([x_2,y_2,z_2]) &= \begin{bmatrix} \frac{\sqrt{2}}{2}x_1 - \frac{\sqrt{2}}{2}y_1 \\ \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2}x_2 - \frac{\sqrt{2}}{2}y_2 \\ \frac{\sqrt{2}}{2}x_2 + \frac{\sqrt{2}}{2}y_2 \\ z_2 \end{bmatrix} \\
&= \left(\tfrac{\sqrt{2}}{2}x_1 - \tfrac{\sqrt{2}}{2}y_1\right)\left(\tfrac{\sqrt{2}}{2}x_2 - \tfrac{\sqrt{2}}{2}y_2\right) \\
&\qquad + \left(\tfrac{\sqrt{2}}{2}x_1 + \tfrac{\sqrt{2}}{2}y_1\right)\left(\tfrac{\sqrt{2}}{2}x_2 + \tfrac{\sqrt{2}}{2}y_2\right) + z_1z_2 \\
&= \tfrac{1}{2}x_1x_2 - \tfrac{1}{2}x_1y_2 - \tfrac{1}{2}x_2y_1 + \tfrac{1}{2}y_1y_2 \\
&\qquad + \tfrac{1}{2}x_1x_2 + \tfrac{1}{2}x_1y_2 + \tfrac{1}{2}x_2y_1 + \tfrac{1}{2}y_1y_2 + z_1z_2 \\
&= x_1x_2 + y_1y_2 + z_1z_2 \\
&= [x_1,y_1,z_1]\cdot[x_2,y_2,z_2].
\end{aligned}$$

■

The proof of the following corollary is easy, and is left for you to do in Exercise 7.

COROLLARY 2
Let $L\colon \mathcal{V} \to \mathcal{W}$ is an isometry between *real* inner product spaces. Then the measure of the angle between any two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathcal{V}$ is equal to the measure of the angle between $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ in $\mathcal{W}$.

Corollary 2 shows that isometries preserve the geometry of real inner product spaces by leaving the angle between corresponding nonzero vectors unchanged. For example, we have already seen that the isometry in Example 4 preserves angles, illustrating this corollary.

Note that Corollary 2 only applies to *real* inner product spaces, not to complex inner product spaces. This is because the angle between two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in a complex inner product space is not defined, since, in a complex vector space, $\frac{\langle \mathbf{v}_1,\mathbf{v}_2\rangle}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|}$ might be a complex number.[9]

An important property of isometries for both real and complex inner product spaces is given in the next theorem.

THEOREM 3
Suppose the linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is an isometry between inner product spaces. Then L is one-to-one. Furthermore, if $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional with $\dim(\mathcal{V}) = \dim(\mathcal{W})$, then L is an isomorphism.

Proof Let L be an isometry between $\mathcal{V}$ and $\mathcal{W}$, and let $\mathbf{v} \in \ker(L)$. Then

$$\begin{aligned}
L(\mathbf{v}) = \mathbf{0}_{\mathcal{W}} &\implies \|L(\mathbf{v})\|_{\mathcal{W}} = 0 \\
&\implies \|\mathbf{v}\|_{\mathcal{V}} = 0 \qquad \text{by part (2) of Theorem 1} \\
&\implies \mathbf{v} = \mathbf{0}_{\mathcal{V}}.
\end{aligned}$$

Therefore, L is one-to-one by part (1) of Theorem 5.12 (or its complex analog).

If $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional with $\dim(\mathcal{V}) = \dim(\mathcal{W})$, then the fact that L is one-to-one together with Corollary 5.13 (or its complex analog) implies that L must be an isomorphism. **QED**

[9] Even though angles between vectors are not defined in a complex inner product space, recall that orthogonality *is* defined in both real and complex inner product spaces, since, in both cases, we can check whether $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$. Hence, part (4) of Theorem 1 *does* make sense in a complex inner product space.

For instance, we have already seen that the coordinatization transformation from $\mathcal{P}_n$ to $\mathbb{R}^{n+1}$ in Example 3 is an isometry. Because the dimensions of $\mathcal{P}_n$ and $\mathbb{R}^{n+1}$ agree, this isometry is an isomorphism by Theorem 3. (In fact, it was shown in general in the proof of Theorem 5.19 that any coordinatization transformation on a finite dimensional vector space is an isomorphism.)

▶ All n-Dimensional Inner Product Spaces are Isometric

DEFINITION

Two inner product spaces $\mathcal{V}$, $\mathcal{W}$ are **isometric** if and only if there is an isomorphism L: $\mathcal{V} \to \mathcal{W}$ that is also an isometry.

The next theorem generalizes Example 3, showing that every n-dimensional real [complex] inner product space is isometric to $\mathbb{R}^n$ [$\mathbb{C}^n$]. You are asked to prove this in Exercise 11.

THEOREM 4
Let $\mathcal{V}$ be a real [complex] n-dimensional inner product space, and let B be an ordered orthonormal basis for $\mathcal{V}$. Then the isomorphism L: $\mathcal{V} \to \mathbb{R}^n[\mathbb{C}^n]$ given by $L(\mathbf{v}) = [\mathbf{v}]_B$ for all $\mathbf{v} \in \mathcal{V}$ is an isometry. That is, $\mathcal{V}$ and $\mathbb{R}^n$ [$\mathbb{C}^n$] are isometric.

This theorem shows that $\mathbb{R}^n$ [$\mathbb{C}^n$] is the "model" n-dimensional real [complex] inner product space because there is a distance-preserving isomorphism between any real [complex] inner product space and $\mathbb{R}^n$ [$\mathbb{C}^n$]. Thus, any n-dimensional real [complex] inner product space essentially "behaves" like $\mathbb{R}^n$ [$\mathbb{C}^n$] not only with respect to its addition and scalar multiplication, but also as far as its inner product operation is concerned!

The next corollary follows easily from Theorem 4. You are asked to prove it in Exercise 13. (The proof is similar to the proof of Corollary 5.20.)

COROLLARY 5
Let $\mathcal{V}$ and $\mathcal{W}$ be real [complex] n-dimensional inner product spaces. Then there is an isomorphism from $\mathcal{V}$ to $\mathcal{W}$ that is an isometry. That is, $\mathcal{V}$ and $\mathcal{W}$ are isometric.

The fact from part (3) of Theorem 1 that isometries preserve the inner product is especially interesting. We know from Section 5.5 that isomorphic vector spaces are essentially equivalent, with properties in one mirroring the corresponding properties in the other. But an isomorphism that is an isometry also preserves the inner product, so all of the properties depending upon the inner product are preserved as well. Thus, up to isomorphism, all n-dimensional *inner product* spaces essentially behave the same, just as all n-dimensional vector spaces behave the same! We saw this in Example 3, in which the "behavior" of $\mathcal{P}_n$ and $\mathbb{R}^{n+1}$ as real inner product spaces is identical. In the next example, we consider $\mathcal{P}_3$ with, seemingly, a different inner product. However, we know from Theorem 4 that this inner product space is actually isometric to $\mathbb{R}^4$ with the standard dot product.

EXAMPLE 5

Consider $\mathcal{P}_3$ as a subspace of the inner product space of all continuous real-valued function on the interval $[-1, 1]$, where

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} f(t)g(t)\, dt.$$

(See Example 11 in Section 7.5.)[10] In that example, we found the mutually orthogonal polynomials $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = t$, $\mathbf{v}_3 = t^2 - \frac{1}{3}$, and $\mathbf{v}_4 = t^3 - \frac{3}{5}t$. (These are multiples of the first four Legendre polynomials.) In part (a) of Exercise 12, you are asked to verify that normalizing these vectors produces $\mathbf{u}_1 = \frac{1}{\sqrt{2}}$, $\mathbf{u}_2 = \left(\sqrt{\frac{3}{2}}\right)t$, $\mathbf{u}_3 = \sqrt{\frac{45}{8}}\left(t^2 - \frac{1}{3}\right)$, and $\mathbf{u}_4 = \sqrt{\frac{175}{8}}\left(t^3 - \frac{3}{5}t\right)$. Hence, $B = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ is an ordered orthonormal basis for $\mathcal{P}_3$. Thus, by Theorem 4, the isomorphism $L\colon \mathcal{P}_3 \to \mathbb{R}^4$ given by $L(\mathbf{p}) = [\mathbf{p}]_B$ is an isometry, implying that $\mathcal{P}_3$ with this inner product is isometric to $\mathbb{R}^4$. You are asked to verify that this isomorphism is an isometry in a particular case in part (b) of Exercise 12. ■

▸ The Matrix of an Isometry

The next theorem gives an easy way to determine whether a given linear transformation is an isometry.

> THEOREM 6
> Let $L\colon \mathcal{V} \to \mathcal{W}$ be a linear transformation between two n-dimensional real [complex] inner product spaces. Let B and C be ordered orthonormal bases for $\mathcal{V}$ and $\mathcal{W}$, respectively. Then L is an isometry if and only if the matrix $\mathbf{A}_{BC}$ for L is an orthogonal [unitary] matrix.

This theorem shows that multiplication by an orthogonal or unitary matrix essentially leaves the geometry (length, distance) of vectors unchanged. This is why the orthogonal [unitary] diagonalization process of Section 6.3 [Section 7.4] has advantages over ordinary diagonalization.

Proof Let L, $\mathcal{V}$, $\mathcal{W}$, B, C, and $\mathbf{A}_{BC}$ be as given in the statement of the theorem.

Suppose, first, that L is an isometry. We want to show that $\mathbf{A}_{BC}$ is orthogonal [unitary]. Let $B = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$. By assumption, B is orthonormal in $\mathcal{V}$. Then $L(B) = (L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n))$ is an orthonormal set in $\mathcal{W}$, by part (4) of Theorem 1. But then $L(B)$ is a linearly independent set by Theorem 7.15, and therefore is an orthonormal basis for $\mathcal{W}$ (since $\mathcal{W}$ is n-dimensional). By Theorem 4, the mapping from $\mathcal{W}$ to $\mathbb{R}^n$ [$\mathbb{C}^n$] that takes $\mathbf{w}$ to $[\mathbf{w}]_C$ is an isometry. Hence, by part (4) of Theorem 1, $([L(\mathbf{u}_1)]_C, \ldots, [L(\mathbf{u}_n)]_C)$ is an orthonormal basis for $\mathbb{R}^n$ [$\mathbb{C}^n$]. But the ith column of $\mathbf{A}_{BC}$ is equal to $[L(\mathbf{u}_i)]_C$, for $1 \le i \le n$. Thus, by part (2) of Theorem 6.7 [part (2) of Theorem 7.7], $\mathbf{A}_{BC}$ is an orthogonal [unitary] matrix.

Conversely, suppose that the matrix $\mathbf{A}_{BC}$ for L is orthogonal. (An identical argument works for the unitary case using the conjugate transpose everywhere below in place of the transpose.) Now, for all $\mathbf{v} \in \mathcal{V}$, we have $\|L(\mathbf{v})\|_{\mathcal{W}}^2 = \langle L(\mathbf{v}), L(\mathbf{v})\rangle_{\mathcal{W}}$. By Theorem 4, the mapping $\mathbf{w} \to [\mathbf{w}]_C$ is an isometry from $\mathcal{W}$ to $\mathbb{R}^n$, and so $\langle L(\mathbf{v}), L(\mathbf{v})\rangle_{\mathcal{W}} = [L(\mathbf{v})]_C \cdot [L(\mathbf{v})]_C$ (in $\mathbb{R}^n$) $= \mathbf{A}_{BC}[\mathbf{v}]_B \cdot \mathbf{A}_{BC}[\mathbf{v}]_B$. Rewriting this vector dot product as a matrix multiplication, we have

$$\begin{aligned} \|L(\mathbf{v})\|_{\mathcal{W}}^2 &= (\mathbf{A}_{BC}[\mathbf{v}]_B)^T \mathbf{A}_{BC}[\mathbf{v}]_B \\ &= [\mathbf{v}]_B^T \mathbf{A}_{BC}^T \mathbf{A}_{BC}[\mathbf{v}]_B \\ &= [\mathbf{v}]_B^T \mathbf{I}_n [\mathbf{v}]_B \qquad \text{since } \mathbf{A}_{BC} \text{ is orthogonal} \\ &= [\mathbf{v}]_B^T [\mathbf{v}]_B = [\mathbf{v}]_B \cdot [\mathbf{v}]_B = \|[\mathbf{v}]_B\|_{\mathbb{R}^n}^2 . \end{aligned}$$

But the mapping $\mathbf{v} \to [\mathbf{v}]_B$ from $\mathcal{V}$ to $\mathbb{R}^n$ is an isometry, and so, part (2) of Theorem 1 implies that

$$\|[\mathbf{v}]_B\|_{\mathbb{R}^n}^2 = \|\mathbf{v}\|_{\mathcal{V}}^2 .$$

Hence, $\|L(\mathbf{v})\|_{\mathcal{W}} = \|\mathbf{v}\|_{\mathcal{V}}$, which, by Theorem 1, makes L an isometry. **QED**

[10] Even though this is the same *vector space* $\mathcal{P}_n$ as in Example 3, it is a different *inner product space*. The inner products used are distinct, although they are equivalent through isomorphism.

EXAMPLE 6 We can show that the matrix (using the standard basis) for a linear operator on $\mathbb{R}^2$ that is an isometry has one of two forms. Since the columns of the corresponding matrix for the operator are unit vectors (by Theorem 6), the first column has the form $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, for some angle θ (why?). A little thought will convince you that the second column must be either $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ or $\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$ in order for it to be a unit vector orthogonal to the first column. Hence, the two possible types of matrices corresponding to an isometry of $\mathbb{R}^2$ are

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

We have already seen that $\mathbf{A}$ is the matrix for a counterclockwise rotation of vectors in the plane through the angle θ. The matrix $\mathbf{B}$ represents a reflection through the line through the origin determined by the vector $\left[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right]$. (This is left for you to show in Exercise 3.) Notice that $|\mathbf{A}| = 1$, while $|\mathbf{B}| = -1$. Thus, we can characterize all linear operators on $\mathbb{R}^2$ that are isometries: those whose matrix has determinant 1 are rotations, while those that have determinant -1 are reflections. ■

▶ Isometries That Are Not Linear Transformations

We saw in Example 1 that there are isometries between inner product spaces that are not linear transformations. For example, a translation function on $\mathbb{R}^n$ preserves distances between vectors, but is not a linear transformation. Also, for complex inner product spaces, the fact that the conjugation operation preserves distances in $\mathbb{C}$ but is not a linear operator on $\mathbb{C}$ provides the framework for many other examples of isometries that are not linear transformations. However, in the case of real inner product spaces, the final theorem of this section essentially shows that all isometries between real inner product spaces consist of a linear transformation followed by a translation. Exercise 14 provides an outline for the proof of this theorem.

THEOREM 7
Let $\mathcal{V}$ and $\mathcal{W}$ be real inner product spaces and let $f\colon \mathcal{V} \to \mathcal{W}$ be an isometry. Then the function $L\colon \mathcal{V} \to \mathcal{W}$ given by $L(\mathbf{v}) = f(\mathbf{v}) - f(\mathbf{0}_{\mathcal{V}})$ for all $\mathbf{v} \in \mathcal{V}$ is a linear transformation. Furthermore, L is an isometry.

EXAMPLE 7 The function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} 3x - 4y + 8 \\ 4x + 3y - 5 \end{bmatrix}$$

is an isometry because

$$\begin{aligned}
\left\| f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) - f\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)\right\| &= \left\| \tfrac{1}{5}\begin{bmatrix} 3a-4b+8 \\ 4a+3b-5 \end{bmatrix} - \tfrac{1}{5}\begin{bmatrix} 3c-4d+8 \\ 4c+3d-5 \end{bmatrix}\right\| \\
&= \tfrac{1}{5}\left\| \begin{matrix} 3(a-c)-4(b-d) \\ 4(a-c)+3(b-d) \end{matrix} \right\| \\
&= \tfrac{1}{5}\sqrt{(3(a-c)-4(b-d))^2 + (4(a-c)+3(b-d))^2} \\
&= \tfrac{1}{5}\sqrt{25(a-c)^2 + 25(b-d)^2} \\
&= \left\| \begin{bmatrix} a-c \\ b-d \end{bmatrix} \right\| = \left\| \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} \right\|.
\end{aligned}$$

Now $f(\mathbf{0}) = \frac{1}{5}\begin{bmatrix} 8 \\ -5 \end{bmatrix}$, and so if we define $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\mathbf{v}) = f(\mathbf{v}) - f(\mathbf{0}_{\mathcal{V}})$, then

$$\begin{aligned}
L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \tfrac{1}{5}\begin{bmatrix} 3x-4y+8 \\ 4x+3y-5 \end{bmatrix} - \tfrac{1}{5}\begin{bmatrix} 8 \\ -5 \end{bmatrix} \\
&= \tfrac{1}{5}\begin{bmatrix} 3x-4y \\ 4x+3y \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},
\end{aligned}$$

a linear transformation. Because the matrix for L is orthogonal, we see that L is indeed an isometry by Theorem 6. ■

▸ New Vocabulary

isometric inner product spaces
isometry between inner product spaces
rigid motion of the plane
rigid motion of 3-space

▸ Highlights

- A linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is an isometry between inner product spaces $\mathcal{V}$ and $\mathcal{W}$ if and only if $\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}} = \|L(\mathbf{v}_1) - L(\mathbf{v}_2)\|_{\mathcal{W}}$ for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$.
- A linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is an isometry between inner product spaces $\mathcal{V}$ and $\mathcal{W}$
  $\iff \|L(\mathbf{v})\|_{\mathcal{W}} = \|\mathbf{v}\|_{\mathcal{V}}$ for all $\mathbf{v} \in \mathcal{V}$
  $\iff \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathcal{V}} = \langle L(\mathbf{v}_1), L(\mathbf{v}_2) \rangle_{\mathcal{W}}$ for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$
  $\iff$ the image under L of every orthonormal set in $\mathcal{V}$ is an orthonormal set in $\mathcal{W}$.
- If a linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is an isometry between *real* inner product spaces, then the angle between two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathcal{V}$ equals the angle between $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ in $\mathcal{W}$.
- If a linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is an isometry between inner product spaces, then L is one-to-one.
- If a linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ is an isometry between two n-dimensional inner product spaces, then L is an isomorphism.
- Inner product spaces $\mathcal{V}$ and $\mathcal{W}$ are isometric if and only if there is an isomorphism $L\colon \mathcal{V} \to \mathcal{W}$ that is also an isometry.

- If $\mathcal{V}$ is an n-dimensional real [complex] inner product space, and B is an ordered orthonormal basis for $\mathcal{V}$, then the isomorphism $\mathbf{v} \to [\mathbf{v}]_B$ from $\mathcal{V}$ to $\mathbb{R}^n$ [$\mathbb{C}^n$] is an isometry.
- If $\mathcal{V}$ and $\mathcal{W}$ are two n-dimensional real [complex] inner product spaces, then $\mathcal{V}$ and $\mathcal{W}$ are isometric.
- Isometric inner product spaces are not only equivalent as isomorphic vector spaces, but also possess equivalent inner products.
- If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation between two n-dimensional real [complex] inner product spaces having orthonormal bases B and C, respectively, then L is an isometry if and only if the matrix $\mathbf{A}_{BC}$ for L is orthogonal [unitary].
- Every linear operator on $\mathbb{R}^2$ that is an isometry is either a rotation about the origin or a reflection about a line through the origin.
- Every isometry $f\colon \mathcal{V} \to \mathcal{V}$ on a *real* inner product space represents a linear transformation L on $\mathcal{V}$ followed by a translation of the vectors in $\mathcal{V}$. Furthermore, the linear transformation L is an isometry.

▶ EXERCISES

1. Which of the following linear operators are isometries?

★ **a)** $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{bmatrix}$.

★ **b)** $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2y + 13z \\ 3x - 2y - 4z \\ 2x + 3y + 6z \end{bmatrix}$.

c) $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{17}\begin{bmatrix} x + 12y + 12z \\ 12x + 8y - 9z \\ 12x - 9y + 8z \end{bmatrix}$.

d) $L\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{7}\begin{bmatrix} 3x + 6y \\ -2x + 3y \\ -6x + 2y \end{bmatrix}$.

★ **e)** $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{15}\begin{bmatrix} 2x + 14y + 5z \\ 11x + 2y - 10z \end{bmatrix}$.

2. Let $\mathbf{A} = \begin{bmatrix} \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{i}{\sqrt{6}} & \frac{i}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

a) Show that $\mathbf{A}$ is a unitary matrix.

b) By Theorem 6, the mapping $L\colon \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$L\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \mathbf{A}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

is an isometry. Verify directly that L preserves the norm of the vector $[1-i,\ -2,\ 3i]$ by showing that

$$\left\| L\left(\begin{bmatrix} 1-i \\ -2 \\ 3i \end{bmatrix}\right) \right\| = \left\| \begin{bmatrix} 1-i \\ -2 \\ 3i \end{bmatrix} \right\|.$$

★ **3.** Verify the claim made in Example 5 that multiplying $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ times $\begin{bmatrix} x \\ y \end{bmatrix}$ gives the reflection of $\begin{bmatrix} x \\ y \end{bmatrix}$ about the line through the origin making an angle of $\frac{\theta}{2}$ with the positive x-axis. (Hint: Use Exercise 21 in Section 1.2.)

4. Each of the following matrices is orthogonal, and so its associated linear operator is an isometry of $\mathbb{R}^2$. In each case, determine whether the isometry represents a rotation (about the origin through an angle θ) or a reflection (about a line through the origin). Then give either the angle θ of rotation, or a vector in the direction of the line of reflection. (See Example 6.)

★ **a)** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

d) $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

★ **e)** $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

5. If $L_1\colon \mathcal{V} \to \mathcal{V}$ and $L_2\colon \mathcal{V} \to \mathcal{V}$ are both isometries, prove that $L_2 \circ L_1$ is also an isometry.

6. If $\mathcal{V}$ is a finite dimensional inner product space and the linear operator $L\colon \mathcal{V} \to \mathcal{V}$ is an isometry, show that L^{-1} exists and that L^{-1} is an isometry.

▶ **7.** Prove Corollary 2.

8. This exercise shows that the converse of Corollary 2 fails to hold.

a) Show that the linear operator $L\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{v}) = c\mathbf{v}$, where $|c| \neq 1$ is not an isometry.

b) Show that the operator L in part (a) preserves angles; that is, the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ equals the angle between $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$, for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$.

9. Show that if a linear operator L on a real inner product space $\mathcal{V}$ is angle preserving and fixes at least one nonzero vector, then L is an isometry. (By "L fixes at least one nonzero vector" we mean that there is a nonzero vector $\mathbf{v}_0$ such that $L(\mathbf{v}_0) = \mathbf{v}_0$.) (Hint: First prove that if $\mathbf{v}_1$ is perpendicular to $\mathbf{v}_0$, then $\|L(\mathbf{v}_1) + \mathbf{v}_0\|_{\mathcal{V}} = \|\mathbf{v}_1 + \mathbf{v}_0\|_{\mathcal{V}}$ by using the fact that the angle between $\mathbf{v}_0$ and $(\mathbf{v}_1 + \mathbf{v}_0)$ is preserved. Square both sides of this expression, then expand using the inner product to show that $\|L(\mathbf{v}_1)\|^2 = \|\mathbf{v}_1\|^2$. Finally, for any vector $\mathbf{v} \in \mathcal{V}$, use the Projection Theorem to decompose $\mathbf{v}$ into the sum of a vector parallel to $\mathbf{v}_0$ and a vector orthogonal to $\mathbf{v}_0$. Use this decomposition to compute $\|L(\mathbf{v})\|^2$, and show that it equals $\|\mathbf{v}\|^2$.)

10. Let $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $(\mathbf{y}_1, \ldots, \mathbf{y}_n)$ be ordered orthonormal bases for $\mathbb{R}^n$. Show that the linear operator L on $\mathbb{R}^n$ determined by $L(\mathbf{x}_i) = \mathbf{y}_i$, for $1 \le i \le n$, is an isometry.

▶ **11.** Prove Theorem 4.

12. This exercise asks you to verify specific claims made in Example 5.

a) Prove that $B = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ given in Example 5 is an ordered orthonormal basis for $\mathcal{P}_3$.

b) For the isomorphism $L\colon \mathcal{P}_3 \to \mathbb{R}^4$ in Example 5, verify that

$$\|\mathbf{p}\|_{\mathcal{P}_3} = \|L(\mathbf{p})\|_{\mathbb{R}^4} = \sqrt{\tfrac{34}{15}}$$

for the vector $\mathbf{p} = 2t^2 + t$. (This verifies that L is an isometry in one particular case.)

▶ **13.** Prove Corollary 5. (Hint: Use Theorem 4 to create appropriate isomorphisms $L\colon \mathcal{V} \to \mathbb{R}^n\,[\mathbb{C}^n]$ and $M\colon \mathcal{W} \to \mathbb{R}^n\,[\mathbb{C}^n]$. Show that M^{-1} exists and is an isometry. Consider $M^{-1} \circ L$.)

▶ **14.** The purpose of this exercise is to prove Theorem 7. Suppose $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $f\colon \mathcal{V} \to \mathcal{W}$ is an isometry; that is, $\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}} = \|f(\mathbf{v}_1) - f(\mathbf{v}_2)\|_{\mathcal{W}}$ for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$. Let $L\colon \mathcal{V} \to \mathcal{W}$ be given by $L(\mathbf{v}) = f(\mathbf{v}) - f(\mathbf{0}_{\mathcal{V}})$ for all $\mathbf{v} \in \mathcal{V}$. Our goal is to prove that L is both a linear transformation and an isometry.

a) Show that $L(\mathbf{0}_{\mathcal{V}}) = \mathbf{0}_{\mathcal{W}}$.

b) Show that L is an isometry; that is, prove that, for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$, $\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}} = \|L(\mathbf{v}_1) - L(\mathbf{v}_2)\|_{\mathcal{W}}$.

c) Prove that L preserves lengths; that is, prove that $\|\mathbf{v}\|_{\mathcal{V}} = \|L(\mathbf{v})\|_{\mathcal{W}}$ for every $\mathbf{v} \in \mathcal{V}$. (Hint: Use parts (a) and (b). Note that Theorem 1 cannot be used here since we do not yet know that L is a linear transformation.)

d) Show that $\|L(\mathbf{v}_1 - \mathbf{v}_2)\|_{\mathcal{W}} = \|L(\mathbf{v}_1) - L(\mathbf{v}_2)\|_{\mathcal{W}}$ for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$. (Hint: Use parts (b) and (c).)

e) Prove that $\|L(c\mathbf{v})\|_{\mathcal{W}} = |c|\,\|L(\mathbf{v})\|_{\mathcal{W}}$ for every $\mathbf{v} \in \mathcal{V}$ and every $c \in \mathbb{R}$. (Hint: Use part (c).)

f) Show that both $\|L(\mathbf{v}) - L(-\mathbf{v})\|_{\mathcal{W}}$ and $(\|L(\mathbf{v})\|_{\mathcal{W}} + \|L(-\mathbf{v})\|_{\mathcal{W}})$ equal $2\,\|L(\mathbf{v})\|_{\mathcal{W}}$ for all $\mathbf{v} \in \mathcal{V}$. (Hint: Use parts (d) and (e).)

g) Use part (f) to show that the angle between $(-L(\mathbf{v}))$ and $L(-\mathbf{v})$ is zero, and, hence, these two vectors are in the same direction. (Hint: Start with $\|L(\mathbf{v}) - L(-\mathbf{v})\|_{\mathcal{W}} = (\|L(\mathbf{v})\|_{\mathcal{W}} + \|L(-\mathbf{v})\|_{\mathcal{W}})$. Square both sides of the equation, then simplify, using the inner product in $\mathcal{W}$ to further expand the left side. The idea here is analogous to Exercise 4 in Section 1.3; that is, that the Triangle Inequality is an equality if and only if the two vectors are in the same direction.)

h) Prove that $L(-\mathbf{v}) = -L(\mathbf{v})$ for every $\mathbf{v} \in \mathcal{V}$. (Hint: Use part (e) to show that the two vectors have the same length, and part (g) to show that they are in the same direction.)

i) Prove that $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathcal{V}} = \langle L(\mathbf{v}_1), L(\mathbf{v}_2) \rangle_{\mathcal{W}}$, for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$. (Hint: Follow the proof of (1) $\Longrightarrow$ (3) in the proof of Theorem 1, using the Polarization Identity. In the step that requires L to be a linear transformation, substitute the result in part (h) as the reason that the step is valid.)

j) Prove that $L(c\mathbf{v}) = cL(\mathbf{v})$ for every $\mathbf{v} \in \mathcal{V}$ and every $c \in \mathbb{R}$. (Hint: Show that $\|L(c\mathbf{v}) - cL(\mathbf{v})\|_{\mathcal{W}} = 0$, implying the desired result. Do this by computing $\|L(c\mathbf{v}) - cL(\mathbf{v})\|_{\mathcal{W}}^2$ by expanding this expression using the inner product in $\mathcal{W}$. Then, after some simplification, use part (i) to convert inner products in $\mathcal{W}$ to inner products in $\mathcal{V}$.)

k) Prove that $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2)$, for every $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$. Note that this result together with part (j) concludes the proof that L is a linear transformation. (Hint: Show that $\|L(\mathbf{v}_1 + \mathbf{v}_2) - (L(\mathbf{v}_1) + L(\mathbf{v}_2))\|_{\mathcal{W}} = 0$, using a strategy analogous to that used in part (j).)

15. Prove that if $f\colon \mathcal{V} \to \mathcal{W}$ is an isometry between two inner product spaces, then f is one-to-one. (Do not assume that f is a linear transformation.)

★ **16.** True or False:

a) If $\mathcal{V}$ and $\mathcal{W}$ are isometric finite dimensional inner product spaces, then $\dim(\mathcal{V}) = \dim(\mathcal{W})$.

b) If $L\colon \mathcal{V} \to \mathcal{W}$ is a linear transformation between two finite dimensional real inner product spaces that preserves lengths of vectors, and if B and C are ordered orthonormal bases for $\mathcal{V}$ and $\mathcal{W}$, respectively, then the matrix for L with respect to B and C is an orthogonal matrix.

c) If a linear operator L on $\mathbb{C}^n$ preserves norms of vectors, then it is an isometry.

d) If a linear operator L on $\mathbb{R}^n$ is an isometry, then it is orthogonally diagonalizable.

e) Every linear operator L on $\mathbb{R}^2$ that is an isometry is either a rotation about the origin or a reflection about a line through the origin.

f) If $\mathcal{V}$ is an n-dimensional real inner product space and B is an ordered basis for $\mathcal{V}$, then the linear transformation $\mathcal{V} \to \mathbb{R}^n$ given by $\mathbf{v} \to [\mathbf{v}]_B$ for every $\mathbf{v} \in \mathcal{V}$ is an isometry.

g) If $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional inner product spaces, and a linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ exists that is also an isometry, then $\dim(\mathcal{V}) \leq \dim(\mathcal{W})$.

▶ Answers to Selected Exercises

(1) Part (a) is an isometry; parts (b) and (e) are not isometries.

(3) Use the fact that the vector $\mathbf{r} = [\cos\frac{\theta}{2}, \sin\frac{\theta}{2}]$ makes an angle of $\frac{\theta}{2}$ with the positive x-axis. Let $\mathbf{x} = [x, y]$. We compute $2(\mathbf{proj}_\mathbf{r}\mathbf{x}) - \mathbf{x}$ for the desired reflection of $\mathbf{x}$, as suggested in Exercise 21 in Section 1.2:

$$\begin{aligned} 2(\mathbf{proj}_\mathbf{r}\mathbf{x}) - \mathbf{x} &= 2\left(\left(\frac{\mathbf{x}\cdot\mathbf{r}}{\|\mathbf{r}\|^2}\right)\mathbf{r}\right) - \mathbf{x} \\ &= 2\left(\frac{x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2}}{1}\right)\left[\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\right] - [x, y] \\ &= \left[2x\cos^2\left(\frac{\theta}{2}\right) + 2y\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) - x,\right. \\ &\qquad \left. 2x\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + 2y\sin^2\left(\frac{\theta}{2}\right) - y\right] \end{aligned}$$

Now use the trigonometric identities

$$2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = \sin\theta, \quad 2\cos^2\left(\frac{\theta}{2}\right) - 1 = \cos\theta, \text{ and } \quad 2\sin^2\left(\frac{\theta}{2}\right) - 1 = -\cos\theta$$

to simplify the above to

$$[x\cos\theta + y\sin\theta,\ x\sin\theta - y\cos\theta],$$

which is the same result obtained from the matrix product

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

(4) (a) Counterclockwise rotation about the origin through an angle of 45°.

(e) Reflection about the line through the origin in the direction of $\left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$.

(16) (a) T (b) F (c) T (d) F (e) T (f) F (g) T

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